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ZBIGNIEW ZARZYCKI[1]

Friction factor for accelerated turbulent flow through long smooth pipes

The paper presents a mathematical model of unsteady turbulent pipe flow developed using the four-layer distribution of the coefficient of turbulent viscosity. This model is used for determining the hydraulic losses. A detailed discussion deals with the instantaneous friction factor for accelerated liquid flow. This coefficient increases with increasing acceleration of liquid flow and decreases with increasing Reynolds number. It is demonstrated that the instantaneous friction factor considerably differs from the quasi-steady value.

1. Introduction

In calculating the transients that occur during unsteady turbulent pipe flow it is very frequently assumed that the hydraulic losses are of quasi-steady nature (e. g. [11]), what is justifiable when real distribution of the velocity field over the pipe cross-section slightly differs from that of a quasi-steady state. The works, which deal with hydraulic losses in accelerated turbulent flow – contrary to those devoted to the pulsating flow (e. g. [3, 8÷9, 12÷13]) – are rare [1, 2, 5] and mainly deal with experimental studies. The results of these studies, regardless of differences existing among them, explicitly show that, during accelerating of the flow, the instantaneous friction factor substantially differs from its quasi-steady value.

The purpose of this work is to analytically determine the instantaneous friction factor, and to investigate its course as a function of acceleration and Reynolds number. Presented is mathematical model of unsteady turbulent flow of liquid through smooth pipes. This model is based upon the Reynolds equation and a supplementary equation which describes the distribution of the coefficient of turbulent viscosity over the pipe cross-section for the four-layer model of the flow region [12, 15] . This model was adapted for determination of the instantaneous friction factor.

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2. Mathematical model of unsteady turbulent flow of liquid in long smooth pipes

Unsteady, axisymmetrical turbulent flow of a Newtonian liquid in a long pipe with a constant internal radius and rigid walls is considered. Moreover, the following assumptions are taken:

- constant distribution of pressure in the pipe cross-section,
- body forces and thermal effects are negligible,
- mean velocity in a pipe cross-section is considerably smaller than the sound velocity in the liquid.

The following approximate equation, shown in the previous paper [12], which omits small terms in the fundamental equation for unsteady flow, is used:

$$\frac{\partial \overline{v}_z}{\partial t} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \overline{v}_z}{\partial r}\right) + \frac{1}{r}\frac{\partial}{\partial r}\left(r\nu_t\frac{\partial \overline{v}_z}{\partial r}\right), \tag{1}$$

where: $\overline{v}_z, \overline{p}$ – averaged in time, respectively: velocity component in the axial direction and pressure (the overscore denotes the short-time averaged value), ρ_0 – density of the liquid (constant), ν – kinematic coefficient of viscosity, v_t – kinematic coefficient of turbulent viscosity, t – time, z – distance along the pipe axis, r – radial distance from the pipe axis.

It is assumed that changes of the coefficient ν_t resulting from the flow change in time are negligible, which means that ν_t is only a function of the radial distance r.

2.1. Model of turbulent viscosity

In order to describe the distribution of the coefficient ν_t over the pipe cross-section, the flow region is to be divided into several layers and, for each of them, the function $\nu_t(r)$ is to be determined. By analogy to the steady pipe flow the following regions of flow can be distinguished [10]: a viscous sublayer (VS), a buffer layer (BL), a developed turbulent flow layer (DTL) and a turbulent core (TC). This is shown in Fig. 1 .

The radial distances r_1, r_2 and r_3 from the pipe centre line are as follows:

$$r_j = R - y(y_j^+) \qquad j = 1, 2, 3 \tag{2}$$

$$y_1^+ = 0,2R^+, \qquad y_2^+ = 35 \qquad y_3^+ = 5, \tag{3}$$

where:

$y^+ = yv^*/\nu$	–	dimensionless distance y,
y	–	distance from the pipe wall,
$v^* = \sqrt{\tau_{ws}/\rho_0}$	–	dynamic velocity,
τ_{ws}	–	wall shear stress for a steady flow,
$R^+ = Rv^*/\nu$	–	dimensionless radius R of the pipe.

