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FINITE ROTATIONS IN THE NONLINEAR THEORY OF THIN SHELLS

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Abstract

The theory of finite rotations in thin shells is developed and many shell relations in terms of finite rotations are presented. Three forms of geometric boundary conditions and energetically compatible static boundary conditions are constructed. Various sets of Eulerian and Lagrangean shell equations are discussed and their consistent simplification within the first-approximation geometrically non-linear theory of isotropic elastic shells is given. A classification of shell problems with small, moderate, large and finite rotations is proposed and appropriate sets of simplified shell equations are presented.

FINITE ROTATIONS IN THE NON-LINEAR THEORY OF THIN SHELLS

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1. INTRODUCTION

The appearance of finite rotations of material elements is the fundamental kinematic property of any non-linear theory of shells. Finite rotations may appear even when strains are small everywhere in the shell, which is obvious from a trivial example of rolling of a sheet of paper into a cylinder.

According to the Cauchy theorem, the deformation of a neighbourhood of any material particle of a continua can be decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation of the principal directions. Within the continuum mechanics the rotational part of deformation is conventionally described by a rotation tensor $\underset{\sim}{R}$ as defined, for example, by Truesdell and Noll [1]. An equivalent description of rotations, by means of a finite rotation vector $\underset{\sim}{\Omega}$, was used by Shamina [2] in discussion of continuum compatibility conditions. Rotation parameters appear explicitly in many three - dimensional relations, being particularly important in the theory of constitutive equations. They play also a fundamental role in analytic mechanics of a rigid - body motion [3].

The rotational part of deformation should play even more important role in any non-linear theory of thin bodies, in particular in the non-linear theory of thin shells, what was recognized long ago. Unfortunately, the shell literature is not free from confusions about analytic representation of the finite rotations. Usually the linearized rotations or displacement gradients are used, apparently on the intuitive grounds, to describe rotations of material elements also within the non-linear range of the shell deformation.

Within the Kirchhoff-Love type non-linear theory of shells Simmonds and Danielson [4,5] employed the finite rotation vector $\underset{\sim}{\Omega}$ of the principal directions of strain as the basic kinematic variable of the shell theory. Novozhilov and Shamina [6] used the notion of the total finite rotation vector $\underset{\sim}{\Omega}_t$ of the boundary to derive various forms of geometric boundary conditions. However, in these works the finite rotation vectors were introduced in a descriptive manner, without giving the analytic relations for them in terms of basic kinematical parameters of the shell deformation, such as displacements of the shell middle surface or components of the deformation gradient tensor.

A general theory of finite rotations in shells was developed in the author's thesis [7] (see also [8]). Many simplified relations , valid under the Kirchhoff-Love constraints, were derived in [7] as particular cases of the exact relations. Their independent derivation was also presented in [9] . The notion of a finite rotation prooved to be very helpful in deriving alternative forms of geometric and static boundary conditions, in obtaining various modified forms of equilibrium equations and in providing a new consistent classification of various approximate variants of geometrically non-linear theory of elastic shells and plates with restricted rotations.

In this report we shall develop the theory of finite rotations in thin shells and discuss some associated problems of shell theory , in which the rotations play an important role. It is assumed that the behaviour of a thin shell can be described with a sufficient accuracy by the behaviour of the shell middle surface. This is accomplished by imposing the Kirchhoff-Love constraints on the shell deformation. The change of the shell thickness during deformation is taken into account only in constitutive equations of an isotropic elastic shell under small strains. Such simplified approach is justified within the first - approximation geometrically non-linear theory of thin elastic shells [10], which is the ultimate goal of this work.

2. STRAINS AND ROTATIONS IN THIN SHELLS

2.1. Notation and preliminary relations

We shall adopt here, as far as possible, the system of notations used by Koiter [11] and by the author [7,9,12].

Let $\underset{\sim}{r}(\theta^\alpha) = x^k(\theta^\alpha)\underset{\sim}{i}_k$ and $\bar{\underset{\sim}{r}}(\theta^\alpha) = \bar{x}^k(\theta^\alpha)\underset{\sim}{i}_k$, $k = 1,2,3$, be position vectors of the shell middle surface in the reference and deformed configurations, respectively. Here θ^α , $\alpha = 1,2$, is a pair of surface convected coordinates and x^k and $\bar{x}^k$ are components of $\underset{\sim}{r}$ and $\bar{\underset{\sim}{r}}$ with respect to a Cartesian frame.

With the reference surface M we associate a standard surface covariant base vectors $\underset{\sim}{a}_\alpha = \underset{\sim}{r}_{,\alpha}$, covariant components of the metric tensor $a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta$ with determinant $a = |a_{\alpha\beta}|$, a unit vector normal to M , $\underset{\sim}{n} = \frac{1}{2}\,\epsilon^{\alpha\beta}\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta$, and covariant components of the curvature tensor $b_{\alpha\beta} = \underset{\sim}{a}_{\alpha,\beta} \cdot \underset{\sim}{n}$. Here comma $(\;)_{,\alpha}$ denotes partial differentiation with respect to surface coordinates θ^α , while $\epsilon^{\alpha\beta}$ are contravariant components of the permutation tensor. Contravariant components $a^{\alpha\beta}$ of the metric tensor satisfying the relations $a^{\alpha\gamma}a_{\beta\gamma} = \delta^\alpha_\beta$ are used to raise the indices at M . Similar geometric quantities associated with deformed surface $\bar{M}$ are marked by a dash: $\bar{\underset{\sim}{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{\underset{\sim}{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\epsilon}^{\alpha\beta}$, $\bar{a}^{\alpha\beta}$ etc. The surface covariant differentiation at M or $\bar{M}$ is denoted by $(\ )_{|\alpha}$ or $(\ )_{||\alpha}$, respectively.

After the surface deformation the basic vectors $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ are expressed in terms of the geometry of M and displacement vector $\mathbf{u} = u_\alpha \mathbf{a}^\alpha + w\mathbf{n}$ by the relations [9,11,12]

$$\bar{\mathbf{a}}_\alpha = l_{\lambda\alpha}\mathbf{a}^\lambda + \phi_\alpha \mathbf{n} = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha} \quad , \quad \bar{\mathbf{n}} = n_\alpha \mathbf{a}^\alpha + n\mathbf{n} \tag{2.1.1}$$

where

$$l_{\alpha\beta} = a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta}$$

$$\theta_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w \quad , \quad \phi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda \tag{2.1.2}$$

$$\omega_{\alpha\beta} = \frac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) = \epsilon_{\alpha\beta}\phi \quad , \quad \phi = \frac{1}{2}\epsilon^{\alpha\beta}u_{\beta|\alpha}$$

$$n_\mu = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}\phi_\alpha l^\lambda_{\cdot\beta} \quad , \quad n = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}l^\lambda_{\cdot\alpha}l^\mu_{\cdot\beta} \tag{2.1.3}$$

The components of the Lagrangean surface strain tensor and the tensor of change of surface curvature are defined by

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \theta_{\alpha\beta} + \frac{1}{2}(\theta^\lambda_\alpha - \omega^\lambda_{\cdot\alpha})(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + \frac{1}{2}\phi_\alpha\phi_\beta \tag{2.1.4}$$

$$\begin{aligned}\kappa_{\alpha\beta} &= -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \\ &= -\{n(\phi_{\alpha|\beta} + b^\lambda_\beta l_{\lambda\alpha}) + n_\lambda(l^\lambda_{\cdot\alpha|\beta} - b^\lambda_\beta\phi_\alpha) - b_{\alpha\beta}\}\end{aligned} \tag{2.1.5}$$

These strain measures satisfy the following compatibility conditions

$$\begin{aligned}&\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu} + \bar{a}^{\kappa\nu}(b_{\kappa\lambda} - \kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}] = 0 \\ &K\gamma^\kappa_\kappa + \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\kappa_{\beta\lambda} + \frac{1}{2}(\kappa_{\alpha\mu}\kappa_{\beta\lambda} + \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda})] = 0\end{aligned} \tag{2.1.6}$$

Here K is the Gaussian curvature of M and

$$\gamma_{\nu\beta\mu} = \gamma_{\nu\beta|\mu} + \gamma_{\nu\mu|\beta} - \gamma_{\beta\mu|\nu} \tag{2.1.7}$$

The base vectors $\underset{\sim}{a}_\alpha$ and $\underset{\sim}{a}^\alpha$ span at $M \in \mathcal{M}$ a two-dimensional vector space V. The tensor product of two surface vectors $\underset{\sim}{u}, \underset{\sim}{v} \in V$ defines the second-order surface tensor $\underset{\sim}{u} \otimes \underset{\sim}{v} = \underset{\sim}{T} \in T_2 = V \otimes V$. Tensor products of the base vectors $\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$, $\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}_\beta$ etc. are bases of the tensor space T_2. For $\underset{\sim}{T} = T^{\alpha}_{\cdot\beta}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\beta$ the transposition is defined by $\underset{\sim}{T}^T = T^{\alpha}_{\cdot\beta}\underset{\sim}{a}^\beta \otimes \underset{\sim}{a}_\alpha$, the contraction by $\mathrm{tr}\underset{\sim}{T} = T^{\alpha}_{\cdot\alpha}$, the multiplication by $\underset{\sim}{T}\underset{\sim}{S} = T^{\alpha}_{\cdot\lambda}S^{\lambda}_{\cdot\beta}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}^\beta$ and the scalar product by $\underset{\sim}{T} \cdot \underset{\sim}{S} = T^{\alpha}_{\cdot\lambda}S^{\lambda}_{\cdot\alpha}$. The detailed presentation of tensor algebra and analysis in absolute notation is given by Lichnerowicz [13], Bowen and Wang [14] and Truesdell [15]. In the notation used here the metric and curvature tensors of $\mathcal{M}$ are given by

$$\underset{\sim}{a} = a_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \quad , \quad \underset{\sim}{b} = b_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \tag{2.1.8}$$

In the neighbourhood of $\mathcal{M}$ let us introduce a spatial normal coordinate system θ^i, $i = 1,2,3$, such that at $\mathcal{M}$ there is $\theta^3 = 0$ and $\theta^3 \equiv \zeta$ is the distance to a point $P \in \mathcal{P} \subset \mathcal{E}$ of the three - dimensional Euclidean point space $\mathcal{E}$ and $-h/2 \le \zeta \le +h/2$, where h is the constant shell thickness in the reference configuration. Then for the position vector of P and for the spatial base vectors at P we have

$$\begin{aligned} \underset{\sim}{p} &= \underset{\sim}{r} + \zeta\underset{\sim}{n} \quad , \quad \underset{\sim}{g}_\phi = \underset{\sim}{p}_{,\phi} = \mu^\alpha_\phi \underset{\sim}{a}_\alpha \quad , \quad \underset{\sim}{g}_3 = \underset{\sim}{p}_{,3} = \underset{\sim}{n} \\ \mu^\alpha_\phi &= \underset{\sim}{a}^\alpha \cdot \underset{\sim}{g}_\phi = \delta^\alpha_\phi - \zeta\delta^\beta_\phi b^\alpha_\beta \end{aligned} \tag{2.1.9}$$

The base vectors $\underset{\sim}{g}_i$ span at $P \in \mathcal{P}$ a three - dimensional vector space W. The tensor product of two space vectors defines a second - - order space tensor $\underset{\sim}{L} \in L_2 = W \otimes W$. The space metric and translation tensors are given by

$$\begin{aligned} \underset{\sim}{1} &= \underset{\sim}{1}^T = g_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j = \underset{\sim}{a} + \underset{\sim}{n} \otimes \underset{\sim}{n} = \mu^\alpha_\phi \underset{\sim}{a}_\alpha \otimes \underset{\sim}{g}^\phi + \underset{\sim}{n} \otimes \underset{\sim}{n} \\ \underset{\sim}{g} &= \underset{\sim}{g}^T = \underset{\sim}{1} - \zeta\underset{\sim}{b} \end{aligned} \tag{2.1.10}$$

This leads to the relations

$$\underset{\sim}{g}_\phi = \delta^\alpha_\phi \underset{\sim}{g}\underset{\sim}{a}_\alpha \quad , \quad \underset{\sim}{g}_3 = \underset{\sim}{g}\underset{\sim}{n} \; . \tag{2.1.11}$$

which allow to express in coordinate - free notation quantities at $P \in P$ in terms of those defined at $M \in M$ and the distance ζ from M .

2.2. Deformation of thin shells under Kirchhoff - Love constraints

Consider a shell S consisting of particles $X, Y, \ldots$. A one-to-one correspondence between particles $X \in S$ and points $P \in P \subset E$ in three-dimensional Euclidean space is the configuration of the shell, $P = \kappa(X)$, $\kappa : S \to P$. In the Lagrangean description the displacement of a particle $X \in S$ from its position P in the reference configuration to a position $\bar{P} = \bar{\kappa}(X)$ in the deformed one, $\bar{P} = \chi(P)$, $\chi = \bar{\kappa} \circ \kappa^{-1}$, is given by a displacement vector

$$\underset{\sim}{v} = \underset{\sim}{v}(P) = \bar{\underset{\sim}{p}}[\chi(P)] - \underset{\sim}{p}(P) = v_i \underset{\sim}{g}^i \tag{2.2.1}$$

Thus for differentials

$$d\bar{\underset{\sim}{p}} = \bar{\underset{\sim}{p}}_{,i} d\theta^i = \underset{\sim}{F} d\underset{\sim}{p} \tag{2.2.2}$$

where the tensor $\underset{\sim}{F} \in L_2$ has the form

$$\underset{\sim}{F} = \underset{\sim}{F}(P) = \underset{\sim}{1} + \operatorname{grad} \underset{\sim}{v} = \underset{\sim}{g}_i \otimes \underset{\sim}{g}^i + \underset{\sim}{v}_{,i} \otimes \underset{\sim}{g}^i = \bar{\underset{\sim}{g}}_i \otimes \underset{\sim}{g}^i \tag{2.2.3}$$

This tensor used by Truesdell and Noll [1] is known as the spatial deformation gradient tensor. It fully describes the state of deformation in a neighbourhood of the particle $X \in S$ during the shell deformation from the reference to deformed configurations.

In general, material fibres , normal to the reference surface M, may after deformation become neither straight nor normal to the surface $\bar{M} = \chi(M)$. In the general case the deformation function χ may be expanded into Taylor series in the vicinity of the shell middle surface. The linear part of the expansion describes exactly the shell deformation at its middle surface. Describing shell deformation only by the linear part of the expansion (or, equivalently, assuming the linear distribution of displacements in the shell space) various relations of the Reissner-type non-linear theory of shells may be discussed [7,16]. In particular, the assumption of the linear distribution of displacements enabled to develope the exact theory of finite rotations in shells [7,8].

In this work we are interested in constructing various relations for the first - approximation geometrically non-linear theory of thin isotropic elastic shells. In such a case the stress state in the shell space is approximately plane and parallel to the shell middle surface [17]. The shell strain energy function depends explicitly only upon the stretching and bending of the shell middle surface. Its dependence upon the transverse strains appear only implicitly, through the modified shell elasticity tensor [10]. In order to simplify all intermediate transformations, we initially impose here the Kirchhoff - Love constraints on the shell deformation. The change of the shell thickness during deformation will, however, be taken into account in the constitutive equations.

According to the Kirchhoff - Love constraints, material fibres that are normal to the reference shell middle surface M remain, after the shell deformation, normal to the deformed surface $\bar{M}$ and do not change their length. Under these constraints the shell deformation gradient tensor $\underset{\sim}{G} \in L_2$ takes at M the following form

$$\underset{\sim}{G} = \underset{\sim}{F}(P')\Big|_{\zeta = 0} = \bar{\underset{\sim}{a}}_\alpha \otimes \underset{\sim}{a}^\alpha + \bar{\underset{\sim}{n}} \otimes \underset{\sim}{n} \qquad (2.2.4)$$

Under (2.2.4) and (2.1.10) for the spatial quantities we obtain

$$\underset{\sim}{F} = \bar{\underset{\sim}{g}}\underset{\sim}{G}\underset{\sim}{g}^{-1} = (\underset{\sim}{G} - \zeta\bar{\underset{\sim}{b}}\underset{\sim}{G})\underset{\sim}{g}^{-1} \quad , \quad \underset{\sim}{v} = \underset{\sim}{u} + \zeta\underset{\sim}{\beta} \qquad (2.2.5)$$

where under K - L constraints $\underset{\sim}{\beta}$ depends upon $\underset{\sim}{u}$ according to

$$\underset{\sim}{\beta} = (\underset{\sim}{G} - \underset{\sim}{1})\underset{\sim}{n} = \bar{\underset{\sim}{n}} - \underset{\sim}{n} = n_\alpha \underset{\sim}{a}^\alpha + (n - 1)\underset{\sim}{n} \qquad (2.2.6)$$

In the Lagrangean description of strain we use the Green strain tensor $\underset{\sim}{E} \in L_2$ defined by

$$\underset{\sim}{E} = \underset{\sim}{E}(P) = \frac{1}{2}(\underset{\sim}{F}^T\underset{\sim}{F} - \underset{\sim}{1}) = E_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \qquad (2.2.7)$$

In viev of (2.2.5) this tensor may be expressed in terms of the surface strain measures according to

$$\underset{\sim}{E} = \underset{\sim}{g}^{-1}(\underset{\sim}{\gamma} + \zeta\underset{\sim}{\kappa} + \zeta^2\underset{\sim}{\nu})\underset{\sim}{g}^{-1} \tag{2.2.8}$$

where

$$\begin{aligned} \underset{\sim}{\gamma} &= \frac{1}{2}(\underset{\sim}{G}^T\underset{\sim}{G} - \underset{\sim}{1}) = \gamma_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \\ \underset{\sim}{\kappa} &= -(\underset{\sim}{G}^T\bar{\underset{\sim}{b}}\underset{\sim}{G} - \underset{\sim}{b}) = \kappa_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \\ \underset{\sim}{\nu} &= \frac{1}{2}(\underset{\sim}{G}^T\bar{\underset{\sim}{b}}^2\underset{\sim}{G} - \underset{\sim}{b}^2) = \nu_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = \frac{1}{2}[(\underset{\sim}{b} - \underset{\sim}{\kappa})(\underset{\sim}{1} + 2\underset{\sim}{\gamma})^{-1}(\underset{\sim}{b} - \underset{\sim}{\kappa}) - \underset{\sim}{b}^2] \end{aligned} \tag{2.2.9}$$

In view of (2.2.5) the deformation gradient tensor $\underset{\sim}{G}$ provides complete information about the shell non-linear deformation compatible with K - L constraints. Since $\underset{\sim}{G}$ is nonsingular then, according to the polar decomposition theorem [1], it can be represented uniquely by two following formulae

$$\underset{\sim}{G} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R} \;, \quad \underset{\sim}{G}^{-1} = \underset{\sim}{U}^{-1}\underset{\sim}{R}^T = \underset{\sim}{R}^T\underset{\sim}{V}^{-1} \tag{2.2.10}$$

Here $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are the right and left stretch tensors, respectively, and $\underset{\sim}{R}$ is the finite rotation tensor. The tensors $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are positive definite and symmetric, while $\underset{\sim}{R}$ is the proper orthogonal tensor. In terms of $\underset{\sim}{G}$ we have

$$\underset{\sim}{U} = \sqrt{\underset{\sim}{G}^T\underset{\sim}{G}} \;, \quad \underset{\sim}{V} = \sqrt{\underset{\sim}{G}\underset{\sim}{G}^T} \;, \quad \underset{\sim}{U} = \underset{\sim}{R}^T\underset{\sim}{V}\underset{\sim}{R} \tag{2.2.11}$$

$$\underset{\sim}{R} = \underset{\sim}{G}\underset{\sim}{U}^{-1} = \underset{\sim}{V}^{-1}\underset{\sim}{G} \;; \quad \underset{\sim}{R}^T = \underset{\sim}{R}^{-1} \;, \quad \det\underset{\sim}{R} = +1 \tag{2.2.12}$$

By the formulae (2.2.1), (2.2.5) and (2.2.10) the deformation of a neighbourhood about a point of the shell middle surface is decomposed into a rigid - body translation, a pure stretch along principal directions of strain and a rigid - body rotations of the principal directions. Decomposition of $\underset{\sim}{G}$ in terms of $\underset{\sim}{U}$ in (2.2.10) is compatible with the Lagrangean description, in terms of $\underset{\sim}{V}$ it is compatible with the Eulerian description.

