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exist for the publication of theoretical and experimental investigations of all aspects of the mechanics and thermodynamics of fluid-flow with special reference to fluid-flow machines

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poświęcone są publikacjom naukowym z zakresu teorii i badań doświadczalnych w dziedzinie mechaniki i termodynamiki przepływów, ze szczególnym uwzględnieniem problematyki maszyn przepływowych

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Conical and cylindrical axisymmetrical flow under influence of non-potential body forces and dissipation effects

The paper is devoted to the axisymmetrical stationary flow influenced by a non-potential force and dissipation effects introduced in the form of losses distributed in the flow domain. The axisymmetric flow is bordered by the inner and outer, conical or cylindrical surfaces. Three effects on the flow parameters have been investigated, namely, divergence of conuses (outer border of the flow), increased losses at the borders and influence of immersed body thickness on the solution existence. Conservation equations of mass, momentum and energy are written in the non-orthogonal system of coordinates based on axisymmetric flow surfaces. It has been shown that the system is of hyperbolic character, with two families of characteristics. This enables introducing a simple algorithm of the problem and defines the formulation of boundary conditions. The examples of solution are presented.

1. Introduction

The model of non-potential body force can be applied to the cases where the flow details may be disregarded from the engineering point of view. This model covers the situations where the existence of bodies immersed in the flow can be replaced by the distribution of a specific body force in the flow [1-2]. Focusing the attention on the body force distribution, one has to keep in mind that this is the modelling of the momentum equation. Mass conservation equation can take into account the fact that part of the space is occupied by the immersed body. Energy equation can model two effects namely:

a) energy subtraction or addition if any,

b) an increase of entropy production.

Such conceptualization help us to construct the closed system of equations, provided that the information on existence of the body has been properly introduced into the model.

As an example of the application one can point out the axisymmetrical model

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of turbomachinery stages. Assumption of the axial symmetry is equivalent to the assumption that blading becomes invisible. Its existence in different conservation equations is represented by means of properly chosen parameters.

2. The concept of non-potential body force in energy equation

The body force of potential character appears in the energy equation as potential energy denoted as π. Non-potential body force $\vec{f}$ can not be converted into a form of energy but it can change the amount of total energy along the path of fluid element, according to the equation

$$\frac{d}{dt}\left(\frac{U^2}{2}+e+\frac{p}{\rho}+\pi\right)=\vec{f}\cdot\vec{U} \tag{1}$$

here U – velocity, p – pressure, ρ – density, e – internal energy. Let us assume that we can split $\vec{f}$ into two components. The first is responsible for the momentum change $\vec{f_p}$ and the second for the entropy production $\vec{f_\mu}$.The right hand side of equation (1) can be rewritten in the form

$$\vec{f}\cdot\vec{U}=(\vec{f_p}+\vec{f_\mu})\cdot\vec{U}. \tag{2}$$

Scalar product (2) can be either negative or positive or zero.

The case when

$$(\vec{f_p}+\vec{f_\mu})\cdot\vec{U}<0 \tag{3}$$

means subtraction of total energy from the fluid element according to (1).
If

$$(\vec{f_p}+\vec{f_\mu})\cdot\vec{U}>0 \tag{4}$$

then the fluid element gains energy.

For

$$(\vec{f_p}+\vec{f_\mu})\cdot\vec{U}=0 \tag{5}$$

one can also distinguish two possibilities either

$$(\vec{f_p}+\vec{f_\mu})\perp\vec{U}, \tag{6}$$

or

$$\vec{f_p}+\vec{f_\mu}=0. \tag{6a}$$

In both of the latter cases the total energy of fluid particle is conserved. We will confine our further consideration to the case (5).

The form (1) of energy equation can be justified under certain conditions. Let us consider the momentum conservation equation for Newtonian liquid in the form

$$\rho\frac{d\vec{U}}{dt}=-\rho\mathrm{grad}\pi-\mathrm{grad}p+\vec{L_\mu}+\rho\vec{f_n} \tag{7}$$

where $\vec{L}_\mu$ is the viscous force expressed as below

$$\vec{L}_\mu = \frac{1}{3}\mu \mathrm{grad}(\mathrm{div}\vec{U}) + \mu\nabla^2\vec{U} \tag{8}$$

and vector $\vec{f}_n$ is a non-potential body force which appears here as a carrier of dimension only without specifying precisely what it is.