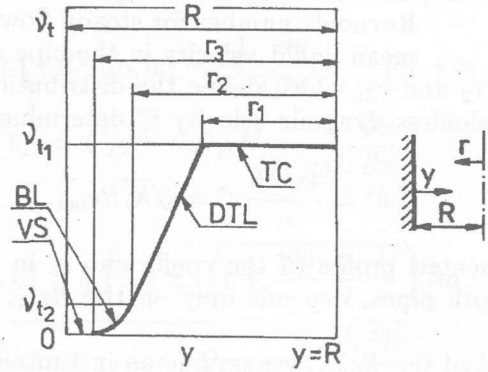

Fig. 1. Schematic representation of four-region model of ν_t

Table 1. Quantities α_i and β_i

Region	α_i	β_i
$VS,\ i = 1$ $r_3 \leq r \leq R$	0	0
$DTL,\ i = 3$ $r_1 \leq r < r_2$	$-\left(\frac{\nu_{t1}-\nu_{t2}}{r_2 - r_1}\right)$	$\nu_{t2} + \left(\frac{\nu_{t1}-\nu_{t2}}{r_2 - r_1}\right) r_2$
$TC,\ i = 4$ $0 \leq r < r_1$	0	ν_{t1}

The coefficient of turbulent viscosity ν_t , is expressed, for particular layers (except the buffer layer) as follows:

$$\nu_t = \alpha_i r + \beta_i, \qquad i = 1, 3, 4. \tag{4}$$

The quantities α_i and β_i are given in Table 1 below.

For the buffer layer $(r_2 \leq r < r_3)$ ν_t is expressed as follows:

$$\nu_t = 0.01(y^+)^2 .\nu \tag{5}$$

Expressions, which define the quantities $\nu_{t1}, \nu_{t2}, r_1, r_2$ and r_3 have the following forms :

$$\nu_{t1} = 0.016\sqrt{\frac{\lambda_s}{2}} = Re_{ms}\nu, \qquad \nu_{t2} = 12.25\nu, \tag{6}$$

$$r_1 = 0.8, \qquad r_2 = \left(1 - \frac{140}{Re_{ms}}\sqrt{\frac{2}{\lambda_s}}\right) R, \qquad r_3 = \left(1 - \frac{20}{Re_{ms}}\sqrt{\frac{2}{\lambda_s}}\right) R, \tag{7}$$

where:

λ_s	–	friction factor for steady flow,
$Re_{ms} = 2Rv_{ms}/\nu$	–	Reynolds number for steady flow,
v_{ms}	–	mean liquid velocity in the pipe cross-section.

The quantities ν_{t1}, r_2 and r_3, which define the distribution of the coefficient ν_t, depend on a dimensionless dynamic velocity $\hat{v}^*$ determined in the following way:

$$\hat{v}^* = \frac{4\sqrt{2}R}{\nu} v^* = \sqrt{\lambda_s} Re_{ms}. \tag{8}$$

Therefore, the presented profile of the coefficient ν_t in the pipe cross-section, in the case of smooth pipes, depends only on the Re_{ms} number since we have $\lambda_s = \lambda_s(Re_{ms})$.

However, if, instead of the Re_{ms}, we assume an instantaneous Reynolds number $Re_m = 2Rv_m/\nu$ (v_m – instantaneous, average in the pipe cross-section, liquid velocity) and, instead of λ_s, assume a quasi-steady friction factor $\lambda_q = \lambda_q(Re_m)$, then the presented distribution of the coefficient ν_t will show itself as the quasi-steady one.

2.2. Momentum equations for particular layers

Knowing the distribution of the coefficient of turbulent viscosity in the pipe cross-section one can break up the equation (1) into four equations that describe the liquid flow in particular layers of the flow region. Introducing the differential operator $D = \partial/\partial t$ into the equation (1) we obtain:

– for the viscous sublayer:

$$D\overline{v}_{z1} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu\left(\frac{\partial^2 \overline{v}_{z1}}{\partial r^2} + \frac{1}{r}\frac{\partial \overline{v}_{z1}}{\partial r}\right), \tag{9}$$