From (2.2.4) and (2.2.10) we obtain

$$\bar{\mathbf{a}}_\alpha = \mathbf{G}\mathbf{a}_\alpha = \mathbf{R}\overset{\vee}{\mathbf{a}}_\alpha = \mathbf{V}\overset{*}{\mathbf{a}}_\alpha \quad , \quad \bar{\mathbf{n}} = \mathbf{G}\mathbf{n} = \mathbf{R}\mathbf{n}$$
$$\bar{\mathbf{a}}^\alpha = (\mathbf{G}^{-1})^T\mathbf{a}^\alpha = \mathbf{R}\overset{\vee}{\mathbf{a}}{}^\alpha = \mathbf{V}^{-1}\overset{*}{\mathbf{a}}{}^\alpha \tag{2.2.13}$$

where

$$\overset{\vee}{\mathbf{a}}_\alpha = \mathbf{U}\mathbf{a}_\alpha \,, \quad \overset{\vee}{\mathbf{a}}{}^\alpha = \mathbf{U}^{-1}\mathbf{a}^\alpha \,, \quad \overset{*}{\mathbf{a}}_\alpha = \mathbf{R}\mathbf{a}_\alpha \,, \quad \overset{*}{\mathbf{a}}{}^\alpha = \mathbf{R}\mathbf{a}^\alpha \tag{2.2.14}$$

The intermediate Lagrangean basis $\overset{\vee}{\mathbf{a}}_\alpha$, $\mathbf{n}$ is obtained by stretching the reference basis $\mathbf{a}_\alpha$, $\mathbf{n}$ along the principal directions of the stretch tensor $\mathbf{U}$. The intermediate Eulerian basis $\overset{*}{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ is obtained by a rigid - body rotation of the reference basis $\mathbf{a}_\alpha$, $\mathbf{n}$ by means of the finite rotation tensor $\mathbf{R}$. The basis $\overset{\vee}{\mathbf{a}}_\alpha$, $\mathbf{n}$ was used by Novozhilov and Shamina [6] and by the author [7,9]. The basis $\overset{*}{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ was used by Simmonds and Danielson [4,5].

Using (2.2.13) and (2.2.14) we may construct exact formulae for various tensors

$$\mathbf{U} = \overset{\vee}{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \mathbf{n}\otimes\mathbf{n} \quad , \quad \mathbf{U}^{-1} = \mathbf{a}_\alpha \otimes \overset{\vee}{\mathbf{a}}{}^\alpha + \mathbf{n}\otimes\mathbf{n}$$
$$\mathbf{V} = \bar{\mathbf{a}}_\alpha \otimes \overset{*}{\mathbf{a}}{}^\alpha + \bar{\mathbf{n}}\otimes\bar{\mathbf{n}} \quad , \quad \mathbf{V}^{-1} = \overset{*}{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}^\alpha + \bar{\mathbf{n}}\otimes\bar{\mathbf{n}} \tag{2.2.15}$$
$$\mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \overset{\vee}{\mathbf{a}}{}^\alpha + \bar{\mathbf{n}}\otimes\mathbf{n} = \overset{*}{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}}\otimes\mathbf{n}$$

Since $\mathbf{U} = \mathbf{U}(M)$ is positive definite and symmetric it has [18] three real and positive eigenvalues U_r in three orthogonal principal directions defined by a triad of unit vectors $\mathbf{f}_r$, which satisfy the set of equations

$$\mathbf{U}\mathbf{f}_r - U_r\mathbf{f}_r = \mathbf{0} \quad , \quad \mathbf{f}_r \cdot \mathbf{f}_s = \delta_{rs} \tag{2.2.16}$$

Under K - L constraints the vector $\mathbf{n}$ coincides with one of the principal directions, say $\mathbf{f}_3 \equiv \mathbf{n}$, with eigenvalue equal to +1. Thus $\mathbf{U}$ can be presented in the following diagonal form

$$\mathbf{U} = U_1\mathbf{f}_1 \otimes \mathbf{f}_1 + U_2\mathbf{f}_2 \otimes \mathbf{f}_2 + \mathbf{n}\otimes\mathbf{n} \tag{2.2.17}$$

According to $(2.2.9)_1$, (2.2.10) and (2.2.17) we have

$$\underset{\sim}{\gamma} = \frac{1}{2}(\underset{\sim}{U}^2 - \underset{\sim}{1}) = \gamma_1 \underset{\sim}{\ell}_1 \otimes \underset{\sim}{\ell}_1 + \gamma_2 \underset{\sim}{\ell}_2 \otimes \underset{\sim}{\ell}_2$$
$$\gamma_\rho = \frac{1}{2}(U_\rho^2 - 1) \quad , \quad \gamma_3 = 0 \tag{2.2.18}$$

In what follows we shall frequently use a modified Lagrangean strain tensor $\overset{\vee}{\underset{\sim}{\gamma}}$ defined by

$$\overset{\vee}{\underset{\sim}{\gamma}} = \underset{\sim}{U} - \underset{\sim}{1} = \sqrt{\underset{\sim}{1} + 2\underset{\sim}{\gamma}} - \underset{\sim}{1} = \overset{\vee}{\gamma}_{\alpha\beta} \underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta \tag{2.2.19}$$

This tensor depends upon displacements through the non-rational relations. On the other hand, numerous geometric relations, which are non-rational in terms of $\underset{\sim}{\gamma}$, become polynomials when expressed in terms of $\overset{\vee}{\underset{\sim}{\gamma}}$. For example, using (2.2.14), (2.2.19) and (2.2.18) we obtain

$$\overset{\vee}{\underset{\sim}{a}}_\alpha = \underset{\sim}{a}_\alpha + \overset{\vee}{\gamma}_{\alpha\beta} \underset{\sim}{a}^\beta \quad , \quad \overset{\vee}{\underset{\sim}{a}}^\alpha = \sqrt{\frac{a}{\bar{a}}}\, \epsilon^{\alpha\beta} \epsilon_{\lambda\mu} (\delta^\mu_\beta + \overset{\vee}{\gamma}{}^\mu_\beta) \underset{\sim}{a}^\lambda$$
$$\bar{a}_{\alpha\beta} = (\delta^\lambda_\alpha + \overset{\vee}{\gamma}{}^\lambda_\alpha)(\delta^\mu_\beta + \overset{\vee}{\gamma}{}^\mu_\beta) a_{\lambda\mu} \quad , \quad 2\gamma_{\alpha\beta} = 2\overset{\vee}{\gamma}_{\alpha\beta} + \overset{\vee}{\gamma}{}^\lambda_\alpha \overset{\vee}{\gamma}_{\lambda\beta} \tag{2.2.20}$$

Similar polynomial relations for $\bar{\epsilon}_{\alpha\beta}$, $\bar{\epsilon}^{\alpha\beta}$, $\sqrt{\frac{a}{\bar{a}}}$, $\bar{a}^{\alpha\beta}$ expressed in terms of $\overset{\vee}{\gamma}_{\alpha\beta}$ and the reference surface geometry are derived in [7,9], together with some inverse formulae.

2.3. Finite rotation tensor and vector

It follows from the first of $(2.2.15)_3$ with (2.1.1) and (2.2.20) that we obtain the following general formulae for the finite rotation tensor in terms of displacements

$$\underset{\sim}{R} = \bar{a}^{\alpha\beta}(\underset{\sim}{a}_\alpha + \underset{\sim}{u}_{,\alpha}) \otimes (\underset{\sim}{a}_\beta + \overset{\vee}{\gamma}_{\beta\lambda} \underset{\sim}{a}^\lambda) + (n_\alpha \underset{\sim}{a}^\alpha + n\underset{\sim}{n}) \otimes \underset{\sim}{n} \tag{2.3.1}$$

According to the spectral decomposition theorem [15,18] the proper orthogonal tensor $\underset{\sim}{R}$ has only one real eigenvalue equal to +1 and two complex conjugate eigenvalues $\cos\omega \pm i \sin\omega$. Let $\underset{\sim}{e}_1$ be a unit vector of the corresponding first principal direction, satisfying $\underset{\sim}{R}\underset{\sim}{e}_1 = \underset{\sim}{e}_1$. Taking arbitrarily a unit vector $\underset{\sim}{e}_2 \perp \underset{\sim}{e}_1$ we obtain the third unit vector defined by $\mathbf{e}_3 = \mathbf{e}_1 \times \mathbf{e}_2$. In this orthonormal basis $\mathbf{e}_r$ the tensor $\mathbf{R}$ takes the form [7,9]

$$\mathbf{R} = \mathbf{e}_1 \otimes \mathbf{e}_1 + \cos\omega\,(\mathbf{e}_2 \otimes \mathbf{e}_2 + \mathbf{e}_3 \otimes \mathbf{e}_3) - \\ - \sin\omega\,(\mathbf{e}_2 \otimes \mathbf{e}_3 - \mathbf{e}_3 \otimes \mathbf{e}_2) \tag{2.3.2}$$

The direction defined by $\mathbf{e}_1$ is called the axis of rotation and the angle ω , $|\omega| < \pi$, is called the angle of rotation about the axis of rotation.

Let us decompose a vector $\mathbf{w} \in W$ in the basis $\mathbf{e}_r$ to obtain

$$\mathbf{w} = w_1\mathbf{e}_1 + w_p(\cos\alpha\,\mathbf{e}_2 + \sin\alpha\,\mathbf{e}_3) \tag{2.3.3}$$

If the vector $\mathbf{w}$ is acted on by the tensor $\mathbf{R}$ then we obtain a new vector $\overset{*}{\mathbf{w}}$, which in the basis $\mathbf{e}_r$ becomes

$$\overset{*}{\mathbf{w}} = \mathbf{R}\mathbf{w} = w_1\mathbf{e}_1 + w_p[\cos(\alpha + \omega)\mathbf{e}_2 + \sin(\alpha + \omega)\mathbf{e}_3] \tag{2.3.4}$$

It is evident from (2.3.4) that the tensor $\mathbf{R}$ rotates the vector $\mathbf{w}$ through the angle ω about the axis of rotation defined by the vector $\mathbf{e}_1$.

In what follows it is convenient to describe the rotational part of shell deformation by means of an equivalent finite rotation vector $\mathbf{\Omega}$ used extensively in analytic mechanics of a rigid - body motion [3] . The direction of $\mathbf{\Omega}$ is defined by $\mathbf{e}_1 \equiv \mathbf{e}$ and the length of it is taken here to be equal to $|\sin\omega|$. Therefore, for $|\omega| < \pi$ we define here the finite rotation vector by the formulae

$$\mathbf{\Omega} = \sin\omega\,\mathbf{e} \tag{2.3.5}$$

Note that $\mathbf{\Omega}$ as defined here is not a vector in the usual sense. In particular, the rules of superposition of finite rotation vectors as discussed, for example, in [3] are different from the usual addition rules of the linear vector space W .

It is easy to show [9] that the rotated vector $\overset{*}{\mathbf{w}}$ is calculated by means of $\mathbf{\Omega}$ according to the following formulae

$$\overset{*}{\mathbf{v}} = \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{v} + \frac{1}{2\cos^2\omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{v}) = \qquad (2.3.6)$$

$$= \cos\omega\, \mathbf{v} + \boldsymbol{\Omega} \times \mathbf{v} + \frac{1}{2\cos^2\omega/2}\, (\boldsymbol{\Omega} \cdot \mathbf{v})$$

In particular, from (2.2.13) it follows that

$$\bar{\mathbf{a}}_\alpha = \overset{\vee}{\mathbf{a}}_\alpha + \boldsymbol{\Omega} \times \overset{\vee}{\mathbf{a}}_\alpha + \frac{1}{2\cos^2\omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \overset{\vee}{\mathbf{a}}_\alpha)$$

$$\bar{\mathbf{n}} = \mathbf{n} + \boldsymbol{\Omega} \times \mathbf{n} + \frac{1}{2\cos^2\omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{n}) \qquad (2.3.7)$$

$$\overset{*}{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \boldsymbol{\Omega} \times \mathbf{a}_\alpha + \frac{1}{2\cos^2\omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{a}_\alpha)$$

The vector $\boldsymbol{\Omega}$ is uniquely defined by the tensor $\mathbf{R}$. Recalling that in Cartesian frame

$$\mathbf{r} = x^k \mathbf{i}_k \quad , \quad \mathbf{a}_\alpha = x^k{}_{,\alpha} \mathbf{i}_k \quad , \quad \mathbf{n} = \frac{1}{2}\, \epsilon^{\alpha\beta} x^k{}_{,\alpha} x^l{}_{,\beta} e_{klm} \mathbf{i}^m \qquad (2.3.8)$$

the tensor $\mathbf{R}$ given by (2.3.1) may easily be expressed as $\mathbf{R} = R_{kl}\mathbf{i}^k \otimes \mathbf{i}^l$ where R_{kl} depend only upon $\mathbf{u}$ and the geometry of M. Then for $\boldsymbol{\Omega}$ we have

$$\boldsymbol{\Omega} = -\frac{1}{2}\, e^{klm} R_{kl} \mathbf{i}_m \quad , \quad \cos\omega = \frac{1}{2}\,(\mathrm{tr}\mathbf{R} - 1) = \frac{1}{2}(R_{11} + R_{22} + R_{33} - 1) \qquad (2.3.9)$$

which defines the vector $\boldsymbol{\Omega}$ by means of its components with respect to the Cartesian basis $\mathbf{i}_k$. Its components with respect to the reference surface basis may be found by using (2.3.8).

It is possible to express $\boldsymbol{\Omega}$ in terms of $\mathbf{u}$ in equivalent alternative forms, directly with respect to the surface basis. If we multiply (2.3.7) by $\overset{\vee}{\mathbf{a}}_\alpha$ or $\mathbf{n}$ and use (2.3.6) we obtain

$$\bar{\mathbf{a}}_\alpha \cdot \overset{\vee}{\mathbf{a}}_\beta = \cos\omega\, \bar{a}_{\alpha\beta} + \boldsymbol{\Omega} \cdot (\overset{\vee}{\mathbf{a}}_\alpha \times \overset{\vee}{\mathbf{a}}_\beta) + \frac{1}{2\cos^2\omega/2}\, (\boldsymbol{\Omega} \cdot \overset{\vee}{\mathbf{a}}_\alpha)(\boldsymbol{\Omega} \cdot \overset{\vee}{\mathbf{a}}_\beta)$$

$$\bar{\mathbf{n}} \cdot \overset{\vee}{\mathbf{a}}_\alpha = \boldsymbol{\Omega} \cdot (\mathbf{n} \times \overset{\vee}{\mathbf{a}}_\alpha) + \frac{1}{2\cos^2\omega/2}\, (\boldsymbol{\Omega} \cdot \mathbf{n})(\boldsymbol{\Omega} \cdot \overset{\vee}{\mathbf{a}}_\alpha)$$

$$\bar{\mathbf{a}}_\alpha \cdot \mathbf{n} = \boldsymbol{\Omega} \cdot (\overset{\vee}{\mathbf{a}}_\alpha \times \mathbf{n}) + \frac{1}{2\cos^2\omega/2}\, (\boldsymbol{\Omega} \cdot \overset{\vee}{\mathbf{a}}_\alpha)(\boldsymbol{\Omega} \cdot \mathbf{n}) \; .$$

which leads to

$$\frac{1}{2}\bar{\epsilon}^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\cdot\check{\mathbf{a}}_{\beta}=\boldsymbol{\Omega}\cdot\mathbf{n}\;,\qquad \frac{1}{2}\bar{\epsilon}^{\alpha\beta}(\bar{\mathbf{n}}\cdot\check{\mathbf{a}}_{\alpha}-\bar{\mathbf{a}}_{\alpha}\cdot\mathbf{n})=\boldsymbol{\Omega}\cdot\check{\mathbf{a}}^{\beta}$$

Now the general expression for $\boldsymbol{\Omega}$ takes the form

$$\boldsymbol{\Omega}=\frac{1}{2}\bar{\epsilon}^{\alpha\beta}[(\bar{\mathbf{n}}\cdot\check{\mathbf{a}}_{\alpha}-\bar{\mathbf{a}}_{\alpha}\cdot\mathbf{n})\check{\mathbf{a}}_{\beta}+(\bar{\mathbf{a}}_{\alpha}\cdot\check{\mathbf{a}}_{\beta})\mathbf{n}] \qquad (2.3.10a)$$

An equivalent alternative formula for $\boldsymbol{\Omega}$ has also been found in [7,9] to be

$$\boldsymbol{\Omega}=\frac{1}{2}(\check{\mathbf{a}}_{\alpha}\times\bar{\mathbf{a}}^{\alpha}+\mathbf{n}\times\bar{\mathbf{n}}) \qquad (2.3.10b)$$

When expressed in the common reference basis $\mathbf{a}_{\alpha}$, $\mathbf{n}$ both forms (2.3.10) lead, after some transformation, to the following formula

$$2\boldsymbol{\Omega}=\epsilon_{\lambda\mu}[n^{\lambda}-\bar{a}^{\alpha\beta}(\delta^{\lambda}_{\alpha}+\check{\gamma}^{\lambda}_{\alpha})\phi_{\beta}]\mathbf{a}^{\mu}+\epsilon_{\lambda\mu}\bar{a}^{\alpha\beta}(\delta^{\lambda}_{\alpha}+\check{\gamma}^{\lambda}_{\alpha})l^{\mu}_{\cdot\beta}\mathbf{n} \qquad (2.3.11)$$