Energy conservation equation, with the help of Newtonian stress tensor and Fourier law of heat transport, can be written in the form

$$\frac{d}{dt}\left(\frac{U^2}{2} + e + \frac{p}{\rho} + \pi\right) = \vec{f}_n \cdot \vec{U} + \frac{1}{\rho}\vec{L}_\mu \cdot \vec{U} + \frac{1}{\rho}\frac{\partial p}{\partial t} + \varepsilon + \frac{1}{\rho}\lambda\nabla^2 T \tag{9}$$

where ε is the dissipation rate, λ heat transport coefficient, T temperature. Let us introduce the following:

- heat transfer is negligible $\lambda \approx 0$,
- flow is stationary $\frac{\partial}{\partial t} = 0$,
- exists a vector $\vec{U}_s = \frac{\alpha_s}{U_x}\vec{\imath} + \frac{\beta_s}{U_y}\vec{\jmath} + \frac{\gamma_s}{U_z}\vec{k}$,
  for $\alpha_s + \beta_s + \gamma_s = 1$ where $\vec{U}(U_x, U_y, U_z)$ and $(\vec{\imath}, \vec{\jmath}, \vec{k})$ are the versors of vector $\vec{U}$.

Then one can rewrite equation (9)

$$\frac{d}{dt}\left(\frac{U^2}{2} + e + \frac{p}{\rho} + \pi\right) = (\vec{f}_n + \frac{1}{\rho}\vec{L}_\mu + \varepsilon \cdot \vec{U}_s) \cdot \vec{U}. \tag{10}$$

If one fixes

$$\vec{f}_n + \frac{1}{\rho}\vec{L}_\mu \equiv \vec{f}_p, \tag{11}$$

$$\varepsilon\vec{U}_s \equiv \vec{f}_\mu \tag{12}$$

then one gets exactly the form (1) of energy equation with the non-potential body forces in the form of (2). This equation

$$\frac{d}{dt}\left(\frac{U^2}{2} + e + \frac{p}{\rho} + \pi\right) = (\vec{f}_p + \vec{f}_\mu) \cdot \vec{U} \tag{1a}$$

coincides with momentum equation written as

$$\rho\frac{d\vec{U}}{dt} = -\rho \mathrm{grad}\pi - \mathrm{grad}p + \rho\vec{f}_p \tag{7a}$$

where appears the non-potential body force (11). This force can be interpreted as a reaction force of immersed bodies in the flow.

3. Conical coordinates

It is convenient to investigate the axisymmetrical flow in an axisymmetrical system of coordinates. We introduce a system based on the family of cones shown in Fig. 1 following [3].

The cross-section of the cone given by $\mathrm{tg}\,\gamma \equiv x^{(1)}$ with the plane $\varphi \equiv x^{(2)}$

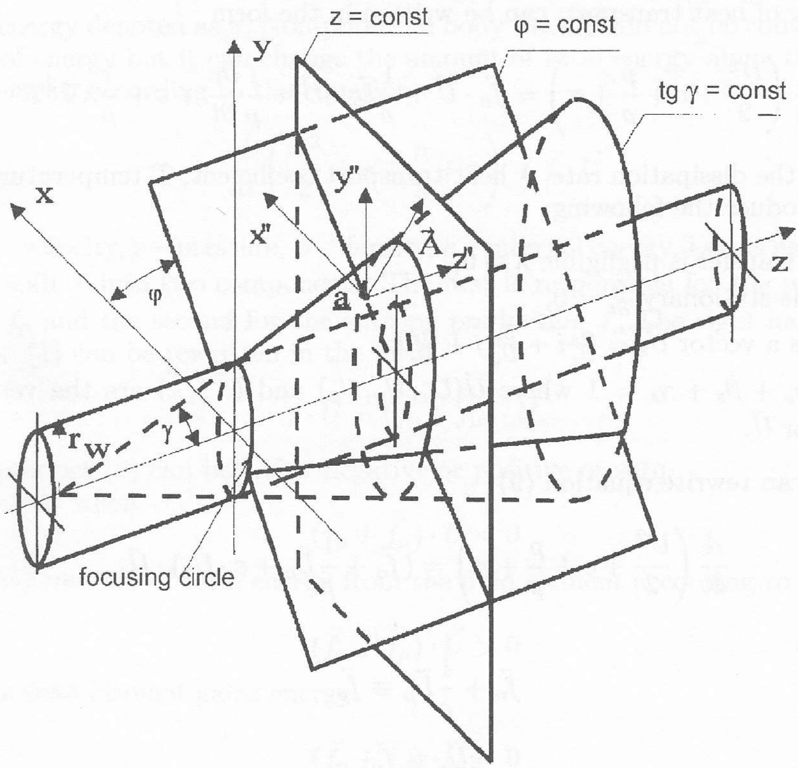

Fig. 1. Conical coordinate system.

