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Foundations of the Navier-Stokes boundary conditions in fluid mechanics

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Abstract

Until recently it was believed that Navier’s boundary condition could be given as a rigorous foundation for slip phenomena. Due to the latest measurements in the mass flow rate of a gas flowing through nano- and microchannels, several discrepancies in the mathematical modelling have been found. Thus, in the literature, the opinion persists for the Navier slip condition to be correct only under certain circumstances, particularly those restricted to the first order boundary conditions. One of many ways to eliminate this discrepancy, which is extensively employed in the contemporary literature, is to develop a variety of the so-called second order boundary conditions. This path, however, seems incorrect since it lacks consistency between the bulk stress tensor and its boundary representation.

In the paper we propose to replace the classical Navier slip condition with the new, more general Navier-Stokes slip boundary condition. Instead of the usual method of consideration, the boundary condition is presented as following from the mass and momentum balances within a thin, shell-like moving layer. Owing to this, the problem of consistency between the internal and external friction in a viscous fluid is solved within the framework of new layer balances, and a proper form of constitutive relations for friction and mobility forces. Finally, the common features of the Navier, Stokes, Maxwell and Reynolds concepts of a boundary slip layer are compared and revalorized. The classifications of different mobility mechanisms, important for flows in nano-, microchannels are also discussed.

Keywords: Enhancement; Slip; Navier number; Navier-Stokes; Nanoflow

1 Background of mathematical modeling

Flows through nano- and microchannels have recently gained increased attention, both because of the strive for fundamental knowledge and due to its signif-

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icant applicational array in chemical devices, energy converters, micro: -sensors, -actuators, -reactors, just to name a few [33]. There exists a wide spectrum of micro-electro-mechanical systems (MEMS) like electrostatic comb micro drives [20, 30] or electrostatically side-driven micromotors [21, 23]. The phenomena of external friction between two continuum phases is the physical reason for the conventional macroscopic approaches to fail at small scales. Thus, the best known no-slip boundary condition between a fluid and solid phase, which is the fundamental notion in the standard fluid mechanics, must not be applied when the scale of the flow problem is changed significantly. Strictly speaking, the no-slip boundary condition is valid only in the case of thermodynamic equilibrium between a fluid flow and an adjacent solid surface. To provide this, speaking in the tone of statistical mechanics, very high rate of collisions of fluid molecules into the solid surface needs to occur. However, in the case of the small-scale flow system, the rate of collisions is insufficient to ensure the condition of thermodynamic equilibrium; a certain amount of tangential velocity should be allowed. Indeed, there exist numerous physical causes of the slip over the solid surface, which are: molecular slip, small dipole moment of polar liquids, trapping the naturally present gas bubbles at the solid surface (see [33, 36]). The slip velocity may not be arbitrary, and must obey the laws of external friction between three-dimensional continua possessing a material surface.

According to the Newton hypothesis ([47], book II, p. 676), the external friction between bodies depends chiefly on three mechanisms: a) pressure within the contacting layer; b) relative velocity of the bodies in contact; c) square of the relative velocity, today known as the relative kinetic energy. Two kinds of the external friction were proposed by Leonhard Euler [17]: one “static”, which is predominant at low relative velocities, and the other “dynamic”, that dominates in the fully developed flow. This line of reasoning appears in the present literature and is more commonly known as the “rate dependent slip” [71, 50]. The concept of the “external friction” between laminas of different bodies, and the “internal friction” between laminas of one body, has been introduced by Coulomb, who conducted the first repeatable experiment that allowed him for verification of his own laws of friction [9].

To satisfy the history, it was Navier in 1822, who proposed to use both: the idea of the dissipative kinetic energy of the external friction (see Eq. (68) below), and the idea of external friction force between fluid \mathcal{A} and the surface of a fixed body \mathcal{B} , which he defined to be: $\mathbf{f}_{AB} = \nu \mathbf{v}$. He called the coefficient of the external friction, ν , “la résistance provenant du glissement” – the external viscosity [46]. The tangential component of the fluid velocity \mathbf{v} on the surface he named the “slip velocity”. The magnitude of the slip velocity is influenced by the value of the external friction coefficient: if $\nu \rightarrow \infty$, then there is no-slip condition; if $\nu = 0$

the slip condition is ideal, which corresponds to the Euler model of internally perfect fluid.

In special applications, as is the case for micro- and nanofluidic devices, when the surface-to-volume ratio is large, the velocity slip occurs naturally, and the “slip hydrodynamic boundary condition” is usually employed. Regardless of the physical mechanism responsible for the slip, the magnitude of the slip may be quantitatively described by the “slip length”, l_s , or a dimensionless slip number, Na (Navier number):

$$l_s = \frac{\mu}{\nu}, \quad \text{Na} = \frac{l_s}{L}, \quad (1)$$

being the ratio of the internal viscosity of the fluid, μ , and the external viscosity, ν , and where L is a characteristic dimension of a channel (say, hydraulic diameter). The coefficient ν always depends on both the fluid and the solid in contact with it. In 1876, Butcher geometrically interpreted the slip length as the distance extending into the solid wall, where the extrapolated velocity profile vanishes (see Fig. 3) [7]. The mass flow rate, due to the presence of the velocity slip, increases. According to Butcher’s interpretation, the slip length is the local equivalent distance below the solid surface at which the no-slip boundary condition is satisfied, if the flow field was hypothetically extended linearly outside the physical domain.

For the sake of consistency, let the history of the measurement be briefly recalled, where the external viscosity coefficient, ν , sometimes called the “external friction coefficient” was investigated. If the linear (Newtonian) shear viscosity of a fluid, μ , is known, then the problem of measuring ν may be reduced to the determination of the slip length. The first technical closure for ν of water over glass has been proposed by Navier in the form varied with water density [46]:

$$\nu = \rho \times 0.0023 \quad (2)$$

and was rigorously verified by Helmholtz and von Piotrowski (see [29]). The first closure on external friction of a rarefied air over a glass was determined by Kundt and Warburg in the form [37, 38]:

$$l_s = \frac{\mu}{\nu} = 0.7122l = 0.7122l_0 \frac{760}{p} \text{ [mm]}, \quad (3)$$

where l , l_0 are the mean-free-paths of a molecule under actual and atmospheric pressure, respectively. The first systematic experimental study reporting the external viscosity for liquids and gases has been made by Wiedemann, who also invented technical devices allowing for measurement of ν and l_s [66]. In 1890 Whetham made several other systematic and critical experimental studies [67]. Many years after, Schnell in 1956, measured the flow rate of water in glass capillaries with a radius of the order of $100 \mu\text{m}$ [54]. Trying to control the external friction by any “interfacial lubrication”, Schnell treated the capillaries with

dimethyldichlorosilane, making its surface a hydrophobic one. Then larger flow rates have been obtained and they were interpreted as a lowering of friction and extension of the slip length at the wall. Schnell's slip length data are consistent with the analytical solution of Navier that predicts the enhancement of the mass flow rate within the capillary tube. Historically, many of the pioneering investigations of non-continuum flows were conducted by researchers in the rarefied gas community who were primarily interested in low-pressure applications [14, 15, 16, 45, 34, 35, 27, 63, 22].

Contemporarily, the modeling of the Navier number (or the dimensionless slip length) is separated into two branches. The first, and predominant one, deals with modeling of the external friction for different gases on solid surfaces. Mathematical formulae on the Navier number and the dimensionless slip length, trying to be in agreement with the experimental measurements, are the crucial point of many wall-slip models of varying complexity. As usual, the main problem is: how many adjustable parameters ought to be proposed, and which of those parameters are truly predictive [62, 69]. Let us recall that the first closure for the Navier number was proposed by Maxwell in terms of dimensionless mean free path of molecules (nowadays the Knudsen number Kn) to be

$$\text{Na} = \left(\frac{2-f}{f} \right) \text{Kn} , \quad (4)$$

where f is the tangential momentum accommodation coefficient (TMAC) [44]. It depends strongly on the kind of gas and solid surface. Independently from the peculiarity of the definition of the Knudsen number, the Maxwell closure (4) expresses the inverse proportionality of the external friction on gas pressure, that was found earlier by Helmholtz and von Piotrowski, and also by Kundt and Warburg. The coefficient f has been determined for manifold materials by fitting the experimental data to the simulation results. These data are presented by Karniadakis *et al.* in a complete report [33].

Beskok and Karniadakis proposed a model of a two-parametrical dimensionless slip length, that strongly depends on rarefaction and pressure [3]. This closure is claimed to be rather successful in correlating the experimental data:

$$\text{Na} = \left(\frac{2-f}{f} \right) \frac{\text{Kn}}{1-b\text{Kn}} . \quad (5)$$

The additional constant b is coming from the analysis of the generalized slip layer model [11, 39], and should be experimentally fitted. An extension of Beskok and Karniadakis's model to a different range of the Knudsen number, and presence of a constant acceleration gradient in the boundary layer – consistent with the Reynolds acceleration gradient [53] – has been proposed by To *et. al.* in the

form [62]:

$$\text{Na} = \left(\frac{2-f}{f} \right) \text{Kn} \left[\beta \frac{\text{Kn}}{\text{Re}} + 1 \right], \quad (6)$$

where for a helium and cooper pair two slip coefficients are: $f = 0.71$, $\beta = 0.76$ and Re is a dimensionless inverse of the gas shear viscosity. Here, a fruitful analogy in behavior of the Reynolds and the Navier number appears.

Second branch of application for the Navier number deals with flows of liquids. Here fundamental properties related to the turbulent slip length have been discovered in the original paper by Thompson and Trojan on shear flow of simple liquids [61]. They found that the slip length is quite independent from the rate of deformation $\dot{\gamma} = d_{xy}$. But after crossing the characteristic value, say $\dot{\gamma}_c$, it was observed that it increases nonlinearly with the shear rate. They have proposed the following closure for dimensionless slip length:

$$\text{Na} = \text{Na}^0 \left[1 - \frac{\dot{\gamma}}{\dot{\gamma}_c} \right]^{-0.5}, \quad (7)$$

where Na^0 is a classical rate-independent Navier number and $\dot{\gamma}_c = 0.093\tau^{-1}$ is a limiting value of the deformation rate, where τ is the characteristic time calculated from the Lennard-Jones potential. A gradual transition in rate dependence of the slip length, from linear to highly nonlinear, firstly observed numerically via molecular dynamics (MD) simulation in the Poiseuille flows, has been discovered experimentally by Zhu and Granick for water flow against a methyl-terminated self-assembled monolayer (SAM) [71]. De Gennes *et al.* proposed, that the rate dependency of the external friction can occur due to formation on the surfaces of some additional structures like the nucleation of vapor bubbles [10]. When the surface nucleation barrier is exceeded at some inception point, the bubbles grow and cover the surface, and the liquid flow is over a thin gas film rather than over the solid surface itself¹. On the other hand, this gradual transition in rate dependency of the slip length, from linear to highly nonlinear, is observed also upon reducing the strength of wall-fluid interactions when in a liquid/solid interface layer some incommensurable vortex structures are formed [50]. Then, after the transition from laminar to layer turbulence, enhancement of the external friction may be observed in a manner proposed by Thomson and Trojan:

$$\text{Na}_{\text{turb}} = \text{Na}_{\text{lam}} (1 + \tau^2 \Pi_d)^\beta, \quad (8)$$

where Π_d is the second invariant of the surface deformation rate \mathbf{d}_s , τ is an intermittency time and $\beta \approx -0.5$ [61]. In analogy to the fluid flow in the bulk, on the

¹Other physical mechanisms which can govern velocity slip are discussed in the papers by Zhang and Yarin [70] and Sinha Ray *et al.* [57].

surface some transition from laminar to turbulent motion should be postulated, and now not the critical Reynolds number Re^{crit} , but rather a critical Navier number Na^{crit} plays the main role.

Various values of the slip length and the methods of their determination have been applied [69, 41, 43, 42, 40, 3]. Fitting the experimental results, with the theoretical results based on slip models, it is often necessary to postulate various dependences of the slip coefficient on the Knudsen number, and the geometry of a considered channel. A number of these slip coefficients are summarized in [33, 36, 20].

It should be noted that the external friction between surfaces of immiscible liquids is more difficult to measure than the internal one. Therefore we usually say that the true slip of velocity occurs only at the liquid/solid interface, whereas an apparent slip occurs at a liquid/gas interface. Von Rybczyński indicated an importance of the external friction coefficient between bubbles of air and water, [63]. Of special interest in nanotechnology is the corresponding reduction in drag, which can be achieved by partial substitution of a liquid/solid contact area (small-slip) with a liquid/gas contact area (finite-slip). It is important especially for biomedical and chemical engineering applications, for instance, in microfilters.

Now a question arises: where is the main scientific issue? Leaders of nanotechnology agree that mathematical modeling of reactive, diffusive, turbulent flows within nanoscale systems, where the surface-to-volume ratio is very high, and the characteristic dimensions are comparable with the length of the molecular free path, should be quite new. When devices are scaled down, the surface-to-volume ratio increases dramatically, and the surface related phenomena become increasingly dominant. Therefore a precise modeling of surface properties of fluid become much more important as opposed to the modeling of bulk properties. Such a situation leads to the radical change of our interest. Instead of mathematical models that concentrate merely on the complexity of phenomena within the bulk (far away from the wall), we now develop models that are simple (or even extremely simple) in the bulk, but are apparently “rich” in governing unknown fields prescribed in the boundary layer.

In the MEMS literature [21, 23], it is assumed that the external friction between solid and liquid surface, cannot be further neglected. It involves the velocity slip, which is now important not only in the surface momentum transfer, but also in surface mass, heat transfer, and reactive processes coupled with an interfacial transport. Transport phenomena, which undergo within a thin shell-like domain, require much more complex, surface like mechanisms of interchanging mass, momentum and entropy. There are also other types of layers as opposed to Navier’s or Stokes’s mechanical one. One of them is the von Smoluchowski thermal layer important for describing the temperature slip [33, 36, 28, 31]. In a nano-heat-

exchanger it is a basic issue to find the proper value of dimensionless thermal slip length (the Nusselt number), usually as a function of other dimensionless surface parameters like the Navier, Sherwood, Levis, Weber numbers, and the bulk numbers like the Euler, Reynolds, Prandtl, Peclet, Schmidt, etc. It is well known that the mechanical properties of a fluid layer at small scales can overlap the thermal, diffusive, absorptive properties of a thin but of finite thickness layer. The problem of the general statement of interrelation between the slip velocity layer, temperature slip layer, concentration slip layer, radiation slip layer, electric potential slip layer, and others, in this paper, due to lack of space, is omitted.

In this article we study a consistency of the Navier-Stokes model for the fluid in the bulk of a flow domain as well as in the boundary layer. Our subject of investigation coming from the fact that the classical fluid dynamics is usually identified with the three-dimensional Navier-Stokes equations in bulk, and with the no-slip boundary conditions. However, studying more precisely both pioneering papers [46, 59], one would find that there exist also some additional developments for the slip-boundary layer, which were initially made by Navier and Stokes. Unfortunately, these concepts were overlooked in most literature. Therefore, according to this scientific truth the Navier-Stokes equations are found in the bulk as well as in the boundary slip layer.

Yet another problem needs to be explained. Since Navier's contribution to the modeling of viscous flow is usually defined as the incompressible viscous fluid model, then the Navier slip boundary conditions are treated to be restricted only to the incompressible Navier-Stokes equations. Quite a different approach to the problem of consistency for bulk equations and their boundary conditions has been presented in the monograph [33]. It was recognized that the slip boundary condition should be used only in conjunction with the compressible Navier-Stokes equations. It means that the combination of Navier slip boundary conditions and the "incompressible Navier-Stokes" equations, which is often used because of convenience, is theoretically inconsistent. In our article, we explain this problem in detail. Generally, one can say that the incompressible Navier bulk equations are consistent with the Navier boundary slip layer concept, and the "compressible Stokes" bulk equations are consistent with the Stokes boundary layer model.

In this article we distinguish Stokes' generalization of the Navier boundary slip layer, since it is assumed the contribution of Stokes is far more significant in the boundary modeling than in the bulk. Strictly speaking, a great step in the development of fluid bulk modeling was proposed in the seminal Cauchy paper in 1827 — it was a beneficial replacement for Navier's equations of a fluid motion in the bulk with an equation of motion expressed in terms of a fluid "nonspherical pressure diade". This diade nowadays is called "the Cauchy stress tensor" or "the momentum flux diade". An important achievement of Stokes was quite similar.

In 1845, he reorganized the Navier boundary model in terms of the Cauchy stress tensor, using additionally the Cauchy concept of a “surface traction force”. The above justifies the introduction of a new notion — the Navier-Stokes boundary layer.

Our paper is devoted to studying some extension of the concept for the Stokes boundary slip layer. At this stage of understanding in the mathematical modeling for interfacial nanoproperties of fluids, we have restricted our interest only to Stokes’ boundary layer, treating it to be the “momentum layer” that accompanies the mass and momentum balance in the bulk of fluid. Therefore, we discuss only the mass boundary layer which is an external boundary condition for the balance of mass. Our another paper suggests for the possibility of an extension on Stokes’ concept of a boundary layer onto the thermal energy, entropy, chemical affinity, etc. [2].

This paper is organized as follows. In the next section, we describe details of kinematics of the middle surface of the layer. The balances of the layer mass and momentum are introduced in Sections 3 and 4. Constitutive relations for the quantities of layer are briefly summarized in Section 5. Details of Navier, Stokes, Maxwell, Reynolds slip layer are presented in Sections 6–9, respectively. Classification of a different mobility mechanism is presented in Section 10. Since the surface mobility forces are linear, like driving forces in 3D irreversible Onsager-like mechanics, then a surface coupling phenomena should be observed. The summary and conclusions are given in the last section.

