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exist for the publication of theoretical and experimental investigations of all aspects of the mechanics and thermodynamics of fluid-flow with special reference to fluid-flow machines

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poświęcone są publikacjom naukowym z zakresu teorii i badań doświadczalnych w dziedzinie mechaniki i termodynamiki przepływów, ze szczególnym uwzględnieniem problematyki maszyn przepływowych

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Geometrically nonlinear equations of a spatially curved and pre-twisted beam with account of warping of the transverse cross-section

In this work equations of motion are given for the one-dimensional elastic body. They are derived from three-dimensional theory of continuous media with the assumption of the finite displacements and deformations. This model incorporates the constrained cross-sectional warping, due to the torsion and shearing, and coupled bending, torsion and stretching. Complex geometry of the body was taken into consideration which is spatial curved, pre-twisted and tapered, and having non-symmetrical cross-sections.

1. Introduction

The structural elements of machines, devices and buildings are spatial bodies, frequently of complicated shapes. To carry out full analysis of their dynamical behavior requires application of the three-dimensional continuum theory.

Application of this general theory to slender bodies is cumbersome and usually redundant. For bodies with two transversal dimensions significantly smaller than their length it is convenient to use special methods that reduce description of motion of such bodies to equations dependent on a single spatial variable and time. Then we deal with one-dimensional theories of continua – the theories of rods and beams. In general, two different approaches are used to construct theories of this type.

In the first case the beam is being replaced with an a priori one-dimensional material continuum whose particles are endowed with some definite vector structure. There is a series of papers that make use of this spatial line model. Ericksen and Truesdell [1] treated an oriented curve as continuum embedded into the

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three-dimensional Euclidean space. Research on this topic was carried out by Whitman and De Silva [2], Green and Laws [3] as well as others. In particular, the authors of [4] established nonlinear equations of equilibrium for thin spatial rods based on the assumption that the displacements are large, but the strains small. They accounted for stretching as well as contraction. Review of the theories of this kind is given by Antmann in [5].

In the second case, one-dimensional models are constructed starting from the theory of three-dimensional continua. This task is carried out with the use of diverse methods. In general, these theories fall into one of the two major categories: mathematically exact or technical. Starting from the three-dimensional theory of elasticity Saint-Venant [6] gave an exact solution for displacements and stresses arising in a cylindrical beam with non-curved axis and simply-connected cross-sections, assuming that the lateral surfaces of the beam are traction-free. The solutions obtained are valid for small displacements and strains. Nevertheless, they supply solid foundation for further development of the theory of rods and beams. Another approach consists in prescribing distributions of displacements, strains and stresses as polynomials of the coordinates of points from transverse cross-sections of the beam. This leads to infinite sequences of equations giving various approximations of the motion of the body. They are functions of the spatial variable along the length of the beam and time. Research on this topic was carried out by Novozhilov [7], Volterra [8, 9], Medick [10]. Generalization of these results may be found in the paper by Gamby [11]. Asymptotic methods are applied to spatial rods and beams undergoing large displacements and rotations, too. These methods consist in seeking solutions with respect to a fixed, small geometrical parameter. In effect one obtains theories of different orders. Examples of such solutions may be found in the papers by Antmann and Warner [12], Parker [13] and Pleus and Sayir [14].

Technical theories of rods and beams are founded on assumptions concerning displacements, strains or stresses arising in these bodies. One can distinguish two principal approaches here. In the first case one assumes, on the a priori basis, the order of magnitude of some definite geometric or static parameters and discards, in the process of constructing the theory, quantities of order higher than assumed. Examples of such theories may be found in the papers by Houbolt and Brooks [15] and Rosen and Friedmann [16]. Another way of constructing technical theories of rods and beams is based on adopting hypotheses determining distributions of displacements and stresses. These hypotheses are treated as internal constraints imposed on the motion and the state of stress of a really three-dimensional body — a beam or rod. The hypotheses of Euler-Bernoulli [17], Kirchhoff [18], Clebsh [20] and Love [21] supply examples of internal constraints of this kind. It is assumed that displacements of the material points from the cross-section under consideration may be described by the rotational rigid-body motion of this cross-section and the displacements normal to the rotated cross-section. In classical formulations small strains and rotations, and hence also small displacements, are assumed. These formulations are then extended to finite rotations, shear deformations and the effects coming from free and constrained warping of the transverse cross-sections as well as more advanced constitutive equations for the material.

Within the linearized theory, Timoshenko examined in [19] the influence of strains generated by unconstrained shear, whereas Janecki [22, 23] analyzed the effects of constrained warpings of transverse cross-sections of a bended beam. Extensive review of articles wherein shear effects were considered may be found in [23]. Dzhanelidze [24] and Vlasov [25] assessed the effects of constrained torsion – a matter of importance in the theory of thin-walled structures. The equations of statics accounting for finite rotations of the transverse cross-sections in a thin beam were considered by a number of authors. Among the earliest important contributions to the nonlinear theory of thin beams undergoing large displacements, rotations and strains are the papers by Reissner [26–28]. His work was later extended via imposition of constraints on the three-dimensional continuum by Jura [29], Jura and Atluri [30, 31], Hodges [32], Danielsen and Hodges [33, 34], Simo [35], Simo and Vu Quock [36–38], Hegemier [39]. Extensive review of papers discussing the role of finite rotations may be found in [32]. Besides finite rotations additional effects, connected with shear deformations, finite elongations and constrained warping of the cross-sections coming from torsion [40], were included in the process of construction of the nonlinear theory of rods and beams.

Many structural elements, like compressor and gas or steam turbine blades under rotational motion, the blades of the wind-driven machines and helicopter propellers, have complicated geometry. They are curved and twisted. In the natural state their cross-sections are asymmetric and strongly tapered along their axis. Therefore, in theoretical considerations on spatially curved and twisted beams a lot of importance is attached to the problem of coupling between bending, twisting and stretching, including also warping of their cross-sections. Formulations confined to small strains and small or moderate rotations, accounting also for the effects from coupled bending, twisting and stretching as well as the effects from the pre-twisting and warping of the transverse cross-sections, were considered in many papers focusing on applicative aspect – among the others in the publications by Hodges and Dowell [41], Hodges [42], Janecki [43, 44], Krenk [45], Krenk and Gunneskov [46], Reissner [47–49], Rosen [16, 50], Vorobev [51] and others.

In a number of dynamical problems – like stability of motion analysis, transient vibrations and aeroelasticity of structural elements under rotational motion – application of the fully linearized theories leads to considerable errors in evaluation of displacements, vibration parameters and stability [37]. Problems of this kind were considered in the papers by Janecki [43] and Reissner [48].

Two different approaches are used in dynamics of beams under rotational motion. In the first case deformations of a beam are considered in a non-inertial system rotating with the body with respect to a stationary inertial system. This yields an uncomplicated expression for the strain energy. Large displacements connected with the rotational motion of the beam are eliminated in this approach. This procedure has been used in many papers, e.g. in [41, 44, 45, 46, 51]. In the second case dynamics of a beam is described in a stationary inertial system. This simplifies the expression for the kinetic energy of the rotating body significantly. The strain energy of the body is more complicated, instead. The equations describing the motion of a beam are less complicated in this approach than in the first case. This method has been used generally in the papers by Simo and Vu Quock [36–38].

This paper will focus on dynamics of one-dimensional model of an elastic body based on exact, geometrically nonlinear description of deformation and nonlinear equations of the three-dimensional continuum mechanics. The model will include constrained warping of the cross-sections arising from twisting and shear as well as mutual coupling between bending, twisting and stretching. Complicated geometry of a curved, twisted and tapered body with asymmetrical transverse cross-sections will also be included.

Essential elements in these considerations are:

a) adoption of a fully geometrically nonlinear model;
b) inclusion of constrained warping of the transverse cross-sections, which is important for thin-walled or bulky structures;
c) inclusion of the inertial effects connected with rotational motion and effects arising from the complexity of geometry of the body; this is of particular importance in the description of blade dynamics of wind-driven machines, helicopters and fluid-flow machines.

2. Kinematics of a beam

2.1. Basic assumptions

1. A beam is a slender body, which in the natural state is twisted and curved, and has variable, asymmetrical cross-sections.
2. During the motion initially flat cross-sections warp and form curved surfaces.
3. Warping of the transverse cross-sections is constrained and arises from twisting and shear.
4. The in-plane strains of the transverse cross-sections are neglected.
5. The axis of the beam remains a smooth spatial curve all throughout the deformation.
6. The material of the beam is linearly elastic.

2.2. Geometry of the undeformed beam

In continuum mechanics a beam is treated as the set $\mathcal{B}$ of material points $\mathcal{X}$ that in the natural state occupy some initial configuration $B_0 \subset \mathcal{E}_3$ in the physical space $\mathcal{E}_3$. This configuration is determined by the Cartesian product

$$B_0 = \underset{0}{A} \times \langle 0, L \rangle \,, \tag{1}$$

where $\underset{0}{A}$ is a flat cross-section of the beam, L is the length of the beam measured along the continuous spatial curve called the axis of the beam. It is assumed that this axis is the line that connects the gravity centers of the transverse cross-sections. In general, transverse cross-sections perpendicular to the axis of the beam at any point may alter their shape, area and position. The position of a cross-section is determined by the angle its principal axes of inertia make with the normal and binormal of the beam axis. Location of an arbitrary point on the axis may be determined by prescribing the length s_0 measured from some chosen, fixed point on this axis.

The beam in its initial configuration is placed in a global inertial system of Cartesian coordinates $(0_I, X_I^m)$ with basis $\{\mathbf{i}_m\}$, (Fig. 1). To describe geometry, a local orthogonal curvilinear coordinate system $(0_0, X_0^m)$ is introduced. The orthogonal basis connected with this system is $\{\underset{0}{\mathbf{e}}_m\}$. The origin of this system is located on the spatially curved axis of the beam. The coordinates (x_0^1, x_0^2) of the system are located in the plane A_0 of the transverse cross-section, perpendicular to the axis and coincide with its principal axes of inertia. The coordinate x_0^3 is measured along the beam axis and may be identified with the arc length s_0. The axis of curved and twisted beam induces in a natural way yet another system of unit orthogonal vectors $\{\underset{0}{\mathbf{n}}_i\}$: the normal, binormal and tangent vector, respectively. Due to pre-twisting of the beam in the natural state, the following relation exists between the base vectors $\{\underset{0}{\mathbf{e}}_m\}$ and $\{\underset{0}{\mathbf{n}}_i\}$:

$$\underset{0}{\mathbf{e}}_i = \boldsymbol{Q}_0(s_0)\underset{0}{\mathbf{n}}_i \ , \tag{2}$$

where

$$\boldsymbol{Q}_0 = \cos\vartheta_0 \mathbf{1} + \sin\vartheta_0(\underset{0}{\mathbf{n}}_3 \times \mathbf{1}) + (1 - \cos\vartheta_0)(\underset{0}{\mathbf{n}}_3 \otimes \underset{0}{\mathbf{n}}_3) \ , \tag{3}$$

is the tensor that determines the position of the base vectors $\{\underset{0}{\mathbf{e}}_i\}$, $\mathbf{1} = \underset{0}{\mathbf{n}}_i \times \underset{0}{\mathbf{n}}_i$ is the unit tensor, $\vartheta_0(s_0)$ is the local angle of the pre-twisting of the beam.