and the plane $z \equiv x^{(3)}$ uniquely determines the position of the point a. The rules of transformation between conical and Cartesian coordinates may have the form

$$\begin{aligned} x &= (r_w + x^{(1)}x^{(3)}) \cos x^{(2)}, \\ y &= (r_w + x^{(1)}x^{(3)}) \sin x^{(2)}, \\ z &= x^{(3)}. \end{aligned} \tag{13}$$

The example of the description of the axisymmetrical channel borders are shown in Fig. 2. A_0A_1 is the inner border, B_0B_1 is the outer border of the channel where the axisymmetrical flow along the cones will be investigated. For the very small taper angle γ the outer border is nearly cylindrical and so will be the

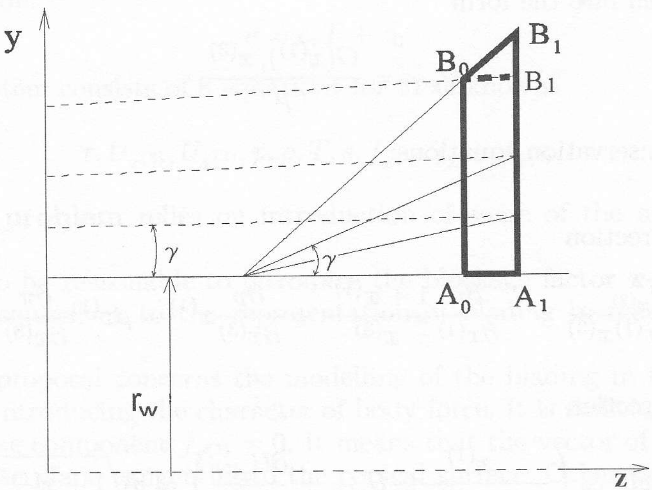

Fig. 2. Notation for conical coordinate system.

intermediate cone surfaces. It is therefore feasible to approximate a cylinder with a cone of a very small taper angle, and so there is no need to introduce a cylindrical system of coordinates.

The problem is formulated as follows:

What should be the curvature of the fluid elements following over the cones in order to keep the flow in accordance with governing equations? Is the system of equations closed or what kind of additional conditions have to be introduced?

Given that the conical stream surfaces are coordinate surfaces as well, we are allowed the velocity vector with only two non-zero components like

$$U_{x^{(1)}} \equiv 0, \quad U_{x^{(2)}} \neq 0, \quad U_{x^{(3)}} \neq 0. \tag{14}$$

The conical system of coordinates is not orthogonal. To derive the system of equations in such a system the easiest way is to use the Christoffel symbols technique. Details of derivation will be omitted here, details are presented in [4-5].

4. System of equations

Let us set down the governing equations in conical system of coordinates taking into account (14), axisymmetrical condition $\frac{\partial}{\partial x^{(2)}} = 0$ and stationary condition $\frac{\partial}{\partial t} = 0$.

Mass conservation equation

$$(1 - \tau(x^{(1)}, x^{(3)}))(r_w + x^{(1)}x^{(3)})x^{(3)}\rho U_{x^{(3)}} = m(x^{(1)}) \tag{15}$$

can be rewritten into the form

$$U_{x^{(3)}} = \frac{G(x^{(1)}), x^{(3)}}{\rho}. \tag{15a}$$

Momentum conservation equations:

- in $x^{(1)}$ direction

$$-\frac{\rho U_{x^{(2)}}^2}{r_w + x^{(1)}x^{(3)}} = -\frac{\partial p}{\partial x^{(1)}}\frac{1+x^{(1)^2}}{x^{(3)}} + \frac{\partial p}{\partial x^{(3)}}x^{(1)} - \rho x^{(1)}\frac{\partial \pi}{\partial x^{(3)}} + \rho f_{x^{(1)}}, \tag{16}$$

- in $x^{(2)}$ direction

$$\rho U_{x^{(3)}}\left(\frac{x^{(1)}}{r_w + x^{(1)}x^{(3)}}U_{x^{(2)}} + \frac{\partial U_{x^{(2)}}}{\partial x^{(3)}}\right) = \rho\sqrt{1+x^{(1)^2}}f_{x^{(2)}}, \tag{17}$$