– for the remaining layers of the flow region:

$$D\overline{v}_{zi} = -\frac{1}{\rho_0}\frac{\partial \overline{p}}{\partial z} + \nu_\Sigma \frac{\partial^2 \overline{v}_{zi}}{\partial r^2} + \left(\frac{\partial \nu_\Sigma}{\partial r} + \frac{\nu_\Sigma}{r}\right)\frac{\partial \overline{v}_{zi}}{\partial r}, \tag{10}$$

where:

$i = 2, 3, 4$		($i = 2$ for the BL, $i = 3$ for the DTL and $i = 4$ for the TC),
$\overline{v}_{zi}$	–	axial component of the velocity for particular layers of the region,
$\nu_\Sigma = \nu + \nu_t$	–	effective coefficient of turbulent viscosity.

The equations (9) and (10) can be resolved, for $i = 3, 4$, and using the relationship (4), into the modified Bessel equations [7] whereas the equation (10), for $i = 2$ and taking into account the following relationship :

$$\frac{r}{R - r} \gg \frac{1}{2} \qquad \text{for } r_2 \le r < r_3$$

can assume the form of a differential equation of the Euler type.

Then, the solutions of the momentum equations for particular layers are as follows:

$$\overline{v}_{zi} = C_i I_0(\eta_i) + D_i K_0(\eta_i) - \frac{1}{\rho_0 D}\frac{\partial \overline{p}}{\partial z} \qquad \text{for } i = 1, 3, 4, \tag{11}$$

$$\overline{v}_{z2} = C_2 e^{n_2'} + D_2 e^{e_2''} - \frac{1}{\rho_0 D}\frac{\partial p}{\partial z}, \tag{12}$$

where:

$$\left.\begin{aligned} &\eta_1 = \sqrt{\frac{r^2 D}{\nu}}; \quad \eta_2', \eta_2'' = \tfrac{1}{2}\left(-1 \pm \sqrt{1 + \frac{400 \cdot \nu \cdot D}{v^{*2}}}\right) ln(R - r), \\ &\eta_3 = 2\sqrt{\frac{\Delta r}{\Delta \nu_t}\left(\frac{\nu_{t2}}{\Delta \nu_t}\Delta r + r_2 - r\right) D}; \quad \eta_4 = \sqrt{\frac{r^2 D}{\nu_{t1}}}, \end{aligned}\right\} \tag{13}$$

$$\Delta r = r_2 - r_1; \quad \Delta \nu_t = \nu_{t1} - \nu_{t2}. \tag{14}$$

I_0, K_0 – modified Bessel functions of the first and second kind of zero order. Integration constants $C_1, ..., C_4$ and $D_1, ..., D_4$, which are present in the equations (11) and (12), were determined from the boundary conditions resulting from the continuity of velocity and shear stress at the point of contact between layers. These conditions can be written in the following way:

$$\left.\begin{aligned} &\overline{v}_{zi}(r = r_i) = \overline{v}_{z_{i+1}}(r = r_i), \\ &\frac{\partial \overline{v}_{zi}}{\partial r}(r = r_i) = \frac{\partial \overline{v}_{z_{i+1}}}{\partial r}(r = r_i) \quad \text{for } i = 1, 2, 3 \end{aligned}\right\} \tag{15}$$

and $\overline{v}_{z1}(r = R) = 0$.

The eighth boundary condition is a finite value of the liquid velocity at the pipe centre line; hence, one can conclude that $D_4 = 0$ since the function $K_0(0)$ assumes an infinitely great value.

For a steady flow, omitting the left sides of the equations (9) and (10), we obtain the following solutions for particular layers:

$$\left.\begin{aligned} &\overline{v}_{z1} = \frac{1}{4\rho_0 \nu}\frac{d\overline{p}}{dz} r^2 + C_1 ln\left(\frac{r}{R}\right) + D_1, \\ &\overline{v}_{z2} = \frac{1}{A}\frac{d\overline{p}}{dz} ln(R - r) + \frac{C_2}{R - r} + D_2, \\ &\overline{v}_{z3} = \frac{r_2 - r_1}{\nu_{t1}\rho_0}\frac{d\overline{p}}{dz} n_2 + C_3 ln\left(\frac{n_2}{R}\right) + D_3, \\ &\overline{v}_{z4} = \frac{1}{4\rho_0 \nu_{t1}}\frac{d\overline{p}}{dz} r^2 + C_4 ln\left(\frac{r}{R}\right) + D_4, \end{aligned}\right\} \tag{16}$$