Many geometric relations, which are expressed in terms of displacements, can also be presented in terms of $\check{\gamma}_{\alpha\beta}$ and $\boldsymbol{\Omega}$. For example, from (2.1.2), (2.1.3) and (2.3.7) it follows that

$$l_{\alpha\beta}=(\delta^{\lambda}_{\beta}+\check{\gamma}^{\lambda}_{\beta})[a_{\lambda\alpha}+\epsilon_{\lambda\alpha}(\boldsymbol{\Omega}\cdot\mathbf{n})-\frac{1}{2\cos^{2}\omega/2}(\boldsymbol{\Omega}\times\mathbf{a}_{\lambda})(\boldsymbol{\Omega}\times\mathbf{a}_{\alpha})]$$

$$\phi_{\beta}=(\delta^{\lambda}_{\beta}+\check{\gamma}^{\lambda}_{\beta})[\epsilon_{\alpha\lambda}(\boldsymbol{\Omega}\cdot\mathbf{a}^{\alpha})-\frac{1}{2\cos^{2}\omega/2}(\boldsymbol{\Omega}\times\mathbf{n})(\boldsymbol{\Omega}\times\mathbf{a}_{\lambda})]$$

$$n_{\alpha}=\epsilon_{\alpha\lambda}(\boldsymbol{\Omega}\cdot\mathbf{a}^{\lambda})-\frac{1}{2\cos^{2}\omega/2}(\boldsymbol{\Omega}\times\mathbf{a}_{\alpha})(\boldsymbol{\Omega}\times\mathbf{n}) \qquad (2.3.12)$$

$$n=1-\frac{1}{2\cos^{2}\omega/2}(\boldsymbol{\Omega}\times\mathbf{n})(\boldsymbol{\Omega}\times\mathbf{n})$$

$$\mathbf{u}_{,\beta}=\check{\gamma}^{\alpha}_{\beta}\mathbf{a}_{\alpha}+(\delta^{\lambda}_{\beta}+\check{\gamma}^{\lambda}_{\beta})[\boldsymbol{\Omega}\times\check{\mathbf{a}}_{\lambda}+\frac{1}{2\cos^{2}\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\check{\mathbf{a}}_{\lambda})]$$

Differentiation of $\boldsymbol{\Omega}$ and $\mathbf{R}$ along surface convected coordinate lines follows from the rules given in [2,20] for a three - dimensional continua. In particular, at the shell middle surface where $\zeta = 0$ we obtain [7] the following formula for differentiation of the finite rotation vector

$$\underset{\sim}{\Omega}_{,\beta} = \cos\omega\,\underset{\sim}{k}_\beta + \frac{1}{2}\,\underset{\sim}{\Omega} \times \underset{\sim}{k}_\beta - \frac{1}{4\cos^2\omega/2}\,\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \underset{\sim}{k}_\beta) \qquad (2.3.13)$$

Here $\underset{\sim}{k}_\beta$ may be called the vector of change of curvature along the coordinate line θ^α = const. , since multiplying (2.3.13) by $\underset{\sim}{\Omega}$ we obtain

$$\underset{\sim}{k}_\beta \cdot \underset{\sim}{\Omega} = \omega_{,\beta}\sin\omega \quad , \qquad \underset{\sim}{k}_\beta \cdot \underset{\sim}{e} = \omega_{,\beta} \qquad (2.3.14)$$

Differentiating $(2.3.14)_1$ with respect to θ^α and using (2.3.13) we arrive after some transformations at

$$\epsilon^{\alpha\beta}(\underset{\sim}{k}_{\beta|\alpha} + \frac{1}{2}\,\underset{\sim}{k}_\alpha \times \underset{\sim}{k}_\beta) = \underset{\sim}{0} \qquad (2.3.15)$$

These are integrability conditions of differential equations (2.3.13).

If we invert (2.3.13) the following formula for $\underset{\sim}{k}_\beta$ in terms of $\underset{\sim}{\Omega}$ may be derived

$$\underset{\sim}{k}_\beta = \underset{\sim}{\Omega}_{,\beta} + \frac{1}{2\cos^2\omega/2}\,\underset{\sim}{\Omega}_{,\beta} \times \underset{\sim}{\Omega} + \omega_{,\beta}\,\mathrm{tg}\,\omega/2\,\underset{\sim}{\Omega} \qquad (2.3.16)$$

It is possible to express $\underset{\sim}{k}_\beta$ in terms of the surface strain measures. Appropriate transformations given by the author [7,8] lead to

$$\underset{\sim}{k}_\beta = \bar{\epsilon}^{\lambda\mu}[(\kappa_{\beta\lambda} + b^{\kappa\nu}_{\beta}\gamma_{\kappa\lambda})\overset{\vee}{\underset{\sim}{a}}_\mu + (\gamma_{\beta\mu|\lambda} - \frac{1}{2}\,\overset{\vee}{\gamma}{}^{\kappa\nu}_{\mu}\gamma_{\kappa\lambda|\beta})\underset{\sim}{n}] \qquad (2.3.17)$$

When (2.3.17) and $(2.2.20)_1$ are introduced into (2.3.15) we obtain three compatibility conditions expressed in terms of $\overset{\vee}{\gamma}_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. They assure the existence of displacement field $\underset{\sim}{u}$ compatible with these strain measures. If (2.3.16) is introduced into (2.3.15), we obtain three compatibility conditions expressed entirely in terms of $\underset{\sim}{\Omega}$.

By solving (2.3.17) with respect to symmetric $\kappa_{\alpha\beta}$ we obtain

$$\kappa_{\alpha\beta} = \frac{1}{2}\,(\bar{\epsilon}_{\alpha\lambda}\underset{\sim}{k}_\beta + \bar{\epsilon}_{\beta\lambda}\underset{\sim}{k}_\alpha) \cdot \overset{\nu\lambda}{\underset{\sim}{a}} - \frac{1}{2}\,(b^{\lambda\nu}_{\alpha}\gamma_{\lambda\beta} + b^{\lambda\nu}_{\beta}\gamma_{\lambda\alpha}) \qquad (2.3.18)$$

This relation together with (2.3.16) gives also general formula for $\kappa_{\alpha\beta}$ in terms of $\underset{\sim}{\Omega}$ and $\overset{\vee}{\gamma}_{\alpha\beta}$.

The rotation of shell material fibres coinciding with principal directions of strain are described completely by $\underset{\sim}{R}$ or $\underset{\sim}{\Omega}$. Other shell fibres may suffer a rotation also during the pure stretch along principal directions of strain. Sometimes it is convenient to replace these two rotations by one equivalent total rotation. This approach is used below to describe the total rotation of the shell boundary element.

3. DEFORMATION OF SHELL BOUNDARY

Deformation of a shell near the lateral boundary is essentialy three - dimensional. Therefore, only by using three - dimensional analysis we are able to obtain the complete information as to the stress and strain state in this shell region. However, our goal is to derive adequate boundary conditions for interior shell equations within the first - approximation theory of thin isotropic elastic shells. Only for this limited goal it is justified to discuss deformation of the shell boundary under the Kirchhoff - Love constraints.

3.1. Total rotation of a boundary

Let C be a boundary curve at M defined by $\theta^\alpha = \theta^\alpha(s)$, where s is the length parameter along C. We assume here, that in the reference shell configuration the lateral boundary surface is rectilinear and orthogonal to M along C. The position of any $P \in \partial P$ is given by

$$\underset{\sim}{p} = \underset{\sim}{p}(s,\zeta) = \underset{\sim}{r}(s) + \zeta\underset{\sim}{n}(s) \qquad (3.1.1)$$

With each point M of C we associate vectors: $\underset{\sim}{t} = \frac{d\underset{\sim}{r}}{ds}$, the unit tangent to C, and $\underset{\sim}{\nu} = \underset{\sim}{t} \times \underset{\sim}{n}$, the outward unit normal.

After the shell deformation compatible with K - L constraints C transforms into $\bar{C}$ and ∂P into $\partial\bar{P}$, which still remains orthogonal to $\bar{M}$, and for any $\bar{P} \in \partial\bar{P}$ we have

$$\bar{\underset{\sim}{p}} = \bar{\underset{\sim}{p}}(s,\zeta) = \bar{\underset{\sim}{r}}(s) + \zeta\bar{\underset{\sim}{n}}(s) \tag{3.1.2}$$

During the shell deformation the orthonormal triad $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ transforms into an orthogonal triad $\bar{\underset{\sim}{a}}_\nu$, $\bar{\underset{\sim}{a}}_t$, $\bar{\underset{\sim}{n}}$, where

$$\begin{aligned} \bar{\underset{\sim}{a}}_t &= \frac{d\bar{\underset{\sim}{r}}}{ds} = t^\alpha\bar{\underset{\sim}{a}}_\alpha = \underset{\sim}{t} + \frac{d\underset{\sim}{u}}{ds} \\ \bar{\underset{\sim}{a}}_\nu &= \bar{\underset{\sim}{a}}_t \times \bar{\underset{\sim}{n}} = \bar{\epsilon}_{\lambda\alpha}t^\alpha\bar{\underset{\sim}{a}}^\lambda = \sqrt{\frac{\bar{a}}{a}}\,\nu_\alpha\bar{\underset{\sim}{a}}^\alpha = (\underset{\sim}{t} + \frac{d\underset{\sim}{u}}{ds}) \times (\underset{\sim}{n} + \underset{\sim}{\beta}) \end{aligned} \tag{3.1.3}$$

$$|\bar{\underset{\sim}{a}}_t| = |\bar{\underset{\sim}{a}}_\nu| = \bar{a}_t = \sqrt{1 + 2\gamma_{tt}} \quad , \quad \gamma_{tt} = \gamma_{\alpha\beta}t^\alpha t^\beta \tag{3.1.4}$$

According to the polar decomposition theorem (2.2.10) the boundary deformation can also be decomposed into a rigid - body translation, a pure strain performed by means of $\underset{\sim}{U}$, $\underset{\sim}{\gamma}$ or $\check{\underset{\sim}{\gamma}}$ and a rigid - body rotation performed by means of $\underset{\sim}{R}$ or $\underset{\sim}{\Omega}$. Therefore, there should be such intermediate vectors $\check{\underset{\sim}{a}}_t$, $\check{\underset{\sim}{a}}_\nu$ for which

$$\begin{aligned} \bar{\underset{\sim}{a}}_t &= \check{\underset{\sim}{a}}_t + \underset{\sim}{\Omega} \times \check{\underset{\sim}{a}}_t + \frac{1}{2\cos^2\omega/2}\,\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \check{\underset{\sim}{a}}_t) \\ \bar{\underset{\sim}{a}}_\nu &= \check{\underset{\sim}{a}}_\nu + \underset{\sim}{\Omega} \times \check{\underset{\sim}{a}}_\nu + \frac{1}{2\cos^2\omega/2}\,\underset{\sim}{\Omega} \times (\underset{\sim}{\Omega} \times \check{\underset{\sim}{a}}_\nu) \end{aligned} \tag{3.1.5}$$

In view of (2.2.20) these vectors are calculated according to

$$\check{\underset{\sim}{a}}_t = \check{\gamma}_{\nu t}\underset{\sim}{\nu} + (1 + \check{\gamma}_{tt})\underset{\sim}{t} \quad , \quad \check{\underset{\sim}{a}}_\nu = (1 + \check{\gamma}_{tt})\underset{\sim}{\nu} - \check{\gamma}_{\nu t}\underset{\sim}{t} \tag{3.1.6}$$

where

$$\check{\gamma}_{\nu t} = \check{\gamma}_{\alpha\beta}\nu^\alpha t^\beta \quad , \quad \check{\gamma}_{tt} = \check{\gamma}_{\alpha\beta}t^\alpha t^\beta \tag{3.1.7}$$

are physical components of $\check{\underset{\sim}{\gamma}}$ at C .

The directions defined by $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ do not coincide, in general, with principal directions of strain at $M \in C$ defined by $\underset{\sim}{f}_1$ and $\underset{\sim}{f}_2$ in (2.2.17). Since during the pure strain only the principal directions are stretched without rotation, the vectors $\underset{\sim}{\nu}$ and $\underset{\sim}{t}$ not only change their lengths but, in general, suffer rotation as well.

Let us denote by $\check{\boldsymbol{\Omega}}_t$ the finite rotation vector of the boundary caused by the pure strain. Applying $(2.3.10)_2$ we obtain

$$\check{\boldsymbol{\Omega}}_t = \sin\check{\omega}_t\,\boldsymbol{n} \quad , \quad \sin\check{\omega}_t = -\frac{\check{\gamma}_{\nu t}}{\bar{a}_t}$$

$$\cos\check{\omega}_t = \frac{1}{\bar{a}_t}(1 + \check{\gamma}_{tt}) \quad , \quad 2\cos^2\check{\omega}_t/2 = \frac{1}{\bar{a}_t}(\bar{a}_t + 1 + \check{\gamma}_{tt}) \tag{3.1.8}$$

and transformation of $\boldsymbol{\nu}$ and $\boldsymbol{t}$ into $\check{\boldsymbol{a}}_\nu$ and $\check{\boldsymbol{a}}_t$ may be expressed in the form

$$\check{\boldsymbol{a}}_\nu = \bar{a}_t[\boldsymbol{\nu} + \check{\boldsymbol{\Omega}}_t \times \boldsymbol{\nu} + \frac{1}{2\cos^2\check{\omega}_t/2}\check{\boldsymbol{\Omega}}_t \times (\check{\boldsymbol{\Omega}}_t \times \boldsymbol{\nu})]$$

$$\check{\boldsymbol{a}}_t = \bar{a}_t[\boldsymbol{t} + \check{\boldsymbol{\Omega}}_t \times \boldsymbol{t} + \frac{1}{2\cos^2\check{\omega}_t/2}\check{\boldsymbol{\Omega}}_t \times (\check{\boldsymbol{\Omega}}_t \times \boldsymbol{t})] \tag{3.1.9}$$

It is seen from (3.1.9) and (3.1.5) that transformation of $\boldsymbol{\nu}$ and $\boldsymbol{t}$ into $\bar{\boldsymbol{a}}_\nu$ and $\bar{\boldsymbol{a}}_t$ consists of extension by the factor $\bar{a}_t$ and two successive rotations performed by $\check{\boldsymbol{\Omega}}_t$ and $\boldsymbol{\Omega}$. In what follows it is convenient to replace these two rotations by a single equivalent rotation performed by an equivalent total rotation vector $\boldsymbol{\Omega}_t$. Adopting the superposition rule given, for example, in [3] to our definition (2.3.5) of the finite rotation vector we obtain

$$\boldsymbol{\Omega}_t = (1 - \frac{\check{\boldsymbol{\Omega}}_t \cdot \boldsymbol{\Omega}}{4\cos^2\check{\omega}_t/2\cos^2\omega/2})[\cos^2\omega/2\,\check{\boldsymbol{\Omega}}_t + \cos^2\check{\omega}_t/2\,\boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\Omega} \times \check{\boldsymbol{\Omega}}_t] \tag{3.1.10}$$

$$\boldsymbol{\Omega}_t = \sin\omega_t\,\boldsymbol{e}_t \tag{3.1.11}$$

Now the transformation of $\boldsymbol{\nu}$, $\boldsymbol{t}$, $\boldsymbol{n}$ into $\bar{\boldsymbol{a}}_\nu$, $\bar{\boldsymbol{a}}_t$, $\bar{\boldsymbol{n}}$ becomes

$$\bar{\boldsymbol{a}}_\nu = \bar{a}_t[\boldsymbol{\nu} + \boldsymbol{\Omega}_t \times \boldsymbol{\nu} + \frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \boldsymbol{\nu})]$$

$$\bar{\boldsymbol{a}}_t = \bar{a}_t[\boldsymbol{t} + \boldsymbol{\Omega}_t \times \boldsymbol{t} + \frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \boldsymbol{t})] \tag{3.1.12}$$

$$\bar{\boldsymbol{n}} = \boldsymbol{n} + \boldsymbol{\Omega}_t \times \boldsymbol{n} + \frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \boldsymbol{n})$$

Note the formal similarity of (3.1.12) and (3.1.9).

An alternative equivalent relation for $\underset{\sim}{\Omega}_t$ can be found directly, by using the cross product as in (2.3.10b) to obtain

$$2\underset{\sim}{\Omega}_t = \frac{1}{\bar{a}_t}\left(\underset{\sim}{\nu} \times \bar{\underset{\sim}{a}}_\nu + \underset{\sim}{t} \times \bar{\underset{\sim}{a}}_t\right) + \underset{\sim}{n} \times \bar{\underset{\sim}{n}} \tag{3.1.13}$$

which gives much simpler formula for $\underset{\sim}{\Omega}_t$ in terms of $\underset{\sim}{u}$.

3.2. Differentiation along the boundary

Differentiating $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ along C we obtain [9]

$$\begin{aligned} &\frac{d\underset{\sim}{\nu}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{\nu} \quad , \quad \frac{d\underset{\sim}{t}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{t} \quad , \quad \frac{d\underset{\sim}{n}}{ds} = \underset{\sim}{\omega}_t \times \underset{\sim}{n} \\ &\underset{\sim}{\omega}_t = \sigma_t \underset{\sim}{\nu} + \tau_t \underset{\sim}{t} + \kappa_t \underset{\sim}{n} \end{aligned} \tag{3.2.1}$$

where in terms of the surface geometry

$$\begin{aligned} &\sigma_t = t^\alpha t^\beta b_{\alpha\beta} \quad , \quad \tau_t = -\nu^\alpha t^\beta b_{\alpha\beta} \\ &\kappa_t = t_\alpha \nu^\alpha{}_{|\beta} t^\beta = -\nu_\alpha t^\alpha{}_{|\beta} t^\beta \end{aligned} \tag{3.2.2}$$

Here σ_t is the normal curvature, τ_t is the geodesic torsion and κ_t is the geodesic curvature of C , respectively.