The position vector of an arbitrary material point located on the undeformed axis of the beam is given by

$$\mathbf{r}_0 = x_I^m \mathbf{i}_m \ . \tag{4}$$

Whereas in the natural reference system it is:

$$\mathbf{r}_0 = \mathbf{r}_0(s_0) \ . \tag{5}$$

Then, the vectors of the natural system of the undeformed axis of the beam follow from the relations

$$\underset{0}{\mathbf{n}}_1 = \frac{1}{\kappa_0}\frac{\mathrm{d}\underset{0}{\mathbf{n}}_3}{\mathrm{d}s_0}, \qquad \underset{0}{\mathbf{n}}_2 = \underset{0}{\mathbf{n}}_3 \times \underset{0}{\mathbf{n}}_1, \qquad \underset{0}{\mathbf{n}}_3 = \frac{\mathrm{d}\mathbf{r}_0}{\mathrm{d}s_0} \ , \tag{6}$$

where $\kappa_0 = |\mathrm{d}\underset{0}{\mathbf{n}}_3/\mathrm{d}s_0|$ is the curvature of the beam axis.

The directions of vectors belonging to the natural system vary from point to point along the beam axis. These variations may be determined with the aid of

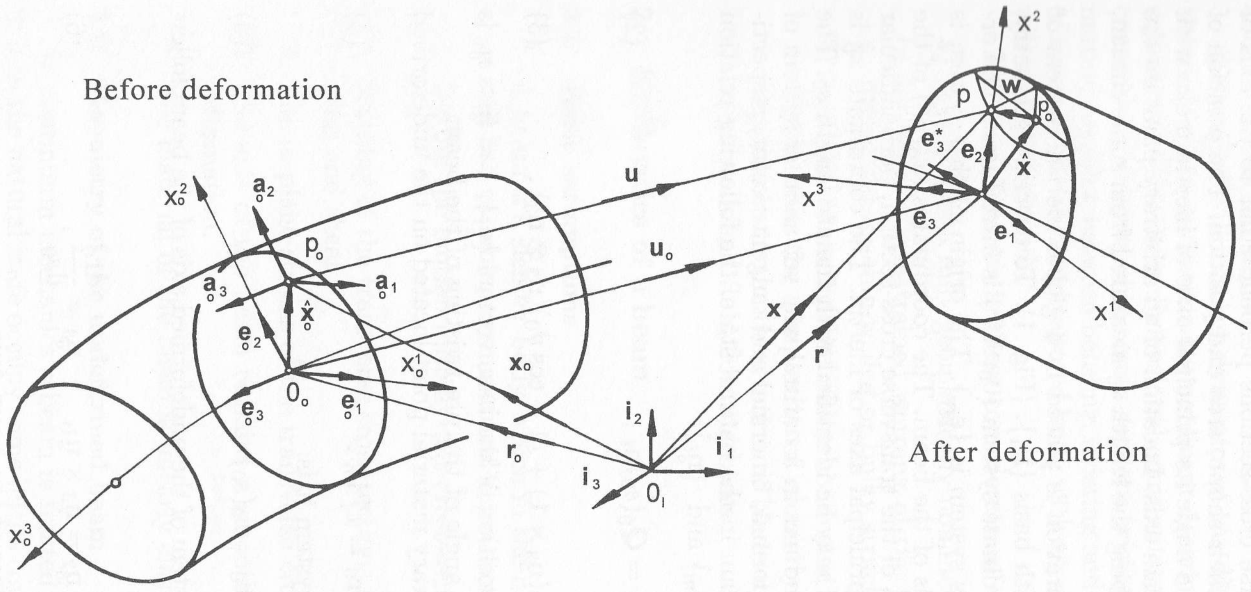

Fig. 1. Diagram illustrating kinematics of deformation of a beam.

$\mathbf{i}_m$ – basis of the global reference system,
$\underset{o}{\mathbf{e}}_m, \mathbf{e}_m$ – basis of the local reference system, related to the transverse cross-section of the beam in the initial and current configuration,
$\underset{o}{\hat{\mathbf{x}}}, \hat{\mathbf{x}}$ – position vectors of a material point of the beam from the transverse cross-section in the initial and current configuration,
$\mathbf{w}$ – the warping vector of the transverse cross-section of the beam,
$\mathbf{e}_3^*$ – the vector tangent to the axis of the deformed beam,

$\mathbf{x}_0, \mathbf{x}$ – position vectors of an arbitrary material point of the beam from the transverse cross-section in the initial and current configuration,
$\mathbf{r}_0, \mathbf{r}$ – position vectors of the material point located on the axis of the beam in the initial and current configuration,
$\mathbf{u}_0, \mathbf{u}$ – the displacement vectors of the material point located of the beam and of an arbitrary point on the axis.

their derivatives with respect to the arc length s_0 of the beam axis. They follow from the Frenet-Serret formulas

$$\frac{\mathrm{d}\underset{\circ}{\mathbf{n}}_i}{\mathrm{d}s_0} = \bar{\mathbf{K}}_0 \underset{\circ}{\mathbf{n}}_i \quad \text{or} \quad \frac{\mathrm{d}\underset{\circ}{\mathbf{n}}_i}{\mathrm{d}s_0} = \bar{\boldsymbol{\kappa}}_0 \times \underset{\circ}{\mathbf{n}}_i \, . \tag{7}$$

Components of the tensor $\bar{\mathbf{K}}_0$ and the coordinates of the vector of curvature $\bar{\boldsymbol{\kappa}}_0$ in the basis $\{\underset{\circ}{\mathbf{n}}_i\}$ are given by the matrices

$$\left[\bar{K}_0^{ij}\right] = \begin{bmatrix} 0 & -\tau_0 & \kappa_0 \\ \tau_0 & 0 & 0 \\ -\kappa_0 & 0 & 0 \end{bmatrix}, \qquad [\bar{\kappa}_0^i] = [0, \kappa_0, \tau_0] \, , \tag{8}$$

where τ_0 is the torsion of the beam with undeformed axis. Using the relations (2) and (8) we obtain

$$\frac{\mathrm{d}\underset{\circ}{\mathbf{e}}_i}{\mathrm{d}s_0} = \mathbf{K}_0 \, \underset{\circ}{\mathbf{e}}_i \quad \text{or} \quad \frac{\mathrm{d}\underset{\circ}{\mathbf{e}}_i}{\mathrm{d}s_0} = \boldsymbol{\kappa}_0 \times \underset{\circ}{\mathbf{e}}_i \, , \tag{9}$$

where

$$\mathbf{K}_0 = -\kappa_0^i \, \epsilon_{ijk} \, \underset{\circ}{\mathbf{e}}_j \otimes \underset{\circ}{\mathbf{e}}_k \, . \tag{10}$$

Components of the curvature tensor $\mathbf{K}_0$ and the coordinates of the vector $\boldsymbol{\kappa}_0$ in the basis $\{\underset{\circ}{\mathbf{e}}_i\}$ are given by the matrices

$$[K_0^{ij}] = \begin{bmatrix} 0 & -\kappa_0^3 & \kappa_0^2 \\ \kappa_0^3 & 0 & -\kappa_0^1 \\ -\kappa_0^2 & \kappa_0^1 & 0 \end{bmatrix},$$

$$[\kappa_0^i] = [\kappa_0 \sin\vartheta_0, \kappa_0 \cos\vartheta_0, \tau_0 + \vartheta_0'], \qquad (\cdot)' = \frac{\partial}{\partial s_0} \, . \tag{11}$$

The position of an arbitrary point located in a transverse cross-section of the beam is determined by the position vector (Fig. 1).

$$\mathbf{x}_0 = \mathbf{r}_0 + \hat{\mathbf{x}}_0 \, , \tag{12}$$

where

$$\hat{\mathbf{x}}_0 = x_0^1 \, \underset{\circ}{\mathbf{e}}_1 + x_0^2 \, \underset{\circ}{\mathbf{e}}_2 = \underset{\circ}{x}^\alpha \, \underset{\circ}{\mathbf{e}}_\alpha, \qquad (\alpha = 1, 2) \, , \tag{13}$$

is the position vector of the point located in a transverse cross-section of the beam.

The base vectors $\{\underset{\circ}{\mathbf{a}}_i\}$ in an arbitrary material point of the beam follow from differentiation of the position vector $\mathbf{x}_0$ with respect to the variables x_0^i. Then, we get

$$\underset{\circ}{\mathbf{a}}_i = \mathbf{A}_0 \, \underset{\circ}{\mathbf{e}}_i \, , \tag{14}$$

where the components of the tensor $\mathbf{A}_0$ in the basis $\{\underset{\circ}{\mathbf{e}}_i\}$ are

$$\Big[A_0^{ij}\Big] = \begin{bmatrix} 1 & 0 & -x_0^2(\tau_0+\vartheta_0') \\ 0 & 1 & x_0^1(\tau_0+\vartheta_0') \\ 0 & 0 & g_0 \end{bmatrix}, \tag{15}$$

$$g_0 = 1 - x_0^1\kappa_0\cos\vartheta_0 + x_0^2\kappa_0\sin\vartheta_0$$

or (16)

$$\underset{\circ}{\mathbf{a}}_\alpha = \underset{\circ}{\mathbf{e}}_\alpha, \qquad \underset{\circ}{\mathbf{a}}_3 = \underset{\circ}{\mathbf{e}}_3 + \boldsymbol{\kappa}_0 \times \hat{\mathbf{x}}_0$$

The vectors $\underset{\circ}{\mathbf{a}}_i$ do not form an orthonormal basis, because the vector $\underset{\circ}{\mathbf{a}}_3$ is not a unit one and perpendicular to the vectors $\underset{\circ}{\mathbf{a}}_\alpha$ The contravariant basis may be determined from the relation $\underset{\circ}{\mathbf{a}}^m\ \underset{\circ}{\mathbf{a}}_n = \delta_n^m$, where δ_n^m is the Kronecker symbol. Then

$$\begin{aligned} \underset{\circ}{a}^1 &= \underset{\circ}{\mathbf{e}}_j + \frac{1}{g_0}\, x_0^2\, \underset{\circ}{\kappa}{}_0^3\, \underset{\circ}{\mathbf{e}}_3\,, \\ \underset{\circ}{a}^2 &= \underset{\circ}{\mathbf{e}}_2 - \frac{1}{g_0}\, x_0^1\, \underset{\circ}{\kappa}{}_0^3\, \underset{\circ}{\mathbf{e}}_3\,, \\ \underset{\circ}{a}^3 &= \frac{1}{g_0}\, \underset{\circ}{\mathbf{e}}_3\,. \end{aligned} \tag{17}$$

2.3. Geometry of the deformed beam

Under the action of external loads the beam deforms. Its configuration in the physical space alters. The transverse cross-section of the beam, made up of material points initially located in a plane perpendicular to the axis, rotates and warps forming a curved surface (Fig. 1). Then, location of an arbitrary material point of the deformed beam may be given by the position vector [44]

$$\mathbf{x} = \mathbf{r}(s_0) + \hat{\mathbf{x}}(x^1, x^2, s_0) + \mathbf{w}(x^1, x^2, s_0)\,, \tag{18}$$

where $\mathbf{r}$, $\hat{\mathbf{r}}$ and $\mathbf{w}$ are: the position vector of a fixed material point located on the axis of the beam, the position vector of the projection of a selected point on the plane arisen from a rigid-body rotation of the transverse cross-section cutting the undeformed beam, and the warp vector normal to the rotated plane, respectively. Warping of the transverse cross-sections arises as a result of twisting, bending and shearing of the beam. The rotated material plane is not perpendicular to the axis of the deformed beam because of shearing.