- in $x^{(3)}$ direction

$$\frac{\rho U_{x^{(3)}}}{\sqrt{1+x^{(1)^2}}}\frac{\partial U_{x^{(3)}}}{\partial x^{(3)}} =$$

$$\frac{\partial p}{\partial x^{(1)}}\frac{x^{(1)}}{x^{(3)}}\sqrt{1+x^{(1)^2}} - \frac{\partial p}{\partial x^{(3)}}\sqrt{1+x^{(1)^2}} + \rho\sqrt{1+x^{(1)^2}}\frac{\partial \pi}{\partial x^{(3)}} + \rho f_{x^{(3)}}. \tag{18}$$

Energy equation:

$$\frac{U_{x^{(2)}}^2 + U_{x^{(3)}}^2}{2} + e + \frac{p}{\rho} + \pi = E_0(x^{(1)}) - a(x^{(1)}, x^{(3)}) \tag{19}$$

where the function $a(x^{(1)}, x^{(3)})$ can model energy subtraction or addition; according to Euler's formula it is

$$a(x^{(1)}, x^{(3)}) = (U_{x^{(2)}}U_{rot})_0 - (U_{x^{(2)}}U_{rot})$$

where U_{rot} is the rotor circumferential velocity. For the nozzle flow $a(x^{(1)}, x^{(3)}) \equiv 0$.
Gibbs equation:

$$\frac{\partial \rho}{\partial x^{(3)}} - \frac{\rho}{kp}\frac{\partial p}{\partial x^{(3)}} = -\frac{k-1}{k}\frac{\rho}{p}\frac{\sqrt{1+x^{(1)^2}}}{U_{x^{(3)}}}T \cdot s \tag{20}$$

with entropy production term s and isentropic exponent k.
State equation:

$$\frac{p}{\rho} = RT. \tag{21}$$

Caloric equation:

$$e = c_v T + e_0. \tag{22}$$

The above system consists of 8 equations for 11 unknowns

$$\tau, U_{x^{(2)}}, U_{x^{(3)}}, p, \rho, T, s, f_{x^{(1)}}, f_{x^{(2)}}, f_{x^{(3)}}, e \tag{23}$$

The **closure problem** relies on introduction of some of the above values as known.

It seems to be reasonable to introduce the blockage factor $\tau = \tau(x^{(1)}, x^{(3)})$. This will be equivalent to the representation of blading in mass conservation equation.

The next proposal concerns the modelling of the blading in the momentum equations by introducing the character of body force. It is sufficient to introduce for example the component $f_{x^{(1)}} = 0$. It means that the vector of the body force is placed in the plane tangential to the conical surface. Other possibilities may be put forth, but here only this one is being considered.

The third assumption to close the system is the entropy production along the flow given by the function $s = s(x^{(1)}, x^{(3)})$. This is also the information about the existing cascade, which is otherwise invisible because of the axisymmetry assumption. Entropy production can be determined by so called loss coefficient $\zeta = \zeta(x^{(1)}, x^{(3)})$ according to the approximate formulae

$$s = s_0 + \zeta R \frac{k}{k-1}\Big(\big(\frac{p_0}{p}\big)^{\frac{k-1}{k}} - 1\Big). \tag{23}$$

5. Method of solution

Since the system is closed one can investigate the character of the system in order to formulate the boundary conditions necessary to integrate the system. One can rearrange the system into the following form of (24)

$$\begin{gathered}
\frac{1+x^{(1)^2}}{x^{(3)}}\frac{\partial p}{\partial x^{(1)}} - x^{(1)}\frac{\partial p}{\partial x^{(3)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}} + 0\frac{\partial \rho}{\partial x^{(1)}} + 0\frac{\partial \rho}{\partial x^{(3)}} = \frac{\rho U^2_{x^{(2)}}}{r_w + x^{(1)}x^{(3)}},\\
0\frac{\partial p}{\partial x^{(1)}} - \frac{\rho}{kp}\frac{\partial p}{\partial x^{(3)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}} + 0\frac{\partial \rho}{\partial x^{(1)}} + 1\frac{\partial \rho}{\partial x^{(3)}} = -\frac{k-1}{k}\frac{s}{R}\frac{\sqrt{1+x^{(1)^2}}}{U_{x^{(3)}}},\\
0\frac{\partial p}{\partial x^{(1)}} + \frac{k}{k-1}\frac{\partial p}{\partial x^{(3)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + \rho U_{x^{(2)}}\left(1 - \frac{U_{rot}}{U_{x^{(2)}}}\right)\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}} + \\
+0\frac{\partial \rho}{\partial x(1)} - \left(\frac{k}{k-1}\frac{p}{\rho} + U^2_{x^{(3)}}\right)\frac{\partial \rho}{\partial x^{(3)}} = \rho U_{x^{(2)}}\frac{\partial U_{rot}}{\partial x^{(3)}} - U_{x^{(3)}}\frac{\partial G}{\partial x^{(3)}}.
\end{gathered} \tag{24}$$