Similar rules hold when differentiating $\underset{\sim}{\nu}$, $\underset{\sim}{t}$, $\underset{\sim}{n}$ in direction of the outward unit normal

$$\begin{aligned} &\frac{d\underset{\sim}{\nu}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{\nu} \quad , \quad \frac{d\underset{\sim}{t}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{t} \quad , \quad \frac{d\underset{\sim}{n}}{ds_\nu} = -\underset{\sim}{\omega}_\nu \times \underset{\sim}{n} \\ &\underset{\sim}{\omega}_\nu = \tau_\nu \underset{\sim}{\nu} + \sigma_\nu \underset{\sim}{t} + \kappa_\nu \underset{\sim}{n} \end{aligned} \tag{3.2.3}$$

where

$$\begin{aligned} &\sigma_\nu = \nu^\alpha \nu^\beta b_{\alpha\beta} \quad , \quad \tau_\nu = -t^\alpha \nu^\beta b_{\alpha\beta} = \tau_t \\ &\kappa_\nu = \nu_\alpha t^\alpha{}_{|\beta} \nu^\beta = -t_\alpha \nu^\alpha{}_{|\beta} \nu^\beta \end{aligned} \tag{3.2.4}$$

Let us define unit vectors at the deformed boundary $\bar{C}$

$$\bar{\nu} = \frac{1}{\bar{a}_t}\bar{a}_\nu \quad , \quad \bar{t} = \frac{1}{\bar{a}_t}\bar{a}_t \tag{3.2.5}$$

Differentiating the orthonormal triad $\bar{\nu}$, $\bar{t}$, $\bar{n}$ along C we obtain

$$\frac{d\bar{\nu}}{ds} = \bar{\omega}_t \times \bar{\nu} \quad , \quad \frac{d\bar{t}}{ds} = \bar{\omega}_t \times \bar{t} \quad , \quad \frac{d\bar{n}}{ds} = \bar{\omega}_t \times \bar{n}$$

$$\bar{\omega}_t = \bar{a}_t(\bar{\sigma}_t\bar{\nu} + \bar{\tau}_t\bar{t} + \bar{\kappa}_t\bar{n}) \tag{3.2.6}$$

where $\bar{\sigma}_t$, $\bar{\tau}_t$ and $\bar{\kappa}_t$ are the normal curvature, the geodesic torsion and the geodesic curvature of $\bar{C}$, respectively

Using (3.1.12) and (3.2.5) the following identity may be derived

$$\bar{n}\cdot\nu - \bar{\nu}\cdot n = 2\Omega_t\cdot t \tag{3.2.7}$$

which differentiated with respect to s gives

$$2\frac{d\Omega_t}{ds}\cdot t = \bar{\omega}_t\cdot(\bar{n}\times\nu - \bar{\nu}\times n) -$$

$$- \omega_t\cdot(n\times\bar{\nu} - \nu\times\bar{n}) - 2(\Omega_t\times\omega_t)\cdot t \tag{3.2.8}$$

Since the vector $\bar{\omega}_t$ is defined in deformed shell configuration we use the polar decomposition theorem (2.2.10) in order to express it in terms of a vector $\check{\omega}_t$ according to

$$\bar{\omega}_t = \check{\omega}_t + \Omega_t\times\check{\omega}_t + \frac{1}{2\cos^2\omega_t/2}\Omega_t\times(\Omega_t\times\check{\omega}_t)$$

$$\check{\omega}_t = \bar{a}_t(\bar{\sigma}_t\nu + \bar{\tau}_t t + \bar{\kappa}_t n) \tag{3.2.9}$$

Using (3.2.6), (3.2.1) and (3.2.9), together with some vector and trigonometric identities, from (3.2.8) we arrive at the following formula for differentiation of the total finite rotation vector

$$\frac{d\Omega_t}{ds} = \cos\omega_t k_t + \frac{1}{2}\Omega_t\times k_t - \frac{1}{4\cos^2\omega_t/2}\Omega_t\times(\Omega_t\times k_t) \tag{3.2.10}$$

where

$$\underset{\sim}{k}_t = \overset{\vee}{\underset{\sim}{\omega}}_t - \underset{\sim}{\omega}_t = - k_{tt}\underset{\sim}{\nu} + k_{\nu t}\underset{\sim}{t} - k_{nt}\underset{\sim}{n} \tag{3.2.11}$$

$$- k_{tt} = \bar{a}_t\bar{\sigma}_t - \sigma_t \quad , \quad k_{\nu t} = \bar{a}_t\bar{\tau}_t - \tau_t \quad , \quad - k_{nt} = \bar{a}_t\bar{\kappa}_t - \kappa_t \tag{3.2.12}$$

Here $\underset{\sim}{k}_t$ defined by (3.2.11) is called the vector of change of curvature of the shell boundary contour.

Note that all differentiation formulae depend only upon $\underset{\sim}{k}_t$ and γ_{tt}. If these parameters are known then $\overset{\vee}{\underset{\sim}{\omega}}_t$ follows from (3.2.11), $\bar{\underset{\sim}{\omega}}_t$ follows from (3.2.9) and derivatives of $\bar{\underset{\sim}{\nu}}$, $\bar{\underset{\sim}{t}}$, $\bar{\underset{\sim}{n}}$ become definite as well, according to (3.2.6).

In order to calculate $\underset{\sim}{k}_t$ let us differentiate $\bar{\underset{\sim}{a}}_t$ and $\bar{\underset{\sim}{n}}$ with respect to s to obtain

$$\begin{aligned} \frac{d}{ds}\bar{\underset{\sim}{a}}_t &= (t^{\alpha}{}_{|\beta} + \bar{a}^{\alpha\nu}\gamma_{\nu\lambda\beta}t^{\lambda})t^{\beta}\bar{\underset{\sim}{a}}_{\alpha} + (\sigma_t - \kappa_{tt})\bar{\underset{\sim}{n}} \\ \frac{d}{ds}\bar{\underset{\sim}{n}} &= -(b_{\alpha\beta} - \kappa_{\alpha\beta})t^{\beta}\bar{\underset{\sim}{a}}^{\alpha} \end{aligned} \tag{3.2.13}$$

From (3.2.5) we also have

$$\frac{d\bar{\underset{\sim}{t}}}{ds} = \frac{1}{\bar{a}_t}\frac{d\bar{\underset{\sim}{a}}_t}{ds} - \frac{1}{\bar{a}_t^2}\frac{d\gamma_{tt}}{ds}\bar{\underset{\sim}{t}} \tag{3.2.14}$$

It follows now from (3.2.6), (3.2.13) and (3.2.14) that

$$\begin{aligned} \bar{a}_t\bar{\sigma}_t &= \frac{1}{\bar{a}_t}(\sigma_t - \kappa_{tt}) \quad , \quad \bar{a}_t\bar{\tau}_t = -\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\,\nu_{\kappa}\bar{a}^{\kappa\alpha}(b_{\alpha\beta} - \kappa_{\alpha\beta})t^{\beta} \\ \bar{a}_t\bar{\kappa}_t &= +\frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}\,(\kappa_t - \nu_{\kappa}\bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}t^{\alpha}t^{\beta}) \end{aligned} \tag{3.2.15}$$

When all tensor components in (3.2.16) are expressed in terms of physical components [9] from (3.2.12) we obtain

$$\begin{aligned} k_{tt} &= \sigma_t\left(1 - \frac{1}{\bar{a}_t}\right) + \frac{\kappa_{tt}}{\bar{a}_t} \\ k_{\nu t} &= \sqrt{\frac{a}{\bar{a}}}\left[\bar{a}_t(\tau_t + \kappa_{\nu t}) + \frac{2\gamma_{\nu t}}{\bar{a}_t}(\sigma_t - \kappa_{tt})\right] - \tau_t \end{aligned} \tag{3.2.16}$$

$$k_{nt} = \kappa_t\left(1 - \frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}\right) - \frac{2\gamma_{\nu t}}{\bar{a}_t^2}\sqrt{\frac{a}{\bar{a}}}\left(\frac{d\gamma_{tt}}{ds} + 2\kappa_t\gamma_{\nu t}\right) + \tag{3.2.16}$$
$$+\sqrt{\frac{a}{\bar{a}}}\left[2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_\nu} + 2\kappa_\nu\gamma_{\nu t} + 2\kappa_t(\gamma_{\nu\nu} - \gamma_{tt})\right]$$

Using the results of [7] an equivalent expression for $k_{\nu t}$ is derived

$$k_{\nu t} = \tau_t\left(\frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}} - 1\right) + \frac{1}{\bar{a}_t}\sqrt{\frac{a}{\bar{a}}}\left[\bar{a}_t^2(\kappa_{\nu t} + 2\sigma_t\gamma_{\nu t} - 2\tau_t\gamma_{\nu\nu}) - \right. \tag{3.2.17}$$
$$\left. - 2\gamma_{\nu t}(\kappa_{tt} + 2\sigma_t\gamma_{tt} - 2\tau_t\gamma_{\nu t})\right]$$

Therefore, $\underset{\sim}{k}_t$ has been expressed entirely in terms of physical components of the surface strain measures at the boundary.

3.3. Geometric boundary conditions

According to (3.1.2), (2.2.1) and $(2.2.5)_2$ the deformed lateral boundary surface $\partial\bar{P}$ is uniquely defined by assuming two vector functions

$$\underset{\sim}{u}(s) = \underset{\sim}{A}(s) \quad , \quad \underset{\sim}{\beta}(s) = \underset{\sim}{B}(s) \quad \text{at } C \tag{3.3.1}$$

If we express $\underset{\sim}{\beta}$ by components according to

$$\underset{\sim}{\beta} = \beta_\nu\bar{\underset{\sim}{a}}_\nu + \beta_t\bar{\underset{\sim}{a}}_t + \beta\underset{\sim}{\bar{n}} \tag{3.3.2}$$

then under K - L constraints

$$\beta_\nu = -\frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}\,\nu_\alpha\bar{a}^{\alpha\beta}(\underset{\sim}{u}_{,\beta}\cdot\underset{\sim}{n})$$
$$\beta_t = -\frac{1}{\bar{a}_t^2}\frac{d\underset{\sim}{u}}{ds}\cdot\underset{\sim}{n} \quad , \quad \beta = 1 - \sqrt{1 - \bar{a}_t^2(\beta_\nu^2 + \beta_t^2)} \tag{3.3.3}$$

It is seen that only $\underset{\sim}{u}$ and β_ν appear as independent parameters in the non-linear K - L type theory of shells.

The conditions of the type

$$\mathbf{u}(s) = \mathbf{A}(s) \quad , \quad \beta_\nu(s) = b(s) \quad \text{at} \quad C \tag{3.3.4}$$

are called displacement boundary conditions of the K - L non-linear shell theory.

Let us differentiate (3.1.2) to obtain

$$\frac{\partial \bar{\mathbf{p}}}{\partial s} = \bar{\mathbf{a}}_t + \zeta \frac{d}{ds} \bar{\mathbf{n}} \quad , \quad \frac{\partial \bar{\mathbf{p}}}{\partial \zeta} = \bar{\mathbf{n}} \quad , \quad \frac{d\bar{\mathbf{r}}}{ds} = \bar{\mathbf{a}}_t \tag{3.3.5}$$

These differential equations define the same boundary surface $\partial \bar{P}$ implicitly, with accuracy up to a constant translation in space. In order to obtain $\partial \bar{P}$ explicitly the equation (3.3.5) should be solved.

The right-hand sides of (3.3.5) are given if values of $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$ are assumed. In (3.1.12) the vectors $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$ were expressed in terms of $\mathbf{\Omega}_t$ and γ_{tt} . Therefore, in order to establish differential equations (3.3.5) it is enough to assume only $\mathbf{\Omega}_t$ and γ_{tt} at C .

Conditions of the type

$$\mathbf{\Omega}_t(s) = \mathbf{m}(s) \quad , \quad \gamma_{tt}(s) = l(s) \quad \text{at} \quad C \tag{3.3.6}$$

are called kinematical boundary conditions of the K - L non-linear theory of shells.

For values of the assumed vectors we have

$$\begin{aligned} \mathbf{m} &= \frac{1}{2\sqrt{1+2l}} \{\boldsymbol{\nu} \times [\mathbf{t} \times \mathbf{B} + \frac{d\mathbf{A}}{ds} \times (\mathbf{n} + \mathbf{B})] + \mathbf{t} \times \frac{d\mathbf{A}}{ds}\} + \mathbf{n} \times \mathbf{B} \\ l &= \mathbf{t} \cdot \frac{d\mathbf{A}}{ds} + \frac{1}{2} \frac{d\mathbf{A}}{ds} \cdot \frac{d\mathbf{A}}{ds} \end{aligned} \tag{3.3.7}$$

Let us differentiate again (3.3.5) to obtain

$$\frac{\partial^2 \bar{\mathbf{p}}}{\partial s^2} = \frac{d}{ds} \bar{\mathbf{a}}_t + \zeta \frac{d^2}{ds^2} \bar{\mathbf{n}} \quad , \quad \frac{\partial^2 \bar{\mathbf{p}}}{\partial s \partial \zeta} = \frac{d}{ds} \bar{\mathbf{n}} \quad , \quad \frac{d^2 \bar{\mathbf{r}}}{ds^2} = \frac{d}{ds} \bar{\mathbf{a}}_t \tag{3.3.8}$$

These differential equations also define the same boundary surface $\partial \bar{P}$ implicitly, with accuracy up to a translation linearly varying with s (that means, up to a rigid - body motion in space). The right-hand sides of (3.3.8) are established if values of $\frac{d}{ds}\bar{\mathbf{a}}_t$ and $\frac{d}{ds}\bar{\mathbf{n}}$ are assumed. However, according to (3.2.6), (3.2.9) and (3.2.11) these vectors may be expressed in terms of $\mathbf{k}_t$ and γ_{tt} , which are sufficient for establishing the equations (3.3.8).

Conditions of the type

$$\mathbf{k}_t(s) = \mathbf{q}(s) \quad , \quad \gamma_{tt}(s) = l(s) \quad \text{at } C \qquad (3.3.9)$$

are called deformational boundary conditions of the K - L non-linear theory of shells. For $\mathbf{q}$ here we have the relation

$$\mathbf{q} = \frac{d\mathbf{m}}{ds} - \frac{1}{1 + \sqrt{1 - \mathbf{m}\cdot\mathbf{m}}}\left[\mathbf{m}\frac{d}{ds}\sqrt{1 - \mathbf{m}\cdot\mathbf{m}} + \mathbf{m}\times\frac{d\mathbf{m}}{ds}\right] \qquad (3.3.10)$$

When values of $\mathbf{u}$ and β_ν are known along C , then $\mathbf{\Omega}_t$, γ_{tt} and $\mathbf{k}_t$ are easily calculated by using (3.3.7) and (3.3.10). If only $\mathbf{k}_t$ is known in advance, in order to obtain $\mathbf{\Omega}_t$ the differential equation (3.2.10) should be solved. Note that the structure of (3.2.10) is analogous to the one describing the motion of a rigid body about a fixed point [3] and the methods of solution developed in analytic mechanics [21] may be of assistance in calculating $\mathbf{\Omega}_t$ from the known $\mathbf{k}_t$.

If values of $\mathbf{\Omega}_t$ and γ_{tt} are known along C , then β_ν follows from (2.3.7) and (3.3.2). In order to obtain $\mathbf{u}$ at C the following differential equation should be solved

$$\frac{d\mathbf{u}}{ds} = \check{\gamma}_{tt}\mathbf{t} + \bar{a}_t\left[\mathbf{\Omega}_t \times \mathbf{t} + \frac{1}{2\cos^2\omega_t/2}\mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \mathbf{t})\right] \qquad (3.3.11)$$

Most shell problems are solved in terms of displacements and then displacement boundary conditions are used. The kinematical boundary conditions are adequate for the shell problems formulated by means of the finite rotation vector [4]. Particularly interesting seem to be deformational boundary conditions, since they are expressed entirely in terms of the shell strain measures. This allows us to formulate the shell problems directly in terms of strain (or stress) measures.

4. BASIC SHELL EQUATIONS

The two-dimensional equilibrium equations and natural boundary conditions, in terms of symmetric internal force and moment resultants, may be obtained most easily by using the virtual work principle [11]. Here one should clearly distinguish between the Eulerian and Lagrangean descriptions [7,9]. When expressed in vector form the Eulerian and Lagrangean equilibrium equations and natural boundary conditions are related through the deformation gradient tensor $\underset{\sim}{G}$. By decomposing $\underset{\sim}{G}$ according to (2.2.10) we may construct various component forms of shell equations with respect to bases $\underset{\sim}{a}_\alpha$, $\underset{\sim}{n}$ or $\check{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ or $\overset{*}{\underset{\sim}{a}}_\alpha$, $\bar{\underset{\sim}{n}}$ or $\bar{\underset{\sim}{a}}_\alpha$, $\bar{\underset{\sim}{n}}$, respectively. Some of these equations and their consistent simplification under small elastic strains and under additionally restricted rotations will be discussed below.

4.1. Equilibrium equations and static boundary conditions

Let a shell with simply connected middle surface be in equilibrium, under the surface force $\bar{\underset{\sim}{p}} = \bar{p}^\alpha \bar{\underset{\sim}{a}}_\alpha + \bar{p}\bar{\underset{\sim}{n}}$, per unit area of deformed surface $\bar{M}$, and under the boundary force $\bar{\underset{\sim}{F}} = \bar{F}^\alpha \bar{\underset{\sim}{a}}_\alpha + \bar{F}\bar{\underset{\sim}{n}} = \bar{F}_\nu \bar{\underset{\sim}{\nu}} + \bar{F}_t \bar{\underset{\sim}{t}} + \bar{F}\bar{\underset{\sim}{n}}$ and the boundary couple $\bar{\underset{\sim}{K}} = \bar{\epsilon}_{\lambda\mu} \bar{K}^\lambda \bar{\underset{\sim}{a}}^\mu = - \bar{K}_t \bar{\underset{\sim}{\nu}} + \bar{K}_\nu \bar{\underset{\sim}{t}}$, per unit length of deformed boundary $\bar{C}$. For completeness, we could also introduce here the surface moments $\bar{\underset{\sim}{m}}$ and the normal component of the boundary couple $\bar{K} = \bar{\underset{\sim}{K}} \cdot \bar{\underset{\sim}{n}}$. Since for thin shells, within the first approximation theory, these loadings are of secondary importance, we assume here at once $\bar{\underset{\sim}{m}} \equiv \underset{\sim}{0}$ and $\bar{K} \equiv 0$.

For any additional virtual displacement field $\delta\underset{\sim}{\bar{u}} = \delta\bar{u}_\alpha \bar{\underset{\sim}{a}}^\alpha + \delta\bar{w}\bar{\underset{\sim}{n}}$, subject to geometrical constraints, there should be symmetric Eulerian stress and couple resultant tensors

$$\bar{\underset{\sim}{N}} = \bar{N}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} \otimes \bar{\underset{\sim}{a}}_{\beta} \quad , \quad \bar{\underset{\sim}{M}} = \bar{M}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} \otimes \bar{\underset{\sim}{a}}_{\beta} \tag{4.1.1}$$

such that the Eulerian virtual work principle IVW = EVW takes the form

$$\iint_{\bar{M}}(\bar{N}^{\alpha\beta}\delta\bar{\gamma}_{\alpha\beta} + \bar{M}^{\alpha\beta}\delta\bar{\kappa}_{\alpha\beta})d\bar{A} = \iint_{\bar{M}}\bar{\underset{\sim}{p}}\cdot\delta\bar{\underset{\sim}{u}}d\bar{A} + \int_{\bar{C}}(\bar{\underset{\sim}{F}}\cdot\delta\bar{\underset{\sim}{u}} + \bar{\underset{\sim}{K}}\cdot\delta\bar{\underset{\sim}{\Omega}}_{t})d\bar{s} \tag{4.1.2}$$

where in deformed surface geometry

$$\begin{aligned}
\delta\bar{\gamma}_{\alpha\beta} &= \frac{1}{2}(\delta\bar{u}_{\alpha||\beta} + \delta\bar{u}_{\beta||\alpha}) - \bar{b}_{\alpha\beta}\delta\bar{w} \\
\delta\bar{\kappa}_{\alpha\beta} &= -\delta\bar{w}_{||\alpha\beta} - \bar{b}^{\lambda}_{\alpha}\delta\bar{u}_{\lambda||\beta} - \bar{b}^{\lambda}_{\beta}\delta\bar{u}_{\lambda||\alpha} - \bar{b}^{\lambda}_{\alpha||\beta}\delta\bar{u}_{\lambda} + \bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta}\delta\bar{w} \\
\delta\bar{\underset{\sim}{\Omega}}_{t} &= \bar{\epsilon}^{\beta\alpha}(\delta\bar{\phi}_{\alpha}\bar{\underset{\sim}{a}}_{\beta} + \frac{1}{2}\delta\bar{\omega}_{\beta\alpha}\bar{\underset{\sim}{n}}) - \delta\bar{\theta}_{\nu t}\bar{\underset{\sim}{n}}
\end{aligned} \tag{4.1.3}$$

and $\bar{s}$ is the length parameter along $\bar{C}$.