2 Moving shell-like region in a fluid continuum

We assume that the Navier-Stokes boundary slip layer (denoted as $\mathcal{M}^+\mathcal{M}^-$) can be treated as thin domain moving in a space with a given migration velocity \mathbf{w} . This shell-like domain divides the continuum into a continuum \mathcal{A} – that is a fluid under consideration, and a continuum \mathcal{B} which can be a free surface, solid body or second fluid, as in Fig. 1. If both \mathcal{A} and \mathcal{B} are fluids then the Navier-Stokes boundary layer represents the moving interfacial region, where physical properties change in a radical manner. For instance in a thin transition layer between liquid and vapor, the change of density is so noticeable, that it looks like a jump over the layer thickness. Therefore, we assume that in the layer we observe so-called “apparent” material properties, quite different than in bulk continuum \mathcal{A} and \mathcal{B} . Thus we define an excess of layer density ρ_s [kgm^{-2}], the particle velocity in the layer \mathbf{v}_s [ms^{-1}], an excess of layer momentum density $\rho_s\mathbf{v}_s$, and a surface excess of momentum flux \mathbf{p}_s [1].

In general, this layer moves with the migration velocity \mathbf{w} that differs from material velocity \mathbf{v}_A in continuum \mathcal{A} , velocity \mathbf{v}_B in continuum \mathcal{B} , and velocity \mathbf{v}_s

in $\mathcal{M}^+\mathcal{M}^-$. In particular case, the velocity \mathbf{w} denotes the rate of changing a phase transition surface within the fluid being at rest [12, 56]. Usually, the component w_n normal to the moving middle surface \mathcal{M} , differs from normal components of \mathbf{v}_A , \mathbf{v}_B and \mathbf{v}_s . It practically means that there is also a mass transport across the layer. Indeed, the geometrical velocity field is not *a priori* known, and can be determined from a special evolution equation [55]. If $\mathbf{w} = \mathbf{v}_s$ then the moving layer is material, if $\mathbf{w} = \mathbf{v}_s \mathbf{I}_s + w_n \mathbf{n}$ the surface is semi-coherent (Fig. 1).

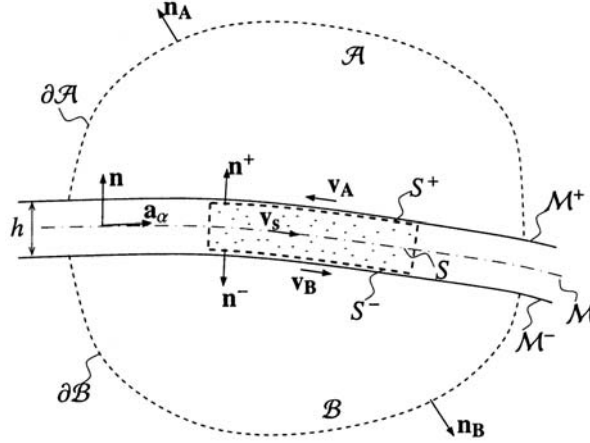


Figure 1. Outline of the Navier-Stokes boundary layer.

Navier and Stokes have intentionally assumed, that the layer density is equal to zero $\rho_s \equiv 0$. Now, we want to determine the slip velocity \mathbf{v}_s from an independent balance of the layer momentum. In special cases however, it simplifies to the well-known balance of the boundary traction forces. For immiscible liquids being in contact, the tangential components $\mathbf{v}_s \mathbf{I}_s$ can be approximately described to be $\frac{1}{2} (\mathbf{v}_A + \mathbf{v}_B) \mathbf{I}_s$. Quite similarly, only a special case is $\rho_s = \frac{1}{2} (\rho_A + \rho_B) h$, where h is a finite thickness of the layer².

We introduce an original concept of an “excess of momentum flux” within the Navier-Stokes layer, which is mathematically represented by the surface symmetrical diade \mathbf{p}_s . It governs the momentum transport within the layer, and therefore it has a tangential and normal components. We postulate the surface momentum flux in a following form:

$$\mathbf{p}_s(\xi) = p^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta + p^{n\alpha} \mathbf{n} \otimes \mathbf{a}_\alpha + p^{\alpha n} \mathbf{a}_\alpha \otimes \mathbf{n} + p^{nn} \mathbf{n} \otimes \mathbf{n}, \quad (9)$$

²We are based here on a general surface kinematics elaborated by [49, 51]. The general form of the surface balances of mass, momentum, angular momentum, energy, entropy, etc. is given by [72, 58, 8, 12].

where ξ^α ($\alpha = 1, 2$) are local surface curvilinear coordinates on \mathcal{M} , and \mathbf{a}_α , \mathbf{n} ($\alpha = 1, 2$) are the base vectors on the middle surface of the layer \mathcal{M} . Since the physical properties of the layer are unknown *a priori*, they depend on the resulting apparent properties in both continua \mathcal{A} and \mathcal{B} . For example, elastic recoverable properties of \mathbf{p}_s depend on an actual shape of the surface \mathcal{M} . Many authors postulate that, due to strong induced elasticity of the fluid layer, it changes from the elastic fluid (only recoverable spherical deformations) into an elastic fluid with recoverable shape deformations [69]. Similarly, owing to induced strong anisotropy, the internal viscosity of the fluid layer can be described by four apparent viscosity coefficients [24].

Let us now recall a few mathematical relations required for establishing of the balance of the layer mass and momentum. At first the Weatherburn surface fundamental diades can be introduced [65]:

$$\mathbf{I}_s = \mathbf{I} - \mathbf{n} \otimes \mathbf{n} = \text{grad}_s \mathbf{x}_s = a^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \quad (10)$$

$$\mathbf{II}_s = -\text{grad}_s \mathbf{n} = b^{\alpha\beta} \mathbf{a}_\alpha \otimes \mathbf{a}_\beta, \quad (11)$$

which are called the first and second fundamental forms of the surface \mathcal{M} . As far as the surface gradient acts also on the coordinate dependent base \mathbf{a}_α , \mathbf{n} , then the surface gradient of velocity is calculated to be

$$\begin{aligned} \text{grad}_s \mathbf{v}_s &= (v^\alpha \mathbf{a}_\alpha + v_n \mathbf{n}) \otimes \nabla_\beta \mathbf{a}^\beta = \\ &= (v_{\alpha|\beta} - v_n b_{\alpha\beta}) \mathbf{a}^\alpha \otimes \mathbf{a}^\beta + (v^\alpha b_{\alpha\beta} + v_{n,\beta}) \mathbf{n} \otimes \mathbf{a}^\beta, \end{aligned} \quad (12)$$

and the surface divergence of velocity vector is based on the contraction $\mathbf{C}_{1,2}$:

$$\begin{aligned} \text{div}_s \mathbf{v}_s &= \mathbf{C}_{1,2} \text{grad}_s \mathbf{v}_s = (v_{\alpha|\beta} - v_n b_{\alpha\beta}) a^{\alpha\beta} = \\ &= v^\alpha{}_{|\alpha} - v_n b^\alpha{}_\alpha = \text{div}_s (\mathbf{v}_{s\parallel}) - v_n \mathbf{I}_b. \end{aligned} \quad (13)$$

where the invariants of the second fundamental form of the curvature diade are: $\mathbf{I}_b = \text{tr} \mathbf{II}_s = b^\alpha{}_\alpha = b^1{}_1 + b^2{}_2 = \left(\frac{1}{r_1} + \frac{1}{r_2} \right)$, $\mathbf{II}_b = \det \mathbf{II}_s = \det (b_{\alpha\beta})$. In analogy to the three-dimensional case, the rate of surface deformation is defined as a symmetric part of the surface gradient of velocity [60]:

$$\begin{aligned} \mathbf{d}_s &= \frac{1}{2} (\text{grad}_s \mathbf{v}_s + \text{grad}_s^T \mathbf{v}_s) = \\ &= \left[\frac{1}{2} (v_{\alpha|\beta} + v_{\beta|\alpha}) - v_n b_{\alpha\beta} \right] \mathbf{a}^\alpha \otimes \mathbf{a}^\beta + \frac{1}{2} (v^\alpha b_{\alpha\beta} + v_{n,\beta}) (\mathbf{n} \otimes \mathbf{a}^\beta + \mathbf{a}^\beta \otimes \mathbf{n}). \end{aligned} \quad (14)$$

The first invariant of \mathbf{d}_s is in analogy to 3D:

$$\mathbf{I}_{\mathbf{d}_s} = \text{tr} \mathbf{d}_s = \mathbf{C}_{1,2} \mathbf{d}_s = v^\alpha{}_{|\alpha} - v_n \mathbf{I}_b. \quad (15)$$

Similarly, the surface gradient of the flux of momentum is

$$\begin{aligned} \text{grad}_s \mathbf{p}_s &= \mathbf{p}_s \otimes (\nabla_\gamma \mathbf{a}^\gamma) = p^{\alpha\beta} |_\gamma \mathbf{a}_\alpha \otimes \mathbf{a}_\beta \otimes \mathbf{a}^\gamma + p^{\alpha\beta} b_{\alpha\gamma} \mathbf{n} \otimes \mathbf{a}_\beta \otimes \mathbf{a}^\gamma + \\ &+ p^{\alpha\beta} b_{\beta\gamma} \mathbf{a}_\alpha \otimes \mathbf{n} \otimes \mathbf{a}^\gamma + p^{n\alpha} |_\gamma (\mathbf{n} \otimes \mathbf{a}_\alpha \otimes \mathbf{a}^\gamma + \mathbf{a}_\alpha \otimes \mathbf{n} \otimes \mathbf{a}^\gamma) + \\ &+ \left(2p^{n\alpha} b_{\alpha\gamma} + p^{nn} |_\gamma \right) \mathbf{n} \otimes \mathbf{n} \otimes \mathbf{a}^\gamma - p^{n\alpha} b^\varepsilon_\gamma (\mathbf{a}_\varepsilon \otimes \mathbf{a}_\alpha \otimes \mathbf{a}^\gamma + \mathbf{a}_\alpha \otimes \mathbf{a}_\varepsilon \otimes \mathbf{a}^\gamma) + \\ &- p^{nn} b^\varepsilon_\gamma (\mathbf{a}_\varepsilon \otimes \mathbf{n} \otimes \mathbf{a}^\gamma + \mathbf{n} \otimes \mathbf{a}_\varepsilon \otimes \mathbf{a}^\gamma), \end{aligned} \quad (16)$$

and its divergence

$$\begin{aligned} \text{div}_s \mathbf{p}_s &= \mathbf{C}_{2,3} \text{grad}_s \mathbf{p}_s = \left(p^{\alpha\beta} |_\beta - p^{n\beta} b^\alpha_\beta - \mathbf{I}_b p^{\alpha n} \right) \mathbf{a}_\alpha + \\ &+ \left(p^{\alpha\beta} b_{\alpha\beta} + p^{n\alpha} |_\alpha - \mathbf{I}_b p^{nn} \right) \mathbf{n}. \end{aligned} \quad (17)$$

Now it is useful to recall the following identities ($d_t = \frac{d}{dt} |_{X=\text{const}}$; $\partial_t = \frac{\partial}{\partial t} |_{\mathbf{x}=\text{const}}$; $\partial_n = \frac{\partial}{\partial \mathbf{n}}$). We can write the Reynolds transport theorem for mass in continuum \mathcal{A} with the moving subsurface $S^+ \subset \mathcal{M}^+$:

$$d_t \int_A \rho_A dv = \int_A \partial_t \rho_A dv + \int_{\partial A} \rho_A \mathbf{v}_A \cdot \mathbf{n}_A ds + \int_{S^+} \rho_A \mathbf{w} \cdot \mathbf{n}^+ ds, \quad (18)$$

which after using of 3D Green-Ostrogradski theorem:

$$\int_A \text{div}(\rho_A \mathbf{v}_A) dv = \int_{\partial A} \rho_A \mathbf{v}_A \cdot \mathbf{n}_A ds + \int_{S^+} \rho_A \mathbf{v}_A \cdot \mathbf{n}^+ ds, \quad (19)$$

takes the following form

$$d_t \int_A \rho_A dv = \int_A [\partial_t \rho_A + \text{div}(\rho_A \mathbf{v}_A)] dv + \int_{S^+} \rho_A (\mathbf{w} - \mathbf{v}_A) \cdot \mathbf{n}^+ ds. \quad (20)$$

The Slattery transport theorem for the surface mass [72, 58]:

$$d_t \int_S \rho_s ds = \int_S (\partial_t \rho_s - \mathbf{I}_b w_n \rho_s) ds + \int_{\partial S} \rho_s \mathbf{v}_{s\parallel} \cdot \mathbf{n}_l dl + \int_l \rho_s \mathbf{w}_\parallel \cdot \mathbf{n}_l dl, \quad (21)$$

together with 2D Green-Ostrogradski identity:

$$\int_S \text{div}(\rho_s \mathbf{v}_{s\parallel}) ds = \int_{\partial S} \rho_s \mathbf{v}_{s\parallel} \cdot \mathbf{n}_l dl + \int_l \rho_s \mathbf{v}_{s\parallel} \cdot \mathbf{n}_l dl, \quad (22)$$

takes the form

$$d_t \int_S \rho_s ds = \int_S [\partial_t \rho_s + \text{div}(\rho_s \mathbf{v}_{s\parallel}) - \mathbf{I}_b w_n \rho_s] ds + \int_l \rho_s (\mathbf{w}_\parallel - \mathbf{v}_{s\parallel}) \cdot \mathbf{n}_l dl, \quad (23)$$

where $\mathbf{w}_{\parallel} = \mathbf{w}\mathbf{I}_s$ and $\mathbf{v}_{s\parallel} = \mathbf{v}\mathbf{I}_s$ are tangential velocities.

Repeating that reasoning for the momentum vector, in a quite similar way, we can obtain the Reynolds transport theorem for momentum in continuum \mathcal{A} with the moving surface S^+ :

$$\begin{aligned} d_t \int_A \rho_A \mathbf{v}_A dv &= \int_A \partial_t (\rho_A \mathbf{v}_A) dv + \int_{\partial A} \rho_A \mathbf{v}_A \otimes \mathbf{v}_A \cdot \mathbf{n}_A ds + \int_{S^+} \rho_A \mathbf{v}_A \otimes \mathbf{w} \cdot \mathbf{n}^+ ds = \\ &= \int_A [\partial_t (\rho_A \mathbf{v}_A) + \operatorname{div} (\rho_A \mathbf{v}_A \otimes \mathbf{v}_A)] dv + \int_{S^+} \rho_A \mathbf{v}_A \otimes (\mathbf{w} - \mathbf{v}_A) \cdot \mathbf{n}^+ ds, \end{aligned} \quad (24)$$

and the Slattery transport theorem for the surface momentum³ [72, 58]:

$$\begin{aligned} d_t \int_S \rho_s \mathbf{v}_s ds &= \int_S [\partial_t (\rho_s \mathbf{v}_s) - \rho_s \mathbf{v}_s w_n \mathbf{I}_b] ds + \\ &\quad + \int_{\partial S} \rho_s \mathbf{v}_s \otimes \mathbf{v}_{s\parallel} \cdot \mathbf{n}_l dl + \int_l \rho_s \mathbf{v}_s \otimes \mathbf{w}_{\parallel} \cdot \mathbf{n}_l dl, \end{aligned} \quad (25)$$

or

$$\begin{aligned} d_t \int_S \rho_s \mathbf{v}_s ds &= \int_S [\partial_t (\rho_s \mathbf{v}_s) + \operatorname{div}_s (\rho_s \mathbf{v}_s \otimes \mathbf{v}_{s\parallel}) - \rho_s \mathbf{v}_s w_n \mathbf{I}_b] ds + \\ &\quad + \int_l \rho_s \mathbf{v}_s \otimes (\mathbf{w}_{\parallel} - \mathbf{v}_{s\parallel}) \cdot \mathbf{n}_l dl. \end{aligned} \quad (26)$$

The infinitesimal domain is quite different for 3D balance and for a layer balance. In a case of usual 3D balance it is sufficient to assume the infinitesimal Euler-Cauchy cube $dv = dx dy dz$. However in a layer, we cut out a part of layer with an infinitesimal volume $dv = h dS^+ \times dS^-$, that is usually called a ‘‘pillbox’’. According to the Fried-Gurtin rule of the pillbox balance [18], we distinguish the integration over the surfaces S^+ and S^- from an integration over $da \cong dh dl$ surface with $\mathbf{n}_l(h) \approx \mathbf{n}_l$ (see Fig. 2)⁴.

Looking for a Green-like transformation for this curved surface, and bearing in mind that \mathbf{p}_s has the form (9), we obtain

$$\begin{aligned} \int_l \left[\int_h \mathbf{p}_s(h, l) \mathbf{n}_l(l) dh \right] dl &\cong \int_l \left[\int_{h^-}^{h^+} \mathbf{p}_s(h, l) \mathbf{n}_l(l) \right] dl = \\ &= \int_l \mathbf{p}_s \mathbf{n}_l dl = \int_S [\operatorname{div}_s \mathbf{p}_s + (\partial_n \mathbf{p}_s \mathbf{n})] ds. \end{aligned} \quad (27)$$

³Petryk and Mróz have discovered yet more general form of time derivative which appears in (26) [49].

⁴The precise integration with the lifting tensors fields along normal coordinates has been elaborated in [12].