We shall describe the geometry of the deformed beam in a local coordinate system $(0, X^m)$ arisen from a rigid-body finite rotation of the original local system $(0_0, X_0^m)$. The basis of this system is made up of the orthonormal vectors $\{\mathbf{e}_m\}$, connected with the rotated plane (treated as the transverse cross-section of the deformed beam). The basis $\{\mathbf{e}_m\}$ may be obtained from a rigid-body rotation of the basis $\{\underset{\circ}{\mathbf{e}}_m\}$.

Thus, at every point of the beam axis there exists an orthogonal transformation [52]

$$\mathbf{e}_m = \mathbf{Q}(s_0)\,\underset{\circ}{\mathbf{e}}_m\,, \tag{19}$$

such that

$$\mathbf{Q} = \cos\omega\mathbf{1} + \sin\omega(\mathbf{k}\times\mathbf{1}) + (1-\cos\omega)(\mathbf{k}\otimes\mathbf{k}) \tag{20}$$

or

$$\mathbf{Q} = \cos\omega\mathbf{1} + \frac{\sin\omega}{\omega}(\boldsymbol{\omega}\times\mathbf{1}) + \frac{1-\cos\omega}{\omega^2}(\boldsymbol{\omega}\otimes\boldsymbol{\omega})\,, \tag{21}$$

where $\boldsymbol{\omega} = \omega\mathbf{k}$, $\mathbf{k}$ is the versor of rotation axis, ω is the angle of rotation, $\mathbf{1}$ is the unit tensor.

In the basis $\underset{\circ}{\mathbf{e}}_m$ we have

$$\boldsymbol{\omega} = \omega_i\underset{\circ}{\mathbf{e}}_i\,, \tag{22}$$

and then, according to (19) and (20), the components of the tensor $\mathbf{Q}$ in the basis $\underset{\circ}{\mathbf{e}}_m$ are given by the matrix

$$[Q^{ij}] =$$

$$\begin{bmatrix} \cos\omega+\omega_1^2\frac{1-\cos\omega}{\omega^2}, & -\omega_3\frac{\sin\omega}{\omega}+\omega_1\omega_2\frac{1-\cos\omega}{\omega^2}, & \omega_2+\frac{\sin\omega}{\omega}+\omega_1\omega_3\frac{1-\cos\omega}{\omega} \\ \omega_3+\frac{\sin\omega}{\omega}+\omega_2\omega_1\frac{1-\cos\omega}{\omega^2}, & \cos\omega+\omega_2^2\frac{1-\cos\omega}{\omega^2}, & -\omega_1+\frac{\sin\omega}{\omega}+\omega_2\omega_3\frac{1-\cos\omega}{\omega} \\ -\omega_2+\frac{\sin\omega}{\omega}+\omega_3\omega_1\frac{1-\cos\omega}{\omega^2}, & \omega_1+\frac{\sin\omega}{\omega}+\omega_3\omega_2\frac{1-\cos\omega}{\omega^2}, & \cos\omega+\omega_3^2\frac{1-\cos\omega}{\omega} \end{bmatrix} \tag{23}$$

and

$$\det\mathbf{Q} = 1, \qquad \mathrm{tr}\mathbf{Q} = 1+2\cos\omega, \qquad \mathbf{Q}^T = -\mathbf{Q}\,. \tag{24}$$

Since $|\mathbf{k}| = 1$, $|\omega| = \omega$, we deal with three independent parameters determining the matrix $[\mathbf{Q}^{ij}]$.

One should notice that $\mathbf{e}_1\underset{\circ}{\mathbf{e}}_1 \neq \mathbf{e}_2\underset{\circ}{\mathbf{e}}_2$. This means that the angles that the principal axes of inertia make with each other before and after the deformation are not identical, unless $\omega_1 = \omega_2$.

By the hypothesis about the in-plane undeformability of the cross-section during deformation of the beam $|\hat{\mathbf{x}}| = |\underset{\circ}{\hat{\mathbf{x}}}|$, the coordinates (x^1, x^2) and (x_0^1, x_0^2) are the same, though related to the basis $\{\mathbf{e}_i\}$ or $\{\underset{\circ}{\mathbf{e}}_i\}$, respectively. As a result of deformation, the axis of the beam curves and elongates. Due to shearing the vector $\mathbf{e}_3^*$ tangent to the deformed axis differs from the vector $\mathbf{e}_3$ normal to the conventional transverse cross-section of the deformed beam. The vector $\mathbf{e}_3^*$ determines a plane additionally rotated with respect to the plane of the transverse cross-section. It is given by the formula

$$\mathbf{e}_3^* = \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}s_0}\,, \tag{25}$$

where $\mathbf{r} = \mathbf{r}_0 + \mathbf{u}_0$ and $\mathbf{u}_0 = u_0^m \underset{0}{\mathbf{e}}_m$ are the vectors of position and displacement of a material point located on the axis of the beam. Using relations (9) we obtain

$$e_3^* = g^i \underset{0}{\mathbf{e}}_i \tag{26}$$

Using (11), we may represent the components g^i of the vector $\mathbf{e}_3^*$ in the basis $\underset{0}{\mathbf{e}}_i$ as

$$\begin{aligned} g^1 &= \frac{du_0^1}{ds_0} - u_0^2(\tau_0 + \vartheta_0') + u_0^3\kappa_0 \cos\vartheta_0 \ , \\ g^2 &= \frac{du_0^2}{ds_0} + u_0^1(\tau_0 + \vartheta_0') - u_0^3\kappa_0 \sin\vartheta_0 \ , \\ g^3 &= 1 + \frac{du_0^3}{ds_0} - u_0^1\kappa_0 \cos\vartheta_0 + u_0^2\kappa_0 \sin\vartheta_0 \ . \end{aligned} \tag{27}$$

The cosines of the angles that the vector $\mathbf{e}_3^*$ makes with the vectors of the rotated basis $\{\mathbf{e}_n\}$ are

$$\cos\beta_m = \frac{\mathbf{e}_3^*\mathbf{e}_m}{|\mathbf{e}_3^*|} = (g^i/g)\, \underset{0}{\mathbf{e}}_i\, \mathbf{Q}\, \underset{0}{\mathbf{e}}_m, \qquad g = |\mathbf{e}_3^*| \ . \tag{28}$$

Hence, the vector $\mathbf{e}_3^*$ may be written in the basis $\{\mathbf{e}_m\}$

$$\mathbf{e}_3^* = g\cos\beta_m \mathbf{e}_m \ . \tag{29}$$

We may determine the spatial derivatives of the basis using the transformation (19) as

$$\frac{\mathrm{d}\mathbf{e}_m}{\mathrm{d}s_0} = \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}s_0}\, \underset{0}{\mathbf{e}}_m + \mathbf{Q}\, \frac{\mathrm{d}\underset{0}{\mathbf{e}}_m}{\mathrm{d}s_0} \ . \tag{30}$$

The tensor of rotation $\mathbf{Q}$, given in the basis $\underset{0}{\mathbf{e}}_m$, is the function of the place a material particle located on the axis of the beam occupies in space. It may be thought of as a line in the space $SO(3)$ of rotation group of ortogonal tensors. To determine the tangent to this line we need to differentiate the relation

$$\mathbf{Q}^T\,\mathbf{Q} = \mathbf{1} \tag{31}$$

Introducing a skew-symmetric tensor $\boldsymbol{\Lambda}_0$ defined by the relations

$$\boldsymbol{\Lambda}_0 = \mathbf{Q}^T\, \frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}s_0}, \qquad \boldsymbol{\Lambda}_0^T + \boldsymbol{\Lambda}_0 = \mathbf{0} \ , \tag{32}$$

we get

$$\frac{\mathrm{d}\mathbf{Q}}{\mathrm{d}s_0} = \mathbf{Q}\boldsymbol{\Lambda}_0 \ . \tag{33}$$

The tensor $\mathbf{\Lambda}_0$ from the manifold $so(3)$ of skew-symmetric tensors, written in the basis $\underset{0}{\mathbf{e}}_m$ represents rotations of a material line. It may be brought to the form

$$\mathbf{\Lambda}_0 = \boldsymbol{\lambda}_0 \times \mathbf{1}\ , \tag{34}$$

where [52]

$$\begin{aligned} \boldsymbol{\lambda}_0 &= \sin\omega\,\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}s_0} + (1-\cos\omega)\frac{\mathrm{d}\mathbf{k}}{\mathrm{d}s_0}\times\mathbf{k} + \frac{\mathrm{d}\omega}{\mathrm{d}s_0}\mathbf{k} \\ \text{or} & \\ \boldsymbol{\lambda}_0 &= \frac{\sin\omega}{\omega}\left(\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}s_0} - \frac{\mathrm{d}\omega}{\mathrm{d}s_0}\frac{\boldsymbol{\omega}}{\omega}\right) + \frac{1-\cos\omega}{\omega^2}\frac{\mathrm{d}\boldsymbol{\omega}}{\mathrm{d}s_0}\times\boldsymbol{\omega} + \frac{\mathrm{d}\omega}{\mathrm{d}s_0}\frac{\boldsymbol{\omega}}{\omega}\ , \end{aligned} \tag{35}$$

is an axial vector. Applying the formulas (9) and (33) to (30) we obtain

$$\frac{\mathrm{d}\mathbf{e}_m}{\mathrm{d}s_0} = \mathbf{K}\,\mathbf{e}_m \qquad \text{or} \qquad \frac{\mathrm{d}\mathbf{e}_m}{\mathrm{d}s_0} = \boldsymbol{\kappa}\times\mathbf{e}_m \tag{36}$$

where

$$\begin{aligned} \mathbf{K} &= \mathbf{Q}(\mathbf{\Lambda}_0 + \mathbf{K}_0)\mathbf{Q}^T = \mathbf{\Lambda} + \mathbf{Q}\mathbf{K}_0\mathbf{Q}^T\ , \\ \mathbf{\Lambda} &= \mathbf{\Lambda}_0\mathbf{Q}^T\ , \end{aligned} \tag{37}$$

is a skew-symmetric tensor in the basis $\{\mathbf{e}_m\}$. This tensor describes the axis of the deformed beam. Its axial vector may be represented as

$$\boldsymbol{\kappa} = \kappa^m\mathbf{e}_m \tag{38}$$

and hence

$$\kappa^m = \frac{1}{2}\epsilon_{mjk}\left(\mathbf{e}_{j,3}\,\mathbf{e}_k\right)\,. \tag{39}$$

The base vectors at an arbitrary material point of the deformed beam may be determined by differentiating the position vector of this point with respect to the coordinates x^m. The position of this point follows from the general formula (18), where $\hat{\mathbf{x}} = x^\alpha\mathbf{e}_\alpha$, $(\alpha = 1, 2)$. To make its description precise we need to fix the warping vector $\mathbf{w}$. We assume that it is normal to the conventional transverse cross-section. Then

$$\mathbf{w} = w(x^1, x^2, x^3)\mathbf{e}_3\ , \tag{40}$$

where w is the amplitude of the distribution of warping. With this choice we have

$$\begin{aligned} \mathbf{a}^*_\alpha &= \mathbf{e}_\alpha + w_{,\alpha}\,\mathbf{e}_3\ , \\ \mathbf{a}^*_3 &= \mathbf{e}^*_3 + \boldsymbol{\kappa}\times(\hat{\mathbf{x}} + \mathbf{w}) + w'\mathbf{e}_3\ , \end{aligned} \tag{41}$$

where $(\cdot)_{,\alpha} = \frac{\mathrm{d}}{\mathrm{d}x^\alpha}$, $(\cdot)' = \frac{\mathrm{d}}{\mathrm{d}s_0}$, $\mathbf{a}^*_\alpha = \partial\mathbf{x}/\partial x^\alpha_0$. Since $\boldsymbol{\kappa} = \kappa^m\mathbf{e}_m$, the latter formula may be rewritten as follows

$$\begin{aligned} \mathbf{a}^*_3 = (g\cos\beta_1 - \kappa^3x^2_0 + w\kappa^2)\mathbf{e}_1 + (g\cos\beta_2 + \kappa^3x^1_0 - w\kappa^1)\mathbf{e}_2 + \\ + (g\cos\beta_3 + w' + \kappa^1x^2_0 - \kappa^2x^1_0)\mathbf{e}_3\,. \end{aligned} \tag{42}$$