By applying the Stockes' theorem the virtual work principle can be transformed [9] into

$$\begin{aligned}
-\iint_{\bar{M}}\bar{\underset{\sim}{N}}^{\beta}{}_{||\beta}\cdot\delta\bar{\underset{\sim}{u}}d\bar{A} + J_i &= \iint_{\bar{M}}\bar{\underset{\sim}{p}}\cdot\delta\bar{\underset{\sim}{u}}d\bar{A} + J_e \\
J_i &= \int_{\bar{C}}(\bar{\underset{\sim}{P}}_{\nu}\cdot\delta\bar{\underset{\sim}{u}} + \bar{M}_{\nu\nu}\delta\bar{\beta}_{\nu})d\bar{s} + \sum_n \Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}}\cdot\delta\bar{\underset{\sim}{u}} \\
J_e &= \int_{\bar{C}}(\bar{\underset{\sim}{R}}\cdot\delta\bar{\underset{\sim}{u}} + \bar{K}_{\nu}\delta\bar{\beta}_{\nu})d\bar{s} + \sum_n \Delta\bar{K}_{t}\bar{\underset{\sim}{n}}\cdot\delta\bar{\underset{\sim}{u}}
\end{aligned} \tag{4.1.4}$$

where

$$\begin{aligned}
\bar{\underset{\sim}{N}}^{\beta} &= \bar{Q}^{\alpha\beta}\bar{\underset{\sim}{a}}_{\alpha} + \bar{Q}^{\beta}\bar{\underset{\sim}{n}} \quad , \quad \delta\bar{\beta}_{\nu} = (\delta\bar{\underset{\sim}{\Omega}}_{t} \times \bar{\underset{\sim}{n}})\cdot\bar{\underset{\sim}{\nu}} = \delta\bar{\underset{\sim}{\Omega}}_{t}\cdot\bar{\underset{\sim}{t}} \\
\bar{Q}^{\alpha\beta} &= \bar{N}^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta} \quad , \quad \bar{Q}^{\beta} = \bar{M}^{\alpha\beta}{}_{||\alpha} \\
\bar{\underset{\sim}{P}}_{\nu} &= \bar{\underset{\sim}{N}}^{\beta}\bar{\nu}_{\beta} + \frac{d}{d\bar{s}}(\bar{M}_{t\nu}\bar{\underset{\sim}{n}}) \quad , \quad \bar{\underset{\sim}{R}} = \bar{\underset{\sim}{F}} + \frac{d}{d\bar{s}}(\bar{K}_{t}\bar{\underset{\sim}{n}})
\end{aligned} \tag{4.1.5}$$

$$\begin{aligned}
\sum_n \Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}}\cdot\delta\bar{\underset{\sim}{u}} &= \sum_{\bar{M}_n}[\bar{M}_{t\nu}(\bar{s}_n + 0) - \bar{M}_{t\nu}(\bar{s}_n - 0)]\bar{\underset{\sim}{n}}(\bar{s}_n)\cdot\delta\bar{\underset{\sim}{u}}(\bar{s}_n) \\
\sum_n \Delta\bar{K}_{t}\bar{\underset{\sim}{n}}\cdot\delta\bar{\underset{\sim}{u}} &= \sum_{\bar{M}_n}[\bar{K}_{t}(\bar{s}_n + 0) - \bar{K}_{t}(\bar{s}_n - 0)]\bar{\underset{\sim}{n}}(\bar{s}_n)\cdot\delta\bar{\underset{\sim}{u}}(\bar{s}_n)
\end{aligned} \tag{4.1.6}$$

and $\bar{M}_n$, $n = 1,2,\ldots,N$ are corners of $\bar{C}$ labelled by $\bar{s} = \bar{s}_n$.

The local form of (4.1.4) gives the Eulerian equilibrium equations and the Eulerian static boundary conditions

$$\bar{\underset{\sim}{N}}^{\beta}\|_{\beta} + \bar{\underset{\sim}{p}} = \underset{\sim}{0} \quad \text{in} \quad \bar{M}$$

$$\bar{\underset{\sim}{P}}_{\nu} = \bar{\underset{\sim}{R}} \quad , \quad \bar{M}_{\nu\nu} = \bar{K}_{\nu} \quad \text{on} \quad \bar{C} \tag{4.1.7}$$

$$\Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}} = \Delta\bar{K}_{t}\bar{\underset{\sim}{n}} \quad \text{at each} \quad \bar{M}_n$$

Usually the undeformed reference configuration is the only known in advance. It is desirable then to express all shell equations in terms of quantities defined in and/or refered to the known geometry of the reference middle surface M .

Transformation rules between deformed and reference geometry of the shell middle surface are [9,12]

$$d\bar{s} = \bar{a}_t ds \quad , \quad d\bar{A} = \sqrt{\frac{\bar{a}}{a}}\, dA$$

$$\bar{\nu}_{\beta} d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, \nu_{\beta} ds \quad , \quad \bar{\nu}^{\beta} d\bar{s} = \sqrt{\frac{a}{\bar{a}}}\, (\delta^{\beta}_{\alpha} + 2\epsilon_{\alpha\lambda}\epsilon^{\beta\mu}\gamma^{\lambda}_{\mu})\nu^{\alpha} ds \tag{4.1.8}$$

$$\bar{t}_{\beta} d\bar{s} = (\delta^{\alpha}_{\beta} + 2\gamma^{\alpha}_{\beta})t_{\alpha} ds \quad , \quad \bar{t}^{\beta} d\bar{s} = t^{\beta} ds$$

Remind also transformation rules for covariant differentiation of a vector v^{β} and a symmetric second-order tensor $T^{\alpha\beta} = T^{\beta\alpha}$ to be [9]

$$\left(\sqrt{\frac{a}{\bar{a}}}\, v^{\beta}\right)\Big\|_{\beta} = \sqrt{\frac{a}{\bar{a}}}\, v^{\beta}\big|_{\beta}$$

$$\left(\sqrt{\frac{a}{\bar{a}}}\, T^{\alpha\beta}\right)\Big\|_{\beta} = \sqrt{\frac{a}{\bar{a}}}\, [T^{\alpha\beta}\big|_{\beta} + \bar{a}^{\alpha\kappa}(2\gamma_{\kappa\lambda|\mu} - \gamma_{\lambda\mu|\kappa})T^{\lambda\mu}] \tag{4.1.9}$$

Let us introduce symmetric Lagrangean stress and couple resultant tensors $\underset{\sim}{N} = N^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta}$ and $\underset{\sim}{M} = M^{\alpha\beta}\underset{\sim}{a}_{\alpha} \otimes \underset{\sim}{a}_{\beta}$ by the relations

$$\bar{\underset{\sim}{N}} = \sqrt{\frac{a}{\bar{a}}}\, \underset{\sim}{G}\underset{\sim}{N}\underset{\sim}{G}^{T} \quad , \quad \bar{\underset{\sim}{M}} = \sqrt{\frac{a}{\bar{a}}}\, \underset{\sim}{G}\underset{\sim}{M}\underset{\sim}{G}^{T}$$

$$\bar{N}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\, N^{\alpha\beta} \quad , \quad \bar{M}^{\alpha\beta} = \sqrt{\frac{a}{\bar{a}}}\, M^{\alpha\beta} \tag{4.1.10}$$

The structure of transformations (4.1.10) is analogous to the one relating the Cauchy stress tensor to the second Piola - Kirchhoff stress tensor in the non-linear continuum mechanics [15,18]. Therefore, the Lagrangean quantities $\underset{\sim}{N}$ and $\underset{\sim}{M}$ may also be called the second Piola - Kirchhoff stress and couple resultant tensors, respectively.

Introducing (2.2.13) and (4.1.8) to (4.1.10) into (4.1.4) we obtain

$$
\begin{aligned}
& -\iint\limits_{M}(\underset{\sim}{G}\underset{\sim}{N}^{\beta})_{|\beta}\cdot\delta\underset{\sim}{u}\,dA + J_i = \iint\limits_{M}\underset{\sim}{p}\cdot\delta\underset{\sim}{u}\,dA + J_e \\
& J_i = \int\limits_{C}(\underset{\sim}{P}_{\nu}\cdot\delta\underset{\sim}{u} + \bar{M}_{\nu\nu}\delta\underset{\sim}{\Omega}_t\cdot\underset{\sim}{\bar{a}}_t)ds + \sum_n \Delta\bar{M}_{t\nu}\underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{u} \qquad (4.1.11) \\
& J_e = \int\limits_{C}(\underset{\sim}{R}\cdot\delta\underset{\sim}{u} + \bar{K}_{\nu}\delta\underset{\sim}{\Omega}_t\cdot\underset{\sim}{\bar{a}}_t)ds + \sum_n \Delta\bar{K}_t\underset{\sim}{\bar{n}}\cdot\delta\underset{\sim}{u}
\end{aligned}
$$

Here the surface force $\underset{\sim}{p}$ is per unit area of M and the boundary force $\underset{\sim}{F}$ and boundary couple $\underset{\sim}{K}$ are per unit length of C. They are supposed to be given through their components with respect to the reference surface geometry

$$
\begin{aligned}
& \underset{\sim}{p} = \sqrt{\frac{\bar{a}}{a}}\,\underset{\sim}{\bar{p}} = p^{\alpha}\underset{\sim}{a}_{\alpha} + p\underset{\sim}{n} \\
& \underset{\sim}{F} = \bar{a}_t\underset{\sim}{\bar{F}} = F^{\alpha}\underset{\sim}{a}_{\alpha} + F\underset{\sim}{n} = F_{\nu}\underset{\sim}{\nu} + F_t\underset{\sim}{t} + F\underset{\sim}{n} \qquad (4.1.12) \\
& \underset{\sim}{K} = \bar{a}_t\underset{\sim}{\bar{K}} = \epsilon_{\alpha\beta}K^{\alpha}\underset{\sim}{a}^{\beta} + K\underset{\sim}{n} = -K_t\underset{\sim}{\nu} + K_{\nu}\underset{\sim}{t} + K\underset{\sim}{n}
\end{aligned}
$$

Since we have assumed $\bar{K} \equiv 0$ then K^{α} and K cannot be independent. After transformation we obtain

$$
\begin{aligned}
& K = -\frac{1}{n}\epsilon_{\lambda\mu}K^{\lambda}n^{\mu} \\
& \hat{K}^{\alpha} = \bar{\epsilon}^{\alpha\beta}\underset{\sim}{K}\cdot\underset{\sim}{\bar{a}}_{\beta} = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}K^{\lambda}(1^{\mu}_{\cdot\beta} - \frac{1}{n}n^{\mu}\phi_{\beta}) \qquad (4.1.13)
\end{aligned}
$$

For the remaining Lagrangean quantities appearing in (4.1.11) we obtain the following relations

$$\underset{\sim}{N}^{\beta} = Q^{\alpha\beta}\underset{\sim}{a}_{\alpha} + Q^{\beta}\underset{\sim}{n} = \sqrt{\frac{\bar{a}}{a}}\,\underset{\sim}{G}^{-1}\bar{\underset{\sim}{N}}^{\beta}$$

$$Q^{\alpha\beta} = N^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda}M^{\lambda\beta} \quad , \quad Q^{\beta} = M^{\alpha\beta}{}_{|\alpha} + \bar{a}^{\beta\kappa}(2\gamma_{\kappa\lambda|\mu} - \gamma_{\lambda\mu|\kappa})M^{\lambda\mu} \qquad (4.1.14)$$

$$\underset{\sim}{P}_{\nu} = \underset{\sim\sim}{GN}^{\beta}\nu_{\beta} + \frac{d}{ds}(\bar{M}_{t\nu}\bar{\underset{\sim}{n}}) \quad , \quad \underset{\sim}{R} = \underset{\sim}{F} + \frac{d}{ds}(\bar{K}_{t}\bar{\underset{\sim}{n}})$$

$$\bar{M}_{t\nu} = \frac{1}{\bar{a}_t^2}M^{\alpha\beta}(\delta^{\lambda}_{\alpha} + 2\gamma^{\lambda}_{\alpha})t_{\lambda}\nu_{\beta} \quad , \quad \bar{K}_{t} = \frac{1}{\bar{a}_t^2}\hat{K}^{\alpha}(\delta^{\lambda}_{\alpha} + 2\gamma^{\lambda}_{\alpha})t_{\lambda}$$

$$\bar{M}_{\nu\nu} = \frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}M^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \quad , \quad \bar{K}_{\nu} = \frac{1}{\bar{a}_t^2}\sqrt{\frac{\bar{a}}{a}}\hat{K}^{\alpha}\nu_{\alpha} \qquad (4.1.15)$$

The relations (4.1.11) are the transformed form of the following Lagrangean virtual work principle

$$\iint_M (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})dA = \iint_M \underset{\sim}{p}\cdot\delta\underset{\sim}{u}dA + \int_C (\underset{\sim}{F}\cdot\delta\underset{\sim}{u} + \underset{\sim}{K}\cdot\delta\underset{\sim}{\Omega}_t)ds \qquad (4.1.16)$$

where now

$$\delta\underset{\sim}{u} = \delta u_{\alpha}\underset{\sim}{a}^{\alpha} + \delta w\underset{\sim}{n} \quad , \quad \delta\gamma_{\alpha\beta} = \frac{1}{2}(l^{\lambda}_{\cdot\alpha}\delta l_{\lambda\beta} + l^{\lambda}_{\cdot\beta}\delta l_{\lambda\alpha} + \phi_{\alpha}\delta\phi_{\beta} + \phi_{\beta}\delta\phi_{\alpha})$$

$$\delta\kappa_{\alpha\beta} = -n(\delta\phi_{\alpha|\beta} + b^{\lambda}_{\beta}\delta l_{\lambda\alpha}) - (\phi_{\alpha|\beta} + b^{\lambda}_{\beta}l_{\lambda\alpha})\delta n - \qquad (4.1.17)$$

$$- n_{\lambda}(\delta l^{\lambda}_{\cdot\alpha|\beta} - b^{\lambda}_{\beta}\delta\phi_{\alpha}) - (l^{\lambda}_{\cdot\alpha|\beta} - b^{\lambda}_{\beta}\phi_{\alpha})\delta n$$

The local form of (4.1.11) or (4.1.16) gives the Lagrangean equilibrium equations and the Lagrangean static boundary conditions

$$(\underset{\sim\sim}{GN}^{\beta})_{|\beta} + \underset{\sim}{p} = \underset{\sim}{0} \quad \text{in } M$$

$$\underset{\sim}{P}_{\nu} = \underset{\sim}{R} \quad , \quad M^{\alpha\beta}\nu_{\alpha}\nu_{\beta} = \hat{K}^{\alpha}\nu_{\alpha} \quad \text{on } C \qquad (4.1.18)$$

$$\Delta\bar{M}_{t\nu}\underset{\sim\sim}{Gn} = \Delta\bar{K}_{t}\underset{\sim\sim}{Gn} \quad \text{at each } M_n$$

By expressing vector equilibrium equations $(4.1.18)_1$ in different shell bases various component forms of it in terms of $N^{\alpha\beta}$ and $M^{\alpha\beta}$ may be derived [7,9]. Let us present some of them here.

When $(4.1.18)_1$ is expressed by components in the base $\mathbf{a}_\alpha$, $\mathbf{n}$ with the help of (2.2.13) and (2.2.1) the following component form of equilibrium equations is obtained [12,25]

$$
\begin{aligned}
&(1^\alpha_{\cdot\lambda} Q^{\lambda\beta} + n^\alpha Q^\beta)_{|\beta} - b^\alpha_\beta(\phi_\lambda Q^{\lambda\beta} + nQ^\beta) + p^\alpha = 0 \\
&(\phi_\lambda Q^{\lambda\beta} + nQ^\beta)_{|\beta} + b_{\alpha\beta}(1^\alpha_{\cdot\lambda} Q^{\lambda\beta} + n^\alpha Q^\beta) + p = 0
\end{aligned}
\qquad (4.1.19)
$$

If we express $(4.1.7)_1$ in the base $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ and transform all components according to (4.1.10) and (4.1.9) then

$$
\begin{aligned}
&Q^{\alpha\beta}{}_{|\beta} + \bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta} Q^{\lambda\beta} - \bar{b}^\lambda_\beta Q^\beta + \sqrt{\frac{\bar{a}}{a}}\,\bar{p}^\alpha = 0 \\
&Q^\beta{}_{|\beta} + \bar{b}_{\alpha\beta} Q^{\alpha\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p} = 0
\end{aligned}
\qquad (4.1.20)
$$

When also the base vectors $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ are transformed according to (2.2.1) from $(4.1.7)_1$ and (4.1.20) we obtain the component form of equilibrium equations equivalent to (4.1.19) to be

$$
\begin{aligned}
&1^\alpha_{\cdot\lambda}(Q^{\lambda\beta}{}_{|\beta} + \bar{a}^{\lambda\kappa}\gamma_{\kappa\mu\beta} Q^{\mu\beta} - \bar{b}^\lambda_\beta Q^\beta) + n^\alpha(Q^\beta{}_{|\beta} + \bar{b}_{\lambda\beta} Q^{\lambda\beta}) + p^\alpha = 0 \\
&\phi_\lambda(Q^{\lambda\beta}{}_{|\beta} + \bar{a}^{\lambda\kappa}\gamma_{\kappa\mu\beta} Q^{\mu\beta} - \bar{b}^\lambda_\beta Q^\beta) + n(Q^\beta{}_{|\beta} + \bar{b}_{\lambda\beta} Q^{\lambda\beta}) + p = 0
\end{aligned}
\qquad (4.1.21)
$$

The transformation formula $(4.1.14)_1$ can also be presented in a more extended form

$$
\begin{aligned}
&\bar{\mathbf{N}}^\beta = \sqrt{\frac{a}{\bar{a}}}\,\mathbf{R}\overset{\vee}{\mathbf{N}}{}^\beta = \sqrt{\frac{a}{\bar{a}}}\,\mathbf{V}\overset{*}{\mathbf{N}}{}^\beta \\
&\overset{\vee}{\mathbf{N}}{}^\beta = Q^{\alpha\beta}\overset{\vee}{\mathbf{a}}_\alpha + Q^\beta\mathbf{n} \quad , \quad \overset{*}{\mathbf{N}}{}^\beta = Q^{\alpha\beta}\overset{*}{\mathbf{a}}_\alpha + Q^\beta\bar{\mathbf{n}}
\end{aligned}
\qquad (4.1.22)
$$

Therefore the equilibrium equations (4.1.7) and (4.1.18) can be presented as follows

$$
(\mathbf{V}\overset{*}{\mathbf{N}}{}^\beta)_{|\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{\mathbf{p}} = \mathbf{0} \quad , \quad (\mathbf{R}\overset{\vee}{\mathbf{N}}{}^\beta)_{|\beta} + \mathbf{p} = \mathbf{0} \qquad (4.1.23)
$$

When written in components with respect to $\overset{*}{\underset{\sim}{a}}_\alpha$, $\bar{\underset{\sim}{n}}$ or $\overset{\vee}{\underset{\sim}{a}}_\alpha$, $\underset{\sim}{n}$ several addition component forms of shell equilibrium equations can be obtained, (see [4,7,9])

4.2. Modified static boundary conditions

It follows from the structure of J_i in $(4.1.11)_2$ that in Lagrangean description the effective internal force $\underset{\sim}{P}_\nu$ and the moment $\bar{a}_t\bar{M}_{\nu\nu}$ are static quantities at C which produce virtual work on variations of displacement parameters $\underset{\sim}{u}$ and β_ν . Assuming at C values for $\underset{\sim}{P}_\nu$ and for $\bar{a}_t\bar{M}_{\nu\nu}$ we obtain the basic variant $(4.1.18)_2$ of static boundary conditions energetically compatible with displacement boundary conditions (3.3.4).