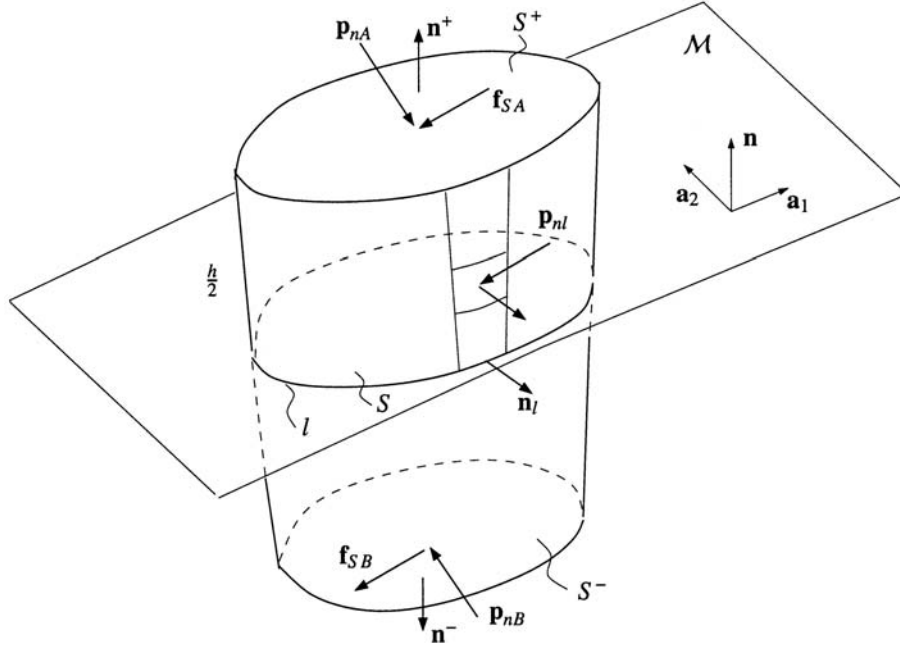


Figure 2. A pillbox domain of the Navier-Stokes layer balances.

It means that the normal change of \mathbf{p}_s deals only with the outer surface components:

$$\mathbf{p}_s = \mathbf{p}_s \mathbf{I}_s + (\mathbf{p}_s \mathbf{n}) \otimes \mathbf{n}. \quad (28)$$

3 Mass balance in the Navier-Stokes layer

Mass of the Navier-Stokes layer in any case cannot be taken to be constant – its rate of changing is so large that cannot be omitted. Such phenomena as drainage, wetting, dewetting, adherence, surface nucleation, catalytic reactions, lubrication, frequently appear in the nature and technology. Therefore we decided to add an additional governing equation, namely a balance of surface mass, in the mathematical model of a generalized Navier-Stokes boundary slip layer. We assumed that sources of mass are the bulk fluid continuum \mathcal{A} and the body \mathcal{B} , or the material line \mathcal{L} that bounds the surface \mathcal{M} , and then the mass balance of the whole system is

$$\begin{aligned}
& d_t \left[\int_A \rho_A dv + \int_B \rho_B dv + \int_S \rho_s ds \right] + \\
& + \int_{S^+} \dot{m}_A ds + \int_{S^-} \dot{m}_B ds + \int_S \dot{m}_s ds + \int_l \dot{m}_l dl = 0,
\end{aligned} \tag{29}$$

where \dot{m}_A , \dot{m}_B are mass influxes [$\text{kg s}^{-1} \text{m}^{-2}$] that come from the continua \mathcal{A} and \mathcal{B} through the surface S^+ and S^- respectively, and \dot{m}_l is a mass outflux [$\text{kg s}^{-1} \text{m}^{-1}$]. We can separate the system onto three sub-systems divided by surfaces S^+ and S^- such that

$$d_t \int_A \rho_A dv + \int_{S^+} \dot{m}_A ds = 0 \quad \text{in } \mathcal{A} \cup \mathcal{M}^+, \tag{30}$$

$$d_t \int_B \rho_B dv + \int_{S^-} \dot{m}_B ds = 0 \quad \text{in } \mathcal{B} \cup \mathcal{M}^-, \tag{31}$$

$$d_t \int_S \rho_s ds + \int_S \dot{m}_s ds + \int_l \dot{m}_l dl = 0 \quad \text{on } \mathcal{M} \cup \mathcal{L}. \tag{32}$$

Balancing only the mass crossing surfaces S^+ and S^- we have a restriction

$$\int_{S^+} \dot{m}_A ds + \int_{S^-} \dot{m}_B ds + \int_S \dot{m}_s ds = 0. \tag{33}$$

Small differences between S^+ , S^- and S can be omitted and therefore we can write

$$\dot{m}_A + \dot{m}_B + \dot{m}_s = 0. \tag{34}$$

The layer mass has a source $\dot{m}_s = -(\dot{m}_A + \dot{m}_B)$ that is equal to contribution coming from \mathcal{A} and \mathcal{B} . Using the Reynolds and Slattery transport theorems (18)–(23) we can obtain from (29) that

$$\begin{aligned}
& \int_A [\partial_t \rho_A + \text{div}(\rho_A \mathbf{v}_A)] dv + \int_B [\partial_t \rho_B + \text{div}(\rho_B \mathbf{v}_B)] dv + \\
& + \int_{S^+} [\dot{m}_A + \rho_A (\mathbf{w} - \mathbf{v}_A) \cdot \mathbf{n}^+] ds + \int_{S^-} [\dot{m}_B + \rho_B (\mathbf{w} - \mathbf{v}_B) \cdot \mathbf{n}^-] ds + \\
& + \int_S [\partial_t \rho_s + \text{div}_s(\rho_s \mathbf{v}_{s\parallel}) - w_n \rho_s \mathbf{I}_b + \dot{m}_s] ds + \int_l [\dot{m}_l + \rho_s (\mathbf{w}_{\parallel} - \mathbf{v}_{s\parallel})] \cdot \mathbf{n} dl = 0.
\end{aligned} \tag{35}$$

The set of equations presented above describes precisely the layer mass balance in the system under consideration. These equations locally read to be

$$\partial_t \rho + \text{div}(\rho \mathbf{v}) = 0 \quad \text{for } \mathcal{A} \cup \mathcal{B}, \tag{36}$$

$$\partial_t \rho_s + \operatorname{div}_s (\rho_s \mathbf{v}_{s\parallel}) - w_n \rho_s \mathbf{I}_b = \dot{m}_A + \dot{m}_B \quad \text{on } \mathcal{M}. \quad (37)$$

From the balance on the surfaces S^+ , S^- and an open part of boundary line, we additionally obtain an accompanying definitions of mass fluxes:

$$\dot{m}_A = -\rho_A (\mathbf{w} - \mathbf{v}_A) \cdot \mathbf{n}^+ \quad \text{on } \mathcal{M}^+, \quad (38)$$

$$\dot{m}_B = -\rho_B (\mathbf{w} - \mathbf{v}_B) \cdot \mathbf{n}^- \quad \text{on } \mathcal{M}^-, \quad (39)$$

$$\dot{m}_s = -\rho_s (\mathbf{w}_{\parallel} - \mathbf{v}_{s\parallel}) \cdot \mathbf{n}_l \quad \text{on } \mathcal{L}. \quad (40)$$

4 Balance of momentum of the Navier-Stokes layer

An idea of common treatment of the external friction phenomena as a true Newtonian “*vis impressa*”, that appears in the continuum mechanics for the first time, can now take part in the balance of momentum, is a great novelty of the Navier-Stokes layer. Therefore, postulated by Navier the friction forces \mathbf{f}_{SA} and \mathbf{f}_{SB} on the surfaces S^+ and S^- respectively, explicitly appear in the balance of system forces. It seems to be a crucial point of our reasoning. Both forces \mathbf{f}_{SA} and \mathbf{f}_{SB} depend on material properties of fluids \mathcal{A} , \mathcal{B} and on apparent, operative properties of the considered layer $\mathcal{M}^+ \mathcal{M}^-$. If the system contains two continua \mathcal{A} , \mathcal{B} and the “pillbox” of the Navier-Stokes layer bounded by material line \mathcal{L} then the condition of a conservation of the total momentum can be written as

$$\begin{aligned} d_t \left[\int_A \rho_A \mathbf{v}_A dv + \int_B \rho_B \mathbf{v}_B dv + \int_S \rho_s \mathbf{v}_s ds \right] + \int_{A \cup B} (\rho_A + \rho_B) \mathbf{b} ds + \int_S \rho_s \mathbf{b}_s ds + \\ + \int_{S^+} (\mathbf{p}_{nA} + \dot{\mathbf{r}}_A) ds + \int_{S^-} (\mathbf{p}_{nB} + \dot{\mathbf{r}}_B) ds + \int_{\partial A} \mathbf{p}_{nA} ds + \int_{\partial B} \mathbf{p}_{nB} ds + \\ + \int_S (\mathbf{p}_{S^+} + \mathbf{p}_{S^-} + \dot{\mathbf{r}}_s) ds + \int_l (\mathbf{p}_{nl} - \dot{\mathbf{r}}_l) + \int_{\partial S} \mathbf{p}_{nl} dl = 0, \end{aligned} \quad (41)$$

where $\dot{\mathbf{r}}_A$, $\dot{\mathbf{r}}_B$ and $\dot{\mathbf{r}}_s$ are the momentum carried with the mass fluxes \dot{m}_A , \dot{m}_B , \dot{m}_s respectively. The traction forces on the contact surfaces are defined to be $\mathbf{p}_{nA} = \mathbf{p}_A \mathbf{n}^+$ and $\mathbf{p}_{nB} = \mathbf{p}_B \mathbf{n}^-$. It means that these forces depend on the stresses \mathbf{p}_A and \mathbf{p}_B in the bulk continua. Additionally, \mathbf{p}_{S^+} and \mathbf{p}_{S^-} are two contact forces which act on the layer side S^+ and S^- , respectively. There is also the surface traction $\mathbf{p}_{nl} = \mathbf{p}_s \mathbf{n}_l$, on the boundary of the Fried-Gurtin “pillbox”, where \mathbf{n}_l is a normal vector to the surface element $dh dl$. Additional momentum

$\dot{\mathbf{r}}_l$ is carried out from the system, if we have the surface mass outflow \dot{m}_l through the line \mathcal{L} . The forces \mathbf{b} and \mathbf{b}_s are the conventional body forces.

Employing the physical relation that restricts momentum transport through the surface \mathcal{M} :

$$\int_{S^+} \dot{\mathbf{r}}_A ds + \int_{S^-} \dot{\mathbf{r}}_B ds + \int_S \dot{\mathbf{r}}_s ds = \int_S (\dot{\mathbf{r}}_A + \dot{\mathbf{r}}_B + \dot{\mathbf{r}}_s) ds \rightarrow \dot{\mathbf{r}}_s = -\dot{\mathbf{r}}_A - \dot{\mathbf{r}}_B, \quad (42)$$

and assuming that the contact forces are restricted by an internal equilibrium, which is some continuum realization of the Newtonian law on “action and reaction”, we obtain

$$\int_{S^+} \mathbf{p}_{nA} ds + \int_{S^-} \mathbf{p}_{nB} ds + \int_S (\mathbf{p}_{S^+} + \mathbf{p}_{S^-}) ds = 0. \quad (43)$$

Assuming further that, not yet defined contact forces \mathbf{p}_{S^+} , \mathbf{p}_{S^-} that depend on the traction forces coming from the continua \mathcal{A} , \mathcal{B} and friction forces \mathbf{f}_{SA} and \mathbf{f}_{SB} , we can write within the contact surfaces a following relation, according to the law of “equality of action and reaction”:

$$\begin{cases} \mathbf{p}_{nA} + \mathbf{f}_{SA} = \mathbf{p}_{S^+} \\ \mathbf{p}_{nB} + \mathbf{f}_{SB} = \mathbf{p}_{S^-} \end{cases}. \quad (44)$$

Employing the above relations (42)–(44) and (24)–(26) we obtain

$$\begin{aligned} & \int_A [\partial_t (\rho_A \mathbf{v}_A) + \operatorname{div} (\rho_A \mathbf{v}_A \otimes \mathbf{v}_A) + \operatorname{div} \mathbf{p}_A - \rho_A \mathbf{b}] dv + \\ & + \int_B [\partial_t (\rho_B \mathbf{v}_B) + \operatorname{div} (\rho_B \mathbf{v}_B \otimes \mathbf{v}_B) + \operatorname{div} \mathbf{p}_B - \rho_B \mathbf{b}] dv + \\ & + \int_S [\partial_t (\rho_s \mathbf{v}_s) + \operatorname{div}_s (\rho_s \mathbf{v}_s \otimes \mathbf{v}_{s\parallel}) - w_n \rho_s \mathbf{v}_s + \operatorname{div}_s \mathbf{p}_s + \partial_n (\mathbf{p}_s \mathbf{n}) + \\ & + (\mathbf{p}_{nA} + \mathbf{p}_{nB} + \mathbf{f}_{SA} + \mathbf{f}_{SB}) - (\rho_s \mathbf{b}_s + \dot{\mathbf{r}}_s)] ds + \\ & + \int_{S^+} [\rho_A \mathbf{v}_A \otimes (\mathbf{w} - \mathbf{v}_A) \mathbf{n}^+ - \dot{\mathbf{r}}_A] dS + \int_{S^-} [\rho_B \mathbf{v}_B \otimes (\mathbf{w} - \mathbf{v}_B) \mathbf{n}^- - \dot{\mathbf{r}}_B] dS + \\ & + \int_l [\rho_s \mathbf{v}_s \otimes (\mathbf{w}_{\parallel} - \mathbf{v}_{s\parallel}) \mathbf{n}_l - \dot{\mathbf{r}}_l] dl = 0. \end{aligned} \quad (45)$$

The rate of momentum carried by the exchange of mass between subsystems is described by three last integrals. Using further (38)–(40) we get

$$\dot{\mathbf{r}}_A = \dot{m}_A \mathbf{v}_A, \quad \dot{\mathbf{r}}_B = \dot{m}_B \mathbf{v}_B, \quad (46)$$

$$\dot{\mathbf{r}}_s = \dot{m}_s \mathbf{v}_s = \dot{m}_A (\mathbf{v}_A - \mathbf{v}_s) + \dot{m}_B (\mathbf{v}_B - \mathbf{v}_s), \quad (47)$$

$$\dot{\mathbf{r}}_l = \dot{m}_l \mathbf{v}_s . \quad (48)$$

The local form of the momentum balance can be finally written as⁵

$$\partial_t (\rho \mathbf{v}) + \operatorname{div} (\rho \mathbf{v} \otimes \mathbf{v} + \mathbf{p}) = \rho \mathbf{b} \quad \text{for } \mathcal{A} \cup \mathcal{B}, \quad (49)$$

$$\begin{aligned} & \partial_t (\rho_s \mathbf{v}_s) + \operatorname{div}_s (\rho_s \mathbf{v}_s \otimes \mathbf{v}_{s\parallel}) - w_n \mathbf{I}_b \rho_s \mathbf{v}_s + \\ & + \operatorname{div}_s \mathbf{p}_s + \partial_n (\mathbf{p}_s \mathbf{n}) + [\mathbf{p}_A \mathbf{n}_A + \mathbf{p}_B \mathbf{n}_B + \mathbf{f}_{SA} + \mathbf{f}_{SB}] = \\ & = \rho_s \mathbf{b}_s + \dot{m}_A (\mathbf{v}_A - \mathbf{v}_s) + \dot{m}_B (\mathbf{v}_B - \mathbf{v}_s) \quad \text{on } \mathcal{M}. \end{aligned} \quad (50)$$

Repeating now the reasoning of d'Alembert and Euler, we can define a surface d'Alembert-Euler acceleration vector to be

$$\mathbf{a}_s = \frac{d_s}{dt} \mathbf{v}_s = \partial_t \mathbf{v}_s + (\operatorname{grad}_s \mathbf{v}_s) \mathbf{v}_{s\parallel} . \quad (51)$$

Employing the surface identity (22), instead of divergence of the convective flux of surface momentum we obtain

$$\rho_s \mathbf{a}_s = \partial_t (\rho_s \mathbf{v}_s) + \operatorname{div}_s (\rho_s \mathbf{v}_s \otimes \mathbf{v}_{s\parallel}) . \quad (52)$$

The Navier-Stokes layer in generalized form is described now by the layer balances of mass (37) and momentum (50). These are two additional nonlinear differential equations for two additional fields of unknowns, i.e. the surface mass density ρ_s and the layer slip velocity \mathbf{v}_s . These equations are both geometrically and physically nonlinear, and should be solved using any discretization method – finite element method (FEM), finite volume method (FVM) – under assumption that the surface \mathcal{M} possesses an independent from the bulk space discretization. In the case when \mathcal{M}^- is a fixed solid surface, the migration velocity $\mathbf{w} = 0$, and then discretization mesh could be fixed in the marching time of numerical solution. Apparently, if $\mathbf{w} \neq 0$, then a moving, the self deforming mesh motion should be resolved together with surface mass and surface momentum equations, and the appropriate set of equations for bulk. There are different cases of using the Navier-Stokes layer balances in the literature. For instance, when \mathcal{A} and \mathcal{B} are ideal, nonviscous Euler fluids, and the surface density is equal to zero $\rho_s = 0$, and the layer momentum flux is omitted $\mathbf{p}_s = 0$, then the surface mass and momentum equations reduce to the generalized form of the Rankine-Hugoniot jump conditions:

$$\left\{ \begin{array}{l} \dot{m}_A = \dot{m}_B \\ \dot{m}_A \mathbf{v}_A + p_A \mathbf{n}_A = \dot{m}_B \mathbf{v}_B + p_B \mathbf{n}_B \end{array} \right. , \quad (53)$$

⁵An example how to define \mathbf{p}_B for the deformable wall is given in the paper by dell'Isola *et al.* ([13], Eq.(40)). For a rigid wall stresses are taken to be zero: $\mathbf{p}_B \equiv 0$.

where p_A, p_B are thermodynamic pressures in the Euler fluids \mathcal{A} and \mathcal{B} , respectively. If, additionally $\mathbf{w} = 0$, and there is an additional contribution to the surface diade $\mathbf{p}_s = \gamma \mathbf{I}_s$, then the layer momentum balance leads to the generalized Young-Laplace equation:

$$\operatorname{div}_s(\gamma \mathbf{I}_s) + p_A \mathbf{n}_A + p_B \mathbf{n}_B = \left[\gamma \left(\frac{1}{r_1} + \frac{1}{r_2} \right) + p_A - p_B \right] \mathbf{n} = 0. \quad (54)$$

If an interfacial density is omitted i.e., $\rho_s = 0$, the difference between the external friction forces \mathbf{f}_{SA} and \mathbf{f}_{SB} simply vanishes then, and a single layer friction force exists:

$$\mathbf{f}_{AB} = \mathbf{f}_{SA} + \mathbf{f}_{SB} = \nu (\mathbf{v}_A - \mathbf{v}_B), \quad (55)$$

where ν is an external viscosity coefficient. It is an exact form of an external friction force proposed by Navier ($\mathbf{v}_B = 0$) and Stokes ($\mathbf{v}_B = \mathbf{v}_{\text{wall}}$). Assuming that the continuum \mathcal{A} is an incompressible viscous fluid: $\mathbf{p}_A = p \mathbf{I} - 2\mu \mathbf{d}$, and the continuum \mathcal{B} is a rigid, fixed solid body: $\mathbf{p}_B = 0, \mathbf{v}_B = 0$, we obtain the Navier slip boundary condition:

$$\mathbf{f}_{AB} + \mathbf{p}_A \mathbf{n}_A = \nu \mathbf{v}_A + (p \mathbf{I} - 2\mu \mathbf{d}) \mathbf{n} = 0 \quad \text{on } \mathcal{M}, \quad (56)$$

where $\mathbf{v}_s = \mathbf{v}_{A|\mathcal{M}}$ is identified with the slip velocity.