The above set is linear with respect to

$$\frac{\partial p}{\partial x^{(1)}}, \frac{\partial p}{\partial x^{(3)}}, \frac{\partial U_{x^{(2)}}}{\partial x^{(1)}}, \frac{\partial U_{x^2}}{\partial x^{(3)}}, \frac{\partial \rho}{\partial x^{(1)}}, \frac{\partial \rho}{\partial x^{(3)}}. \tag{25}$$

Adding additional equations (26)

$$\begin{aligned}
0\frac{\partial p}{\partial x^{(1)}} + 0\frac{\partial p}{\partial x^{(3)}} + dx^{(1)}\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + dx^{(3)}\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}} + 0\frac{\partial \rho}{\partial x^{(1)}} + 0\frac{\partial \rho}{\partial x^{(3)}} &= dU_{x^{(2)}}, \\
dx^{(1)}\frac{\partial p}{\partial x^{(1)}} + dx^{(3)}\frac{\partial p}{\partial x^{(3)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(3)}} + 0\frac{\partial \rho}{\partial x^{(1)}} &= dp, \\
0\frac{\partial p}{\partial x^{(1)}} + 0\frac{\partial p}{\partial x^{(3)}} + 0\frac{\partial U_{x^{(2)}}}{\partial x^{(1)}} + 0\frac{\partial U_x^{(2)}}{\partial x^{(3)}} + dx^{(1)}\frac{\partial \rho}{\partial x^{(1)}} + dx^{(3)}\frac{\partial \rho}{\partial x^{(3)}} &= d\rho.
\end{aligned} \tag{26}$$

the system is closed. It has main determinant in the form

$$W_0 = \begin{Vmatrix}
\frac{1+{x^{(1)}}^2}{x^{(3)}} & -x^{(1)} & 0 & 0 & 0 & 0 \\
0 & -\frac{\rho}{\kappa p} & 0 & 0 & 0 & 1 \\
0 & \frac{\kappa}{\kappa-1} & 0 & \rho U_{x^{(2)}}(1-\frac{U_{rot}}{U_{x^{(2)}}}) & 0 & -(\frac{k}{k-1}\frac{p}{\rho} + U^2_{x^{(3)}}) \\
dx^{(1)} & dx^{(3)} & 0 & 0 & 0 & 0 \\
0 & 0 & dx^{(1)} & dx^{(3)} & 0 & 0 \\
0 & 0 & 0 & 0 & dx^{(1)} & dx^{(3)}
\end{Vmatrix} \tag{27}$$

The condition $W_0 = 0$ leads to two families of real characteristics

$$dx^{(1)} = 0 \tag{28}$$

and

$$\frac{dx^{(1)}}{dx^{(3)}} = \frac{1+{x^{(1)}}^2}{x^{(1)}x^{(3)}}. \tag{29}$$

Along the first family (28) the mass flow rate function $m(x^{(1)})$ and $E_0(x^{(1)})$ are kept constant. For the second family of characteristics (29) the ordinary differential equation has to be fulfilled

$$\frac{dp}{dx^{(1)}} = \frac{\rho U^2_{x^{(2)}}}{r_w + x^{(1)}x^{(3)}}\frac{x^{(3)}}{1+{x^{(1)}}^2}. \tag{30}$$

The problem is reduced to the solution of ordinary equation (30) along the characteristics (29).