Let $\underset{\sim}{F}_\nu$ and $\underset{\sim}{B}_\nu(O)$ be a total force and a total couple, with respect to an origin O in space, of all internal stress and couple resultants acting at a part of the boundary $\bar{C}$. In the Lagrangean description these vectors are defined by

$$\underset{\sim}{F}_\nu = \underset{\sim}{F}^o_\nu + \int_{M_o}^{M} \underset{\sim}{P}_\nu ds \quad , \quad \underset{\sim}{B}_\nu(O) = \underset{\sim}{B}^o_\nu(O) + \int_{M_o}^{M} (\bar{M}_{\nu\nu}\bar{\underset{\sim}{a}}_t + \bar{\underset{\sim}{r}} \times \underset{\sim}{P}_\nu)ds \qquad (4.2.1)$$

where $\underset{\sim}{F}^o_\nu$ and $\underset{\sim}{B}^o_\nu(O)$ are initial values of $\underset{\sim}{F}_\nu$ and $\underset{\sim}{B}_\nu(O)$ at $M = M_o$.

The total couple $\underset{\sim}{B}_\nu(\bar{M}) \equiv \underset{\sim}{B}_\nu$, with respect to a current point $\bar{M}$ of $\bar{C}$, is calculated according to

$$\underset{\sim}{B}_\nu = \underset{\sim}{B}_\nu(O) - \bar{\underset{\sim}{r}} \times \underset{\sim}{F}_\nu \qquad (4.2.2)$$

Differentiating $(4.2.1)_1$ and (4.2.2) we obtain

$$\frac{d\underset{\sim}{F}_\nu}{ds} = \underset{\sim}{P}_\nu \quad , \quad \frac{d\underset{\sim}{B}_\nu}{ds} = \bar{M}_{\nu\nu}\bar{\underset{\sim}{a}}_t - \bar{\underset{\sim}{a}}_t \times \underset{\sim}{F}_\nu \qquad (4.2.3)$$

Let us differentiate $\delta\underset{\sim}{u}$ and $\delta\underset{\sim}{\Omega}_t$ with respect to $\bar{s}$ and take into account $(4.1.8)_1$, which leads to

$$\frac{d}{ds}\delta\underset{\sim}{\Omega}_t = \bar{a}_t \delta\underset{\sim}{k}_t \quad , \quad \frac{d}{ds}\delta\underset{\sim}{u} = \delta\bar{\gamma}_{tt}\bar{\underset{\sim}{a}}_t + \delta\underset{\sim}{\Omega}_t \times \bar{\underset{\sim}{a}}_t \tag{4.2.4}$$

Here $\delta\underset{\sim}{k}_t$ is the vector of virtual change of curvature of the shell boundary contour $\bar{C}$ and $\delta\bar{\gamma}_{tt} = \delta\bar{\gamma}_{\alpha\beta}\bar{t}^\alpha\bar{t}^\beta = \frac{1}{\bar{a}_t^2}\delta\gamma_{tt}$, $\delta\gamma_{tt} = \delta\gamma_{\alpha\beta}t^\alpha t^\beta$, according to $(4.1.8)_3$.

Now it is possible to transform J_i in (4.1.11) as follows

$$\begin{aligned} J_i &= \int_C [\frac{d}{ds}(\underset{\sim}{F}_\nu\cdot\delta\underset{\sim}{u}) - \underset{\sim}{F}_\nu\cdot\frac{d}{ds}\delta\underset{\sim}{u} + \bar{M}_{\nu\nu}\delta\underset{\sim}{\Omega}_t\cdot\bar{\underset{\sim}{a}}_t]ds + \sum_n \Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}}\cdot\delta\underset{\sim}{u} = \\ &= \int_C [(\bar{M}_{\nu\nu}\bar{\underset{\sim}{a}}_t - \bar{\underset{\sim}{a}}_t \times \underset{\sim}{F}_\nu)\cdot\delta\underset{\sim}{\Omega}_t - \frac{1}{\bar{a}_t^2}(\bar{\underset{\sim}{a}}_t\cdot\underset{\sim}{F}_\nu)\delta\gamma_{tt}]ds + \\ &\quad + \sum_n (\Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}} - \Delta\underset{\sim}{F}_\nu)\cdot\delta\underset{\sim}{u} \end{aligned} \tag{4.2.5}$$

By introducing $(4.2.3)_2$ and $(4.2.4)_1$ into (4.2.5) we also get

$$\begin{aligned} J_i &= -\int_C [\bar{a}_t\underset{\sim}{B}_\nu\cdot\delta\underset{\sim}{k}_t + \frac{1}{\bar{a}_t^2}(\bar{\underset{\sim}{a}}_t\cdot\underset{\sim}{F}_\nu)\delta\gamma_{tt}]ds + \\ &\quad + \sum_n [(\Delta\bar{M}_{t\nu}\bar{\underset{\sim}{n}} - \Delta\underset{\sim}{F}_\nu)\cdot\delta\underset{\sim}{u} - \Delta\underset{\sim}{B}_\nu\cdot\delta\underset{\sim}{\Omega}_t] \end{aligned} \tag{4.2.6}$$

where

$$\begin{aligned} \sum_n \Delta\underset{\sim}{F}_\nu\cdot\delta\underset{\sim}{u} &= \sum_{M_n}[\underset{\sim}{F}_\nu(s_n+0) - \underset{\sim}{F}_\nu(s_n-0)]\cdot\delta\underset{\sim}{u}(s_n) \\ \sum_n \Delta\underset{\sim}{B}_\nu\cdot\delta\underset{\sim}{\Omega}_t &= \sum_{M_n}[\underset{\sim}{B}_\nu(s_n+0) - \underset{\sim}{B}_\nu(s_n-0)]\cdot\delta\underset{\sim}{\Omega}_t(s_n) \end{aligned} \tag{4.2.7}$$

The relations (4.2.5) and (4.2.6) show that during virtual deformation some static parameters produce at C a virtual work on variations of geometric parameters $\underset{\sim}{\Omega}_t$, γ_{tt} and $\underset{\sim}{k}_t$, γ_{tt} which establish the deformed lateral boundary surface $\partial\bar{P}$. Therefore, within K – L type non--linear theory of shells to each of displacemental, kinematical and deformational quantity discussed in p.3.3. there corresponds a static quantity according to the following schema

$$
\begin{aligned}
&\mathbf{u} \Longleftrightarrow \mathbf{P}_\nu \quad , \quad \beta_\nu \Longleftrightarrow \bar{a}_t \bar{M}_{\nu\nu} \\
&\boldsymbol{\Omega}_t \Longleftrightarrow \bar{M}_{\nu\nu}\bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{F}_\nu \quad , \quad \gamma_{tt} \Longleftrightarrow -\frac{1}{\bar{a}_t^2}(\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu) \\
&\mathbf{k}_t \Longleftrightarrow -\bar{a}_{.t}\mathbf{B}_\nu \quad , \quad \gamma_{tt} \Longleftrightarrow -\frac{1}{\bar{a}_t^2}(\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu)
\end{aligned}
\tag{4.2.8}
$$

In exactly the same way we can transform in (4.1.11) the integral J_e of the virtual work due to the external boundary loading. Defining a total force $\mathbf{T}$ and a total couple $\mathbf{C}$, with respect to a current point $\bar{M}$, of the external boundary force and boundary couple

$$
\begin{aligned}
&\mathbf{T} = \mathbf{T}^{o} + \int_{M_o}^{M} \mathbf{R}\,ds \quad , \quad \mathbf{C} = \mathbf{C}(0) - \bar{\mathbf{r}} \times \mathbf{T} \\
&\mathbf{C}(0) = \mathbf{C}^{o}(0) + \int_{M_o}^{M} (\bar{K}_\nu \bar{\mathbf{a}}_t + \bar{\mathbf{r}} \times \mathbf{R})\,ds
\end{aligned}
\tag{4.2.9}
$$

the integral J_e is transformed into

$$
\begin{aligned}
J_e = &\int_C \left[(\bar{K}_\nu \bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{T})\cdot\delta\boldsymbol{\Omega}_t - \frac{1}{\bar{a}_t^2}(\bar{\mathbf{a}}_t\cdot\mathbf{T})\delta\gamma_{tt}\right] ds + \\
&+ \sum_n (\Delta\bar{K}_t \bar{\mathbf{n}} - \Delta\mathbf{T})\cdot\delta\mathbf{u}
\end{aligned}
\tag{4.2.10}
$$

$$
\begin{aligned}
J_e = &-\int_C \left[\bar{a}_t \mathbf{C}\cdot\delta\mathbf{k}_t + \frac{1}{\bar{a}_t^2}(\bar{\mathbf{a}}_t\cdot\mathbf{T})\delta\gamma_{tt}\right] ds + \\
&+ \sum_n \left[(\Delta\bar{K}_t \bar{\mathbf{n}} - \Delta\mathbf{T})\cdot\delta\mathbf{u} - \Delta\mathbf{C}\cdot\delta\boldsymbol{\Omega}_t\right]
\end{aligned}
\tag{4.2.11}
$$

Introducing (4.2.5) and (4.2.10) or (4.2.6) and (4.2.11), respectively, into (4.1.11) we obtain the following modified forms of static boundary conditions

$$
\begin{aligned}
&\bar{M}_{\nu\nu}\bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{F}_\nu = \bar{K}_\nu \bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{T} \quad , \quad \bar{\mathbf{a}}_t\cdot\mathbf{F}_\nu = \bar{\mathbf{a}}_t\cdot\mathbf{T} \quad \text{on } C \\
&\Delta\bar{M}_{t\nu}\mathbf{G}\mathbf{n} - \Delta\mathbf{F}_\nu = \Delta\bar{K}_t \mathbf{G}\mathbf{n} - \Delta\mathbf{T} \quad \text{at each } M_n
\end{aligned}
\tag{4.2.12}
$$

$$\underset{\sim}{B}_\nu = \underset{\sim}{C} \quad , \quad \bar{\underset{\sim}{a}}_t \cdot \underset{\sim}{F}_\nu = \bar{\underset{\sim}{a}}_t \cdot \underset{\sim}{T} \quad \text{on} \quad C$$

$$\Delta\bar{M}_{t\nu}\underset{\sim}{Gn} - \Delta\underset{\sim}{F}_\nu = \Delta\bar{K}_t\underset{\sim}{Gn} - \Delta\underset{\sim}{T} \quad \text{and} \quad \Delta\underset{\sim}{B}_\nu = \Delta\underset{\sim}{C} \quad \text{at each} \quad M_n \tag{4.2.13}$$

These static boundary conditions are energetically compatible with kinematical (3.3.6) and deformational (3.3.9) boundary conditions, respectively.

When the shell problems are solved in displacements the basic variant (4.1.18) of static boundary conditions should be used. The modified static boundary conditions (4.2.12) are adequate for shell problems formulated by means of the finite rotation vector $\underset{\sim}{\Omega}$ and stress resultants $N^{\alpha\beta}$. The modified static boundary conditions (4.2.13) should be used when formulating the shell problems entirely in terms of the shell strain or stress measures.

4.3. Simplified shell relations under small elastic strains

The various shell relations discussed so far have purely geometrical character, which follows from the assumption of K - L constraints on deformation process of the shell. The relations still contain unrestricted strains and unrestricted rotations and do not depend upon the shell material properties. In what follows we shall discuss possible simplification of shell relations in the case of a thin shell composed of an isotropic elastic material under the assumption, that strains are small everywhere in the shell space.

For a shell subjected to forces applied only at its lateral boundaries John [22] obtained exact a priori estimates of stresses and their derivatives in the interior domain of the shell. The common measure of small quantities in [22] is the small parameter θ defined by

$$\theta = \max\left(\frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right) \tag{4.3.1}$$

where d is the distance of the shell point from the lateral boundary, R is a large parameter and η is the largest strain in the shell space.

The estimates of [22] may still be used if we admit some smooth surface force $\underset{\sim}{\bar{p}} = \bar{p}^{\alpha}\underset{\sim}{\bar{a}}_{\alpha} + \bar{p}\underset{\sim}{\bar{n}}$, with small variability, to be applied at the upper and/or the lower boundary of the shell space, provided that it does not disturb the approximately plain stress state in the shell [23] and gives the surface components [7]

$$\bar{p}^{\alpha} = O(E\eta\theta) \quad , \quad \bar{p} = O(E\eta\theta^2) \tag{4.3.2}$$

In this case the small parameter θ may be redefined on physical grounds [24] as follows

$$\theta = \max\left(\frac{h}{L}, \frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right) \tag{4.3.3}$$

where L is the smallest wavelength of deformation patterns at $\mathcal{M}$ and R is the smallest principal radius of curvature of $\mathcal{M}$.

Under small strains for the bending theory of shells it is assumed, that strains in the shell space , caused by stretching and bending of the shell middle surface, are of comparable order in the entire shell region. In this case we have the following estimates

$$\begin{aligned}
&a_{\alpha\beta} = O(1) \quad , \quad b_{\alpha\beta} = O(\tfrac{1}{R}) \\
&\gamma_{\alpha\beta} = O(\eta) \quad , \quad h\kappa_{\alpha\beta} = O(\eta) \quad , \quad h^2\nu_{\alpha\beta} = O(\eta\theta^2) \\
&\frac{\bar{a}}{a} = 1 + 2\gamma^{\lambda}_{\lambda} + O(\eta^2) = 1 + O(\eta) \\
&\bar{a}^{\alpha\beta} = a^{\alpha\beta} - 2\gamma^{\alpha\beta} + O(\eta^2) = a^{\alpha\beta} + O(\eta)
\end{aligned} \tag{4.3.4}$$

Within the first - approximation theory of thin isotropic elastic shells, the strain energy function Σ , per unit area of $\mathcal{M}$, can be consistently approximated by the following expression [7,10,24]

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right) + O(Eh\eta^2\theta^2) \tag{4.3.5}$$

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right)$$

The formula (4.3.5) takes into account the main contributions to the elastic strain energy of a shell caused by stretching and bending of the shell middle surface as well as by the transverse strains. The last contribution is taken into account by using in (4.3.5) the modified elasticity tensor $H^{\alpha\beta\lambda\mu}$.

Appropriate constitutive equations of the first - approximation theory compatible with (4.3.5) take the form

$$\begin{aligned} N^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} = C[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}] + O(Eh\eta\theta^2) \\ M^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} = D[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^{\lambda}_{\lambda}] + O(Eh^2\eta\theta^2) \end{aligned} \tag{4.3.6}$$

and their inverse

$$\begin{aligned} \gamma_{\alpha\beta} &= A[(1+\nu)N_{\alpha\beta} - \nu a_{\alpha\beta}N^{\lambda}_{\lambda}] + O(\eta\theta^2) \\ \kappa_{\alpha\beta} &= B[(1+\nu)M_{\alpha\beta} - \nu a_{\alpha\beta}M^{\lambda}_{\lambda}] + O\left(\frac{\eta\theta^2}{h}\right) \end{aligned} \tag{4.3.7}$$

where

$$C = \frac{Eh}{1-\nu^2}\ , \quad D = \frac{Eh^3}{12(1-\nu^2)}\ , \quad A = \frac{1}{Eh}\ , \quad B = \frac{12}{Eh^3} \tag{4.3.8}$$

Under small strains $\gamma_{\alpha\beta}$ is still given by (2.1.4) while for $\kappa_{\alpha\beta}$ in (2.1.5) we should put

$$\begin{aligned} n &= [1 + \theta^{\kappa}_{\kappa} + \frac{1}{2}(\theta^{\kappa}_{\kappa})^2 - \frac{1}{2}\theta^{\kappa}_{\mu}\theta^{\mu}_{\kappa} + \phi^2][1 - \gamma^{\lambda}_{\lambda} + O(\eta^2)] \\ n_{\mu} &= [-(1+\theta^{\kappa}_{\kappa})\phi_{\mu} + \phi^{\lambda}(\theta_{\lambda\mu} - \omega_{\lambda\mu})][1 + O(\eta)] \\ \gamma^{\lambda}_{\lambda} &= \theta^{\lambda}_{\lambda} + \frac{1}{2}\theta^{\lambda}_{\mu}\theta^{\mu}_{\lambda} + \frac{1}{2}\phi^{\lambda}\phi_{\lambda} + \phi^2 \end{aligned} \tag{4.3.9}$$

Note that in n here we have to take into account more accurate estimate for $\sqrt{\frac{a}{\bar{a}}}$, since the leading term of the product $n(\phi_{\alpha|\beta} + b^{\lambda}_{\beta}l_{\lambda\alpha})$

cancels with $-b_{\alpha\beta}$ in (2.1.5). As a result, in geometrically non-linear shell problems (with unrestricted rotations) $\gamma_{\alpha\beta}$ are quadratic polynomials and $\kappa_{\alpha\beta}$ are polynomials of fifth degree in displacements u_α, w and their surface gradients.