5 Recoverable relations for surface momentum flux

The layer flux of momentum is responsible for recoverable capillary and viscous transport: $\mathbf{p}_s = \mathbf{p}_s^{(c)} + \mathbf{p}_s^{(\nu)}$. The first most important part of the elastic recoverable diade $\mathbf{p}_s^{(c)}$, that is known as the capillarity diade, can be described by the surface tension γ . This quantity was introduced to the process of mathematical modeling by Young, Laplace and Poisson. The second contribution comes from the recoverable stresses called the surface bending C_1, C_2 , introduced by Gibbs. There is also a layer ‘‘normal pressure’’ ϖ , introduced by Stokes. These altogether lead to the following definition of the capillarity diade:

$$\mathbf{p}_s^{(c)} = \varpi \mathbf{n} \otimes \mathbf{n} + \gamma \mathbf{I}_s + C \mathbf{II}_s, \quad \partial_n (\mathbf{p}_s \mathbf{n}) = \varpi \mathbf{n}, \quad (57)$$

where $2C = C_1 + C_2$, and $\operatorname{div}_s \mathbf{p}_s^{(c)} = \gamma \mathbf{I}_b \mathbf{n} + C (\mathbf{I}_b^2 - 2\mathbf{II}_b) \mathbf{n}$. A quite general form of the capillarity diade has been proposed recently in [1] as

$$\mathbf{p}_s^{(c)} = \gamma_0 - \mathbf{II}_s \gamma_1 + \mathbf{n} \otimes \mathbf{I}_s \operatorname{div}_s (\gamma_1 - \mathbf{II}_s \gamma_2), \quad (58)$$

where the surface capillary measures can be defined to be spherical

$$\gamma_0 = \gamma \mathbf{I}_s, \quad \gamma_1 = C \mathbf{II}_s, \quad \gamma_2 = K \mathbf{III}_s. \quad (59)$$

These capillary measures are expressed in terms of the first, second and third fundamental surface forms, and γ , C , K are the surface tension, bending and torque coefficients, respectively. If we suppose however, that the surface tension γ is a surface dependent quantity, as for example in [4]:

$$\gamma = \sigma (\rho_s - \rho_{s0}) , \quad (60)$$

where σ is a constant tension, and ρ_{s0} is an equilibrium-like surface density, then we have: $\text{div}_s (\gamma \mathbf{I}_s) = (\text{grad}_s \gamma) \mathbf{I}_s + \gamma \text{div}_s b f \mathbf{I}_s$. From $\text{grad}_s \gamma \neq 0$ it follows that the Marangoni ripples and other capillary effects can additionally be described, [24]. Additionally, if the rate of change of ρ_s is given as the result of mass flow rate coming from the fluid \mathcal{A} :

$$\dot{m}_A = \rho_A \mathbf{v}_A \cdot \mathbf{n}_A = \tau^{-1} (\rho_s - \rho_{s0}) , \quad (61)$$

where τ is the Bilicki-Kestin relaxation time [4], and employing the surface mass balance (37): $\text{div}_s (\rho_s \mathbf{v}_s) = -\tau^{-1} (\rho_s - \rho_{s0})$, we can find the tangential components of surface velocity \mathbf{v}_s . It could be done by using of a simplified, steady-state form of the momentum balance (50):

$$\mu \mathbf{I}_s (\text{grad} \mathbf{v}_A + \text{grad}^T \mathbf{v}_A) \mathbf{n} + \text{div}_s (\gamma \mathbf{I}_s) + \nu (\mathbf{v}_A - \mathbf{v}_s) = \dot{m}_A \mathbf{v}_A , \quad (62)$$

which in special cases can be solved analytically.

The viscous properties of the Navier-Stokes layer depend on the so-called ‘‘apparent viscosity’’ which, in general, possesses a ‘‘transversal’’ anisotropy [24]. One can define the viscous surface stresses by using the surface diade of the rate of deformation (14), and a normal change $v_{n,n}$:

$$\mathbf{p}_s^{(\nu)} = \lambda' (\text{tr} \mathbf{d}_s) \mathbf{I}_s + \lambda'' v_{n,n} \mathbf{n} \otimes \mathbf{n} + 2\mu' \mathbf{I}_s \mathbf{d}_s \mathbf{I}_s + 2\mu'' (\mathbf{d}_s - \mathbf{I}_s \mathbf{d}_s \mathbf{I}_s) . \quad (63)$$

This diade does not undergo the classical 3D de Saint-Venant condition, saying that the viscous stresses must be traceless. For a special case when $\lambda'' = \mu'' = 0$, this constitutive relation leads to that proposed by Boussinesq (1913) [5, 56]:

$$\mathbf{p}_s^{(\nu)} = (\lambda' - \mu') (\text{tr} \mathbf{d}_s) \mathbf{I}_s + 2\mu' \mathbf{I}_s \mathbf{d}_s \mathbf{I}_s . \quad (64)$$

The formula for surface viscosity coefficients λ' , μ' needs extended investigations. Both coefficients have nothing in common with the internal viscosity μ of fluid \mathcal{A} . It should be underlined that another contributions for \mathbf{p}_s , that come from additional surface fields, like the chemical potential, phase transition parameter, turbulent intermittency parameter, surface entropy and temperature, surface electric potential, etc., are also possible, however this issue needs more elaboration.

If the Navier-Stokes layer is formulated as a thin film of second phase, then a serious increase of the slip length is observed. De Gennes *et al.* [10] have found

that, if the film thickness is h and the shear interfacial viscosity is μ' , then the slip length is determined to be

$$l_s = h \left(\frac{\mu}{\mu'} - 1 \right). \quad (65)$$

6 Navier boundary layer

Let us now turn to Navier's original line of reasoning that, unlike our own, is based on the argumentation elaborated by molecular dynamics. Let us recall that the molecular dynamics approach is based on the precise equation of motion for every molecule and less precise description of frictional interaction between molecules, [48, 50]:

$$m_{(k)} \ddot{\mathbf{x}}_{(k)} + \chi_{(k)} \dot{\mathbf{x}}_{(k)} + \sum_{l \neq k}^n \frac{\partial V_{(k,l)}}{\partial \mathbf{x}_{(k)}} = \mathbf{f}_{(k)} \quad k = 1, \dots, n. \quad (66)$$

Usually, the elastic recoverable, interaction potential $V_{(k,l)}$ for the molecules (k) and (l) is taken in MD as the pairwise Lennard-Jones potential with two constants. Friction in the above model is described by the dumping coefficient $\chi_{(k)}$. The forces $\mathbf{f}_{(k)}$, should be known, however sometimes are defined in a manner consistent with the continuum approach – for instance, by the inlet and outlet difference of pressure [36, 28]. On the solid boundary both the elastic potential and the dumping coefficient should take into account a specific interaction with hot boundary solid molecules.

On the other hand, molecular dynamics of Navier is a little different. Since it is based on variational formulation. He proposes a quantity, which nowadays is identified with the total energy of continuum:

$$0 = \iiint_V dx dy dz \{ \delta \mathcal{W}_{\text{ext}} + \delta \mathcal{W}_{\text{rev}} + \delta \mathcal{W}_{\text{d'Alem}} + \delta \mathcal{W}_{\text{visc}} \} + \iint ds \delta \mathcal{W}_{\text{separ}}, \quad (67)$$

where particular elements of the total energy variation are⁶ expressed in terms of virtual displacement $\delta \mathbf{x} = \delta \mathbf{v} dt$:

$$\begin{aligned} \delta \mathcal{W}_{\text{ext}} &= \mathbf{f}_{\text{body}} \cdot \delta \mathbf{x}, \\ \delta \mathcal{W}_{\text{rev}} &= p \left(\frac{d\delta x}{dx} + \frac{d\delta y}{dy} + \frac{d\delta z}{dz} \right) = p \text{div}(\delta \mathbf{x}), \\ \delta \mathcal{W}_{\text{d'Alem}} &= \rho \mathbf{a} \cdot \delta \mathbf{x}, \\ \delta \mathcal{W}_{\text{visc}} &= \mu [(\text{grad} \mathbf{v}) + (\text{grad}^T \mathbf{v})] \cdot [\delta \text{grad} \mathbf{v}] + \mu [\text{tr}(\text{grad} \mathbf{v})] [\text{tr} \delta(\text{grad} \mathbf{v})] dt, \\ \delta \mathcal{W}_{\text{separ}} &= \nu \mathbf{v} \cdot \delta \mathbf{v} dt. \end{aligned} \quad (68)$$

⁶See ([46], pp. 395, 412, 404, 411), respectively.

Integrating by parts, in an analogy to Lagrange's reasoning, Navier got not only a local form of the equation of motion (nowadays identified with the momentum balance) but also an adequate boundary conditions:

$$\begin{aligned} & \iiint_V dx dy dz \left\{ -\mathbf{f}_{\text{body}} - \operatorname{div}(-p\mathbf{I}) + \rho\mathbf{a} - \operatorname{div} \left\{ 2\mu \left[\mathbf{d} + \frac{1}{3}(\operatorname{tr}\mathbf{d})\mathbf{I} \right] \right\} \right\} \cdot [\delta\mathbf{v}] + \\ & + \iint_{\mathcal{M}} ds \left[-\nu\mathbf{v} + \left\{ 2\varepsilon \left[\mathbf{d} + \frac{1}{3}(\operatorname{tr}\mathbf{d})\mathbf{I} \right] \right\} \mathbf{n} \right] \cdot \delta\mathbf{v} = 0, \end{aligned} \quad (69)$$

which are valid both for compressible and incompressible flows. Here, $\mathbf{a} = \partial_t \mathbf{v} + (\operatorname{grad} \mathbf{v})\mathbf{v}$ is the d'Alembert-Euler acceleration vector. Taking next the incompressibility conditions $\operatorname{tr}\mathbf{d} = 0$, Navier got the local equations of momentum balance⁷:

$$\mathbf{f}_{\text{body}} + \operatorname{div}(-p\mathbf{I}) = \rho\mathbf{a} - \operatorname{div}(2\mu\mathbf{d}), \quad (70)$$

and boundary conditions, nowadays called "the slip boundary conditions"⁸:

$$\nu\mathbf{v} = 2\mu\mathbf{d}\mathbf{n} \quad \text{on} \quad \mathcal{M}. \quad (71)$$

Taking into account that fluid does not penetrate the wall i.e. $\mathbf{v} \cdot \mathbf{n} = 0$, Navier simplifies the above equations to the form:

$$\nu\mathbf{v} = \mu(\operatorname{grad}\mathbf{v})\mathbf{n}. \quad (72)$$

Two independent viscosity coefficients appear in this celebrated equation. The first one is an internal viscosity μ and the second one is an external viscosity ν (ε and E in original Navier notation). Therefore, finishing his own derivations for equations of motion of a viscous fluid, Navier adds comments on a dependency for the coefficient ν on a type of wall material (cooper, glass). Next, trying to compare the model of internally and externally viscous flow with Girard's experiments [46], Navier has prepared three analytical solutions for different flows in pipes and open channels. In his solutions both coefficients μ and ν take part, and what is important for the future definition of the length of slip l_s , the ratio ($l_s = \frac{\mu}{\nu}$) frequently appears. However, a notion for flow enhancement could not be

⁷See ([46], p. 414). Here we touch a problem of consistency of denotations since we face the letters ε , E (Navier) and μ , ν (Stokes) for the same internal viscosity and external viscosity, respectively. We propose to accept Stokes's denotation, since his contribution to the discovery of the boundary layer slip is meaningful and motivated, not by MD, but also within the framework of rational continuum mechanics.

⁸See ([46], p. 415). It should be underlined that this equation is not Galilean invariant. It is true and problematic. Quite recently a consistent approach to two-dimensional surface momentum balance which is Galilean invariant has been given by dell'Isola *et al.* [13].

discovered by Navier, since the referential mass flow rate was discovered later by Poiseuille in 1844. Indeed nowadays, taking Navier’s and Poiseuille’s solution into consideration for the capillary pipe flow, we can easily find the flow enhancement due to the presence of a slip:

$$\text{enh} = \frac{\dot{m}_{\text{Navier}}}{\dot{m}_{\text{Poiseuille}}} = \left(1 + \frac{8l_s}{a}\right) \frac{1}{\left(1 + \frac{l_s}{2a}\right)}, \quad (73)$$

valid for arbitrary fluid and solid materials⁹. In the case of the slip length equal to the pipe’s radius, $l_s = a$, the flow enhancement is $\text{enh} = 4/3$.

7 Stokes boundary layer

Keeping in mind the problem of consistency between the models established within the bulk and at the boundary, let us firstly recall the Newton second law of dynamics proposed for a viscous fluid by Stokes (1845), and treated by him as a continuum version of linear momentum balance. In the case of fluid motion, a balance of the linear momentum is undergoing a vector density per unit of an actual volume of fluid. Assume, for the sake of continuity, that the contact forces between fluid molecules can be simply represented as the divergence of a flux momentum tensor \mathbf{p} :

$$\underbrace{\mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{vis}}}_{\text{Navier}} \equiv \underbrace{\text{div } \mathbf{p}}_{\text{Stokes}}, \quad (74)$$

where, as opposed to Cauchy, Stokes defines the flux of momentum \mathbf{p} , not as a “tension” but as a “pression”. Stokes assumes this pressure diade as an additive composition of the spherical pressure and the “tangential actions” as he thought about viscous stresses. Such an assumption is fundamental for describing the momentum transport within the continua of particles. The same assumption applied for the continua of long material filaments or small shells, cannot be further correct and successful, since only in the continuum of particles the mechanism for the transport of momentum can be correctly described by a tensor of second

⁹It is a strongly nonlinear formula which is able to describe the Knudsen paradox, [35, 22]. It is a commonly observed fact that in all of these microdevices, we measure a serious enhancement of mass flow rates in comparison to corresponding mass flow rates predicted by solution of the classical Navier Stokes equations, or by the turbulent Navier-Stokes-Reynolds equations. From the point of view of rational continuum mechanics, the anomalous mass flow rates through the microchannels are of particular interest. Additionally to the flow enhancement anomaly, we observe also a specific flow paradoxes. The most known one is the Knudsen paradox, where the mass flow is strongly nonlinear and attains some minimum. Early studies on microchannel liquid flow had already concluded that the theoretical predictions, using the Navier-Stokes equations and no slip velocity at the wall as boundary condition, led to underestimation of the mass flow rates [33].

order. Therefore, when the laws of Newtonian mechanics were applied to the substantial continuum of particles, the results were unclear and irrational. Left side of (74), known as Euler's reasoning, was well known to Stokes, as to be the only available way to postulate these fundamental laws of Nature. Even today one can find in the literature similar kind of argumentation: "if formula (74) is always true, the notion of stress tensor is unnecessary". Up to now, there is no other possibility. Everyone who wants to state the momentum balance, from the very beginning, should go back to the Euler elementary parallelepiped and should draw the balance of internal and external forces to be

$$\mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{body}} + \mathbf{f}_{\text{d'Alem}} + \mathbf{f}_{\text{vis}} = 0. \quad (75)$$

It is not necessary to go deep into details in order to see that Stokes repeats precisely the Euler argumentation. He postulated to consider an infinitesimal fluid particle $E = \Delta x \Delta y \Delta z$, taken in the form a rectangular parallelepiped. The following set of "impressa forces" appears in this infinitesimal volume:

- d'Alembert's inertia force:

$$\mathbf{f}_{\text{d'Alem}} = \rho \mathbf{a} \Delta x \Delta y \Delta z,$$

- the body force:

$$\mathbf{f}_{\text{body}} = -\rho \mathbf{b} \Delta x \Delta y \Delta z,$$

- the neighborhood force:

$$\mathbf{f}_{\text{pressure}} + \mathbf{f}_{\text{vis}} = \text{div} \mathbf{p} \Delta x \Delta y \Delta z.$$

According to the Euler postulate, these forces, after a simple adding to themselves, and dividing by the infinitesimal volume, lead to the most celebrated local balance of momentum in the so-called Euler description:

$$\rho (\mathbf{a} - \mathbf{b}) + \text{div} \mathbf{p} = 0. \quad (76)$$

Here the specific density of the body force is \mathbf{b} , and ρ is the fluid density. Now, adopting the d'Alembert-Euler definition of the acceleration vector \mathbf{a} , and using over-dot, instead of Stokes' denotation $\frac{D}{Dt}$ for the material time derivative, we obtain

$$\mathbf{a} = \dot{\mathbf{v}} = \partial_t \mathbf{v} + (\text{grad} \mathbf{v}) \mathbf{v}, \quad (77)$$

what means that the acceleration of a fluid particle is the nonlinear function of velocity. Further, substituting for (76) Stokes's constitutive definition for the pressure diade:

$$\mathbf{p} = p \mathbf{I} - 2\mu (\mathbf{d} - \delta \mathbf{I}) - 3\kappa \delta \mathbf{I} = p \mathbf{I} - 2\mu \mathbf{d} + \left(\frac{2}{3}\mu - \kappa \right) \mathbf{I}_d \mathbf{I}, \quad (78)$$

where the dilatational change of volume is described by $3\delta = \text{I}_d = \text{tr}(\mathbf{d})$. Supposing the bulk viscosity $\kappa = 0$; and the internal friction coefficient μ to be homogeneous, i.e. constant in space, we have

$$\rho(\dot{\mathbf{v}} - \mathbf{b}) + \text{grad}p - \mu [\text{lap}\mathbf{v} + \text{div}(\text{grad}^T\mathbf{v})] + \frac{2}{3}\mu\text{grad}\text{I}_d = 0, \quad (79)$$

where $\text{lap} = \text{div}(\text{grad})$ or within the Cartesian coordinates, [69]:

$$\rho(\dot{\mathbf{v}} - \mathbf{b}) + \text{grad}p - \mu\text{lap}\mathbf{v} - \frac{\mu}{3}\text{grad}\text{I}_d = 0. \quad (80)$$

The equations in the form (80) are usually called the ‘‘Navier-Stokes equations’’. These equations are geometrically nonlinear with respect to the definition of the acceleration vector (77), and physically nonlinear with respect to the relation between pressure, density, temperature and entropy η – however this relation has not been mentioned by Stokes explicitly.