In the technical theory it is also assumed [44] that the amplitude of warping is of the form

$$w(x_0^1, x_0^2, s_0, t) = \sum_{\lambda=0}^{2} \varphi_\lambda(x_0^1, x_0^2)\theta_\lambda(s_0, t)\ . \tag{43}$$

The functions $\theta_\lambda(s_0, t)$ are unknown functions of the distribution of warpings in the cross-sections along the axis of the beam arisen from twisting and shearing. φ_λ are prescribed functions of distribution of warping in the cross-section under consideration and are defined in the local coordinate system connected with the cross-section of the undeformed beam. Another assumption made in dynamics is that the function φ_0 is determinable from a static problem for pure twisting of a prismatic beam, and the functions φ_1 and φ_2 from a static problem for bending of a beam under transverse forces [44]. With these assumptions we may write

$$\frac{dw}{ds_0} = \sum_{\lambda=0}^{2} (\vartheta_0' \varrho_\lambda^* \theta_\lambda + \varphi_\lambda \theta' \lambda) \tag{44}$$

where

$$\varrho_\lambda^* = \epsilon_{\alpha\beta}\, x_0^\beta\, \frac{\partial \varphi_\lambda}{\partial x_0^\alpha} \tag{45}$$

after we use the transformation formulas (2).

If the constraints on warping of the transverse cross-sections are neglected, it is assumed that $\theta'_\lambda = 0$.

3. Measures of deformation

The gradient of deformation

$$\mathbf{F} = \frac{\partial \mathbf{x}}{\partial \mathbf{x}_0} = \frac{\partial \mathbf{x}}{\partial x_0^\alpha} \otimes \underset{\circ}{\mathbf{e}}_\alpha + \frac{\partial \mathbf{x}}{\partial s_0} \otimes \underset{\circ}{\mathbf{e}}_3 = \mathbf{a}_\alpha^* \times \underset{\circ}{\mathbf{e}}_\alpha + \mathbf{a}_3^* \otimes \underset{\circ}{\mathbf{e}}_3 \tag{46}$$

furnishes full information about the strains and rotations of the material line elements in the neighborhood of a material point of a deformable body. In the above formula $\mathbf{x}_0$ and $\mathbf{x}$ are the position vectors of an arbitrary material point in the undeformed and deformed configuration, respectively. In our case of a beam treated as a one-dimensional body, this gradient may be brought to the form

$$\mathbf{F} = \mathbf{Q} + \mathbf{e}_3 \otimes \nabla \mathbf{w} + [(\mathbf{e}_3^* - \mathbf{e}_3) + \boldsymbol{\kappa} \times (\hat{\mathbf{x}} + \mathbf{w}) + w'\mathbf{e}_3] \otimes \underset{\circ}{\mathbf{e}}_3 \tag{47}$$

when we use the relations (18), (40) and (41), and (36). Alternatively, we may write

$$\mathbf{F} = \mathbf{Q}\{1 + \underset{\circ}{\mathbf{e}}_3 \otimes \nabla \mathbf{w} + [(\mathbf{Q}^T \mathbf{e}_3^* - \underset{\circ}{\mathbf{e}}_3) + \mathbf{Q}^T \boldsymbol{\kappa} \times (\underset{\circ}{\hat{\mathbf{x}}} + \underset{\circ}{\mathbf{w}}) + w' \underset{\circ}{\mathbf{e}}_3] \otimes \underset{\circ}{\mathbf{e}}_3\} \tag{48}$$

where

$$\mathbf{Q} = \mathbf{e}_i \otimes \underset{\circ}{\mathbf{e}}_i , \qquad \nabla \mathbf{w} = \frac{\partial w}{\partial x_0^\alpha} \underset{\circ}{\mathbf{e}}_\alpha , \qquad \hat{\mathbf{x}}_0 = x_0^\alpha \underset{\circ}{\mathbf{e}}_\alpha , \qquad \underset{\circ}{\mathbf{w}} = w \underset{\circ}{\mathbf{e}}_3 \, . \tag{49}$$

Since $\underset{\circ}{\hat{\mathbf{x}}} + \underset{\circ}{\mathbf{w}} = \mathbf{Q}^T(\mathbf{x} - \mathbf{r})$, we must also have

$$\mathbf{F} = \mathbf{Q}\{\mathbf{1} + \underset{\circ}{\mathbf{e}}_3 \otimes \nabla \mathbf{w} + [(\mathbf{Q}^T \mathbf{e}_3^* - \underset{\circ}{\mathbf{e}}_3) + \mathbf{Q}^T \boldsymbol{\kappa} \times \mathbf{Q}^T(\mathbf{x} - \mathbf{r}) + w' \underset{\circ}{\mathbf{e}}_3] \otimes \underset{\circ}{\mathbf{e}}_3\} \, . \tag{50}$$

The tensor of rotation $\mathbf{Q}$ appearing in the above formulas determines the mean rotation of the transverse cross-section of the deformed beam. It does not account for the local rotations of the material points connected with the deformation of the beam axis nor for the warpings in these cross-sections. The vectors occurring in these formulas

$$\boldsymbol{\epsilon} = \mathbf{e}_3^* - \mathbf{e}_3 \qquad \text{or} \qquad \boldsymbol{\epsilon}_0 = \mathbf{Q}^T \boldsymbol{\epsilon} = \mathbf{Q}^T \mathbf{e}_3^* - \underset{\circ}{\mathbf{e}}_3 \tag{51}$$

and tensors

$$\boldsymbol{\Lambda} = \mathbf{K} - \mathbf{Q}\,\mathbf{K}_0 \mathbf{Q}^T \qquad \text{or} \qquad \boldsymbol{\Lambda}_0 = \mathbf{Q}^T \mathbf{K}\,\mathbf{Q} - \mathbf{K}_0 \, , \tag{52}$$

describe strains of the beam in the basis $\{\mathbf{e}_m\}$ or the basis $\{\underset{\circ}{\mathbf{e}}_m\}$, respectively. They describe stretching, contraction and alteration of curvature. They were suggested by Reissner [27]. The vector $\boldsymbol{\epsilon}$ determines the difference between the vector tangent to the axis of the deformed beam and the vector normal to the transverse cross-section of this beam.

Other measures of deformations are also introduced in mechanics of deformable body. For large strain problems the Green tensor is often used

$$\mathbf{E} = \frac{1}{2}(\mathbf{F}^T \mathbf{F} - \mathbf{1}) \, , \tag{53}$$

where $\mathbf{F}$ is the gradient of deformation. For a body of initially complex geometry, to grasp alteration of strain resulting from loads, one may introduce a relative measure of deformation

$$\tilde{\mathbf{E}} = \mathbf{E} - \mathbf{E}_0 \tag{54}$$

where

$$\mathbf{E}_0 = \frac{1}{2}(\mathbf{F}_0^T \mathbf{F}_0 - \mathbf{1}) \, . \tag{55}$$

$\mathbf{E}_0$ and $\mathbf{F}_0$ describe the initial geometry of the body. In the case of small strains it is convenient to use a symmetric tensor of strain

$$\boldsymbol{\Gamma}^* = \frac{1}{2}(\boldsymbol{\Gamma} + \boldsymbol{\Gamma}^T) \tag{56}$$

where

$$\boldsymbol{\Gamma} = \mathbf{Q}^T \mathbf{F} - \mathbf{1} \, , \tag{57}$$

is a tensor of deformation introduced by Jaumann [54].

For the model of beam considered in this paper the gradient of full deformation is defined by the formula (48).

On the other hand, the gradient of the natural geometry of the beam

$$\mathbf{F}_0 = \frac{\partial \overset{\circ}{\mathbf{x}}}{\partial \mathbf{r}_0} = \underset{\circ}{\mathbf{a}}_i \otimes \underset{\circ}{\mathbf{e}}_j \,. \tag{58}$$

Then the Green tensor of relative strain is

$$\tilde{\mathbf{E}} = \frac{1}{2}[\mathbf{a}_i^* \mathbf{a}_j^* - \underset{\circ}{\mathbf{a}}_i \, \underset{\circ}{\mathbf{e}}_j](\underset{\circ}{\mathbf{e}} \otimes \underset{\circ}{\mathbf{e}}_j) \,. \tag{59}$$

Using the formulas (18), (41) and (42) we may write the components of this tensor in the material basis as

$$\begin{aligned}
\tilde{E}_{\alpha\beta} &= \frac{1}{2} w_{,\alpha} w_{,\beta}, \qquad (\alpha, \beta = 1, 2) \\
\tilde{E}_{13} &= \Gamma_{13}^* + \frac{1}{2}[(g\cos\beta_3 - 1) - x_0^1 \kappa^2 + x_0^2 \kappa^1 + w_{,3}] w_{,1} \,, \\
\tilde{E}_{23} &= \Gamma_{23}^* + \frac{1}{2}[(g\cos\beta_3 - 1) - x_0^1 \kappa^2 + x_0^2 \kappa^1 + w_{,3}] w_{,2} \,, \\
\tilde{E}_{33} &= \Gamma_{33}^* + \frac{1}{2}(g\cos\beta_1 - x_0^2 \kappa^3 - w\kappa^2)^2 + \frac{1}{2}(g\cos\beta_2 + x_0^1 \kappa^3 - w\kappa^1)^2 + \,, \\
&- \frac{1}{2}[(x_0^1)^2 + (x_0^2)^2](\kappa_0^3)^2 - \frac{1}{2}(-x_0^1 \kappa_0^2 + x_0^2 \kappa_0^1)^2 \,,
\end{aligned} \tag{60}$$

where

$$\begin{aligned}
\Gamma_{13}^* &= \frac{1}{2}(g\cos\beta_1 - x_0^2 \tilde{\kappa}^3 + w\kappa^2 + w_{,1}) \,, \\
\Gamma_{23}^* &= \frac{1}{2}(g\cos\beta_2 + x_0^1 \tilde{\kappa}^3 - w\kappa^1 + w_{,2}) \,, \\
\Gamma_{33}^* &= (g\cos\beta_3 - 1) - x_0^1 \tilde{\kappa}^2 + x_0^2 \tilde{\kappa}^1 + w' \,,
\end{aligned} \tag{61}$$

are the components of a symmetric relative Jaumann tensor and $\tilde{\kappa}^i = \kappa^i - \kappa_0^i$.

Proceeding further we also need to work out a model for warping of the transverse cross-sections.