where

$$\left.\begin{aligned} &A = 0.01\frac{v^{*2}}{\nu}\rho_0 = \frac{0.01}{32}\frac{\rho_0 \nu}{R^2}\lambda_s Re_{ms}^2, \\ &n_2 = \frac{\nu}{\nu_{t1}}(r_2 - r_1) + r_2 - r. \end{aligned}\right\} \tag{17}$$

The integration constants, which are present in the above solutions, were determined from the same boundary conditions as for the equations (11) and (12).

3. Hydraulic resistance. Instantaneous friction factor

Assuming that $\partial\overline{p}/\partial r = 0$ and integrating the equation (1) over the pipe cross-section (from $r = 0$ to $r = R$), and considering that $\partial/\partial t = D$, we obtain:

$$\rho_0 D v_m + \frac{\partial p}{\partial z} + \frac{2}{R}\tau_w = 0. \tag{18}$$

By determining the wall shear stress at the pipe wall τ_w, as follows:

$$\tau_w = -\rho_o \nu \frac{\partial \overline{v}_{z1}}{\partial r}\mid_{r=R} \tag{19}$$

we obtain a relationship of the following type:

$$\tau_w = f(D, Re_m)\frac{\partial p}{\partial z}. \tag{20}$$

The equations (18) and (20) enable us to determine transfer functions that describe the hydraulic resistance. Introducing the following dimensionless quantities:

$$\hat{v}_m = (R/\nu)v_m, \quad \hat{\tau}_w = (R^2/\rho_0\nu^2)\tau_w, \quad \hat{p} = (R^2/\rho_0\nu^2)p \tag{21}$$

we obtain the following transfer functions:
– a transfer function relating the pipe wall shear stress to the mean velocity:

$$\hat{G}_{\tau v}(\hat{D}, Re_m) = \frac{\hat{\tau}_w}{\hat{v}_m}, \tag{22}$$

– a transfer function relating the pressure gradient to the mean velocity:

$$\hat{G}_{pv}(\hat{G}, Re_m) = \frac{\partial\hat{p}/\partial z}{\hat{v}_m}, \tag{23}$$

where the quantity:

$$\hat{D} = (R^2/\nu)\frac{\partial}{\partial t} \tag{24}$$

is the dimensionless differential operator.
Between the functions $\hat{G}_{\tau v}$, and $\hat{G}_{pv}$ there is the following relationship:

$$\hat{G}_{pv} = -(\hat{D} + 2\hat{G}_{\tau v}). \tag{25}$$

The function $\hat{G}_{\tau v}$ is of a complex form and, for that reason, is given in Appendix A.

The function $\hat{G}_{\tau v}$ enables us to determine the instantaneous friction factor λ_n. This was determined from the following relationship:

$$\tau_w = \frac{1}{8}\rho_0 \lambda_n v_m^2. \tag{26}$$

Making use of the relationship (22) we obtain:

$$\lambda_n = \frac{16\hat{G}_{\tau v}(\hat{D}, Re_m)}{Re_m}. \tag{27}$$

Using the differential operator definition, i. e. $Dv_m = \partial v_m/\partial t$ and taking into account the equation (24) we have:

$$\hat{D} = K_n = \left(\frac{R^2}{\nu v_m}\right)\frac{\partial v_m}{\partial t}. \tag{28}$$

The above quantity is presented in the literature [9, 14] as a flow unsteadiness parameter K_n.
This parameter expresses the ratio of inertia forces to viscous forces. Thus, λ_n is determined as a function of the parameter K_n and Reynolds number related to the instantaneous mean velocity v_m:

$$\lambda_n = \frac{16\hat{G}_{\tau v}(K_n, Re_m)}{Re_m}. \tag{29}$$

For accelerated flow the parameter K_n (equation (28)) can be presented in the following form:

$$K_n = \hat{a}/Re_m, \tag{30}$$

where $\hat{a}$ – dimensionless, average over the pipe cross-section, acceleration of the liquid:

$$\hat{a} = (2R^3/\nu^2)a, \quad a = \partial v_m/\partial t. \tag{31}$$

Therefore, the relationship (29) determines λ_n as a function of acceleration $\hat{a}$ and Reynolds number Re_m.