With respect to the internal friction μ , these equations are linear, what is an exact prolongation of the Newtonian one-dimensional case into a three-dimensional one [47]. A simple type of nonlinearity with respect to the internal friction was only postulated by Stokes, who assumed that the coefficient of viscosity, in general, should be a function¹⁰:

$$\mu = \mu(p, \theta, \text{I}_d, \text{II}_d, \text{III}_d). \quad (81)$$

In general, the phenomenon of internal friction between the adjacent layers of fluid could depend on the normal components of stresses or pressure. The internal friction is a phenomenon of interchange of momentum between fluid particles due to the molecular motion of fluid layers. Independently from the mechanism of friction proportional to a relative velocity for the contacting layer, it is worth to consider a mode of friction depending on the normal internal force. According to Coulomb, as the normal stresses increase the frictional resistance increases [9]. Therefore, Stokes clearly recognized that the shearing viscosity, being proportional to relative velocity, is this mode of internal friction that could be considered independent of pressure. He states:

¹⁰Development of this line of reasoning is presented in brilliant Truesdell’s paper [C. Truesdell, A new definition of fluid, I. The Stokesian Fluid, *Journ. de Math.* **29** (1950), 215-243]. A one most known example of this kind of constitutive relations for the shear viscosity is the shear-thinning Carreau model of viscosity:

$$\mu = \mu_0 [1 + (\alpha\text{II}_d)^2]^{\frac{(N-1)}{2}},$$

where α , N are the relaxation time constant and power law-index, respectively [R.B. Bird, R.C. Armstrong, O. Hassanger, *Dynamics of polymer liquids*, Wiley, NY, 1987].

“[...] It has been concluded by Du Buat, from his experiments on the motion of water in pipes and canals, that the total retardation of velocity due to friction is not increased by increasing the pressure I shall therefore suppose that for water, and by analogy for other incompressible fluids, μ is independent of pressure. [...]”

We will now examine the problem of Stokes’ boundary slip layer concept. It is well known that after Stokes’ proposition for a celebrated set of governing equations for the motion of fluid undergoing internal friction, he turned his attention to the problem of external friction for fluid/solid and fluid/fluid interfaces. Assuming Stokes’ reasoning, one can find that there was an unification – from one side, the great researchers by Young, Laplace and Poisson concerning these conditions, which should be fulfilled on a free surface i.e., the surface between water and air, and from the other side, a concept of the slip boundary condition of Navier. In order to do this, he proposed to take into consideration a distinguished thin boundary layer, based on a middle surface \mathcal{M} . In analogy to Cauchy’s infinitesimal cube $dV = dx dy dz$, Stokes proposes to make a balance of momentum in an infinitesimal part of the layer that is bounded from the top via the infinitesimal surface S^+ and from the bottom via S^- [see Fig.1].

He supposed that the fluid under consideration occupies the volume V_A and is bounded by a surface \mathcal{M}^+ , which is oriented outside by a normal unit vector $\mathbf{n}_A = -\mathbf{n}^+$. Thus, the surface \mathcal{M} is a limitation of a material boundary layer – an infinitesimal “pillbox” layer element, that allows for performing proper momentum balance – that he termed $\mathcal{M}^+\mathcal{M}^-$. This layer has a very small but finite thickness h . From the bottom, the layer is limited by a surface \mathcal{M}^- oriented by a normal $\mathbf{n}_B = -\mathbf{n}^-$, which bounds a second arbitrary continuum (having a volume V_B) being in a contact with the fluid under consideration. A form of this arbitrary contacting continuum Stokes takes to be: a solid, a surface of separation and a free surface. Any momentum balances within the thin layer should be referred, not to an infinitesimal cube, but to the infinitesimal layer “pillbox” bounded by a small surface dS^+ from the top and dS^- from the bottom.

In the simplest case, when the thickness of the Stokes layer is going to zero, and when the effect of the curvature of surfaces \mathcal{M}^+ and \mathcal{M}^- can be neglected, then the force boundary condition, that comes from the Cauchy theorem¹¹, is reformed and equivalently described as a balance of momentum, expressed from the point of view of a layer $\mathcal{M}^+\mathcal{M}^-$. Omitting now the inertia forces of the matter in the layer, and its body forces, and any other internal forces like the

¹¹Note that Cauchy theorem says that on the boundary of a body, through the classical “tetrahedron argument”, a connection of the internal stresses \mathbf{p} with the external traction $\mathbf{f}_{\partial V}$ is: $\mathbf{f}_{\partial V} + \mathbf{p}\mathbf{n} = 0$. For a body for which a boundary surface is assumed as a layer, the Cauchy theorem changes into: $\mathbf{f}_{\partial V} + \text{div}_s \mathbf{p}_s + \mathbf{p}\mathbf{n} = 0$ (see [19]).

surface elastic tension and the surface traction tensor, we can write a balance of the layer momentum (50) as a simple sum of the external forces:

$$\operatorname{div}_s \mathbf{p}_s + \mathbf{f}_{AB} + \mathbf{p}_A \mathbf{n}_A + \mathbf{p}_B \mathbf{n}_B - \varpi \mathbf{n} = 0. \quad (82)$$

In practice, because surfaces \mathcal{M}^+ and \mathcal{M}^- are nearly parallel, we have to use only one normal vector $\mathbf{n} = \mathbf{n}^+ = \mathbf{n}^-$.

Let us take, firstly, as an illustration of acting of (82), the case of a free surface and $\mathcal{M}^- = \mathcal{M}'$. Physically, on this side (surface \mathcal{M}') the gas is located, and there is usually a constant pressure $p_B = \pi'$. Then the Cauchy traction force is

$$\mathbf{p}_B \mathbf{n}_B = -\pi' \mathbf{n}. \quad (83)$$

From the side of a fluid under consideration, the force acting on the surface \mathcal{M}^+ , according to Eq. (78) is equal:

$$\mathbf{p}_A \mathbf{n}_A = [p\mathbf{I} - 2\mu(\mathbf{d} - \delta\mathbf{I}) - 3\kappa\delta\mathbf{I}] \mathbf{n}. \quad (84)$$

Next, putting further simplification of no-slip conditions $\mathbf{p}_s = \mathbf{f}_{SA} = \mathbf{f}_{SB} = 0$, we get a first simple boundary condition on a free surface between fluid and gas:

$$(\pi' - p) \mathbf{n} + 2\mu \mathbf{d}\mathbf{n} - 2\mu \delta \mathbf{n} + 3\kappa \delta \mathbf{n} = 0. \quad (85)$$

This equation for an incompressible fluid under consideration $3\delta = \mathbf{I}_d = 0$ reduces to

$$(\pi' - p) \mathbf{n} + 2\mu \mathbf{d}\mathbf{n} = 0. \quad (86)$$

Explicitly, for $\mathbf{n} = l\mathbf{e}_x + m\mathbf{e}_y + n\mathbf{e}_z$, this equation was written by Stokes to be¹²

$$\begin{aligned} l(\pi' - p) + \mu \left\{ 2l \frac{du}{dx} + m \left(\frac{du}{dy} + \frac{dv}{dx} \right) + n \left(\frac{du}{dz} + \frac{dw}{dx} \right) \right\} &= 0, \\ m(\pi' - p) + \mu \left\{ l \left(\frac{du}{dy} + \frac{dv}{dx} \right) + 2m \frac{dv}{dy} + n \left(\frac{du}{dz} + \frac{dw}{dx} \right) \right\} &= 0, \\ n(\pi' - p) + \mu \left\{ l \left(\frac{du}{dz} + \frac{dw}{dx} \right) + m \left(\frac{dv}{dz} + \frac{dw}{dy} \right) + 2n \frac{dw}{dz} \right\} &= 0. \end{aligned} \quad (87)$$

This set of equations is not sufficient for describing a free surface motion. It should be considered together with the condition of materiality of the free surface. The condition of materiality of any surface, according to Lagrange, is described as follows: if $\mathcal{M} = 0$ is an equation of the free surface then $\dot{\mathcal{M}} = 0$ is needed for its materiality¹³. If $\dot{\mathcal{M}} = 0$ is satisfied, then there is no generation or destruction of particles of fluid at the free surface.

Now let us take into account an internal elasticity of the Stokes layer in the form of the Young-Laplace surface capillarity tensor. Thus, the balance (82) takes

¹²See [59], Eq.(14).

¹³See [59], Eq.(15).

an extended form, which is a composition of the Young-Laplace and the stress boundary equations. If $\mathbf{p}_s = \gamma \mathbf{I}_s$ is the two-dimensional diade of surface tension with a constant surface tension $\gamma = \text{const}$; and $\text{div}_s \mathbf{p}_s = \text{div}_s (\gamma \mathbf{I}_s) = \gamma \text{tr}(\mathbf{II}_s) \mathbf{n}$, where the mean curvature of \mathcal{M} is $\text{tr} \mathbf{II}_s = (1/r_1 + 1/r_2)$, we finally obtain instead of Eq. (86):

$$(\pi' - p - \gamma \text{tr} \mathbf{II}_s) \mathbf{n} + 2\mu \mathbf{d}\mathbf{n} - 2\mu \delta \mathbf{n} + 3\kappa \delta \mathbf{n} = 0. \quad (88)$$

This is the boundary condition for the free surface undergoing the surface tension. For incompressible fluid $\delta = 0$ and a normal component of Eq. (88) indeed reduces to the ordinary Young-Laplace equation.

Let us consider a second case when the surface $\mathcal{M}^- = \mathcal{M}'$ is a solid surface which moves with the wall velocity $\mathbf{v}_B = u' \mathbf{e}_x + v' \mathbf{e}_y + w' \mathbf{e}_z = \mathbf{v}_{\text{wall}}$. Then, according to the Cauchy theorem for traction force, we have a representation in terms of boundary stresses as

$$\mathbf{f}_{\partial B} = \begin{cases} \mathbf{f}'_{\partial M} = P' \mathbf{e}_x + Q' \mathbf{e}_y + R' \mathbf{e}_z & : \text{rigid body,} \\ \mathbf{p}' \mathbf{n}' & : \text{deformable body.} \end{cases} \quad (89)$$

In this moment, we can observe the crucial point of Stokes reasoning – it is a definition of the external friction force \mathbf{f}_{SA} of layer that becomes a more complex than Navier's one. At first, let ϖ be a normal pressure at the Stokes layer being in contact with a solid, which would exist (especially close to \mathcal{M}^-), if there were no motion of molecules in its neighborhood¹⁴. It means that a spherical pressure tensor $p \mathbf{I}$ in the bulk, now in a layer, degenerates into an ellipsoid: $\gamma \mathbf{I}_s + \varpi \mathbf{n} \otimes \mathbf{n}$.

In order to understand a kind of external friction, let us consider, according to Newton's hypothesis, three basic mechanisms of the internal friction. We state here a quite similar postulate: the Newton hypothesis is also substantial for the external friction phenomena. Therefore, we postulate the following form of the layer force:

$$\mathbf{f}_{AB} = f_{AB} N \frac{\mathbf{v}_A - \mathbf{v}_{\text{wall}}}{|\mathbf{v}_A - \mathbf{v}_{\text{wall}}|} + \nu (\mathbf{v}_A - \mathbf{v}_{\text{wall}}) + f_\kappa (\mathbf{v}_A - \mathbf{v}_{\text{wall}})^2 \frac{\mathbf{v}_A - \mathbf{v}_{\text{wall}}}{|\mathbf{v}_A - \mathbf{v}_{\text{wall}}|} \quad (90)$$

which depends on the layer pressure ϖ , and three types of external friction forces. The first one is related to a static friction coefficient f_{AB} (like Coulomb linear law of friction) between the fluid \mathcal{A} and the material of layer $\mathcal{M}^+ \mathcal{M}^-$, where N is the pressure coming from the fluid under consideration:

$$N = |\mathbf{n}_A \cdot \mathbf{p}_A \mathbf{n}_A|, \quad (91)$$

¹⁴It is a reminiscence of the old discussion of Clermont, d'Alembert, J. Bernoulli and Euler about the wall pressure in hydrostatics. Let us recall that the question is: How the bulk pressure in static water differs from the pressure in the neighborhood of a solid wall? Yet more complex, nonlocal pressure description one can find in Duhem's proposal – see [14, 15, 16].

and $(\mathbf{v}_A - \mathbf{v}_{\text{wall}}) |\mathbf{v}_A - \mathbf{v}_{\text{wall}}|^{-1}$ is a unit vector in a direction of relative velocity (a direction of the external friction force). The second one is an additional tangential friction force on $\mathcal{M}^+ \mathcal{M}^-$ due to an external viscosity ν and the relative velocity field between \mathcal{M}^+ and \mathcal{M}^- . The third term in Eq. (90) is due to square of relative velocity. If the friction coefficients f_{AB} and f_κ in Eq. (90) become simultaneously zero, then (90) has a form postulated by Stokes:

$$\mathbf{f}_{AB} = \nu (\mathbf{v}_A - \mathbf{v}_{\text{wall}}). \quad (92)$$

The second external force coming from the fluid under consideration is given by Eq. (89). If there is no other additional forces coming from the layer $\mathcal{M}^+ \mathcal{M}^-$, what means that the linear and angular momentum are going to zero (resulting velocities in the layer are null), then, omitting capillary elastic forces of the layer, from the balance (82) we obtain $\mathbf{v}_A = u \mathbf{e}_x + v \mathbf{e}_y + w \mathbf{e}_z$:

$$(\varpi - p) \mathbf{n} + \nu (\mathbf{v}_A - \mathbf{v}_{\text{wall}}) + 2\mu \mathbf{d}\mathbf{n} - 2\mu \delta \mathbf{n} + 3\kappa \delta \mathbf{n} = 0. \quad (93)$$

Explicitly, for $\kappa = 0$, this condition takes the following form¹⁵:

$$\begin{aligned} l(\varpi - p) + \nu(u - u') + \mu \left\{ 2l \left(\frac{du}{dx} - \delta \right) + m \left(\frac{du}{dy} + \frac{dv}{dx} \right) + n \left(\frac{du}{dz} + \frac{dw}{dx} \right) \right\} &= 0, \\ m(\varpi - p) + \nu(v - v') + \mu \left\{ l \left(\frac{du}{dy} + \frac{dv}{dx} \right) + 2m \left(\frac{dv}{dy} - \delta \right) + n \left(\frac{du}{dz} + \frac{dw}{dx} \right) \right\} &= 0, \\ n(\varpi - p) + \nu(w - w') + \mu \left\{ l \left(\frac{du}{dz} + \frac{dw}{dx} \right) + m \left(\frac{dv}{dz} + \frac{dw}{dy} \right) + 2n \left(\frac{dw}{dz} - \delta \right) \right\} &= 0. \end{aligned} \quad (94)$$

Materiality of a contact surface \mathcal{M}^- is satisfied since it is the surface of solids.

It turns out that the friction force \mathbf{f}_{AB} could be known only after a solution of the problem¹⁶, whereas in the case of a free surface this force is a given data. On the other hand, the form of the solid surface \mathcal{M}^- is given, whereas the form of the free surface is known only by the solution of the problem. Equation (94) is a generalization of the Navier boundary conditions and contains two quite new elements: the velocity of wall surface \mathbf{v}_{wall} and a new external layer pressure ϖ , which in general differs from the internal fluid pressure¹⁷ p .

There remains to consider the third case of two fluids in a contact, having a common surface given by $\mathcal{M} = 0$. Fluid velocities are $\mathbf{v}_A \equiv \mathbf{v}$ (side \mathcal{M}^+)

¹⁵See [59],Eq.(17).

¹⁶Let us notice that it leads to a new kind on non-linearity connected with the boundary conditions, i.e., the problem becomes a non-linear one, even if the equations in the bulk are linear.