In the case of constrained twisting of a beam, pre-twisted in the natural state, Dzhanelidze's hypothesis [24] is used. According to this hypothesis the amplitude of warping is governed by the following relation

$$w = \varphi(\hat{\mathbf{x}}_0) \tilde{\kappa}^3(s_0, t) \,, \tag{62}$$

where $\varphi(\hat{\mathbf{x}}_0)$ is the Saint-Venant twisting function defined in the local coordinate system – a system connected with the transverse cross-section of the beam. Hence

$$w_{,3} = \vartheta_0' \varrho_0^* \tilde{\kappa}^3 + \varphi(\hat{x}_0)(\tilde{\kappa}^3)', \qquad (\cdot)' = \frac{\partial}{\partial s_0} \,, \tag{63}$$

where

$$\varrho_0^* = \epsilon_{\alpha\beta}\, x_0^\beta \,\frac{\partial\varphi}{\partial x_0^\alpha}, \qquad (\alpha,\beta = 1,2)\,. \tag{64}$$

Taking into account the foregoing assumptions, and assuming additionally that the warpings of the cross-sections are small, the components of strain Γ_{m3}^* may be represented as follows

$$\begin{aligned}
\Gamma_{13}^* &= \frac{1}{2}\Big[\epsilon_0^1 + (\frac{\partial\varphi}{\partial x_0^1} - x_0^2)\tilde{\kappa}^3\Big]\,,\\
\Gamma_{23}^* &= \frac{1}{2}\Big[\epsilon_0^2 + (\frac{\partial\varphi}{\partial x_0^2} + x_0^1)\tilde{\kappa}^3\Big]\,,\\
\Gamma_{33}^* &= \epsilon_0^3 - x_0^1\tilde{\kappa}^2 + x_0^2\tilde{\kappa}^1 + \vartheta_0'\varrho_0^*\tilde{\kappa}^3 + \varphi(\tilde{\kappa}^3)'\,.
\end{aligned} \tag{65}$$

They are functions of generalized strains

$$\epsilon_0^m = g\cos\beta_m - \delta_{m3}, \qquad \tilde{\kappa}^m, \qquad (\tilde{\kappa}^3)'. \tag{66}$$

where $\delta_{\alpha 3} = 0$ for $\alpha = 1,2$, $\delta_{33} = 1$.

4. Constitutive equations

Determination of the resultant stresses arising in the material of the body, compatible with the model of deformation assumed, is an important issue. A lot of attention was devoted to this question in the theory of beams [35, 53].

For elastic isotropic and homogenous material the strain energy measured per unit volume of the undeformed body is most often assumed to have the form

$$\psi_0 = \frac{1}{2}\lambda \mathrm{tr}\,\tilde{\mathbf{E}} + \mu \mathrm{tr}\,(\tilde{\mathbf{E}}^2)\,. \tag{67}$$

Then the symmetric Piola-Kirchhoff stress tensor is

$$\tilde{\mathbf{T}} = \frac{\partial\psi_0}{\partial\mathbf{E}} = \lambda(\mathrm{tr}\,\tilde{\mathbf{E}})\mathbf{1} + 2\mu\tilde{\mathbf{E}}\,, \tag{68}$$

where λ and μ are the Lame material constants. Analogous relations apply to the tensor of strain Γ^* and the symmetric stress tensor T^*, introduced by Jaumann.

The relations written above are valid for sufficiently small strains, but do not exclude occurrence of finite rotations and displacements for sufficiently slender bodies. In the case of beams undergoing small strains it is assumed that

$$T_{\alpha 3}^* = 2G\Gamma_{\alpha 3}^*, \qquad T_{33}^* = E\Gamma_{33}^*\,, \tag{69}$$

where G and E are the Kirchhoff and Young moduli, respectively. Then, the strain energy for beams may be expressed as follows

$$\Psi_0 = \frac{1}{2}\int_{A_0} \Big[2G\Gamma^*_{\alpha 3}\Gamma^*_{\alpha 3} + E(\Gamma^*_{33})^2\Big]\, dA\,, \tag{70}$$

where A_0 is the transverse cross-section of the beam in the natural configuration. Taking into account (65) and (70) one obtains

$$\Psi_0 = \Psi_0\Big(\epsilon^i_0, \tilde{\kappa}^i_0, (\tilde{\kappa}^3)'\Big)\,, \tag{71}$$

as the function of generalized strains. Then, constitutive equations for internal forces in the material description are

$$Q_i = \frac{\partial \Psi_0}{\partial \epsilon^i_0}, \quad M_i = \frac{\partial \Psi_0}{\partial \tilde{\kappa}^i}, \quad B = \frac{\partial \Psi_0}{\partial (\tilde{\kappa}^3)'}\,. \tag{72}$$

Hence we get

$$\begin{aligned}
Q_1 &= GA\epsilon^1_0\,,\\
Q_2 &= GA\epsilon^2_0\,,\\
Q_3 &= E\Big[A\epsilon^3_0 + S_1\tilde{\kappa}^1 - S_2\tilde{\kappa}^2 + \vartheta'_0 J_\varrho \tilde{\kappa}^3 + J_{\varphi\varphi}(\tilde{\kappa}^3)'\Big],\\
M_1 &= E\Big[S_1\epsilon^3_0 + J_{11}\tilde{\kappa}^1 - J_{12}\tilde{\kappa}^2 + \vartheta'_0 J_{1\varrho}\tilde{\kappa}^3 + J_{1\varphi}(\tilde{\kappa}^3)'\Big]\,,\\
M_2 &= E\Big[-S_2\epsilon^3_0 - J_{12}\tilde{\kappa}^1 + J_{22}\tilde{\kappa}^2 - \vartheta'_0 J_{2\varrho}\tilde{\kappa}^3 - J_{2\varphi}(\tilde{\kappa}^3)'\Big],\\
M_3 &= GJ_v\tilde{\kappa}^3 + \vartheta'_0 E\Big[J_\varrho\epsilon^3_0 + J_{1\varrho}\tilde{\kappa}^1 - J_{2\varrho}\tilde{\kappa}^2 + \vartheta'_0 J_{\varrho\varrho}\tilde{\kappa}^3 + J_{\varrho\varphi}(\tilde{\kappa}^3)'\Big]\,,\\
B &= E\Big[S_\varphi\epsilon^3_0 + J_{1\varphi}\tilde{\kappa}^1 - J_{2\varphi}\tilde{\kappa}^2 + \vartheta'_0 J_{\varrho\varphi}\tilde{\kappa}^3 + J_{\varphi\varphi}(\tilde{\kappa}^3)'\Big]\,,
\end{aligned} \tag{73}$$

where A, S_α, S_φ, J_v, $J_{\alpha\beta}$, J_ϱ, $J_{\alpha\varrho}$, $J_{\varrho\varrho}$, $J_{\alpha\varphi}$, $J_{\varrho\varphi}$ are geometrical characteristics of the cross-sections transverse to the undeformed beam defined in [44], see (117). The forces Q_α are the transverse forces, Q_3 the longitudinal force, M_α are the bending moments, M_3 is the total twisting moment and B is the bending-twisting bimoment.

Relations for the transverse forces Q_α do not account for the decrease of stiffness due to nonuniform distribution of the shearing stresses in the transverse cross-sections of the beam. To include this effect it is necessary to consider warping of the transverse cross-sections caused by shearing. To this end the following hypothesis is assumed [44]

$$w = \varphi_1(\underset{\circ}{\mathrm{x}})\epsilon^1_0(s_0, t) + \varphi_2(\underset{\circ}{\mathrm{x}})\epsilon^2_0(s_0, t)\,, \tag{74}$$

where φ_λ are bending functions of a prismatic beam.

To simplify considerations we shall confine attention to a prismatic beam undergoing merely stretching and shearing. For small strains the components of the deformation tensor take the form

$$
\begin{aligned}
\tilde{\Gamma}_{13} &= \Big(1+\frac{\partial \varphi_1}{\partial \overset{\circ}{x}{}^1}\Big)\epsilon_0^1 + \Big(\frac{\partial \varphi_2}{\partial \overset{\circ}{x}{}^1}\Big)\epsilon_0^2 \,, \\
\tilde{\Gamma}_{23} &= \Big(\frac{\partial \varphi_1}{\partial \overset{\circ}{x}{}^2}\Big)\epsilon_0^1 + \Big(1+\frac{\partial \varphi_2}{\partial \overset{\circ}{x}{}^2}\Big)\epsilon_0^2 \,, \qquad (75)\\
\tilde{\Gamma}_{33} &= \epsilon_0^3 + \varphi_1(\epsilon_0^1)' + \varphi_2(\epsilon_0^2)' \,.
\end{aligned}
$$

Using (70) we determine the elastic energy

$$\Psi_0 = \Psi_0(\epsilon_0^i, (\epsilon_0^\alpha)') \qquad (76)$$

and the resultant internal forces

$$Q_i = \frac{\partial \Psi_0}{\partial \epsilon_0^i}\,, \quad H_\alpha = \frac{\partial \Psi_0}{\partial (\epsilon_0^\alpha)'} \,. \qquad (77)$$

They are given by the relations

$$
\begin{aligned}
Q_1 &= GA(k_{11}\epsilon_0^1 + k_{12}\epsilon_0^2)\,, \\
Q_2 &= GA(k_{12}\epsilon_0^1 + k_{22}\epsilon_0^2)\,, \\
Q_3 &= EA\epsilon_0^3\,, \qquad (78)\\
H_1 &= E\big[S_{\varphi_1}\epsilon_0^3 + J_{\varphi_1\varphi_1}(\epsilon_0^1)' + J_{\varphi_1\varphi_2}(\epsilon_0^2)'\big]\,, \\
H_2 &= E\big[S_{\varphi_2}\epsilon_0^3 + J_{\varphi_1\varphi_2}(\epsilon_0^1)' + J_{\varphi_2\varphi_2}(\epsilon_0^2)'\big]\,,
\end{aligned}
$$

where

$$k_{\alpha\beta} = \frac{1}{A_0}\int_{A_0} \Big(\delta_{\alpha\kappa} + \frac{\partial \varphi_\alpha}{\partial \overset{\circ}{x}_k}\Big)\Big(\delta_{\beta\kappa} + \frac{\partial \varphi_\beta}{\partial \overset{\circ}{x}_k}\Big)\, dA\,, \qquad (79)$$

are the coefficients of shearing and

$$S_{\varphi_\alpha} = \int_{A_0} \varphi_\alpha \, dA \quad, J_{\varphi_\alpha\varphi_\beta} = \int_{A_0} \varphi_\alpha\varphi_\beta \, dA\,, \qquad (80)$$

are the geometrical characteristics of the transverse cross-sections associated with constrained shearing.

The algorithm of determination of the bending function and the geometrical characteristics was presented in [44]. The forces H_α are self-equilibrated moments of constrained shearing.