4. The examination of steady flow friction factor

The above presented method of calculating the friction factor will be verified to the steady flow case. In this case, the function $\hat{G}_{pv}$, taken with negative sign, represents a constant resistance R_{0s} which depends only on the Reynolds number for steady flow – Re_{ms}. This can be written as follows:

$$\hat{R}_{0s} = \hat{0}s(Re_{ms}) = -\frac{\partial\hat{p}/\partial z}{\hat{v}_m}, \tag{32}$$

where $\hat{R}_{0s}$ represents a dimensionless resistance:

$$\hat{R}_{0s} = R_{0s}\frac{\pi R^4}{\rho_{0s}\nu}. \tag{33}$$

The constant resistance was determined using the following relationship:

$$v_m = \frac{2}{R^2}\left(\int_0^{r_1} \overline{v}_{z4} r\, dr + \int_{r_1}^{r_2} \overline{v}_{z3} r\, dr + \int_{r_2}^{r_3} \overline{v}_{z2} r\, dr + \int_{r_3}^{R} \overline{v}_{z1} r\, dr)\right). \tag{34}$$

Quatities $\overline{v}_{z1}, \overline{v}_{z2}, \overline{v}_{z3}$ and $\overline{v}_{z4}$ are described by the relationship (16).
On the basis of Eq. (34) we obtain a relationship of the following type:

$$v_m = f(Re_{ms})\frac{\partial p}{\partial t}. \tag{35}$$

The expression for the resistance R_{0s}, determined in the above way, assumes a complex mathematical form and, therefore, has been approximated with a simple expression of the following form:

$$\hat{R}_{0s} = a\ Re_{mb}^{b} \tag{36}$$

where

$$\left.\begin{array}{lll} \text{for } Re_{ms} \in (10^4 \div 10^5) & a = 0.02628, & b = 0.7830, \\ \text{for } Re_{ms} \in (10^5 \div 10^6) & a = 0.01440, & b = 0.8330, \\ \text{for } Re_{ms} \in (10^6 \div 10^7) & a = 0.00987, & b = 0.8604. \end{array}\right\} \tag{37}$$

As it results from the equation (25) there is the following relationship for the steady flow:

$$\hat{G}_{\tau v} = -0.5\hat{G}_{pv} = 0.5\hat{R}_{0s}.$$

Hence, based on the equation (27), one can obtain the steady flow friction factor relationship:

$$\lambda_s = \frac{8\hat{R}_{0s}}{Re_{ms}}. \tag{38}$$

The results of the calculations done for λ_s, according to Eqs. (38) and (36) are compiled in Table 2.
Also given in Table 2 are values of λ_s calculated according to the Prandtl formula [10]:

$$\frac{1}{\sqrt{\lambda_s}} = 0.869 ln(Re_{ms}\sqrt{\lambda_s}) - 0.8. \tag{39}$$

The differences between the values of λ_s calculated according to Eqs. (38) and (36), and those calculated from Eq. (39) are small. They partially result from the approximation of that expression which describes the constant resistance. This testifies to the correctness of the method assumed for determination of the friction factor.

Table 2. Friction factor values for the steady flow

Re_{ms}	λ_s		error
	Eq. (38), (36)	Eq. (39)	%
10^4	0.028480	0.030860	7.7
$2 \cdot 10^4$	0.024512	0.025869	5.2
$5 \cdot 10^4$	0.020096	0.020875	3.7
10^5	0.017328	0.017976	3.6
$5 \cdot 10^5$	0.012874	0.013147	2.0
10^6	0.011467	0.011635	1.4
$5 \cdot 10^6$	0.009167	0.008974	2.1
10^7	0.008321	0.008096	2.7

5. Results of friction factor calculations for accelerated flow

The course of the instantaneous friction factor for accelerated turbulent flow s presented in Fig. 2 as a function of the Reynolds number Re_m, for various limensionless accelerations $\hat{a}$.