¹⁷Concerning the external viscosity coefficient, Stokes gives some supplementation of the Navier's data. He quoted Dubuat's experiments that have found that when the mean velocity of water flowing through a pipe is less than *circa* one inch per second, the water near the inner surface of the pipe is at rest. It is a case when the external viscosity goes to infinity ($\nu \rightarrow \infty$), i.e. the slip length is going to zero.

and $\mathbf{v}_B \equiv \mathbf{v}'$ (side \mathcal{M}^-). In such case it is important to put the condition of impermeability, saying that the normal velocities should be the same¹⁸:

$$\mathbf{v} \cdot \mathbf{n} + \mathbf{v}' \cdot \mathbf{n}' = 0 \quad \text{or} \quad l(u - u') + m(v - v') + n(w - w') = 0. \quad (95)$$

Let the interface layer $\mathcal{M}^+\mathcal{M}^-$ between both fluids possesses a Navier-like friction force, due to postulated an external viscosity, say ν_{AB} , which in general, differs from the external viscosity between a fluid and a surface¹⁹. Then, in analogy to the previous consideration, we have a following set of governing equations:

- A fundamental momentum balance in the layer (the *vis viva* and the capillarity tensor of $\mathcal{M}^+\mathcal{M}^-$ are omitted here) — under assumption of one common friction force: $\mathbf{f}_{AB} = \mathbf{f}_{SA} + \mathbf{f}_{SB}$ (see Eq. (55)):

$$\mathbf{f}_{AB} + \mathbf{p}_A \mathbf{n}_A + \mathbf{p}_B \mathbf{n}_B = 0. \quad (96)$$

- An external friction force within the layer:

$$\mathbf{f}_{AB} = \nu_{AB} (\mathbf{v}_A - \mathbf{v}_B). \quad (97)$$

- Two external traction forces on the surfaces bounding the layer:

$$\mathbf{p}_A \mathbf{n}_A \equiv \mathbf{p} \mathbf{n} = [p\mathbf{I} - 2\mu(\mathbf{d} - \delta\mathbf{I}) - 3\kappa\delta\mathbf{I}] \mathbf{n}, \quad (98)$$

$$\mathbf{p}_B \mathbf{n}_B \equiv \mathbf{p}' \mathbf{n}' = [p'\mathbf{I} - 2\mu'(\mathbf{d}' - \delta'\mathbf{I}) - 3\kappa'\delta'\mathbf{I}] \mathbf{n}'. \quad (99)$$

If we suppose $\nu_{AB} \rightarrow 0$, as appears most probable, the problem significantly simplifies. Nevertheless we are not sure under which circumstances the external viscosity between two different fluids can be measured, and when its appearance is essentially important²⁰.

¹⁸[59], Eq.(18).

¹⁹Coefficient ν_{AB} was introduced by von Rybczyński [63], who called it “slipping coefficient of fluids”. Nowadays, the general theory of nonequilibrium mixture also takes into account velocity slipping between the components (see: M. Cieszko, J. Kubik, *Derivation of matching conditions at the contact surface between fluid-saturated porous solid and bulk fluid*, Transport in Porous Media, **34**, 319-336, 1999).

²⁰This external viscosity coefficient should be important when gas bubbles flow up in liquid, or when liquid droplets flow down in the rarefied air. This last case was a subject of consideration of von Rybczyński, who was first author who calculated coefficient $\nu_{SS'}$ between water and air. He shown that for respectively slow motion of a water droplet the inner liquid remains the spherical shape and its resistance force is [63]:

$$F = 6\pi\mu Rc \frac{\nu_{SS'}R + 2\mu}{\nu_{SS'}R + 3\mu}, \quad \mu - \text{viscosity of air.}$$

Finishing the reconstruction of the crucial concept of the Stokes boundary layer, it should be mentioned in some important geometrical interpretation of slip velocity condition. It was given by Butcher, who prepared the figure (see Fig. 3) showing a profile of velocity field in a flow with the slip over a solid surface [7]. For the Helmholtz-Piotrowski coefficient G (now l_s ; further discussion one may find in Maxwell [44], Schnell [54], Whetham [67]) Butcher, for the first time in literature, introduced an interpretation of slip length. From a geometrical interpretation, it follows that the slip flow in a given channel is equivalent to a no-slip in a channel with the increased dimension on the “slip length”. Thus the ratio of slip length to a characteristic dimension of the canal is so important and is defined as the dimensionless slip length or the Navier number²¹.

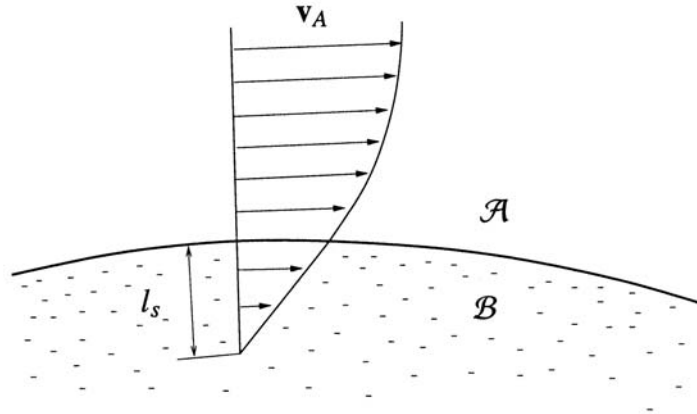


Figure 3. Definition of the slip length l_s for a simple shear flow, according the Butcher concept of equivalent flows. Reconstruction from [7].

²¹Butcher’s geometrical interpretation was widely accepted by English authors. For instance, Maxwell has mentioned that [44]:

“[...] It, therefore, the gas at a finite distance from the surface is moving parallel to the surface, the gas in contact with the surface will be sliding over it with the finite velocity, and the motion of the gas will be nearly the same as if stratum of depth had been removed from the solid and filled with the gas, there being now no slipping between the new surface of the solid and the gas in contact with it [...]”

Butcher’s figure and interpretation has been repeated by Sir Horace Lamb [H. Lamb, *A treatise on the mathematical theory of the motion of fluids*, Cambridge, 1879, edn. 8, (1938), Dover Publ., 1961] who has quoted Maxwell formulae on the ratio of the mass flow rate in slipping flow and no-slip flow in a straight circular pipe with the inner radius a as:

$$\frac{\dot{m}_{\text{slip}}}{\dot{m}_{\text{no-slip}}} = \left(1 + 4 \frac{l_s}{a} \right) .$$

8 Maxwell boundary layer

It is important to make a deeper analysis of the Stokes model of a boundary layer. Therefore, this section is devoted, *inter alia*, to discussing the most important extension of Stokes's concept worked out by Maxwell in 1879 [44]. Study of the background for Maxwell's slip velocity boundary conditions gives a quite new perspective on the complexity of the problem. Regarding the problem of the consistency between an internal model of continua (in bulk) and external model of continua (its boundary), one can notice that it refers actually to the three basic situations:

- The bulk model is much richer than its boundary model — it is a most frequently used case in the continuum mechanics and rational thermodynamics. In this case internal coefficients have important values whereas external coefficients go to infinity, i.e., its characteristic length going to zero.
- Conversely, the internal coefficients are very small, and in the model dominates its external property, which has a dominating influence on the observed phenomena. It is exactly the case discussed here – it appears in the Maxwell slip layer.
- The fully consistent model where every internal (bulk) coefficient of a model has its counterpart in an external coefficient.

However, it is seldom noticed that the above three basic situations should be always stated within the framework of the Stokes layer boundary mechanics, because of its generality. In further explanation of the Maxwell model of a slip layer, we stand on the position that Stokes's reasoning is a superior to Maxwell's reasoning.

The discussed here Maxwell's paper, entitled *On stresses in rarefied gases arising from inequalities of temperature* [44], possesses two distinct parts, based on two quite independent motivations. The first part of the paper, entitled "March 1878", is motivated by an attempt to explain Crookes' discovery of the rotation of "windmill" in a partially evacuated radiometer²². Maxwell put a basic hypothesis that, in Crookes' experiments, since the pressure is very low, the new stresses are growing due to a second gradient of temperature in rarefied gas. These stresses can be capable of producing rapid motion in a radiometer windmill. In other words, in a gaseous medium where there is only linear distribution of temperature, there are no additional thermal stresses. A problem of boundary conditions in this first part of paper [from March 1878] has not been considered.

²²See W. Crookes, *Philos. Trans. Roy. Soc. London* 166, 325 (1876).

But, in that time, Maxwell is conscious that this type of thermal stresses, calculated for a hot solid sphere of uniform temperature and immersed in the colder gas, cannot give rise to any force tending to move the sphere in one direction rather than in another. In the framework of his model of stresses the sphere placed within the finite portion of gas is already in equilibrium. How is it then possible to account for the observed by Tyndall fact that an additional force acts between solid bodies immersed in rarefied gases? This kind of motion, nowadays called thermophoresis [6] is connected with small solid particles, typically spherical (like volcanic ashes, dust, etc.), suspended in a fluid, within which an externally imposed linear temperature difference (constant temperature gradient) induces a force that moves the sphere from hotter to colder places, i.e., the particle moves against the temperature gradient²³.

In order to explain this, even in March 1878, Maxwell turns his attention not to his own model of thermal stresses but to the slip phenomena discovered in liquids by [29] and in rarefied gases by [37, 38]. The mechanical slip phenomenon is related to the finite value of the Navier external viscosity and appears as a sliding of a fluid in contact with the solid surface. Maxwell precisely underlines a difficulty for mathematical treatment, since the gas close to the solid surface is probably in quite a different “state of condensation”. It means that quite a different model is needed for the description of the phenomena of sliding. One example of this extraordinary situation, discovered by Kundt and Warburg, is the fact that the velocity of gas sliding over the surface, induced by given tangential viscous stresses, varies inversely with pressure.

In the first part (March 1878) of his celebrated paper, Maxwell does not attempt to take into account the effect of this mechanical sliding motion, because the main goal of his paper is devoted to the invention of a thermal stress model in the bulk — the consideration of a kinetic relation close to the solid surface is “completely destroying the simplicity of our first solution of the problem”.

Let us briefly recapitulate the Maxwell results concerning the additional stress related to “inequalities of temperature”. These stresses can appear only when the dispose temperature is nonlinear. From his formula, the pressure tensor is reconstructed:

$$\mathbf{p} = p\mathbf{I} - 2\mu\mathbf{d} + \frac{2}{3}\mu I_d\mathbf{I} + \beta_1\frac{1}{2}(\text{grad}\mathbf{g} + \text{grad}^T\mathbf{g}) + \beta_2(\text{div}\mathbf{g})\mathbf{I}, \quad (100)$$

where the temperature gradient is denoted as $\mathbf{g} = \text{grad}\theta$ and β_1, β_2 are the constitutive constants which can be called the thermal transpiration coefficients. After substitution to the Stokes form of the balance of momentum (76):

$$\rho(\dot{\mathbf{v}} - \mathbf{b}) + \text{div}\mathbf{p} = 0, \quad (101)$$

²³See J. Tyndall, Proc. Roy. Inst. 6, 1, 1870; D.A. Edwards, H. Brenner, *Interfacial Transport Processes and Rheology*, Butterworth-Heinemann, Boston 1991.

and after using the tensorial identities $\text{grad} \mathbf{g} = \text{grad}^T \mathbf{g} = \theta_{,ij} \mathbf{e}_i \otimes \mathbf{e}_j$, and $\text{div} \mathbf{d} = \frac{1}{2} (\text{lap} \mathbf{v} + \text{grad} \mathbf{I}_d)$, we obtain an equation of a fluid motion that takes into account the defined above unconventional contribution to the momentum transport:

$$\rho \dot{\mathbf{v}} + \text{grad} p - \mu \text{lap} \mathbf{v} - \frac{1}{3} \mu \text{grad} \mathbf{I}_d + \beta_1 \text{lap} \mathbf{g} + \beta_2 \text{grad}(\text{lap} \theta) = \rho \mathbf{b}. \quad (102)$$

If Maxwell's constitutive constants β_1 and β_2 are going to zero, the model of fluid under consideration becomes identical with the Stokes model. Further taking the following identities: $\text{div} \mathbf{g} = \text{lap} \theta$ and $\text{lap} \mathbf{g} = \text{lap}(\text{grad} \theta) = \text{grad}(\text{lap} \theta)$ we write (102) to be [44]:

$$\rho \dot{\mathbf{v}} + \text{grad} p - \mu \text{lap} \mathbf{v} - \frac{1}{3} \mu \text{grad} \mathbf{I}_d + (\beta_1 + \beta_2) \text{grad}(\text{lap} \theta) = \rho \mathbf{b}. \quad (103)$$

Here, according to the nonequilibrium kinetic treatment, Maxwell was able to estimate the value of the thermal transpiration constants to be [44]:

$$\beta_1 = 3 \frac{\mu^2}{\rho \theta}; \quad \beta_2 = \frac{3 \mu^2}{2 \rho \theta}. \quad (104)$$

These are really very small quantities that depend on viscosity, μ , density of gas, ρ , and its temperature, θ . Let us note that more concise calculations of the Maxwell fundamental equation, which includes also its nonlinear parts, lead to the expression for stress tensor in thermal transpiration phenomena which also depends on a linear distribution of temperature²⁴:

$$\mathbf{p} = p \mathbf{I} - 2\mu \mathbf{d} + \frac{2}{3} \mu \mathbf{I}_d \mathbf{I} + \beta_1 \frac{1}{2} (\text{grad} \mathbf{g} + \text{grad}^T \mathbf{g}) + \beta_2 (\text{div} \mathbf{g}) \mathbf{I} + \beta_3 \mathbf{g} \otimes \mathbf{g}. \quad (105)$$

The boundary condition where the coefficient β_3 and the first gradient of temperature \mathbf{g} appear, is fundamentally quite different from original Maxwell's one, since \mathbf{g} appears within the friction force \mathbf{f}_{SA} — what means quite another physical phenomena. It is also a historical truth that the presence of \mathbf{g} in the boundary layer was firstly postulated by Reynolds, and next proven by Maxwell.

Now, let us consider the second part of Maxwell paper, known as *Appendix May 1879* (few months before Maxwell's death). The direct reason for writing of this appendix was a great Reynolds discovery of thermal transpiration. Maxwell, who was a reviewer of Reynolds seminal paper *Thermal transpiration* [52] has an opportunity to study the principles of the Reynolds thermal transpiration at the manuscript stage, before a formal publication. The thing of a great novelty was Reynolds' proposal of modeling of thermal transpiration, i.e., the motion of

²⁴J. Badur, B.-H. Sun, Some remarks on the structure of evolution equations, *Appl. Math. Mech.* 16, 747–757 (1995), Eq. (17).

gas through the capillary thin tube from the colder to hotter ends, with linear drop of temperature at the walls. Reynolds was able to predict the transport of momentum as a one-dimensional steady-state momentum drift between the hot and cold reservoirs, situated at opposite ends of the capillary, with the resulting constant local temperature gradient in each point of the fluid being in isobaric state. These conditions are quite opposite to the Maxwell bulk model of stresses, Eq. (100), where only a linear distribution of temperature is present. Then, the only possibility to explain Reynolds' discovery was to look at the boundary slip phenomena, where the boundary force, that depends on a thermal gradient, must be postulated. This concept, proposed earlier in Reynolds' manuscript, is a subject of Maxwell's celebrated appendix.

Unfortunately, if the thermal transpiration phenomenon was discovered entirely by Reynolds, its explanation proposed by Maxwell, could not be published earlier than the original Reynolds information. This personal mistake of Maxwell, as well as a mistake of *Philosophical Transaction* editors, should be pointed out. Such a situation, as an example of a non-ethic behaviour in science should be criticized²⁵.

Additionally, Maxwell has rejected a first proposal of description made by the author of discovery, saying²⁶:

“[...] It was not till after I had read Professor Reynolds's paper that I began to reconsider the surface condition of gas, so that what I have done is simply to extend the surface phenomena the method is, in some respects, better than that adopted by Professor Reynolds, while I admit that his method is sufficient to establish the existence of the phenomena, though not to afford an estimate of their amount. [...]”

Maxwell assumes, similarly to Reynolds, that the kinetic theory of gases in the vicinity of a solid surface should be reformulated, and the governing equations should include conditions which must be satisfied on a surface of the solid body. Unfortunately, the solid body surface is absolutely rigid and stress-free state. Its molecules are absolutely fixed what means that the surface temperature is close to absolute zero. A difference in contact of two gases, for instance, hydrogen and carbon dioxide with the same glass surface follows only from a number of absorbed and reflected gas molecules. Maxwell prefers to treat the surface as something intermediate between a perfectly reflecting and a perfectly absorbing

²⁵Those were the last days in Maxwell's life. Thus, when Stokes received the letter from Reynolds describing this case, he did not undertake any action, and after Maxwell's death the problem achieved a different dimension and was no longer important for the young Reynolds. The faith still favoured him and during the following 8 years he made 17 new discoveries (all of them are available to see at an exhibition in the University of Manchester museum laboratory).

²⁶Maxwell, [44], Appendix, p. 249.

surface. Therefore, an experimentally verified portion f describes absorption of all the incident molecules and the portion $1 - f$ describes the perfectly reflected molecules.

Let suppose that the surface is a plane y, z and that the gas is flowing in that side of it, for which x is positive. Next, let v be a main surface component of velocity in the main y direction. The original Maxwell's slip-velocity formula is given by the following expression²⁷:

$$\nu - G \left(\frac{dv}{dx} - \frac{3}{2} \frac{\mu}{\rho\theta} \frac{d^2\theta}{dxdy} \right) - \frac{3}{4} \frac{\mu}{\rho\theta} \frac{d\theta}{dy} = 0, \quad (106)$$

where μ – internal viscosity coefficient, ρ is the gas density, θ is an absolute temperature and G is the Helmholtz-Piotrowski denotation for slip length l_s . To finish the process of reconstruction of Maxwell slip boundary condition (106), let us write the boundary force condition in terms of the Navier-Stokes layer in a general form (see: 7.23):

$$\mathbf{f}_{AB} + \mathbf{p}_A \mathbf{n}_A + \mathbf{p}_B \mathbf{n}_B = 0, \quad (107)$$

where:

- the fluid boundary force:

$$\mathbf{p}_A \mathbf{n}_A = \left[p \mathbf{I} - 2\mu \mathbf{d} + \frac{2}{3} \mu \mathbf{I}_d \mathbf{I} + \beta_1 \frac{1}{2} (\text{grad} \mathbf{g} + \text{grad}^T \mathbf{g}) + \beta_2 (\text{div} \mathbf{g}) \mathbf{I} \right] \mathbf{n}, \quad (108)$$

- the rigid body surface boundary:

$$\mathbf{p}_B \mathbf{n}_B = 0, \quad (109)$$

- the surface friction force:

$$\mathbf{f}_{AB} = \nu (\mathbf{v} - \mathbf{v}_{\text{wall}} - c_{v\theta} \text{grad}_s \theta). \quad (110)$$

The thermo-mobility coefficient $c_{v\theta}$ should be formulated, according to Maxwell's formula (106), as a coefficient that is not dependent on the property of the solid surface:

$$c_{v\theta} = \frac{3}{4} \frac{\mu}{\rho\theta}. \quad (111)$$

²⁷See [44], Appendix, Eq.(68).