Introducing the vector of generalized forces

$$\mathbf{H} = \mathrm{col}\,(Q_i, M_i, B)\,, \qquad (81)$$

and vector of generalized strains

$$\gamma = \operatorname{col}\left(\epsilon_0^i, \tilde{\kappa}_0^i, (\tilde{\kappa}^3)'\right), \tag{82}$$

and using the relations (73) and (78), we may write the following constitutive relation

$$\mathbf{H} = \mathbf{K}\gamma\,, \tag{83}$$

valid within the kinematics assumed and small strains. One should notice that in the case considered here the stiffness matrix $\mathbf{K}$ is symmetric

$$[\mathbf{K}] = \begin{bmatrix} k_{11}GA & k_{12}GA & 0 & 0 & 0 & 0 & 0 \\ k_{21}GA & k_{22}GA & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & EA & ES_1 & -ES_2 & \vartheta_0' EJ_\varrho & ES_\varphi \\ 0 & 0 & ES_1 & EJ_{11} & -EJ_{12} & \vartheta_0' EJ_{1\varrho} & EJ_{1\varphi} \\ 0 & 0 & -ES_2 & -EJ_{21} & EJ_{22} & -\vartheta_0' EJ_{2\varrho} & -EJ_{2\varphi} \\ 0 & 0 & \vartheta_0' EJ_\varrho & \vartheta_0' EJ_{1\varrho} & -\vartheta_0' EJ_{2\varrho} & GJ_\vartheta + \vartheta_0'^2 EJ_{\varrho\varrho} & \vartheta_0' EJ_{\varrho\varphi} \\ 0 & 0 & ES_\varphi & EJ_{1\varphi} & -EJ_{2\varphi} & \vartheta_0' EJ_{\varrho\varphi} & EJ_{\varphi\varphi} \end{bmatrix}. \tag{84}$$

5. Equations of motion

5.1. Internal forces

To derive equations of dynamics of a beam subjected to the action of surface and mass forces, we need to determine the internal forces arising in its transverse cross-sections. We derive them from the variation of work measured per unit undeformed volume performed by the stresses [54]

$$\delta \mathcal{W}_0 = \int_{\mathcal{B}} \mathbf{T}^B : \delta \mathbf{U}\, dV\,, \tag{85}$$

where $\mathbf{T}^B$ is the Biot stress tensor and $\mathbf{U}$ is the stretch tensor; symbol : denotes full contraction of two tensors. For the beam model under consideration, by (50) and (51), we have

$$\mathbf{U} = \mathbf{Q}^T\mathbf{F} = \mathbf{1} + \underset{\circ}{\mathbf{e}}_3 \otimes \nabla\mathbf{w} + [\boldsymbol{\epsilon}_0 + \boldsymbol{\kappa}_0 \times \mathbf{Q}^T(\mathbf{x} - \mathbf{r}) + w'\underset{\circ}{\mathbf{e}}_3] \otimes \underset{\circ}{\mathbf{e}}_3\,, \tag{86}$$

where $\boldsymbol{\kappa}_0 = \mathbf{Q}^T\boldsymbol{\kappa}$.

Taking into account warping of the transverse cross-sections, caused by twisting and bending, and according to the hypothesis (43), we get

$$\begin{aligned} \delta\mathbf{U} &= \delta\boldsymbol{\epsilon}_0 \otimes \underset{\circ}{\mathbf{e}}_3 + \left[\delta\boldsymbol{\kappa} \times \mathbf{Q}^T(\mathbf{x} - \mathbf{r})\right] \otimes \underset{\circ}{\mathbf{e}}_3 + \sum_{\lambda=0}^{2} \delta\,\theta_\lambda' \varphi_\lambda (\underset{\circ}{\mathbf{e}}_3 \otimes \underset{\circ}{\mathbf{e}}_3) + \\ &+ \sum_{\lambda=0}^{2} \delta\theta_\lambda \left[\varphi_{\lambda,\alpha}(\underset{\circ}{\mathbf{e}}_3 \otimes \underset{\circ}{\mathbf{e}}_\alpha) + \varphi_\lambda(\boldsymbol{\kappa}_0 \times \underset{\circ}{\mathbf{e}}_3) + \varphi_\lambda'(\underset{\circ}{\mathbf{e}}_3 \otimes \underset{\circ}{\mathbf{e}}_3)\right], \end{aligned} \tag{87}$$

where $(\cdot)' = \frac{\partial}{\partial s_0}$, $(\cdot)_{,\alpha} = \frac{\partial}{\partial \underset{\circ}{x}_\alpha}$. Since

$$\mathbf{T}^\beta = \mathbf{T}^B_\alpha \otimes \underset{\circ}{\mathbf{e}}_\alpha + \mathbf{T}^B_3 \otimes \underset{\circ}{\mathbf{e}}_3 \,, \tag{88}$$

by (85) we get for the unit undeformed length

$$\begin{aligned}\delta \mathcal{W}_0 = \int_{A_0} (\mathbf{N}\delta\boldsymbol{\epsilon}_0 + \mathbf{M}\delta\boldsymbol{\kappa} \quad &+ \quad M_\varphi \delta\theta_0 + G_1\delta\theta_1 + G_2\delta\theta_2 + \\ &+ \quad B\delta\theta_0' + H_1\delta\theta_1' + H_2\delta\theta_2')\, dA \,, \end{aligned}\tag{89}$$

where

$$\begin{aligned}
\mathbf{N} &= \int_{A_0} \mathbf{T}^B_3 \, dA = N_i \underset{\circ}{\mathbf{e}} \,, \\
\mathbf{M} &= \int_{A_0} \left[\mathbf{Q}^T(\mathbf{x}-\mathbf{r}) \times \mathbf{T}^B_3\right] dA = M_i \underset{\circ}{\mathbf{e}} \,, \\
M_\varphi &= \underset{\circ}{\mathbf{e}}_3 \int_{A_0} \left[\varphi_{0,\alpha}\mathbf{T}^B_\alpha + \varphi_0(\mathbf{T}^B_3 \times \boldsymbol{\kappa}_0) + \varphi_0'\mathbf{T}^B_3\right] dA \,, \\
G_\lambda &= \underset{\circ}{\mathbf{e}}_3 \int_{A_0} \left[\varphi_{\lambda,\alpha}\mathbf{T}^B_\alpha + \varphi_\lambda(\mathbf{T}^B_3 \times \boldsymbol{\kappa}) + \varphi_\lambda'\mathbf{T}^B_3\right] dA \,, \qquad (\lambda = 1,2) \,, \\
B &= \underset{\circ}{\mathbf{e}}_3 \int_{A_0} \varphi_0 \mathbf{T}^B_3 \, dA \,, \\
H_\lambda &= \underset{\circ}{\mathbf{e}}_3 \int_{A_0} \varphi_\lambda \mathbf{T}^B_3 \, dA \,.
\end{aligned}\tag{90}$$

These are material forces, defined in the undeformed configuration, valid for finite deformations. As is visible from the relation (89) for the variation of work performed by the internal forces, the strain measures work-conjugate with the forces $\mathbf{N}$, $\mathbf{M}$, M_φ, G_λ, B and H_λ are ϵ_0, κ_κ, $\delta\theta_0$, $\delta\theta_\lambda$, $\delta\theta_0'$ and $\delta\theta_\lambda'$, respectively.

The components of the vector $\mathbf{N}$ are the transverse forces and the longitudinal force, of the vector $\mathbf{M}$ the bending and twisting moment, M_φ and B is the moment and bending-twisting bimoment, occurring at constrained twisting, and G_λ and H_λ are the components of the transverse force and moment, respectively, caused by constrained shearing.

In the case of a pre-twisted beam, the derivative with respect to the distributions of warping appearing in (90) is equal to

$$\varphi_\lambda' = \frac{\partial \varphi_\lambda}{\partial s_0} = \vartheta_0' \varrho_\lambda^* \,, \tag{91}$$

where ϱ_λ^* is given by the formula (45).

Introducing asymmetrical Piola-Kirchhoff stress tensor $\mathbf{T}^0$ related to the Biot tensor through the formula $\mathbf{T}^0 = \mathbf{Q}\mathbf{T}^B$, one obtains

$$\mathbf{N} \;=\; \mathbf{Q}^T\mathbf{n}, \qquad \mathbf{n} = \int_{A_0} \mathbf{T}^0_3 \, dA \,,$$

$$\begin{aligned} \mathbf{M} &= \mathbf{Q}^T\mathbf{m}, \qquad \mathbf{m} = \int_{A_0} (\mathbf{x}-\mathbf{r}) \times \mathbf{T}_3^0\, dA\,, \\ M_\varphi &= \mathbf{e}_3 \int_{A_0} \left[\varphi_{0,\alpha}\mathbf{T}_\alpha^0 + \varphi_0(\mathbf{T}_3^0 \times \boldsymbol{\kappa}) + \varphi_0'\mathbf{T}_3^0\right] dA\,, \\ B &= \mathbf{e}_3 \int_{A_0} \varphi_0 \mathbf{T}_3^0\, dA\,. \end{aligned} \tag{92}$$

Similar relations hold for the forces Q_λ and H_λ, $(\lambda = 1,2)$.

The spatial vectors of forces $\mathbf{n}$ and $\mathbf{m}$ act in the transverse cross-sections in the deformed configuration. Their components in the basis $\{\mathbf{e}_i\}$ are the same as those of the vectors $\mathbf{N}$ and $\mathbf{M}$ in the basis $\{\underset{\circ}{\mathbf{e}}\}$.

Upon introducing the Saint-Venant's twisting moment

$$M_v = \int_{A_0} \Big(\frac{\partial\varphi}{\partial \underset{\circ}{x}_\alpha} - \epsilon_{\alpha\beta}\underset{\circ}{x}_\beta\Big) T_{\alpha 3}^B\, dA\,, \tag{93}$$

one may represent the bending-twisting moment as follows

$$M_\varphi = M_v - M_3 + \vartheta_0' B_\varrho^* - \epsilon_{\alpha\beta}\kappa_\alpha \int_{A_0} \varphi_0 T_{\beta 3}^B\, dA\,, \tag{94}$$

where

$$B_\varrho^* = \int_{A_0} \varrho_0^* T_{33}^B\, dA \tag{95}$$

and ϱ_0^* is given by the formula (64).

The relation (94) is a generalization of the well-known Vlasov's relation [25] to the case of finite deformations of a spatially pre-twisted and curved beam.

5.2. Equations of dynamics

We shall derive the equations of motion for the model assumed using the material principles of conservation of momentum and moment of momentum of the three-dimensional continuum mechanics.

$$\operatorname{Div}\mathbf{T}^0 + \varrho_0\mathbf{b}_0 = \varrho_0\ddot{\mathbf{x}}, \qquad \frac{\partial\mathbf{x}}{\partial \underset{1}{x}_i} \times \mathbf{T}_i^0 = \mathbf{0}\,, \tag{96}$$

where $\mathbf{T}^0$ is the Piola-Kirchhoff stress tensor, $\mathbf{b}_0$ is the mass load and ϱ_0 is the specific density of the body, all related to the unit undeformed volume of the body.

Using the definition of the vector of forces (92) $\mathbf{n}$ and the principle of conservation of momentum (96) we get

$$\frac{\partial\mathbf{n}}{\partial s_0} = \int_{A_0} \frac{\partial\mathbf{T}_3^0}{\partial s_0}\, dA = -\int_{A_0} (\mathbf{T}_{\alpha,\alpha}^0 + \varrho_0\mathbf{b}_0)\, dA + \int_{A_0} \varrho_0\ddot{\mathbf{x}}\, dA\,. \tag{97}$$

Applying the Gauss-Ostrogradsky formula we obtain

$$\frac{\partial \mathbf{n}}{\partial s_0} + \bar{\mathbf{q}} + \tilde{\mathbf{q}} = \frac{d\mathbf{l}}{dt}, \tag{98}$$

where

$$\bar{\mathbf{q}} = \int_{\partial A_0} \mathbf{T}^0_\alpha n_\alpha \, d(\partial A), \qquad \tilde{\mathbf{q}} = \int_{A_0} \varrho_0 \mathbf{b}_0 \, dA, \tag{99}$$

are the mass and surface loads, respectively, n_α are the direction cosines of the normal to the lateral surface of the beam and

$$\mathbf{l} = \int_{A_0} \varrho_0 \dot{\mathbf{x}} \, dA \tag{100}$$

is the momentum measured per unit length of the undeformed beam.