Within small strains the finite rotation vector $\underset{\sim}{\Omega}$, expressed exactly by (2.3.11), reduces to

$$\underset{\sim}{\Omega} = \{\epsilon^{\beta\alpha}[\phi_\alpha(1 + \frac{1}{2}\theta^\kappa_\kappa) - \frac{1}{2}\phi^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})]\underset{\sim}{a}_\beta + \phi\underset{\sim}{n}\}[1 + O(\eta)]\,. \tag{4.3.10}$$

The total finite rotation vector $\underset{\sim}{\Omega}_t$ of the boundary, defined exactly by (3.1.13), may be simplified according to

$$\underset{\sim}{\Omega}_t = \frac{1}{2}(\underset{\sim}{\nu} \times \bar{\underset{\sim}{a}}_\nu + \underset{\sim}{t} \times \bar{\underset{\sim}{a}}_t + \underset{\sim}{n} \times \bar{\underset{\sim}{n}})[1 + O(\eta)] \tag{4.3.11}$$

where

$$\begin{aligned} \bar{\underset{\sim}{a}}_t &= (\theta_{\nu t} - \phi)\underset{\sim}{\nu} + (1 + \theta_{tt})\underset{\sim}{t} + \phi_t\underset{\sim}{n}\,, \qquad \bar{\underset{\sim}{a}}_\nu = \bar{\underset{\sim}{a}}_t \times \bar{\underset{\sim}{n}} \\ \bar{\underset{\sim}{n}} &= \{[-\phi_\nu(1 + \theta_{tt}) + \phi_t(\theta_{\nu t} + \phi)]\underset{\sim}{\nu} + [\phi_\nu(\theta_{\nu t} - \phi) - \\ &\quad - \phi_t(1 + \theta_{\nu\nu})]\underset{\sim}{t} + [1 + \frac{1}{2}(\phi_\nu^2 + \phi_t^2) + \frac{1}{2}(\theta^\kappa_\kappa)^2 - \theta^\lambda_\kappa\theta^\kappa_\lambda]\underset{\sim}{n}\}[1 + O(\eta)] \end{aligned} \tag{4.3.12}$$

$$\begin{aligned} \phi_t &= \frac{dw}{ds} - \tau_t u_\nu + \sigma_t u_t\,, \qquad \phi = -\frac{du_\nu}{ds} + \kappa_t u_t - \tau_t w + \theta_{\nu t} \\ \theta_{\nu\nu} &= \frac{du_\nu}{ds_\nu} + \kappa_\nu u_t - \sigma_\nu w\,, \qquad \theta_{tt} = \frac{du_t}{ds} + \kappa_t u_\nu - \sigma_t w \\ \theta_{\nu t} &= \frac{1}{2}\left(\frac{du_t}{ds_\nu} + \frac{du_\nu}{ds} - \kappa_t u_t - \kappa_\nu u_\nu\right) + \tau_t w \end{aligned} \tag{4.3.13}$$

Note that here $\underset{\sim}{\Omega}$ is quadratic and $\underset{\sim}{\Omega}_t$ is cubic with respect to displacements and their surface gradients.

When introduced into the Lagrangean virtual work principle (4.1.16) the relations (2.1.4), (2.1.5) with (4.3.9) and (4.3.11) with (4.3.12) would give us the reduced set of Lagrangean shell equations of the type (4.1.18). The set becomes extremely complex when expressed in terms of displacements and we do not elaborate it here. Anyway, in such general case the only reasonable way to obtain a solution at once in terms of displacements is to apply numerical methods directly to a functional based on reduced form of (4.1.16) and not to the resulting set of shell equations.

The error indicated in the constitutive equations (4.3.6) and (4.3.7) allows to make essention reductions also in basic shell equations of the type (4.1.20). Let us present here the reduction procedure leading to the set of bending shell equations [9] formulated by means of the shell strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$.

Within the error indicated in (4.3.6) for $Q^{\alpha\beta}$ and Q^{β} in (4.1.20) we obtain the following estimates

$$
\begin{aligned}
Q^{\alpha\beta} &= C[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}] + O(Eh\eta\theta^2) \\
Q^{\beta} &= D[(1-\nu)\kappa^{\alpha\beta}{}_{|\alpha} + \nu a^{\alpha\beta}\kappa^{\lambda}_{\lambda|\alpha}] + O(Eh^2\frac{\eta\theta^2}{\lambda})
\end{aligned}
\qquad (4.3.14)
$$

where the parameter λ , used here to estimate surface derivatives , is defined by

$$\lambda = \frac{h}{\theta} = \min\left(L\ ,\ d\ ,\ \sqrt{hR}\ ,\ \frac{h}{\sqrt{\eta}} \right) \qquad (4.3.15)$$

Using (4.3.4) for terms appearing in the compatibility conditions (2.1.6) we have the estimates

$$\bar{a}^{\kappa\nu} b_{\kappa\lambda}\gamma_{\nu\beta\mu} = O(\frac{\eta\theta^2}{h\lambda}) \quad , \quad \bar{a}^{\kappa\nu}\kappa_{\kappa\lambda}\gamma_{\nu\beta\mu} = O(\frac{\eta\theta^2}{h\lambda}) \qquad (4.3.16)$$

$$K\gamma^{\kappa}_{\kappa} = O(\frac{\eta\theta^2}{\lambda^2}) \quad , \quad \bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda} = O(\frac{\eta\theta^2}{\lambda^2}) . \qquad (4.3.17)$$

and the compatibility conditions reduce to the form

$$
\begin{aligned}
&\kappa^{\beta}_{\alpha|\beta} - \kappa^{\beta}_{\beta|\alpha} = O(\frac{\eta\theta^2}{h\lambda}) \\
&\gamma^{\beta}_{\alpha}{}^{|\alpha}_{|\beta} - \gamma^{\alpha}_{\alpha}{}^{|\beta}_{|\beta} - (b^{\beta}_{\alpha}\kappa^{\alpha}_{\beta} - b^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) + \frac{1}{2}(\kappa^{\beta}_{\alpha}\kappa^{\alpha}_{\beta} - \kappa^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) = O(\frac{\eta\theta^2}{\lambda^2})
\end{aligned}
\qquad (4.3.18)
$$

The conditions $(4.3.18)_1$ allow to obtain sharper estimate for Q^{β}

$$Q^{\beta} = D\kappa^{\lambda}_{\lambda}{}^{|\beta} + O(Eh^2\frac{\eta\theta^2}{\lambda}) \qquad (4.3.19)$$

When $(4.3.14)_1$ and (4.3.19) are introduced into the equilibrium equations (4.1.20) they reduce within the same error to

$$C[(1-\nu)\gamma^{\beta}_{\alpha|\beta} + \nu\gamma^{\beta}_{\beta|\alpha}] + \bar{p}_{\alpha} = O(Eh\frac{\eta\theta^2}{\lambda})$$
$$D\kappa^{\alpha}_{\alpha}\Big|^{\beta}_{\beta} + C(b^{\alpha}_{\beta} - \kappa^{\alpha}_{\beta})[(1-\nu)\gamma^{\beta}_{\alpha} + \nu\delta^{\beta}_{\alpha}\gamma^{\lambda}_{\lambda}] + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2}) \quad (4.3.20)$$

Under small strains and bending shell theory components (3.2.16) of $\underset{\sim}{k}_t$ can be reduced to the linear form

$$k_{tt} = \kappa_{tt} + O(\frac{\eta\theta^2}{h}) \quad , \quad k_{\nu t} = \kappa_{\nu t} + O(\frac{\eta\theta^2}{h})$$
$$k_{nt} = 2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_{\nu}} + 2\kappa_{\nu}\gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) + O(\frac{\eta\theta^3}{h}) \quad . \quad (4.3.21)$$

The appropriate static boundary conditions may be obtained by consistent reduction of quantities appearing in (4.2.13). In this case

$$\underset{\sim}{P}_{\nu} = (P_{\nu\nu}\bar{\underset{\sim}{\nu}} + P_{t\nu}\bar{\underset{\sim}{t}} + P_{n\nu}\bar{\underset{\sim}{n}})[1 + O(\eta)]$$
$$P_{\nu\nu} = C(\gamma_{\nu\nu} + \nu\gamma_{tt}) + O(Eh\eta\theta^2) \quad , \quad P_{t\nu} = C(1-\nu)\gamma_{\nu t} + O(Eh\eta\theta^2)$$
$$P_{n\nu} = D[\frac{d\kappa_{\nu\nu}}{ds_{\nu}} + \nu\frac{d\kappa_{tt}}{ds_{\nu}} + 2(1-\nu)\frac{d\kappa_{t\nu}}{ds}] + \quad (4.3.22)$$
$$+ D(1-\nu)[\kappa_t(\kappa_{\nu\nu} - \kappa_{tt}) + 2\kappa_{\nu}\kappa_{t\nu}] + O(Eh\eta\theta^3)$$
$$\bar{M}_{\nu\nu} = M_{\nu\nu} + O(Eh^2\eta\theta^2) \quad , \quad \bar{M}_{t\nu} = M_{t\nu} + O(Eh^2\eta\theta^2)$$

Therefore, for the static parameters (4.2.13) to be assumed on C we obtain the following simplified formula

$$\underset{\sim}{B}_{\nu} = \underset{\sim}{B}^{O}_{\nu} + \int_{M_O}^{M} [D(\kappa_{\nu\nu} + \nu\kappa_{tt})\bar{\underset{\sim}{t}} + \bar{\underset{\sim}{r}} \times \underset{\sim}{P}_{\nu}]ds - \bar{\underset{\sim}{r}} \times \int_{M_O}^{M} \underset{\sim}{P}_{\nu} ds$$
$$\bar{\underset{\sim}{a}}_t \cdot \underset{\sim}{F}_{\nu} = \bar{\underset{\sim}{t}} \cdot (\underset{\sim}{F}^{O}_{\nu} + \int_{M_O}^{M} \underset{\sim}{P}_{\nu} ds) \quad (4.3.23)$$

where $\underset{\sim}{P}_{\nu}$ is given by (4.3.22).

The resulting set of six bending shell equations (4.3.18) and (4.3.20) is remarkably simple . Four of them are linear and two are quadratic with respect to the shell strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. Also all the formulae (4.3.21) and (4.3.22) for boundary quantities are linear in the strain measures.

The set of shell equations may be used for solving bending shell problems directly in terms of the strain measures, without having had to calculate displacements first. The resulting strain measures give us also the stress and couple resultants and, therefore, the stress distribution in the shell space. For some shell problems this may end the solution. The displacement field may be obtained, if necessary, by additional integration of the strain - displacement relations.

4.4. Canonical intrinsic shell equations

In many shell problems there may be some regions, in which small strains caused by membrane forces may happen to be of essentially different order (higher or smaller by the factor θ^2) from those caused by moments. Within these regions the reduced bending shell equations $(4.3.18)_1$ and $(4.3.20)_1$ may not be accurate enough, since they contain terms of only one kind: changes of curvatures and membrane strains, respectively.

The refinement of $(4.3.18)_1$ and $(4.3.20)_1$ can be carried out by selecting stress resultants $N^{\alpha\beta}$ and changes of curvatures $\kappa_{\alpha\beta}$ as two independent variables of shell equations. The refinement procedure was originaly suggested by Danielson [26] in terms of slightly different basic variables and with surface force $\bar{p}$ taken into account. It was also applied by Koiter and Simmonds [24], in the absence of surface forces. Here following our earlier results [9] we present the refined shell equations with all surface forces (4.3.2) taken into account.

With $N^{\alpha\beta}$ and $\kappa_{\alpha\beta}$ as independent variables we are able to obtain much better estimate for $Q^{\alpha\beta}$ which, according to $(4.1.14)_2$, (4.3.6) and (4.3.4) becomes

$$Q^{\alpha\beta} = N^{\alpha\beta} - D(b^{\alpha}_{\kappa} - \kappa^{\alpha}_{\kappa})[(1-\nu)\kappa^{\kappa\beta} + \nu a^{\kappa\beta}\kappa^{\lambda}_{\lambda}] + O(Eh\eta\theta^4) \qquad (4.4.1)$$

This estimate introduces an error $O(Eh\frac{\eta\theta^4}{\lambda})$ into the equilibrium equations $(4.1.20)_1$. Within the same error the second term of $(4.1.20)_1$ is reduced with the help of (4.3.4), (4.4.1) and (4.3.7) as follows

$$\begin{aligned}\bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}Q^{\lambda\beta} &= (2\gamma^{\alpha}_{\lambda|\beta} - \gamma_{\lambda\beta}|^{\alpha})N^{\lambda\beta} + O(Eh\frac{\eta\theta^4}{\lambda}) = \\ &= 2A[(1+\nu)N^{\alpha}_{\lambda} - \nu\delta^{\alpha}_{\lambda}N^{\kappa}_{\kappa}]_{|\beta}N^{\lambda\beta} - \\ &\quad - A[(1+\nu)N_{\lambda\beta} - \nu a_{\lambda\beta}N^{\kappa}_{\kappa}]|^{\alpha}N^{\lambda\beta} + O(Eh\frac{\eta\theta^4}{\lambda}) = \\ &= 2A[N^{\alpha}_{\lambda}N^{\lambda\beta} + \nu(N^{\alpha}_{\lambda}N^{\lambda\beta} - N^{\alpha\beta}N^{\lambda}_{\lambda})]_{|\beta} - \\ &\quad - \frac{1}{2}A[(1+\nu)N_{\lambda\beta}N^{\lambda\beta} - \nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]|^{\alpha} + \\ &\quad + 2A[(1+\nu)N^{\alpha}_{\lambda}\bar{p}^{\lambda} - \nu N^{\lambda}_{\lambda}\bar{p}^{\alpha}] + O(Eh\frac{\eta\theta^4}{\lambda})\end{aligned} \tag{4.4.2}$$

Using the identities

$$N^{\alpha}_{\lambda} = \delta^{\alpha}_{\lambda}N^{\kappa}_{\kappa} - \epsilon^{\alpha\mu}\epsilon_{\lambda\nu}N^{\nu}_{\mu}\ , \quad N^{\lambda\beta} = a^{\beta\lambda}N^{\rho}_{\rho} - \epsilon^{\beta\gamma}\epsilon^{\lambda\kappa}N_{\gamma\kappa} \tag{4.4.3}$$

we obtain the relation

$$\begin{aligned}(N^{\alpha}_{\lambda}N^{\lambda\beta} - N^{\alpha\beta}N^{\lambda}_{\lambda})_{|\beta} &= a^{\alpha\beta}(N_{\lambda\kappa}N^{\lambda\kappa} - N^{\lambda}_{\lambda}N^{\kappa}_{\kappa})_{|\beta} - (N^{\alpha}_{\lambda}N^{\lambda\beta} - N^{\alpha\beta}N^{\lambda}_{\lambda})_{|\beta} = \\ &= \frac{1}{2}(N_{\lambda\beta}N^{\lambda\beta} - N^{\lambda}_{\lambda}N^{\beta}_{\beta})|^{\alpha}\end{aligned} \tag{4.4.4}$$

which is also an identity for any continuously differentiable symmetric tensor components.

With the help of (4.4.4) the relation (4.4.2) takes now the form

$$\begin{aligned}\bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta}Q^{\lambda\beta} &= 2A(N^{\alpha}_{\lambda}N^{\lambda\beta})_{|\beta} - \frac{1}{2}A[(1-\nu)N_{\lambda\beta}N^{\lambda\beta} + \nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]|^{\alpha} + \\ &\quad + 2A[(1+\nu)N^{\alpha}_{\lambda}\bar{p}^{\lambda} - \nu N^{\lambda}_{\lambda}\bar{p}^{\alpha}] + O(Eh\frac{\eta\theta^4}{\lambda})\end{aligned} \tag{4.4.5}$$

For the third term in $(4.1.20)_1$, by taking into account (4.3.19) and (4.3.4), we obtain the following estimate

$$-\bar{b}^{\alpha}_{\beta}Q^{\beta} = -D(b^{\alpha}_{\beta} - \kappa^{\alpha}_{\beta})\kappa^{\lambda}_{\lambda}|^{\beta} + O(Eh\frac{\eta\theta^4}{\lambda}) \tag{4.4.6}$$

Taking into account (4.4.1), (4.4.5) and (4.4.6) with $(4.3.20)_2$ the refined equilibrium equations take the following form

$$N^{\beta}_{\alpha|\beta} - D[(1-\nu)b^{\lambda}_{\alpha}\kappa^{\beta}_{\lambda} + \nu b^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}]_{|\beta} - Db^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda|\beta} + D[(1-\nu)\kappa^{\lambda}_{\alpha}\kappa^{\beta}_{\lambda} + \nu\kappa^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda}]_{|\beta} + D\kappa^{\beta}_{\alpha}\kappa^{\lambda}_{\lambda|\beta} + 2A(N^{\lambda}_{\alpha}N^{\beta}_{\lambda})_{|\beta} - \frac{1}{2}A[(1-\nu)N^{\beta}_{\lambda}N^{\lambda}_{\beta} + \nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]_{|\alpha} + A[2(1+\nu)N^{\lambda}_{\alpha}\bar{p}_{\lambda} + (1-3\nu)N^{\lambda}_{\lambda}\bar{p}_{\alpha}] + \bar{p}_{\alpha} = O(Eh\frac{\eta\theta^4}{\lambda}) \tag{4.4.7}$$

$$D\kappa^{\alpha}_{\alpha}\Big|^{\beta}_{\beta} + (b^{\alpha}_{\beta} - \kappa^{\alpha}_{\beta})N^{\beta}_{\alpha} + \bar{p} = O(Eh^2\frac{\eta\theta^2}{\lambda^2})$$

In exactly the same way the reduction of compatibility conditions $(2.1.6)_1$ may be carried out within a smaller error than in $(4.3.18)_1$. Let us replace $\bar{a}^{\kappa\nu}$ by $a^{\kappa\nu}$ in the second term of $(2.1.6)_1$. This, according to (4.3.4), introduces an error $O(\frac{\eta\theta^4}{h\lambda})$. Then using (2.1.7), (2.1.4) and $(4.3.7)_1$ this term can be estimated by

$$\begin{aligned}\epsilon_{\alpha\beta}\epsilon^{\lambda\mu}\bar{b}^{\kappa}_{\lambda}\gamma_{\kappa\cdot\mu}^{\ \ \beta} &= (b^{\kappa}_{\alpha} - \kappa^{\kappa}_{\alpha})(2\gamma^{\beta}_{\kappa|\beta} - \gamma^{\beta}_{\beta|\kappa}) - (b^{\kappa}_{\beta} - \kappa^{\kappa}_{\beta})\gamma^{\beta}_{\kappa|\alpha} + O(\frac{\eta\theta^4}{h\lambda}) = \\ &= -2A(1+\nu)(b^{\beta}_{\alpha} - \kappa^{\beta}_{\alpha})\bar{p}_{\beta} + A\nu(b^{\beta}_{\beta} - \kappa^{\beta}_{\beta})N^{\lambda}_{\lambda|\alpha} - \\ &\quad - A(1+\nu)[(b^{\kappa}_{\beta} - \kappa^{\kappa}_{\beta})N^{\beta}_{\kappa|\alpha} + (b^{\kappa}_{\alpha} - \kappa^{\kappa}_{\alpha})N^{\lambda}_{\lambda|\kappa}] + O(\frac{\eta\theta^4}{h\lambda})\end{aligned} \tag{4.4.8}$$

Taking into account (4.4.8), $(4.3.20)_1$ and $(4.3.7)_1$ the refined compatibility conditions take the following form

$$\kappa^{\beta}_{\alpha|\beta} - \kappa^{\beta}_{\beta|\alpha} - A(1+\nu)[b^{\lambda}_{\beta}N^{\beta}_{\lambda|\alpha} + b^{\beta}_{\alpha}N^{\lambda}_{\lambda|\beta}] + A\nu b^{\beta}_{\beta}N^{\lambda}_{\lambda|\alpha} + A(1+\nu)[\kappa^{\lambda}_{\beta}N^{\beta}_{\lambda|\alpha} + \kappa^{\beta}_{\alpha}N^{\lambda}_{\lambda|\beta}] - A\nu\kappa^{\beta}_{\beta}N^{\lambda}_{\lambda|\alpha} - 2A(1+\nu)b^{\beta}_{\alpha}\bar{p}_{\beta} + 2A(1+\nu)\kappa^{\beta}_{\alpha}\bar{p}_{\beta} = O(\frac{\eta\theta^4}{h\lambda}) \tag{4.4.9}$$

$$AN^{\alpha}_{\alpha}\Big|^{\beta}_{\beta} + (b^{\alpha}_{\beta}\kappa^{\beta}_{\alpha} - b^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) - \frac{1}{2}(\kappa^{\alpha}_{\beta}\kappa^{\beta}_{\alpha} - \kappa^{\alpha}_{\alpha}\kappa^{\beta}_{\beta}) + A(1+\nu)\bar{p}^{\alpha}_{\ |\alpha} = O(\frac{\eta\theta^2}{\lambda^2})$$

The relations (4.4.7) and (4.4.9) are canonical form of intrinsic equations in the first - approximation non-linear theory of thin elastic shells [24]. The appropriate deformational and static boundary conditions expressed in terms of $N^{\alpha\beta}$ and $\kappa_{\alpha\beta}$ follow from the more accurate reduction of (3.2.16) and (4.2.13), respectively.