The full lines show the courses obtained for the presented in this work four-layer nodel of the flow region while the broken lines show those obtained from the fol- owing relationship:

$$\lambda_n = 16\sqrt{\hat{a}}\frac{1}{(Re_m)^{\frac{2}{3}}} \tag{40}$$

vhich is valid for the initial period of acceleration of the liquid in the pipe, and on condition that, at the initial moment, the liquid was at rest. The relationship 40), discussed here, is given in Appendix B. For the initial period of liquid acce- eration the curves obtained for both cases are close to each other. Also plotted n Fig. 2 is the course of the quasi-steady friction factor λ_q, calculated from Eq. 39). The curves given in Fig. 2 display the fact that λ_n increases with increasing limensionless acceleration, and is substantially different from λ_q. In Fig. 3 a com- parison is made between the computational results of λ_n and those experimental taken from the work [4]. These results relate to the instance of acceleration of vater ($\nu = 10^{-6}$ m^{2}s^{-1}) in a pipe with the radius of $R = 0.0305$ m. At the initial noment water was at rest. The presented curves (1, 2 and 3) relate to various ccelerations of the liquid ($a = 0.87; 3.1$ and 11.87 ms^2). Figure 3 also shows the

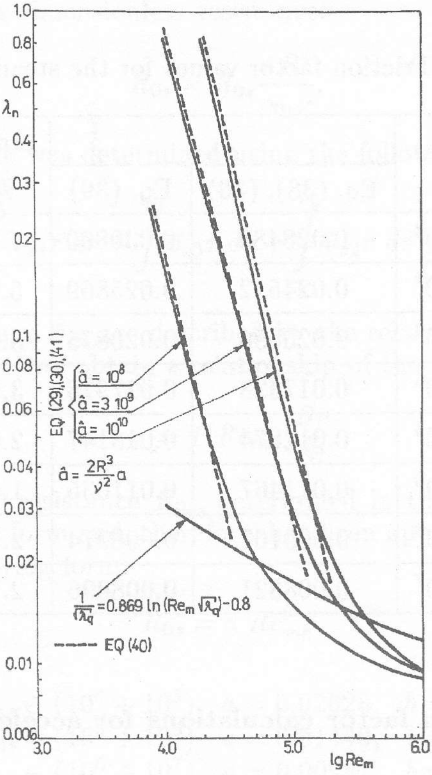

Fig. 2. Instantaneous friction factor λ_n for accelerated flow

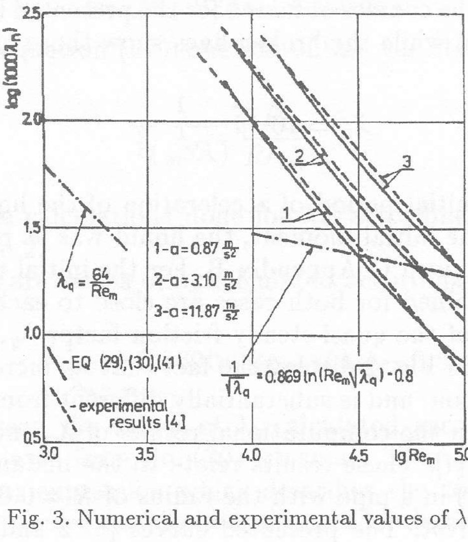

Fig. 3. Numerical and experimental values of λ_n

courses of the quasi-steady friction factor $\lambda_q = 64/Re_m$ for laminar and turbulent flow. The presented courses show that λ_n considerably differs from λ_q and, moreover, (as in Fig. 2) λ_n increases with increasing acceleration, and decreases with increasing Reynolds number Re_m.