In view of the definition of the vector of internal forces (92) and the principle of conservation of momentum we must have

$$\begin{aligned} \frac{\partial \mathbf{m}}{\partial s_0} &= \frac{\partial}{\partial s_0} \int_{A_0} (\mathbf{x} - \mathbf{r}) \times \mathbf{T}^0_3 \, dA = \int_{A_0} \frac{\partial \mathbf{x}}{\partial s_0} \times \mathbf{T}^0_3 \, dA - \frac{\partial \mathbf{r}}{\partial s_0} \times \int_{A_0} \mathbf{T}^0_3 \, dA + \\ &+ \int_{A_0} (\mathbf{x} - \mathbf{r}) \times [\varrho_0 \ddot{\mathbf{x}} - \varrho_0 \mathbf{b}_0 - \mathbf{T}^0_{\alpha,\alpha}] \, dA . \end{aligned}$$

Applying the Gauss-Ostrogradsky formula and using (96) for the moment of momentum we obtain the equation

$$\frac{\partial \mathbf{m}}{\partial s_0} + \mathbf{e}^*_3 \times \mathbf{n} + \bar{\mathbf{m}} + \tilde{\mathbf{m}} = \frac{d\mathbf{h}}{dt}, \tag{101}$$

where

$$\bar{\mathbf{m}} = \int_{\partial A_0} \left[(\mathbf{x} - \mathbf{r}) \times \mathbf{T}^0_\alpha \right] \cdot n_\alpha d(\partial A), \qquad \tilde{\mathbf{m}} = \int_{A_0} \varrho_0 (\mathbf{x} - \mathbf{r}) \times \mathbf{b}_0 \, dA, \tag{102}$$

are the loads from the surface and mass moments, respectively, and

$$\mathbf{h} = \int_{A_0} \varrho_0 [(\mathbf{x} - \mathbf{r}) \times \dot{\mathbf{x}}] \, dA, \tag{103}$$

is the moment of momentum measured per unit length of the underformed beam. We shall now derive the equations relating the generalized forces with constrained twisting. Proceeding in a manner similar as for the vector of forces $\mathbf{n}$ above we obtain

$$\frac{\partial B}{\partial s_0} - M_\varphi + \bar{b} + \tilde{b} = \mathbf{e}_3 \int_{A_0} \varrho_0 \varphi_0 \ddot{\mathbf{x}} \, dA, \tag{104}$$

where

$$\bar{b} = \mathbf{e}_3 \int_{\partial A_0} \varphi_0 \mathbf{T}^0_\alpha n_\alpha \, d(\partial A), \qquad \tilde{b} = \mathbf{e}_3 \int_{A_0} \varrho_0 \varphi_0 \mathbf{b}_0 \, dA, \tag{105}$$

are the bimoment and mass loads. The equation for constrained shearing may be obtained in a similar way

$$\frac{\partial H_\lambda}{\partial s_0} - G_\lambda + \bar{h} + \tilde{h} = \mathbf{e}_3 \int_{A_0} \varrho_0 \varphi_\lambda \ddot{\mathbf{x}}\, dA\,, \tag{106}$$

where

$$\bar{h} = \mathbf{e}_3 \int_{\partial A_0} \varphi_\lambda \mathbf{T}^0_\alpha n_\alpha\, d(\partial A), \qquad \tilde{h} = \mathbf{e}_3 \int_{A_0} \varrho_0\, \varphi_\lambda\, \mathbf{b}_0\, dA\,. \tag{107}$$

The material form of the equations of motion is more convenient in a number of applications. To obtain this form one needs to define the tensor $\boldsymbol{\Omega}_0$ and vector ω_0 of the material angular velocity in the rotational motion.

$$\boldsymbol{\Omega}_0 = \mathbf{Q}^T\dot{\mathbf{Q}} \quad , \qquad \omega_0 \times \mathbf{1} = \boldsymbol{\Omega}_0\,, \tag{108}$$

where $\dot{(\cdot)} = \frac{\partial}{\partial t}$. Using the relations for forces and loads

$$\begin{array}{llll} \mathbf{n} = \mathbf{Q}\mathbf{N}, & \mathbf{m} = \mathbf{Q}\mathbf{M}, & \mathbf{T}^0 = \mathbf{Q}\mathbf{T}^B\,, & \\ \bar{\mathbf{q}} = \mathbf{Q}\bar{\mathbf{q}}_0, & \bar{\mathbf{m}} = \mathbf{Q}\bar{\mathbf{m}}_0, & \tilde{\mathbf{q}} = \mathbf{Q}\tilde{\mathbf{q}}_0, & \tilde{\mathbf{m}} = \mathbf{Q}\tilde{\mathbf{m}}_0\,, \end{array} \tag{109}$$

the equations of motion may be brought to the form

$$\begin{aligned} \frac{\partial \mathbf{N}}{\partial s_0} + \bar{\mathbf{K}}_0\mathbf{N} + \bar{\mathbf{q}}_0 + \tilde{\mathbf{q}}_0 &= \frac{\partial \mathbf{l}_0}{\partial t} + \boldsymbol{\Omega}_0\mathbf{l}_0\,, \\ \frac{\partial \mathbf{M}}{\partial s_0} + \bar{\mathbf{K}}_0\mathbf{M} + (\epsilon_0 + \underset{\circ}{\mathbf{e}}_3) \times \mathbf{N} &= \frac{\partial \mathbf{h}_0}{\partial t} + \boldsymbol{\Omega}_0\mathbf{h}_0\,, \\ \frac{\partial B}{\partial s_0} - M_\varphi + \bar{b} + \tilde{b} &= \underset{\circ}{\mathbf{e}}_3 (\frac{\partial \mathbf{k}^0_0}{\partial t} + \boldsymbol{\Omega}_0\mathbf{k}^0_0)\,, \\ \frac{\partial H_\lambda}{\partial s_0} - G_\lambda + \bar{h} + \tilde{h} &= \underset{\circ}{\mathbf{e}}_3 (\frac{\partial \mathbf{k}^0_\lambda}{\partial t} + \boldsymbol{\Omega}_0\mathbf{k}^0_\lambda)\,. \end{aligned} \tag{110}$$

where, according to (37),

$$\bar{\mathbf{K}}_0 = \mathbf{K}_0 + \boldsymbol{\Lambda}_0\,. \tag{111}$$

Besides

$$\begin{aligned} \mathbf{l}_0 &= \int_{A_0} \varrho_0 \mathbf{v}_0\, dA, \qquad \mathbf{h}_0 = \int_{A_0} \left[\varrho_0(\underset{\circ}{\hat{\mathbf{x}}} + \underset{\circ}{\mathbf{w}}) \times \mathbf{v}_0\right] dA, \\ \mathbf{k}^0_\lambda &= \int_{A_0} \varrho_0 \varphi_\lambda \mathbf{v}_0\, dA, \quad (\lambda = 0, 1, 2)\,, \end{aligned} \tag{112}$$

where

$$\begin{aligned} \mathbf{v}_0 &= \bar{\mathbf{v}}_0 + \boldsymbol{\Omega}_0 \underset{\circ}{\hat{\mathbf{x}}} + (\dot{w} + w\boldsymbol{\Omega}_0)\underset{\circ}{\mathbf{e}}_3\,, \\ \bar{\mathbf{v}}_0 &= \dot{\bar{u}}_i \underset{\circ}{\mathbf{e}}, \qquad \bar{\mathbf{u}} = \mathbf{r} - \mathbf{r}_0, \qquad \bar{\mathbf{u}}_0 = \mathbf{Q}^T\bar{\mathbf{u}}\,. \end{aligned} \tag{113}$$

Using (114) and (43) we may bring the vectors of momentum and moment of momentum to the form

$$
\begin{aligned}
\mathbf{l}_0 &= \varrho_0 \Big[A_0(\bar{\mathbf{v}}_0 + \boldsymbol{\omega}_0 \times \bar{\mathbf{u}}_0) + S_\alpha(\boldsymbol{\omega}_0 \times \underset{\circ}{\mathbf{e}}_\alpha) + \sum_{\lambda=0}^{2} S_{\varphi\lambda}(\dot{\theta}_\lambda + \theta_\lambda \boldsymbol{\omega}_0 \times \mathbf{1})\underset{\circ}{\mathbf{e}}_3 \Big] , \\
\mathbf{h}_0 &= \varrho_0 \Big[S_\alpha(\underset{\circ}{\mathbf{e}}_\alpha \times \bar{\mathbf{v}}_0) + \sum_{\lambda=0}^{2} S_{\varphi_\lambda}\theta_\lambda(\underset{\circ}{\mathbf{e}}_3 \times \bar{\mathbf{v}}_0) + \sum_{\lambda=0}^{2} J_{\alpha\varphi_\lambda}\dot{\theta}_\lambda(\underset{\circ}{\mathbf{e}}_\alpha \times \underset{\circ}{\mathbf{e}}_3) \Big] + \\
&+ \varrho_0 \Big[\mathbf{J} + \sum_{\lambda=0}^{2} \mathbf{J}_{\varphi_\lambda}\theta_\lambda + \sum_{\lambda,\mu=0}^{2} J_{\varphi_\lambda\varphi_\mu}\theta_\lambda\theta_\mu(\underset{\circ}{\mathbf{e}}_\alpha \otimes \underset{\circ}{\mathbf{e}}_\alpha) \Big] \boldsymbol{\omega}_0 , \qquad (114) \\
\mathbf{k}^0_\lambda &= \varrho_0 \Big[S_{\varphi_\lambda}\bar{\mathbf{v}}_0 + J_{\alpha\varphi_\lambda}(\boldsymbol{\omega}_0 \times \underset{\circ}{\mathbf{e}}_\alpha) + \sum_{\mu=0}^{2} J_{\varphi_\lambda\varphi_\mu}\dot{\theta}_\mu \underset{\circ}{\mathbf{e}}_3 + \sum_{\mu=0}^{2} J_{\varphi_\lambda\varphi_\mu}\theta_\mu(\boldsymbol{\omega}_0 \times \underset{\circ}{\mathbf{e}}_3) \Big], \\
&(\lambda = 0, 1, 2)
\end{aligned}
$$

where

$$
\begin{aligned}
\mathbf{J} &= J_{\alpha\beta}\underset{\circ}{\mathbf{e}}_\alpha \otimes \underset{\circ}{\mathbf{e}}_\beta + J_0 \underset{\circ}{\mathbf{e}}_3 \otimes \underset{\circ}{\mathbf{e}}_3 , \\
\mathbf{J}_{\varphi_\lambda} &= J_{\alpha\varphi_\lambda}[\underset{\circ}{\mathbf{e}}_\alpha \otimes \underset{\circ}{\mathbf{e}}_3 + \underset{\circ}{\mathbf{e}}_3 \otimes \underset{\circ}{\mathbf{e}}_\alpha] , \qquad (115)
\end{aligned}
$$

and

$$
\begin{aligned}
J_{\alpha\beta} &= \epsilon_{\alpha\mu}\epsilon_{\beta\mu} \int_{A_0} \underset{\circ}{x}_\mu \underset{\circ}{x}_\mu \, dA, & J_0 &= \int_{A_0} \underset{\circ}{x}_\alpha \underset{\circ}{x}_\alpha \, dA , \\
J_{\alpha\varphi_\lambda} &= \int_{A_0} \underset{\circ}{x}_\alpha \varphi_\lambda \, dA, & J_{\varphi_\lambda\varphi_\mu} &= \int_{A_0} \varphi_\lambda\varphi_\mu \, dA , \qquad (116) \\
S_\alpha &= \int_{A_0} \underset{\circ}{x}_\alpha \, dA, & S_{\varphi_\lambda} &= \int_{A_0} \varphi_\lambda \, dA .
\end{aligned}
$$

The functions φ_λ of distributions of warpings of the transverse cross-sections are determined up to a constant. These constants may be determined so that $S_{\varphi_\lambda} = 0$. Assuming that the line of centers of gravity of the transverse cross-sections coincides with the axis of the beam we have $S_\alpha = 0$. These assumptions simplify the formulas (115) significantly. If the transverse cross-sections are additionally bisymmetric, then $J_{\alpha\varphi_\lambda} = 0$.