6. Examples of solution

As an example a conical nozzle shown in Fig. 2 was chosen. The main geometrical parameters were

$$\begin{aligned} r_w &= 1\text{m}, \\ \Delta z &= z_{A1} - z_{A0} = 0.25 \text{ m}. \end{aligned}$$

At the inlet uniform distribution of pressure, temperature and normal velocity were assumed

$$\begin{aligned} p_0 &= 15000 \quad \text{Pa}, \\ T_0 &= 373.15 \quad \text{K}, \\ U_n &= 100 \quad \text{m/s}. \end{aligned}$$

Along the outer border the increase of circumferential velocity was linear with respect to the coordinate z

$$U_{x(2)} = 400 \frac{z - z_{A0}}{z_{A1} - z_{A0}}.$$

The following influences on the solution are investigated

a) influence of the taper angle from 45^0 to 0^0,

b) influence of loss distribution coefficient from 0 (isentropic flow) to the value $\zeta = 0.0615$ (mean value),

c) influence of the blockage factor given by function

$$\tau = (b + 0.02) * (0.6 - q)/0.6) * \left(\frac{1 - \bar{z}}{0.5}\right)^{1.2} \left(\frac{\bar{z}}{0.5}\right)^{1.2}$$

where

- q (conical coordinate) varies from 0 to 0.6,
- $\bar{z} = \frac{z - z_{A0}}{z_{A1} - z_{A0}}$,
- b varies from 0.028 to 0.048.

The results of calculation are presented as S_2 surfaces evolution in the form of animated movies. Here only the two extreme situations are shown in the following figures.

For the influence of the taper angle Fig. 3 shows S_2 surface for 45^0 and Fig. 4 for 0^0.

The influence of loss coefficient is shown in Fig. 5 for the isentropic situation and Fig. 6 for the maximum loss coefficient.

For the influence of blockage factor we have for $\tau = 0.048$ a situation as in Fig. 7 and for the maximum there is no solution at the root Fig. 8. The 'white spot' occurs at the root, which simply means no solution.

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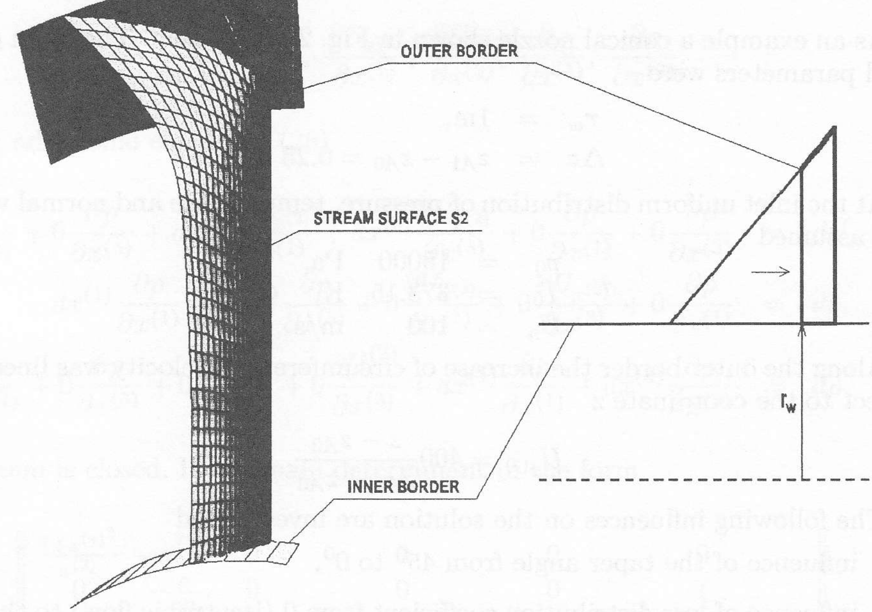

Fig. 3. Stream surface S_2 for taper angle 45 ^{0}.