Using the definition $l_s \equiv G = \mu/\nu$, and dividing the balance (107) by the external viscosity, ν , we obtain a generalization of the Maxwell slip boundary layer Eq. (106) as

$$\begin{aligned} \mathbf{v} - \mathbf{v}_{\text{wall}} = & c_{v\theta} \text{grad}_s \theta + \frac{p}{\nu} \mathbf{n} - 2l_s \mathbf{d}\mathbf{n} + \frac{2}{3} l_s \mathbf{I}_d \mathbf{n} + \\ & + \frac{\beta_1}{\nu} \frac{1}{2} (\text{grad} \mathbf{g} + \text{grad}^T \mathbf{g}) \mathbf{n} + \frac{\beta_2}{\nu} (\text{div} \mathbf{g}) \mathbf{n}. \end{aligned} \quad (112)$$

The ratios $l_{s(\beta_1)} = \beta_1/\nu$ and $l_{s(\beta_2)} = \beta_2/\nu$ can be called the thermal transpiration slip coefficients. It should be noted that despite $l_s = \mu/\nu$ be the main characteristics of the external viscosity, and cannot be further treated as the only one and general characteristic of the Navier-Stokes layer. For example, in the case of the rigid cold particle immersed into a gas at rest ($\mathbf{v} \equiv 0$), and under assumption of the temperature distribution linearity ($\text{grad} \mathbf{g}$), the thermal velocity of a particle is

$$\mathbf{U} = \mathbf{v}_{\text{wall}} = -c_{v\theta} \text{grad}_s \theta. \quad (113)$$

Nowadays, the velocity \mathbf{U} is called the thermophoretic velocity [6]. It characterizes the motion of nanoparticles that follows from the surface gradient of temperature. It should be underlined that Maxwell was able to find an explicit formula for the length of slip, using apparatus of the kinetic theory, ²⁸:

$$l_s = \frac{\mu}{\nu} = \frac{1}{2} \mu (2\pi)^{\frac{1}{2}} (p\rho)^{-\frac{1}{2}} \left(\frac{2}{f} - 1 \right) = \frac{2}{3} \left(\frac{2}{f} - 1 \right) l, \quad (114)$$

where l means the Meyer relation for the mean-free path of a gas molecule, and f is the fraction absorbed. If $f = 1/2$, or the surface acts as if it were half perfectly reflecting and half perfectly absorbing, then we got $l_s = 2l$. For fully absorbing surface is $l_s = 2/3l$. In practice the slip length depends on a kind of surface material and gas. From Kundt and Warburg experimental data it follows that for air on a glass surface at 17 °C the slip length is $l_s = 8/p$, and for hydrogen is $l_s = 15/p$, where the pressure is then given in dynes per square centimeter [35]. The coefficient of a partial absorption f is usually called in contemporary literature as tangential momentum accommodation coefficient (TMAC). It accounts for the average tangential to a surface momentum exchange between the fluid molecules and the solid molecules, and is not dependent on the heat flow. The value of this coefficient should be evaluated experimentally, but it is known that it varies from zero (for specular reflection) up to unity (for complete or diffuse accommodation) [20, 50].

Unfortunately, the second coefficient of the Maxwell model, i.e., the coefficient of thermal mobility $c_{v\theta}$, essential for the thermal transpiration, has been omitted

²⁸See [44], Appendix, Eq.(67).

from any literature discussion. Indeed, it is erroneously defined by Eq. (106), since it suggests its independence on the surface properties. Notice that it is quite opposite situation then in the Reynolds boundary layer model, where Reynolds gives much more correct definition of the thermal mobility coefficient, and no so complex definition of the slip length. Strictly speaking, Eq. (106) implies that the coefficients l_s and $c_{v\theta}$ are dependent on one another.

If one introduce, following Maxwell, a directional derivative, $\frac{d}{d\nu}$, directed along some main flow direction, ν , with the cosines of the normal ν given by l , m , n then the condition (107) can be expressed in the original form ([44], Appendix, Eq. 69):

$$\begin{aligned} u - G \frac{d}{d\nu} [(1 - l^2) u - lmv - lnw] + \frac{3}{4} \frac{\mu}{\rho\theta} \left(\frac{d}{dx} - l \frac{d}{d\nu} \right) (\theta + 4G \frac{d\theta}{d\nu}) &= 0, \\ v - G \frac{d}{d\nu} [(1 - m^2) v - mnu - lmw] + \frac{3}{4} \frac{\mu}{\rho\theta} \left(\frac{d}{dy} - m \frac{d}{d\nu} \right) (\theta + 4G \frac{d\theta}{d\nu}) &= 0, \\ w - G \frac{d}{d\nu} [(1 - n^2) w - nlu - mnw] + \frac{3}{4} \frac{\mu}{\rho\theta} \left(\frac{d}{dz} - n \frac{d}{d\nu} \right) (\theta + 4G \frac{d\theta}{d\nu}) &= 0. \end{aligned} \quad (115)$$

These equations are called the ‘‘Maxwell slip boundary layer’’. Let us note that in each of these equations, i.e., in the first one (106), simplified along the axis y , in Eq. (107), or in the general form of (114) in a special surface direction, the very particular role plays the gradient of temperature. It is a completely external surface effect which is not connected with a form of stress tensor, for instance with (105). It means that the motion of the gas close to a solid surface, in general is governed by two kinds of forces. The first is a mechanical one, which is connected with the external viscosity, and the second one is a temperature gradient which drives of gas particle close to the surface from colder to hotter part. Therefore the coefficient of thermal mobility $c_{v\theta}$ (see: Eq. (111) above) is independent from mechanical layer properties and should be experimentally verified²⁹.

Finally, let us recall Maxwell solution for the flow of a gas in a long capillary tube having inner radius α , which occurs under two driving constant forces. These forces are a pressure transpiration due to difference of pressure at the ends of the tube, and the thermal transpiration due to difference of temperature at the same ends of the tube. Since the gas is flowing from the higher to lower pressure and, simultaneously, from the colder to the hotter end, then these effects can be summarized. In a particular case, where the driving forces are opposite and equal themselves, there is no net outflow of gas from the capillary. Then an

²⁹There are numerous modern papers that mention about the proper experiments. The impressive electrokinetic properties predicted for a carbon nanotube channels have not yet been measured in careful experiments [6, 68].

enhancement of mass flux due to the Maxwell slip is³⁰:

$$\frac{\dot{m}_{\text{Maxwell}}}{\dot{m}_{\text{Poiseuille}}} = \left(1 + 4\frac{l_s}{a}\right) - \frac{8}{\pi}c_{v\theta}\frac{\mu}{\rho a^4}\frac{d\theta}{dz}\left(\frac{dp}{dz}\right)^{-1}. \quad (116)$$

This enhancement is essential only if the inner radius a is small in comparison with the slip length l_s and thermal mobility $c_{v\theta}$ is small. Thermal contribution to the slip is important when the gas is rarefied. Both driving forces (per unit of length of the pipe): dp and $d\theta$, can be in opposition. In a particular case there is no flow in the pipe $\dot{m} = 0$. Then we have³¹

$$\frac{dp}{d\theta} = 6\frac{\mu^2}{\rho\theta}\frac{1}{a^2 + 4l_s a}. \quad (117)$$

For given temperature difference $d\theta = 100$ K, under the pressure of 40 mm of mercury, and assuming $l_s = 0.00016$ cm, this formula leads to the resulting pressure at the hot end which exceed that at the cold end by about 1.2 millionth of the atmosphere [44]. Modern numerical techniques allowed us to reconstruct this experiment by means of finite volume method. Obtained results are however slightly different — see Fig. 4, point b) for which $\dot{m} = 0$.

9 Reynolds boundary layer

Osborne Reynolds is the sole author of the thermal transpiration discovery [52]. He had his own line of explanation for thermal transpiration, and contrary to Maxwell, his own line of reasoning. He asserts that a primary reason for the thermal transpiration in the bulk motion is not the second gradient of temperature along the axis of a capillary, but rather an axial gradient of acceleration — when it acts close to the wall surface it enhances the normal Navier slip. Let us now return to the forgotten Reynolds' paper, that deals mainly with the experimental discovery of thermal transpiration [52]. This paper, after some critics of Reynolds's methodology of mathematical treatment of the subject, was removed from the official scientific practices, and nowadays is formally out of substantial discussion. The reason, for such rejection from scientific considerations, of the

³⁰Another objective for analytical study lies in exploring the underlying physics of the so-called Knudsen paradox [45, 34]. Explanations of this paradox cannot be given by model of Navier slip layer, and needs more advanced method of modeling [33, 43]. Let recall, that the Knudsen paradox relates to the presence of a minimum of mass flow rate in a function of the Knudsen number [35, 22]. Thus, the exploration of Knudsen paradox and its full understanding also require a considerations on the limit of continuum approaches. It is fact, that the Knudsen-Gaede flow should be a fundamental benchmark for nanoflows of rarefied gases like the Pouiseulle or Couette flow at macroscale.

³¹See [44], Appendix, Eq. (81).

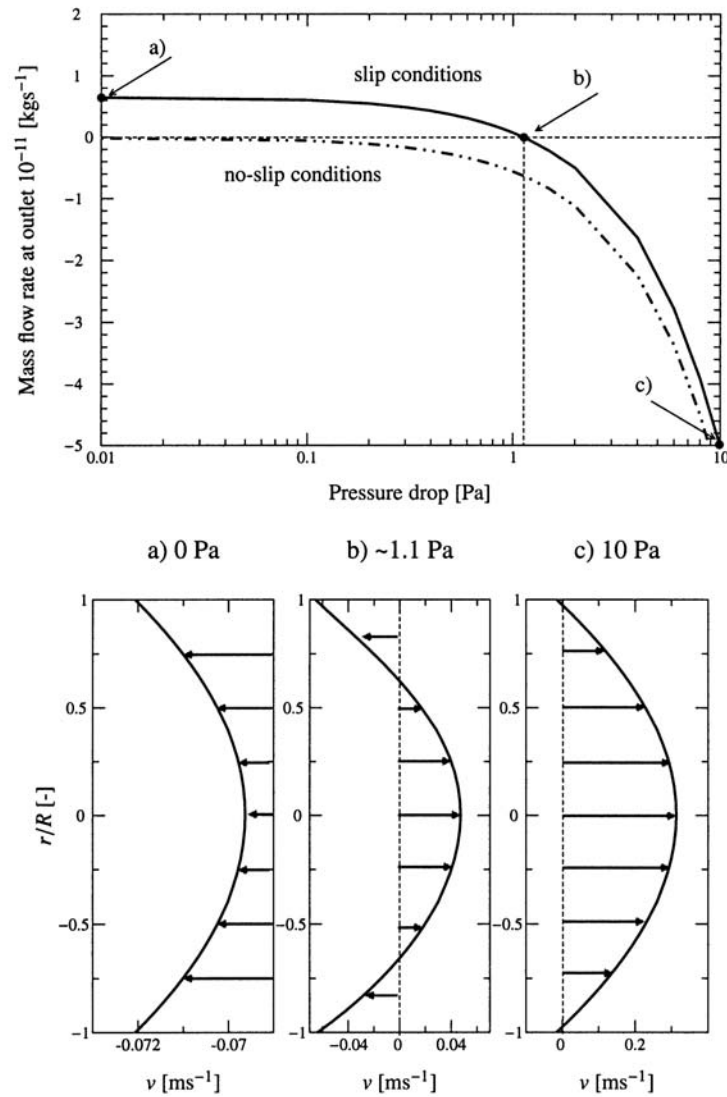


Figure 4. The calculated mass flow rate and relevant velocity profiles in the Maxwell capillary tube for given constant temperature difference $d\theta = 100$ K, and for different dp : a) 0 Pa, b) 1.1 Pa and c) 10 Pa. The case a) describes pure thermal transpiration (no pressure driven flow), where slip velocity $v_s = 0.0077$ m/s drives the bulk flow of a gas.

paper entitled *On certain dimensional properties of matter in the gaseous state*, is probably not connected with the thermal transpiration *per se*, but rather with a great Reynolds' scientific goal which was the unification of three transpiration phenomena into a generalized model³². These phenomena are: the Graham "pressure transpiration", the Graham "concentration transpiration" and, just discovered by Reynolds, the "thermal transpiration". In order to show how Reynolds made this, let (according to Reynolds) denote ϖ , θ , N the normal pressure, temperature and mole concentration of a gaseous component (for instance $N_1 = \rho_1/M_1$), respectively. We can obtain a unified model of pressure, temperature and concentration transpirations in the boundary layer, by adding the particular contributions described by equations (see [52], Eqs. (110), (112) and (117)). Starting from the Stokes boundary layer equation, $-\varpi \mathbf{n} + \mathbf{f}_{AB} + \mathbf{p}_A \mathbf{n}_A = 0$, such a model can be presented shortly to be

$$\begin{aligned} & \nu (\mathbf{v} - \mathbf{v}_{\text{wall}} - c_{v\varpi} \text{grad}_s \varpi - c_{v\theta} \text{grad}_s \theta - c_{vN} \text{grad}_s N) + \\ & + (p - \varpi) \mathbf{n} - 2\mu \mathbf{d}\mathbf{n} + 2\beta \mathbf{d}_{(2)} \mathbf{n} = 0. \end{aligned} \quad (118)$$

Three serious differences with Maxwell's boundary condition (106) can be observed here. The first one is an explicit appearance of the pressure gradient $\text{grad}_s \varpi$, realizing the Graham pressure transpiration phenomena, that is sometimes called the pressure driven motion. For the same material of capillary tube and for different gases, the pressure mobility coefficient $c_{v\varpi}$ varies. Graham found that the ratio of $c_{v\varpi}$ for air and hydrogen is 2.04 [25, 26]. The second difference concerns more correct definition of the thermal mobility coefficient $c_{v\theta}$ than Maxwell has made. For description of the thermal mobility, Reynolds has introduced not a single one, but four accommodation coefficients f_1, f_2, f_3, f_4 . The two first depend on the properties of solid surface material and a kind of gas, and the next two describe a more complete, than Maxwell's, process of interaction of gas with the boundary. The last difference is related to the Graham diffusional transpiration described by the concentration mobility coefficient c_{vN} . These three differences allow us to say that there is a substantial base for distinction between both models of boundary layer. Let us note yet an additional difference in the definition of the Cauchy momentum flux on the boundary. In Maxwell's proposal (see Eq. (100) above), \mathbf{p}_A contains the second gradients of temperature. On the other hand Reynolds' proposition extends definition of \mathbf{p}_A by taking a diade of acceleration of deformation $\mathbf{d}_{(2)}$. Reynolds assumes further that the coefficient β responsible for accelerative contributions should be found on the base of a researches on "dimensional properties of mater". The consistency of the mathematical model of fluid in a bulk, and within the layer boundary was the

³²The most important criticism comes from W. Feddersen, M.J. Violle, and G.F. Fitzgerald (see: Philosophical Magazine, February, 1881).

reason for writing him the paper *On the equations of motion and the boundary conditions for viscous fluid*, [53]. Here, Reynolds proposes a new definition of the Cauchy momentum flux, where an acceleration of deformation diade $\mathbf{d}_{(2)}$ appears. It is defined as a symmetric tensor in terms of the acceleration vector \mathbf{a} :

$$\mathbf{d}_{(2)} = \frac{1}{2} (\text{grad}\mathbf{a} + \text{grad}^T\mathbf{a}) \neq \dot{\mathbf{d}}. \quad (119)$$

Thus, the Stokes form of the momentum flux is extended to be

$$\mathbf{p}_A \equiv \mathbf{p} = p\mathbf{I} - 2\mu\mathbf{d} + \frac{2}{3}\mu I_d\mathbf{I} + 2\beta\mathbf{d}_{(2)}, \quad (120)$$

where the Reynolds coefficient $\beta = r/p$ is inversely proportional to pressure. Thus, substituting the above constitutive equation to the momentum balance, $\rho\dot{\mathbf{v}} + \text{div}\mathbf{p} = \rho\mathbf{b}$, and omitting $\text{div}(\text{grad}^T\mathbf{a})$, we obtain³³

$$\rho\mathbf{a} + \beta\text{lap}\mathbf{a} + \text{grad}p - \mu\text{lap}\mathbf{v} - \frac{1}{3}\mu\text{grad}I_d = \rho\mathbf{b}. \quad (121)$$

The part with β coefficient can further be omitted in the bulk of fluid flow, but on the boundary it represents very important – an accelerative – contribution to the slip conditions.

The problem of consistency of field equations in the bulk with those postulated on the boundary, started by Reynolds, is actual up to day. In 1964 Deissler, on the basis of kinetic theory, has proposed a slip velocity equation where the second and higher derivatives of gas velocities take part [11]. It can be shortly written as ([33], Eq. (2.29)):

$$\nu(\mathbf{v} - \mathbf{v}_{\text{wall}}) = \mu \left[C_1 Kn \frac{\partial \mathbf{v}}{\partial n} + C_2 Kn^2 \frac{\partial^2 \mathbf{v}}{\partial n^2} + \dots \right]. \quad (122)$$

The term argued that higher powers of the Kn and higher order derivatives of velocity must be used in the slip equations. On the other hand, well known kinetic theory models of stresses like the Maxwell (Eq. (100)), Burnett, Chapman, Grad, have no any contribution of the second velocity gradient — probably Reynolds non-explicit contribution via gradient of acceleration $\mathbf{d}_{(2)}$ is much better motivated physically.