6. Conclusions

We have developed a model of a finitely deformable, curved and pre-twisted beam, accomodating the warping distorions of the cross-sections caused by shearing and twisting. The model is based on concepts of 3D material continuum. The mechanical work of the beam is derived exactly from stress work of 3D continuum, and provides an exact identification of the resultant forces and conjugated strain measures. The model incorporates the bi-moment, the bending-twisting moment for the constrained twisting, and the moments connected with the constrained shearing of the beam. A generalization of the Vlasov formula for the bending-twisting moment to the case of finite deformations of curved and pre-twisted beams is given. For the constrained warping the decrease of stiffness of the beam, due to non-uniform distribution of stress in the cross-sections, is naturally accounted. The dynamics equations for the 1D problem, in particular for the bi-moment, and moments connected with constrained shearing, are derived. Properly invariant 1D constitutive relations are developed within the framework of linear elasticity. The proposed formulation can be applied to a broad range of practical problems. For instance, the non-linear treatment of the flexible beams is the basis for analysis of the helicopter, windmill and turbine slender blades, as well as of the rotors with flexible shafts. The non-linear geometric effects are of special importance in dynamic analysis of flexible rotating structures.

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References

[1] Ericksen, J. L., Truesdell, C.: *Exact theory of stress and strain in rods and shells*, Archive for Rational Mech. and Anal., 1, 1958, 295–323.

[2] Whitman, A. B., De Silva C. N.: *A dynamical theory of elastic directed curves*, J. Appl. Math. and Phys., 20, 1969, 200–212.

[3] Green A. E., Laws N.: *A general theory of rods*, Proc. Royal Soc., 239A, 1966, 145–155.

[4] Whitman A. B., De Silva C. N.: *An exact solution in a nonlinear theory of rods*, J. Elasticity, Vol. 4(1974), 4, 265–280.

[5] Antmann S.: *The theory of rods*, Handbuch der Physik, vol. VIa/2, Springer-Verlag 1972.

[6] de Saint Venant A. J. B.: *Mémoire sur l'e'quilibre des corps solides, dans les limites de leur èlasicitè, et sur les conditions de lèur resistance, quand les dèplacements èprouvès par leurs points ne sont pas tres petits*, C. R. Acad. Sci., Paris, 24, 1847.

[7] Novozhilov V. V.: *Osnovy nieliniejnoj teorii uprugosti*, Moskva, 1948 (in Russian).

[8] Volterra E.: *Equations of motion for curved elastic bars by the use of the "method of internal constraints"*, Ingenieur-Archiv, 23, 1955, 402–409.

[9] Volterra E.: *Second approximation of method of internal constraints and its applications*, Int. J. Mech. Sci., 3, 1961, 47–67.

[10] Medick M. A.: *One dimensional theories of vave propagation and vibrations in elastic bars of rectangular cross-section*, J. Appl. Mech., 1966, 489–495.

[11] Gamby D. *One-dimensional theories of motion for beams*, J. Elasticity, Vol. 7(1977), 4, 353–367.

[12] Antmann S. S., Warner W. H.: *Dynamical theory of hyperelastic rods*, Archive for Rational Mech. and Anal., 23, 1966, 135–162.

[13] Parker D. F.: *An asymptotic analysis of large deflections and rotations of elastic rods*, Int. J. Solids and Structures, 15, 1979, 361–377.

[14] Pleus P., Sayir M. *A second order theory for large deflections of slender beams*, J. Appl. Math. and Phys., 34, 1983, 192–217.

[15] Houbolt J. C., Brooks G. W.: *Differential equations of motion for combined flapwise bending, chordwise bending and torsion of twisted nonuniform rotor blades*, NACA Rep.1346, 1958.

[16] Rosen A., Friedmann P.: *The nonlinear behaviour of elastic slender straight beams undergoing small strains and moderate rotations*, J. Appl. Mech., Vol. 46(1979), 161–168.

[17] Timoshenko S.: *History of strenght of materials*, McGraw-Hill Book Co., INC, N-Y, Toronto, London, 1953.

[18] Kirchhoff G.: *Über das Gleichgewicht und die Bewegung eines unendlich dünnen elastischen Stabes*, J. für die reine angewandte Mathematik, Vol. 56(1859), 285–313.

[19] Timoshenko S. P.: *On the correction for shear of the differential equation for transverse vibrations of prismatic bars*, Phil. Mag., 41, 1921, 744–746.

[20] Clebsh A.: *Theorie der Elastizität fester Körper*, Leipzig 1862.

[21] Love A. E.: *A treatise on the mathematical theory of elasticity*, 4^{th} edition 1927, Dover, New York, chapters 16–19.

[22] Janecki S.: *Beam self vibrations with account of shearing*, Theor. and Appl. Mech., 2 (15), 1977, 247–263 (in Polish).

[23] Janecki S.: *The effect of constrained cross-sectional warping on the bending of beams*, J. Theor. and Appl. Mech., 1 (36), 1998, 121–144.

[24] Dzhanelidze G. Yu.: *K teorii tonkich i tonkostennych sterzhniey*, Prikl. Mat. i Mech., XIII(1949), 6, 597–608 (in Russian).

[25] Vlasov V. Z.: *Tonkostennyje uprugie sterzhni*, Gosudarstvennoe Izdatelstvo Fiziko-Matematičskoj Literatury, Moskva 1959 (in Russian).

[26] Reissner E.: *On one-dimensional finite beam theory: the plane problem*, J. Appl. Math. and Phys., 23, 1972, 796–804.

[27] Reissner E.: *On one-dimensional large-displacement finite-strain beam theory*, Stud. Appl. Math., 52, 1973, 87–95.

[28] Reissner E.: *On finite deformations of space-curved beams*, J. Appl. Math. and Phys., 32, 1981, 734–744.

[29] Jura M.: *Finite displacements theory of naturally curved and twisted beams with finite rotations*, Proc. of the Euromech Coll. 197: Finite Rotations in Structural Mech. (ed. W. Pietraszkiewicz), Springer-Verlag 1985, 148–157.

[30] Jura M., Atluri S. N.: *Dynamic analysis of finitely stretched and rotated three-dimensional space-curved beams*, Comp. and Structures, 29 (5), 1988, 875–889.

[31] Jura M., Atluri S. N.: *On a consistent theory and variational formulation of finitely stretched and rotated 3-D space-curved beams*, Comp. Mech., 4, 1989, 73–88.

[32] Hodges D. H.: *Nonlinear beam kinematics for small strains and finite rotations*, Vertica, 11(1987), 3, 573–589.

[33] Danielson D. A., Hodges D. H.: *Nonlinear beam kinematics by decomposition of the rotation tensor*, J. Appl. Mech., 54(1987), 2, 258–262.

[34] Danielson D. A., Hodges D. H.: *A beam theory for large global rotation, moderate local rotation, and small strain*, J. Appl. Mech., 55(1988), 1, 179–184.

[35] Simo J. C.: *A finite strain beam formulation. The three dimensional dynamic problem*, Part I. Comp. Meth. in Appl. Mech. and Eng., 49, 1985, 55–70.

[36] Simo J. C., Vu-Quoc L.: *On the dynamics of flexible beams under large overall motion — the plane case*: part I. J. Appl. Mech., 53, 1986, 849–854.

[37] Simo J. C., Vu-Quoc L.: *The role of non-linear theories in transient dynamic analysis of flexible structures*, J. Sound and Vibr., 119, 1987, 487–508.

[38] Simo J. C., Vu-Quoc L.: *On the dynamics in space of rods undergoing large motions — a geometrically exact approach*, Computer Meth. in Appl. Mech. and Eng., 66, 1988, 125–161.

[39] Hegemier G. A., Nair S.: *A nonlinear dynamical theory for heterogeneous, anisotropic, elastic rods*, AIAA J., 15(1977), 1, 8–15.

[40] Simo J. C., Vu-Quoc L.: *A geometrically-exact rod model incorporating shear and torsion-warping deformation*, Int. J. Solids and Structures, 27(1991), 3, 371–393.

[41] Hodges D. H., Dowell E. H.: *Nonlinear equations of motion for the elastic bending and torsion of twisted nonuniform rotor blades*, NASA TN D-7818, 1974.

[42] Hodges D. H.: *Torsion of pretwisted beams due to axial loading*, J. Appl. Mech., 47, 1980, 393–397.

[43] Janecki S.: *Twisting of rotating and pre-twisted rods with account of geometric non-linearities*, Prace IMP PAN (Transactions of IFFM), 76, 1977, 121–135 (in Polish).

[44] Janecki S.: *Dynamics of cantilever rotor blades of last stages of steam turbines*, Zesz. Nauk. IMP PAN (Bulletin of IFFM PASci.), 105, 1980, 1010 (in Polish).

[45] Krenk S.: *A linear theory for pretwisted elastic beams*, J. Appl. Mech., 50, 1987, 137–142.

[46] Krenk S., Gunneskov O.: *Pretwis and shear flexibility in the vibration of turbine blades*, J. Appl. Mech., 52, 1985, 409–415.

[47] Reissner E.: *Some considerations on the problem of torsion and flexure of prismatic beams*, Int. J. Solids and Structures, 15(1979), 1, 41–54.

[48] Reissner E.: *On a simple variational analysis of small finite deformations of pretwisted elastic beams*, J. Appl. Math. and Phys., 34, 1983, 642–648.

[49] Reissner E.: *A variational analysis of small finite deformations of pretwisted elastic beams*, Int. J. Solids and Structures, 21(1985), 7, 773–779.

[50] Rosen A., Loewy R. G., Mathew M. B.: *Nonlinear dynamics of slender rods*, AIAA J., 25(1987), 4, 611–619.

[51] Vorobev Yu., S. Szorr, B. F.: *Teoria zakruczonnych sterzhniej*, Naukova Dumka 1983, Kiev (in Russian).

[52] Pietraszkiewicz W., Badur J.: *Finite rotations in the description of continuum deformation*, Int. J. Eng. Science, 25(1983), 4, 611–619.

[53] Whitman A. B., Cohen H.: *Constitutive equations for curved and twisted, initially stressed elastic rods*, Acta Mech., 30(1978), 237–257.

[54] Atluri S. N.: *Alternate stress and conjugate strain measures, and mixed variational formulations involving rigid rotations, for computational analyses of finitely deformed solids, with application to plates and shells — I*, Comp. and Structures, 18(1984), 1, 53–116.

Geometrycznie nieliniowe równania przestrzennie zakrzywionej i wstępnie skręconej belki z uwzględnieniem spaczenia przekrojów poprzecznych

Streszczenie

W pracy przedstawiono równania ruchu jednowymiarowego modelu ciała sprężystego. Wyprowadzono je z trójwymiarowej teorii ośrodków ciągłych i przy założeniu skończonych deformacji. W modelu uwzględniono skrępowaną deplanację przekrojów poprzecznych spowodowaną skręcaniem i ścinaniem oraz wzajemne sprzężenie zginania, skręcania i rozciągania. Wzięto pod uwagę skomplikowaną geometrię ciała przestrzennie zakrzywionego, wstępnie skręconego i zbieżnego oraz mającego niesymetryczne przekroje poprzeczne.