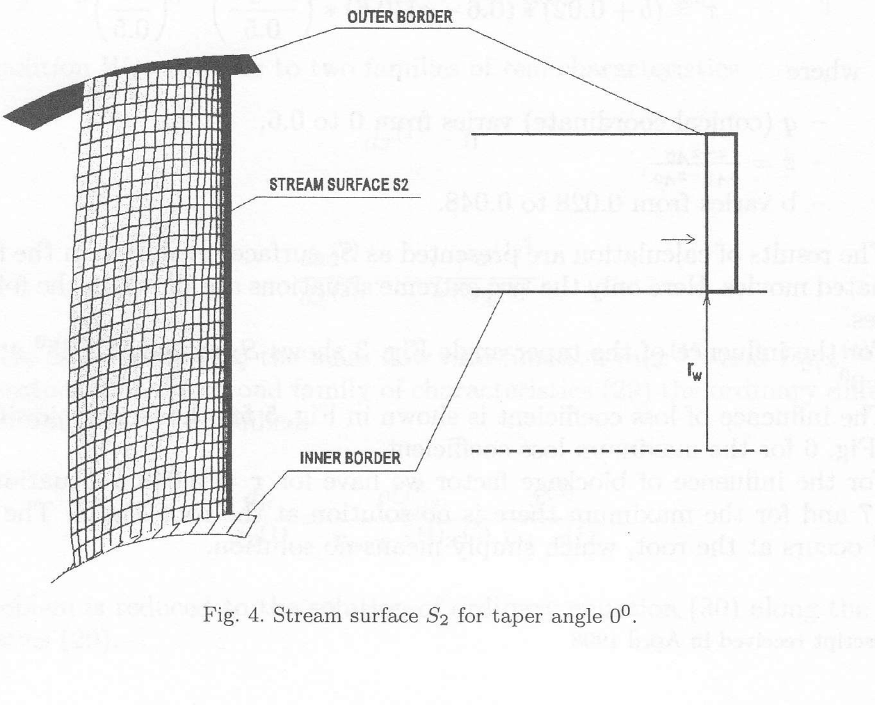

Fig. 4. Stream surface S_2 for taper angle 0^0.

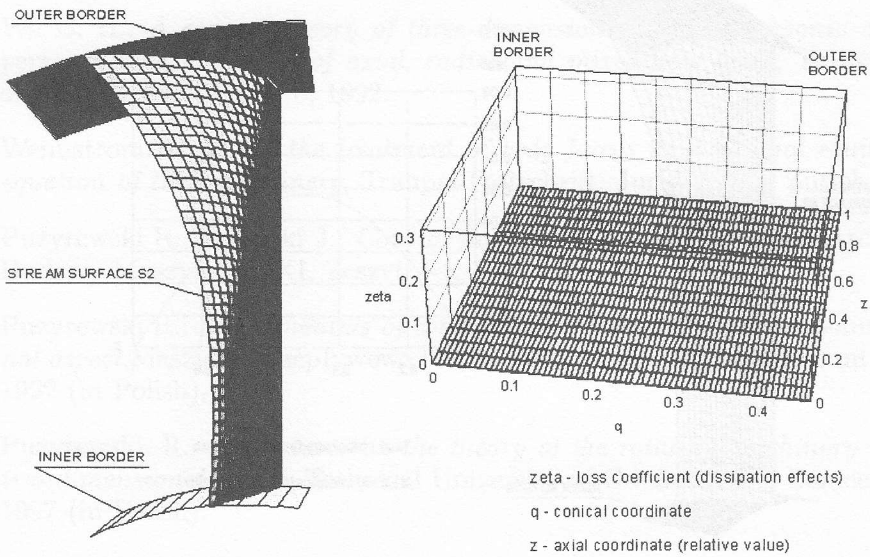

Fig. 5. Stream surface S_2 for isentropic situation.

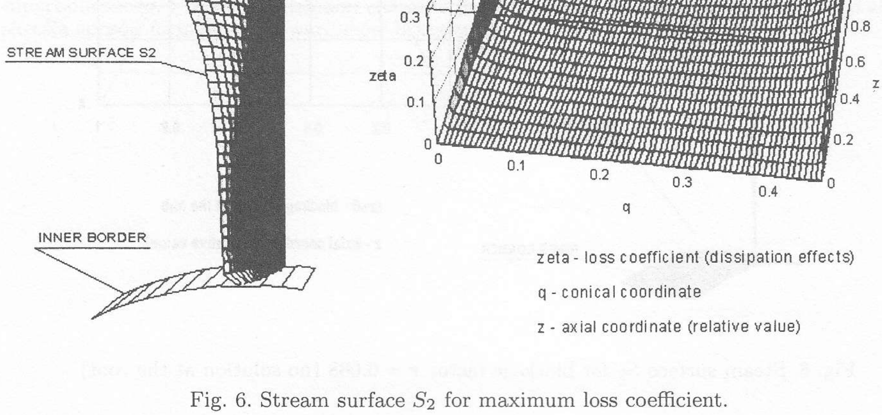

Fig. 6. Stream surface S_2 for maximum loss coefficient.