In general, the consistency of the field equation within the bulk and on its boundary for weakly-nonlocal continua with higher gradient has not been solved completely yet. There are no appropriate scientific tools for setting the mathematically concise system of equations. The only one known and recognized method is

³³See [53], Eq.(12).

the so-called Green transformation (1839), which using extended definition of divergence can find true interrelation between the bulk and the boundary equations. For instance, when a continuum contains the time derivative of rate of deformation $\dot{\mathbf{d}}$ and its gradient $\text{grad } \mathbf{d}$, it is simply to show, using the Green transformation twice, that the balance of momentum takes the Skiba-Pearson form:

$$\rho \frac{d}{dt} \left[\mathbf{v} - \text{div} \left(\frac{\partial \phi}{\partial \dot{\mathbf{d}}} \right) \right] - \rho \mathbf{b} + \text{div} \left[\mathbf{p} - \text{div} \left(\frac{\partial \phi}{\partial \text{grad} \mathbf{d}} \right) \right] = 0, \quad (123)$$

where the additional dependence is described by the Rayleigh-like dissipation function $\phi = \phi(\dot{\mathbf{d}}, \text{grad} \mathbf{d})$, which fulfills, from the definition, the *perpetuum mobile* principles. Fried and Gurtin, assuming only the gradient of \mathbf{d} : $\phi = \phi(\text{grad} \mathbf{d})$, have defined the hyperviscous stress triade $\mathbf{G} = G_{ijk} \mathbf{e}_i \otimes \mathbf{e}_j \otimes \mathbf{e}_k$ [18]:

$$\begin{aligned} \mathbf{G} = \frac{\partial \phi}{\partial \text{grad} \mathbf{d}} = & \eta_1 \text{grad}(\text{grad} \mathbf{v}) + \\ & + \eta_2 [\text{grad}^T(\text{grad}^T \mathbf{v}) + \text{grad}^T(\text{grad} \mathbf{v}) - \mathbf{I} \otimes \text{lap} \mathbf{v}], \end{aligned} \quad (124)$$

with two additional internal viscous friction coefficients η_1 and η_2 . Then they have obtained the following a simplification of the Skiba-Pearson momentum balance in the bulk:

$$\rho(\dot{\mathbf{v}} - \mathbf{b}) + \text{div}(\mathbf{p} - \text{div} \mathbf{G}) = 0, \quad (125)$$

and not one but two boundary conditions coming from applying the Green transformation twice [18]:

$$\begin{aligned} \mathbf{p}_n = \mathbf{p} \mathbf{n} - (\text{div} \mathbf{G}) \mathbf{n} - \text{div}_s(\mathbf{G} \mathbf{n}) - \text{I}_b(\mathbf{G} \mathbf{n}) \mathbf{n}, \\ \mathbf{m}_n = (\mathbf{G} \mathbf{n}) \mathbf{n}. \end{aligned} \quad (126)$$

These are nothing else as an extension of the classical Cauchy definition of traction force $\mathbf{p}_n = \mathbf{p} \mathbf{n}$. The first traction force is working on a field of velocity (like the classical Umov flux of mechanical energy), and the second traction vector \mathbf{m}_n , new in the context of viscous fluids³⁴, will be working on the normal derivative of velocity $\frac{\partial \mathbf{v}}{\partial n}$ [18]. Therefore, Fried and Gurtin postulate the following slip and adherence condition:

$$\nu(\mathbf{v}_A - \mathbf{v}_{\text{wall}}) = \mathbf{p}_n : \quad \text{generalized slip condition}, \quad (127)$$

$$\nu_A \left(\frac{\partial \mathbf{v}_A}{\partial n} - \frac{\partial \mathbf{v}_{\text{wall}}}{\partial n} \right) = \mathbf{m}_n : \quad \text{generalized adherence condition}, \quad (128)$$

³⁴For solid continuum with the second gradient including such a second traction force has been introduced by S. Forte, M. Vianello, *On surface stresses and edge forces*, Rendiconti di Matematica 8, 409-426 (1988).

where $\nu = \mu/l_s$ and $\nu_a = \mu_1/l_a$ stand for the external friction coefficient and external adherence friction coefficient l_s and l_a are the slip length and the adherence length. This model of boundary layer is fully consistent with its bulk model of the second order fluid, and covers the Deissler one, Eq. (122).

10 Interfacial mobility mechanisms – a classification

Let us consider, as an illustrating example of the generality of Stokes's slip condition, a case of a non-Newtonian incompressible fluid. This kind of fluid isn't anything else but the case of nonlinear extension of Newton's hypothesis of internal viscosity, important for stating a constitutive equation for such fluids like cream, oils or bitumen. In those cases the Stokes pressure diade \mathbf{p} is determined by two shearing viscosity coefficients μ_1 and μ_2 :

$$\mathbf{p}_A = p\mathbf{I} - \mu_1\mathbf{d} - \mu_2\mathbf{d}^2. \quad (129)$$

Let us stand a force balance according to the Stokes's concept of a boundary layer (Eq. (96) above). Omitting the capillarity surface stress tensor $\mathbf{p}_s = 0$, and assuming that the solid body is a fixed, rigid and in the stress-free state. i.e. $\mathbf{p}_B\mathbf{n}_B = 0$, we suppose the boundary external friction force $\mathbf{f}_{AB} = \nu(\mathbf{v}_A - \mathbf{v}_B) = \nu(\mathbf{v}_A - \mathbf{v}_{\text{wall}})$, and the external friction force $\mathbf{p}_A\mathbf{n}_A$ according to Eq. (129). After balancing these forces we have

$$\mathbf{f}_{AB} + \mathbf{p}_A\mathbf{n}_A + \mathbf{p}_B\mathbf{n}_B = (p\mathbf{I} - \mu_1\mathbf{d} - \mu_2\mathbf{d}^2)\mathbf{n} + \nu(\mathbf{v}_A - \mathbf{v}_{\text{wall}}). \quad (130)$$

This formulation of a force balance in the Stokes layer mathematically and also physically is inconsistent, since we have two internal viscosity coefficients μ_1 and μ_2 and only one external, Navier viscosity³⁵ ν .

Let us consider now a more consistent velocity slip boundary conditions that should be consistent with the Newton postulate stating, that a friction phenomenon depends on three components: the pressure dependent part, the relative velocity part, and the square velocity dependent part. Let the Newton postulate be true for a fluid in the bulk as well as for the thin layer on a boundary surface realizing a contact with a solid surface. Then taking into account the external friction force in the Newtonian form (Eq. (90) above) and the force of an internal

³⁵We keep Stokes's denotation of the coefficient of external viscosity. There is a subtle point here, allowing us to say that there is no difference between the Stokes coefficient ν and the Navier coefficient E , i.e. $E \equiv \nu$. Unfortunately, this statement is not accepted in the literature since, usually, different coefficients mean different physical quantities.

friction in the form of Eq. (130), we have more consistent boundary conditions:

$$\begin{aligned} (p\mathbf{I} - \mu_1\mathbf{d} - \mu_2\mathbf{d}^2)\mathbf{n} + \varpi\mathbf{n} + f_{SS'}N\frac{\mathbf{v} - \mathbf{v}_{\text{wall}}}{|\mathbf{v} - \mathbf{v}_{\text{wall}}|} + \\ + \nu(\mathbf{v} - \mathbf{v}_{\text{wall}}) + f_\kappa(\mathbf{v} - \mathbf{v}_{\text{wall}})^2\frac{\mathbf{v} - \mathbf{v}_{\text{wall}}}{|\mathbf{v} - \mathbf{v}_{\text{wall}}|} = 0. \end{aligned} \quad (131)$$

Some consistencies of this condition can be simply recognized if we compare the internal and external coefficients that appear in the model. This consistency can even be extended on reversible properties of the model, i.e., the internal (Euler) and the external (Stokes) pressures p and ϖ , respectively. In the Tab. 1 the comparison of these properties is shown .

Table 1. Comparison of a concise model of internal and external friction, according to Newton's postulate. The model ([†]) of a viscous bulk pressure has been proposed by Natanson [45].

	Internal (bulk)	External (boundary)	Characteristic ratio
Elastic pressure	p [Nm ⁻²]	ϖ [Nm ⁻²]	$\lambda_{\text{press}} = p/\varpi$
Frictional pressure	$p_{\text{vis}} = k_{\text{vis}}J^\dagger$ [Pa]	$f_{SS'}$	$\lambda_{\text{vis}} = k_{\text{vis}}/f_{SS'}N$
Linear slip velocity	μ_1 [Nsm ⁻²]	ν [Nsm ⁻³]	$l_{1\nu} = \mu_1/\nu$
Square slip velocity	μ_2 [Ns ² m ⁻²]	f_k [Ns ² m ⁻³]	$l_{2\nu} = \mu_2/f_k$

The better consistency of the above model results from the fact that it needs three coefficients of internal friction (k_{vis} , μ_1 , μ_2) and three coefficients of external friction ($f_{SS'}$, ν , f_k), respectively. Therefore, we can define a ratio between the internal and external friction by a dimensionless coefficient λ_{vis} , and two lengths of velocity slip: $l_{1\nu}$ and $l_{2\nu}$ (see Tab. 1). Having a measure of internal properties of friction, one can connect the external properties of friction at the Stokes boundary layer by appropriate closures written for λ_{vis} , $l_{1\nu}$ and $l_{2\nu}$, respectively.

In turn, going back to the model for an inconsistent layer Eq. (130), and after dividing it by the external Navier viscosity, ν , we obtain the more familiar form of Eq. (97) (its surface part):

$$\mathbf{I}_s(\mathbf{v} - \mathbf{v}_{\text{wall}}) = \mathbf{I}_s(2l_s\mathbf{d} + \dot{l}_s\mathbf{d}^2). \quad (132)$$

In Eq. (132) the slip length $l_s = \mu_1/\nu$ is defined as a ratio between the internal and external viscosity. However, the ratio μ_2/ν cannot be recognized as a slip length but rather as a rate of slip length \dot{l}_s . Let us consider a classification of different phenomenological models for Navier-Stokes boundary layer. Balance of the momentum for the Navier-Stokes layer, generally, has an above postulated

form:

$$\begin{aligned}
 & \partial_t (\rho_s \mathbf{v}_s) + \operatorname{div}_s (\rho \mathbf{v}_s \otimes \mathbf{v}_{s\parallel}) - w_n \mathbb{I}_b \rho_s \mathbf{v}_s + \\
 & + \operatorname{div}_s \mathbf{p}_s + \partial_n (\mathbf{p}_s \mathbf{n}) + [\mathbf{p}_A \mathbf{n}_A + \mathbf{p}_B \mathbf{n}_B + \mathbf{f}_{SA} + \mathbf{f}_{SB}] = \\
 & = \rho_s \mathbf{b}_s + \dot{m}_A (\mathbf{v}_A - \mathbf{v}_s) + \dot{m}_B (\mathbf{v}_B - \mathbf{v}_s) .
 \end{aligned} \tag{133}$$

In many situations the slip velocity \mathbf{v}_s can be calculated from a simplified form of above equation. If we omit the inertia properties of the layer, migration of mass across the layer, and the capillary and viscous surface tensors, Eq. (133) reduces to the simple balance of surface forces:

$$-\varpi \mathbf{n} + (\mathbf{p}_A - \mathbf{p}_B) \mathbf{n} + \mathbf{f}_{AB} = 0 \quad \text{on } \mathcal{M} . \tag{134}$$

Since \mathbf{p}_A and \mathbf{p}_B are well defined for arbitrary fluids, the main problem is a determination of ϖ and the force of external friction \mathbf{f}_{AB} .

Here, we must note that the previous statements of the phenomena of surface mobility, called transpiration, should be added to the proper definition of surface friction. Yet another mobility force, other than the difference of pressure or temperature, was discovered by Graham in 1849. He found a new kind of transpiration called ‘‘atomism’’ [25, 26]. This phenomena is nowadays called ‘‘diffusional transpiration’’ or ‘‘diffusionphoresis’’. It is quite different kind of flow than the classical transpiration flow induced by difference of the normal surface pressures, i.e. ‘‘pressure transpiration’’. The diffusion transpiration deals with a flow of gas mixture by a long capillary pipe, where there is another interaction of every mixture component with a surface. It leads to the mixture separation. In this case the most important is a coefficient of diffusion mobility, c_{vN} . Another type of induced motion is due to the difference of an electric potential, ϕ , on a surface. This phenomenon is called ‘‘electrophoresis’’ and is governed by an electro-mobility coefficient³⁶, $c_{v\phi}$. Other mobility mechanism is connected with the phase transition change [4, 6] and the surface gradient of the phase order parameter x .

Let us note that these all types of mobility, i.e., pressure, thermal, diffusional, phase, and electrical define only an external mobility force in the Stokes layer. This force, partially given by Eqs. (112) and (120), can be generalized to

$$\begin{aligned}
 \mathbf{f}_{AB} = \nu (\mathbf{v} - \mathbf{v}_{\text{wall}} - c_{v\varpi} \operatorname{grad}_s \varpi - c_{v\theta} \operatorname{grad}_s \theta - c_{vN} \operatorname{grad}_s N - \\
 + c_{v\phi} \operatorname{grad}_s \phi - c_{vx} \operatorname{grad}_s x) ,
 \end{aligned} \tag{135}$$

where ν is the coefficient of external viscosity, defined on the mean surface of the Stokes layer, $\mathbf{v}_s = \mathbf{v} - \mathbf{v}_{\text{wall}}$ are velocity vectors on top and bottom surface of the Stokes layer, and N is the concentration of gas in the mixture.

³⁶Electrophoresis was discovered by von Smoluchowski in 1916 [64]. See also H.J. and Keh, J.L. Anderson, *Boundary effects on electrophoretic motion of colloidal sphere*, J. Fluid Mech. **153**, 417-439 (1985).

In a special case, when gas is at rest, we can observe a motion of the particle induced by different surface inequalities. This kind of motion is called in the literature the “phoretic motion” [6]. In general, any nanoparticle immersed in the fluid may undergo simultaneously five types of motions which are shown in Tab. 2³⁷.

Table 2. Five kinds of motions connected with the surface mobility of a particle immersed in a fluid at rest. Here: $c_{v\theta}$ – the thermo-mobility coefficient, c_{vN} – the concentration-mobility coefficient, $c_{v\phi}$ – electro-mobility coefficient, $c_{v\varpi}$ – the pressure-mobility coefficient, c_{vx} – the phase-mobility coefficient.

Phenomena	Corresponding velocity	Driving potential
Thermophoresis	$\mathbf{v}_{\text{wall}} = c_{v\theta} \text{grad}_s \theta$	Temperature θ
Diffusionphoresis	$\mathbf{v}_{\text{wall}} = c_{vN} \text{grad}_s N$	Concentration N
Electrophoresis	$\mathbf{v}_{\text{wall}} = c_{v\phi} \text{grad}_s \phi$	Electric potential ϕ
Pressurephoresis	$\mathbf{v}_{\text{wall}} = c_{v\varpi} \text{grad}_s \varpi$	Pressure ϖ
Phasephoresis	$\mathbf{v}_{\text{wall}} = c_{vx} \text{grad}_s x$	Order parameter x

11 Conclusions

In the paper the applications of the extended Navier-Stokes boundary layer equations, including the different surface mobility mechanisms are presented in order to explain the enhanced flow in microchannels.

Generalization of the Navier-Stokes boundary slip layer, formulated in the present paper, supplements the original Navier-Stokes model by additional surface quantities like the surface mass and the surface momentum flux. In the present case the slip velocity \mathbf{v}_s is determined from the solution of the complete balance of momentum (50) written within the layer. Since the stress tensors \mathbf{p}_A and \mathbf{p}_B are determined in the bulk and cannot be arbitrarily changed at the boundary, such an approach leads to the separation for those constitutive relations which can be imposed to fulfill the surface balance of momentum. There is still an open place for the modeling of the surface momentum diade \mathbf{p}_s and the surface friction force \mathbf{f}_{AB} , where indeed a second gradient of surface velocity can be postulated.

³⁷These phenomena must be distinguished from the motionless phenomena like: “temperature jump”, “concentration jump”, “potential jump” related with the external heat conductivity, external mass diffusivity, and external electric conductivity coefficients, respectively. Recently the phenomenon of jump concentration of salt in a gel mixture has been discovered by [32].

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One of the reviewer's suggestion, that the authors wish to respond in a wider context, is the remark to refuse the basic boundary conditions its fundamental generality and restrict them solely to the range of its application. There are reasons not to do it, however. Firstly, historically the pioneers and veterans of this approach had formulated general boundary conditions not being aware of the nanoflows in nanochannels. In the past, the validity of the classical modelling of viscous fluids, known commonly as the Navier-Stokes model, was often disputed by many researchers [23, 42, 48]. Questioning was based on the differences between experimental and numerical results for models based on the Navier-Stokes equations in the bulk, and the application of no-slip boundary conditions at the walls. It was initially assumed that the value of the internal shear viscosity coefficient, μ , was not suitable to use it in the nano-scale, but seems now that those differences result from the application of the no-slip condition at the walls. Than the old Navier's boundary friction concept, leading to the slip velocity, has been vigorously explored by many scientists [33, 30, 21, 31].

There are few generalisations of the Navier-Stokes slip boundary in the literature (see [33, 36, 62, 11]), that strictly follow the Maxwell's line of reasoning, and attempts to introduce higher order derivatives, thereby called "the second order velocity slip". It should be noted, however, that these second order slip models are inconsistent with the bulk model, where only the first velocity gradient appears. Therefore it should not be solved within the continuum model framework.

Secondly, there is no assurance, that the generalised boundary conditions would not find application in the macroscale. In automotive traction drives, for instance, mechanic power is being transmitted via friction forces, which result from the slip velocity between lubricated and highly loaded contacts between rolling bodies. This example of application for the Navier-Stokes boundary conditions, as has been shown by Pietrov, has little to do with the nanoscale directly. This assertion is true, independently on the lubrication model used (i.e., Reynolds or Johnson-Teraarwerk nonlinear model).

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