6. Conclusions

The paper presents mathematical model of unsteady turbulent flow of liquid in smooth pipes. This model was used for determination of instantaneous hydraulic resistance. Also presented is the method for determining the instantaneous friction factor. This method has been tested on an instance of steady flow and the obtained values of the friction factor have been found to be consistent with those calculated from the Prandtl formula. The main emphasis was laid on the calculation of the instantaneous friction factor for accelerated flow. It has been found that λ_n increases with increasing dimensionless acceleration $\hat{a} = (2R^3/\nu^2 a$, $(a = \partial v_m/\partial t)$, and decreases with increasing Reynolds number Re_m. Some results of numerical calculations of λ_n were compared with those obtained from experiments and relatively good consistency of both results has been found. Moreover, it has been demonstrated that the instantaneous friction factor substantially differs from the quasi-steady value. This fact should be taken into account in the calculations of transients occurring in those systems which include so called hydraulic lines, where calculations are made based on quasi-steady models of hydraulic losses.

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Współczynnik strat tarcia dla turbulentnego przepływu przyspieszonego przez długie rury gładkie

Streszczenie

W pracy przedstawiono model matematyczny niestacjonarnego turbulentnego przepływu cieczy w przewodach zamkniętych z wykorzystaniem czterowarstwowego rozkładu współczynnika lepkości turbulentnej. Model ten wykorzystano do wyznaczenia strat hydraulicznych. Szczegółowe rozważania dotyczą

chwilowego współczynnika strat tarcia dla przepływu przyspieszonego. Współczynnik ten rośnie ze wzrostem przyspieszenia cieczy i maleje ze wzrostem liczby Reynoldsa. Wykazano, że chwilowy współczynnik strat tarcia znacznie odbiega od wartości quasi-ustalonej.

Appendix A

The form of the transfer function $\hat{G}_{\tau v}$ is as follows:

$$\hat{G}_{\tau v} = \frac{\hat{D}}{\beta\sqrt{\hat{D}} - 2}, \tag{41}$$

$$\beta = \beta(\hat{D}, Re_m) = \frac{1}{\phi_1 I_1(n_{1R}) - \phi_2 K_1(n_{1R})}, \tag{42}$$

$$\begin{aligned} \phi_1 &= \frac{c_7 z_1 + 4\sqrt{\hat{D}} z_2 K_1(n_{12})}{q_1 z_1 + g_2 z_2}, \\ \phi_2 &= \frac{-c_6 z_1 + 4\sqrt{\hat{D}} z_2 I_1(n_{13})}{q_1 z_1 + g_2 z_2}, \end{aligned} \tag{43}$$

$$\begin{aligned} z_1 &= a_1(-e_1 n_1 + e_3 n_2) + e_1(a_2 n_1 - a_3 n_2), \\ z_2 &= a_1(e_2 m_1 - e_3 m_2) + e_1(a_2 m_1 - a_3 m_2), \end{aligned} \tag{44}$$

$$\begin{aligned} q_1 &= d_6 c_7 - d_7 c_6, \\ q_2 &= 4\sqrt{\hat{D}}[d_6 K_1(n_{13}) + d_7 I_1(n_{13})], \end{aligned} \tag{45}$$

$$\begin{aligned} n_1 &= b_3 \frac{1}{l_2 l_3} w_1 + \frac{1}{l_3}\sqrt{\hat{D}} K_1(n_{32}) w_2, \\ n_2 &= b_2 \frac{1}{l_2 l_3} w_1 - \frac{1}{l_3}\sqrt{\hat{D}} I_1(n_{32}) w_2, \\ m_1 &= b_3 \frac{1}{l_2} w_3 - \sqrt{\hat{D}} K_1(n_{32}) w_4, \\ m_2 &= b_2 \frac{1}{l_2} w_3 + \sqrt{\hat{D}} I_1(n_{32}) w_4, \end{aligned} \tag{46}$$

$$\begin{aligned} w_1 &= (1 - m^2) sinh\alpha, \\ w_2 &= -(sinh\alpha + mcosh\alpha), \\ w_3 &= -(sinh\alpha - mcosh\alpha), \\ w_4 &= -2sinh\alpha, \end{aligned} \tag{47}$$