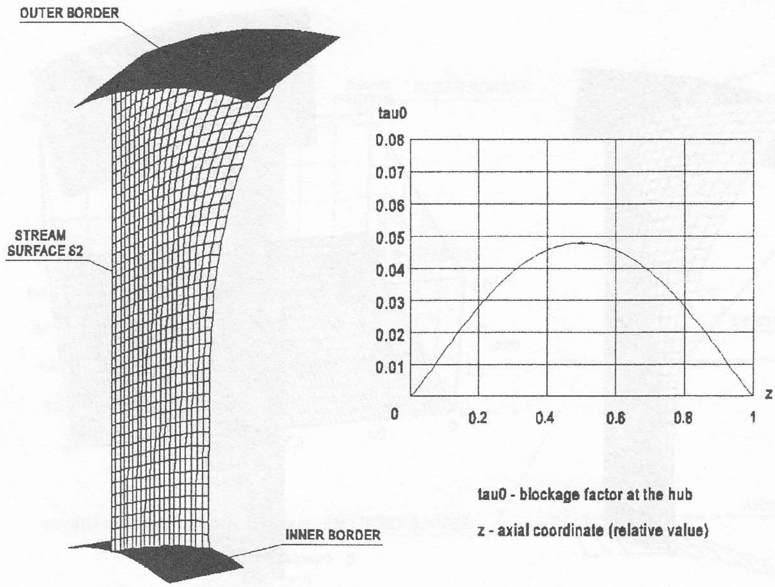

Fig. 7. Stream surface S_2 for blockage factor $\tau = 0.048$.

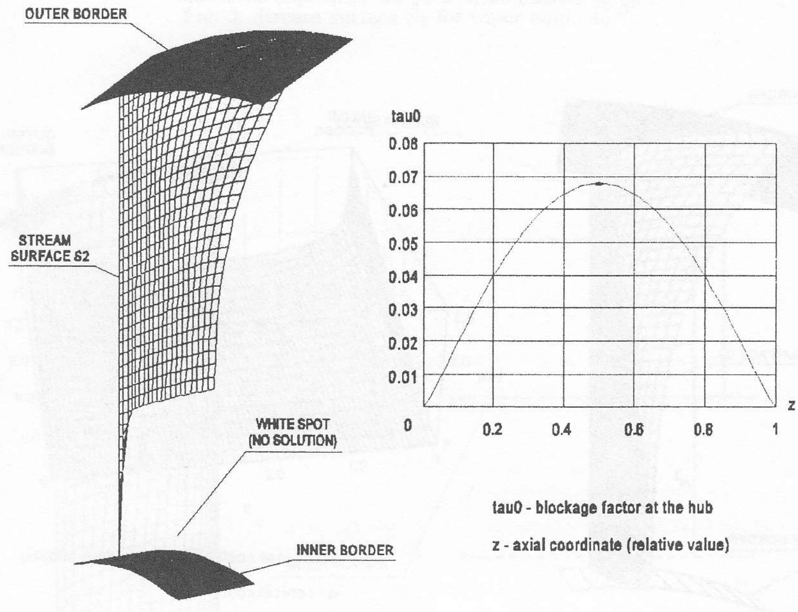

Fig. 8. Steam surface S_2 for blockage factor $\tau = 0.068$ (no solution at the root).

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Stożkowy i cylindryczny przepływ pod wpływem niepotencjalnych sił masowych i efektów dysypacyjnych

Streszczenie

Przedstawiono problem osiowo-symetrycznego stacjonarnego przepływu z uwzględnieniem niepotencjalnych sił masowych oraz procesów dysypacyjnych (w postaci strat entropowych rozmieszczonych w obszarze przepływu). Obszar przepływu jest ograniczony przez wewnętrzną (cylindryczną) i zewnętrzną (cylindryczną lub stożkową) powierzchnię. Rozpatrywany jest wpływ trzech czynników na parametry przepływu: rozwartość stożków (zewnętrzne ograniczenie obszaru), wzrost strat entropowych w obszarach brzegowych oraz wpływ grubości ciał redukujących przekrój przepływu, na istnienie rozwiązań. Równania zachowania masy, ilości ruchu oraz energii zapisane są w krzywoliniowym układzie współrzędnych, opartym na osiowo-symetrycznych powierzchniach prądu. Pokazano, iż rozpatrywany układ równań jest typu hiperbolicznego, z dwoma rodzinami charakterystyk, co umożliwia zbudowanie prostego algorytmu oraz określa sposób formułowania warunków brzegowych. Zamieszczono przykłady rozwiązań.