$$\alpha = \frac{1}{2} m ln(l_2/l_3), \tag{48}$$

$$m = \sqrt{1 + \hat{D}\xi}, \tag{49}$$

$$\xi = \frac{12800}{\lambda Re_m^2}, \tag{50}$$

$$\begin{aligned}
&a_1 = I_0(n_{41}), \quad a_2 = -I_0(n_{31}), \qquad a_3 = K_0(n_{31}),\\
&b_2 = I_0(n_{32}), \quad b_3 = K_0(n_{32}),\\
&c_6 = I_0(n_{13}), \quad c_7 = K_0(n_{13}),\\
&d_6 = I_0(n_{1R}), \quad d_7 = K_0(n_{1R}),\\
&e_1 = I_1(n_{41}), \quad e_2 = I_1 = I_1(n_{31}), \quad e_3 = -K_1(n_{31}),
\end{aligned} \tag{51}$$

$$\begin{aligned}
n_{41} &= \frac{1 - l_1}{\sqrt{p_1}}\sqrt{\hat{D}},\\
n_{31} &= 2\left(\frac{l_1 - l_2}{p_1 - p_2}\right)\sqrt{p_1}\sqrt{\hat{D}},\\
n_{32} &= 2\left(\frac{l_1 - l_2}{p_1 - p_2}\right)\sqrt{p_2}\sqrt{\hat{D}},\\
n_{13} &= (1 - l_3)\sqrt{\hat{D}},\\
n_{1R} &= \sqrt{\hat{D}},
\end{aligned} \tag{52}$$

$$\begin{aligned}
l_1 &= 0.2,\\
l_2 &= \frac{140}{Re_m}\sqrt{\frac{2}{\lambda}} = \frac{140\sqrt{2}}{\hat{v}^*},\\
l_3 &= \frac{20}{Re_m}\sqrt{\frac{2}{\lambda}} = \frac{20\sqrt{2}}{\hat{v}^*},\\
p_1 &= 0.016\sqrt{\frac{\lambda}{2}}Re_m = \frac{0.016}{\sqrt{2}}\hat{v}^*,\\
p_2 &= 12.25..
\end{aligned} \tag{53}$$

Appendix B

For unsteady laminar flow, putting $\nu_t = 0$ in the equation (1) we obtain its solution in the following form:

$$v_z = C_1 I_0(\eta_1) + D_1 K_0(\eta_1) - \frac{1}{\rho_0 D}\frac{\partial p}{\partial z}. \tag{54}$$

where

$$n_1 = \sqrt{\frac{r^2 D}{\nu}}. \tag{55}$$

The boundary conditions are as follows:

1. $r = 0, \quad v_z = a\ finite\ value,$
2. $r = R, \quad v_z = 0.$

From the condition 1 it can be concluded that $D_1 = 0$ since the function $K_0(0)$ assumes an infinitely large value. Using the equations (18) and (19) we obtain a transfer function for laminar flow, as follows:

$$\hat{G}_{\tau v} = \sqrt{\hat{D}}\frac{I_1(\sqrt{\hat{D}})}{I_2(\sqrt{\hat{D}})}, \tag{56}$$

where I_1, I_2 – modified Bessel functions of first kind, of the orders 1 and 2. For very great arguments z the Bessel functions can be approximated with the formulae [6]:

$$I_1(z) \approx I_2(z) \approx \frac{e^z}{\sqrt{2\pi z}}. \tag{57}$$

Calculating values of the Bessel functions from the exact expressions and from the equation (57) one can find that, for $z = 100$, the approximation error is 1.8%. Taking into account the fact that the operator D corresponds to the argument z it can be said that the above approximation is justifiable when t approaches 0. This corresponds to the initial period of liquid motion from standstill. In this case the function $\hat{G}_{\tau v}$ assumes the following form:

$$\hat{G}_{\tau v} = \sqrt{\hat{D}}. \tag{58}$$

If we assume the quantity $K_n = \hat{a}/Re_m$ (according to the formulae (28) and (30)) for the $\hat{D}$, then, based on the formula (29), we have:

$$\lambda_n = \frac{16\sqrt{2a}}{\nu}\left(\frac{R}{Re_m}\right)^{\frac{3}{2}} = 16\sqrt{\hat{a}}\,(Re_m)^{-\frac{3}{2}} \tag{59}$$

and, thus, the relationship (40).