The canonical intrinsic shell equations (4.4.7) and (4.4.9) describe accurately the behaviour of a thin shell in the whole internal region independently of the strain state in the shell. For some special shell problems these equations may be simplified. Some simplified forms of the equations were discussed by the author [25,27] based on restrictions assumed for the bending-to-membrane strain ratio.

5. GEOMETRICALLY NON-LINEAR THEORY OF SHELLS WITH RESTRICTED ROTATIONS

5.1. Classification of rotations

For many engineering purposes it is hardly necessary to allow the rotations of any magnitude. Some shell structures would become unserviceable if really unrestricted rotations were permitted to occure. Therefore it is certainly worthwhile to discuss the possible reduction of geometrically non-linear shell relations resulting from consistently restricted rotations.

The well known classifications of the approximate variants of shell equations were proposed by Mushtari and Galimov [28] and Koiter [11]. In [28] restrictions on components of the linearized rotation vector $\underset{\sim}{\Phi}$ were used to make distinction between three approximate variants of shell equations with "small, medium or large bending". In [11] four approximate variants with "infinitesimal, small finite, moderate or large deflections" were clearly defined by putting various restrictions on displacement gradients and components of $\underset{\sim}{\Phi}$. Note that in the classifications the word "rotation" does not appear at all, since neither $\underset{\sim}{\Phi}$ nor displacement gradients themselves are the finite rotations of the shell material elements.

By the polar decomposition (2.2.10) finite strains and finite rotations have clearly been separated. Within the geometrically non - linear theory we have restricted strains to be small everywhere in the shell. It seems therefore natural to restrict now in a consistent manner either the finite rotation tensor or the equivalent finite rotation vector. Since $\underset{\sim}{R}$ and $\underset{\sim}{\Omega}$ are uniquely described in (2.3.2) and (2.3.5) by the angle of rotation ω and the unit vector $\underset{\sim}{e}$ describing a direction of the rotation axis the both parameters may be restricted independently.

Within geometrically non-linear theory of thin isotropic and elastic shells there exist a small parameter θ defined in (4.3.3). This parameter can be used to introduce the following classification of rotations [7]:

$$\begin{aligned} &\omega \leq O(\theta^2) \text{ - small rotations} \\ &\omega = O(\theta) \text{ - moderate rotations} \\ &\omega = O(\sqrt{\theta}) \text{ - large rotations} \\ &\omega \geq O(1) \text{ - finite rotations} \end{aligned} \qquad (5.1.1)$$

Since for $|\omega| < \pi/2$ we have $O(|\underset{\sim}{\Omega}|) = O(\sin\omega) = O(\omega)$, the classification proposed here restricts only the magnitude $|\underset{\sim}{\Omega}|$ of the finite rotation vector, leaving the direction of the rotation axis to be arbitrary. It is known, however, that many shell structures are manufactured to be quite rigid in the direction tangent to the reference surface M even if they are allowed to be flexible in the direction orthogonal to M. For this reason, when discussing simplified variants of shell equations, there may be of interest to put restrictions of different order on appropriate components $\underset{\sim}{\Omega}\cdot\underset{\sim}{a}_\alpha$ or $\underset{\sim}{\Omega}\cdot\underset{\sim}{n}$. The name "small, moderate, large or finite rotation" may then be associated with the particular component of the finite rotation vector.

As an example, let us discuss here the simplest case of the non-linear theory of shells admiting only small rotations. In this case we have the following estimates

$$\begin{aligned} &|\underset{\sim}{\Omega}| = O(\theta^2) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{a}_\alpha = O(\theta^2) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\theta^2) \\ &\phi_\alpha = O(\theta^2) \quad , \quad \phi = O(\theta^2) \quad , \quad \theta_{\alpha\beta} = O(\theta^2) \end{aligned} \qquad (5.1.2)$$

The shell strain measures become

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + O(\eta\theta^2) \qquad (5.1.3)$$

$$\kappa_{\alpha\beta} = -\frac{1}{2}[\phi_{\alpha|\beta} + \phi_{\beta|\alpha} + b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] + O(\frac{\eta\theta^2}{\lambda})$$

The finite rotation vectors $\underset{\sim}{\Omega}$ and $\underset{\sim}{\Omega}_t$ can be approximated by

$$\underset{\sim}{\Omega} = \underset{\sim}{\phi}[1 + O(\theta^2)] \quad , \quad \underset{\sim}{\Omega}_t = (\underset{\sim}{\phi} - \theta_{\alpha\beta}\nu^{\alpha}t^{\beta}\underset{\sim}{n})[1 + O(\theta^2)]$$
$$\underset{\sim}{\phi} = \epsilon^{\beta\alpha}\phi_{\alpha}\underset{\sim}{a}_{\beta} + \phi\underset{\sim}{n} \tag{5.1.4}$$

where $\underset{\sim}{\phi}$ is the linearized rotation vector.

When introduced into the Lagrangean virtual work principle (4.1.16) these relations lead to the Lagrangean equilibrium equations and static boundary conditions

$$\underset{\sim}{N}^{\beta}|_{\beta} + \underset{\sim}{p} = \underset{\sim}{0} \quad \text{in } M$$
$$\underset{\sim}{N}^{\beta}\nu_{\beta} + \frac{d}{ds}(M_{t\nu}\underset{\sim}{n}) = \underset{\sim}{F} + \frac{d}{ds}(K_t\underset{\sim}{n}) \quad , \quad M_{\nu\nu} = K_{\nu} \quad \text{on } C \tag{5.1.5}$$
$$\Delta M_{t\nu}\underset{\sim}{n} = \Delta K_t\underset{\sim}{n} \quad \text{at each } M_n$$

$$\underset{\sim}{N}^{\beta} = (N^{\alpha\beta} - b^{\alpha}_{\lambda}M^{\lambda\beta})\underset{\sim}{a}_{\alpha} + M^{\alpha\beta}|_{\alpha}\underset{\sim}{n} \tag{5.1.6}$$

Therefore, within small rotations the geometrically non-linear theory of shells reduces to the classical linear theory of shells [29,30,31]. Note that the linearized rotation vector $\underset{\sim}{\phi}$ enters many relations of the linear theory of shells. It was used, in particular, to construct multivalued solutions for shells with multi-connected regions [32,33].

5.2. Moderate rotation theory of shells

When rotations are assumed to be moderate then, using (4.3.10) and (2.1.4), we have the estimates

$$|\underset{\sim}{\Omega}| = O(\theta) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{a}_{\alpha} = O(\theta) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\theta)$$
$$\phi_{\alpha} = O(\theta) \quad , \quad \phi = O(\theta) \quad , \quad \theta_{\alpha\beta} = O(\theta^2) \tag{5.2.1}$$

It follows from the error indicated in the shell strain energy function (4.3.5) that $\gamma_{\alpha\beta}$ may be simplified by omitting terms $O(\eta\theta^2)$ while in $\kappa_{\alpha\beta}$ we can omit terms $O(\frac{\eta\theta}{\lambda})$. Therefore, the shell strain measures take here the following approximate forms

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\phi_\alpha\phi_\beta + \underline{\frac{1}{2}a_{\alpha\beta}\phi^2 - \frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta^2)$$
$$\kappa_{\alpha\beta} = -\frac{1}{2}[\phi_{\alpha|\beta} + \phi_{\beta|\alpha} + \underline{b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})}] + O(\frac{\eta\theta}{\lambda}) \tag{5.2.2}$$

Since $\frac{1}{2}(b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) = O(\frac{\eta\theta}{\lambda})$ we could also omit these terms in $(5.2.2)_2$ within the accuracy of the first - approximation theory. These terms are linear in displacements and it is rather a convention to keep them here together with definition of $\kappa_{\alpha\beta}$ as a shell strain measure.

The finite rotation vectors are approximated by

$$\boldsymbol{\Omega} = (\epsilon^{\beta\alpha}\phi_\alpha + \underline{\frac{1}{2}\phi^\beta\phi})\boldsymbol{a}_\beta + \phi\boldsymbol{n} + O(\eta\theta)$$

$$\boldsymbol{\Omega}_t = \boldsymbol{\Omega} - \gamma_{\nu t}\boldsymbol{n} + O(\theta^3) = \tag{5.2.3}$$
$$= (\phi_t + \underline{\frac{1}{2}\phi_\nu\phi})\boldsymbol{\nu} + (-\phi_\nu + \underline{\frac{1}{2}\phi_t\phi})\boldsymbol{t} + (\phi - \theta_{\nu t} - \frac{1}{2}\phi_\nu\phi_t)\boldsymbol{n} + O(\eta\theta)$$

If we introduce (5.2.2) into the left - hand side of (4.1.16) after transformation we obtain the following formula for the internal virtual work

$$IVW = -\iint_{\mathcal{M}} (\boldsymbol{GN}^\beta)_{|\beta}\cdot\delta\boldsymbol{u}\,dA + J_i$$
$$J_i = \int_C \{[\boldsymbol{GN}^\beta\nu_\beta + \frac{d}{ds}(M_{t\nu}\boldsymbol{n})]\cdot\delta\boldsymbol{u} - M_{\nu\nu}\delta\phi_\nu\}ds + \sum_n \Delta M_{t\nu}\delta w \tag{5.2.4}$$

where $\delta\phi_\nu = -\delta\beta_\nu$ and

$$\boldsymbol{GN}^\beta = [N^{\alpha\beta} - \underline{b^\alpha_\lambda M^{\lambda\beta} - \frac{1}{2}\omega^{\alpha\beta}N^\lambda_\lambda - \frac{1}{2}(\omega^{\alpha\lambda}N^\beta_\lambda + \omega^{\beta\lambda}N^\alpha_\lambda)} +$$
$$+ \underline{\frac{1}{2}(\theta^{\alpha\lambda}N^\beta_\lambda - \theta^{\beta\lambda}N^\alpha_\lambda)}]\boldsymbol{a}_\alpha + (\phi_\alpha N^{\alpha\beta} + M^{\alpha\beta}{}_{|\beta})\boldsymbol{n} \tag{5.2.5}$$

Note that within the shell theory admiting moderate rotations the moment $M_{t\nu}$ appears in $(5.2.4)_2$ multiplied by $\underset{\sim}{n}$ and not by $\underset{\sim}{\bar{n}}$ as in the general case $(4.1.14)_3$. This results from linearity of $\kappa_{\alpha\beta}$ in (5.2.2) and is compatible with accuracy of the shell strain energy (4.3.5). Although the energy arguments cannot be applied directly to the external boundary couple $\underset{\sim}{K}$, it would seem to be inconsistent to use a better approximation for $\underset{\sim}{K}$ in EVW than that resulting for $\underset{\sim}{M}^{\beta} = \epsilon_{\alpha\lambda}M^{\alpha\beta}\underset{\sim}{a}^{\lambda}$ at the boundary in IVW. Since according to $(4.1.13)_1$, (5.2.1) and (4.3.9) there is $K = K^{\alpha}\cdot O(\theta)$ the dominant terms in $\underset{\sim}{K}$ and $\underset{\sim}{\Omega}_t$ are

$$\begin{aligned} \underset{\sim}{K} &= (-K_t\underset{\sim}{\nu} + K_\nu\underset{\sim}{t})[1+O(\theta)] \\ \underset{\sim}{\Omega}_t &= (\phi_t\underset{\sim}{\nu} - \phi_\nu\underset{\sim}{t} + \phi\underset{\sim}{n})[1+O(\theta)] \end{aligned} \tag{5.2.6}$$

Introducing (5.2.2) and (5.2.6) into (4.1.16) we obtain the Lagrangean equilibrium equations and the Lagrangean static boundary conditions

$$\begin{aligned} &(\underset{\sim}{G}\underset{\sim}{N}^{\beta})_{|\beta} + \underset{\sim}{p} = \underset{\sim}{0} \quad \text{in } M \\ &\underset{\sim}{G}\underset{\sim}{N}^{\beta}\nu_\beta + \frac{d}{ds}(M_{t\nu}\underset{\sim}{n}) = \underset{\sim}{F} + \frac{d}{ds}(K_t\underset{\sim}{n}) \ , \quad M_{\nu\nu} = K_\nu \quad \text{on } C \\ &\Delta M_{t\nu}\underset{\sim}{n} = \Delta K_t\underset{\sim}{n} \quad \text{at each } M_n \end{aligned} \tag{5.2.7}$$

The component form of $(5.2.7)_1$ with (5.2.5) reads

$$\begin{aligned} &[N^{\alpha\beta} - \underline{b^{\alpha}_{\lambda}M^{\lambda\beta}} - \tfrac{1}{2}\omega^{\alpha\beta}N^{\lambda}_{\lambda} - \underline{\tfrac{1}{2}(\omega^{\alpha\lambda}N^{\beta}_{\lambda} + \omega^{\beta\lambda}N^{\alpha}_{\lambda})} + \\ &\quad + \underline{\tfrac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} - \theta^{\beta\lambda}N^{\alpha}_{\lambda})}]_{|\beta} - b^{\alpha}_{\beta}(\phi_\lambda N^{\lambda\beta} + M^{\lambda\beta}{}_{|\lambda}) + p^{\alpha} = 0 \\ &(\phi_\alpha N^{\alpha\beta} + M^{\alpha\beta}{}_{|\alpha})_{|\beta} + b_{\alpha\beta}[N^{\alpha\beta} - \underline{b^{\alpha}_{\lambda}M^{\lambda\beta}} - \underline{\tfrac{1}{2}(\omega^{\alpha\lambda}N^{\beta}_{\lambda} + \omega^{\beta\lambda}N^{\alpha}_{\lambda})}] + p = 0 \end{aligned} \tag{5.2.8}$$

The equations of moderate rotation theory of shells are quite complex, partialy due to the presence of the last two terms (underlined by dots) in the expression $(5.2.2)_1$ for $\gamma_{\alpha\beta}$. Since these terms are $O(\eta\theta)$ by neglecting them in $(5.2.2)_1$ we would introduce an additional error $O(Eh\eta^2\theta)$ into the strain energy (4.3.5), which sometimes may cause some decrease in accuracy of the solution. At the expence of indicated loss in accuracy, we still may neglect [11,9] these terms in (5.2.2). As a result terms underlined by dots will not appear in (5.2.5) and (5.2.8).

In many engineering structures only rotations about a tangent to the shell middle surface are allowed to be moderate, while rotations about a normal are supposed to be small. In such a case we have

$$\begin{aligned} &\underset{\sim}{\Omega}\cdot\underset{\sim}{a}_\alpha = O(\theta) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\theta^2) \\ &\phi_\alpha = O(\theta) \quad , \quad \phi = O(\theta^2) \quad , \quad \theta_{\alpha\beta} = O(\theta^2) \end{aligned} \tag{5.2.9}$$

and all the relations (5.2.2) to (5.2.8) can be considerably simplified, without any loss in accuracy, by omitting there all terms underlined by a solid line.

Further possible simplifications of shell equations, within moderate rotation theory of shells, were discussed by Koiter [11] and the author [9].

5.3. Large rotation theory of shells

When rotations are allowed to be large then from (4.3.10) and (2.1.4) we have

$$\begin{aligned} &|\underset{\sim}{\Omega}| = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{a}_\alpha = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\sqrt{\theta}) \\ &\phi_\alpha = O(\sqrt{\theta}) \quad , \quad \phi = O(\sqrt{\theta}) \quad , \quad \theta_{\alpha\beta} = O(\theta) \\ &\underset{\sim}{\Omega} = \epsilon^{\beta\alpha}[\phi_\alpha(1 + \tfrac{1}{2}\theta^\kappa_\kappa) - \tfrac{1}{2}\phi^\lambda(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})]\underset{\sim}{a}_\beta + \phi\underset{\sim}{n} + O(\eta\sqrt{\theta}) \end{aligned} \tag{5.3.1}$$

The strain tensor cannot be simplified here. In order to reduce the tensor of change of curvature we use the relations [9]

$$\begin{aligned} &\omega_{\lambda\alpha|\beta} = \theta_{\alpha\beta|\lambda} - \theta_{\lambda\beta|\alpha} + b_{\alpha\beta}\phi_\lambda - b_{\lambda\beta}\phi_\alpha \\ &\phi_\lambda\theta^\kappa_\kappa - \phi^\mu\theta_{\mu\lambda} = -\tfrac{1}{2}\phi_\lambda\phi^2 + O(\theta^2) \end{aligned} \tag{5.3.2}$$

The relation $(5.3.2)_1$ is an identity, while $(5.3.2)_2$ is satisfied within the large rotation theory of shells.

When (4.3.9), (5.3.2) together with (5.3.1) are introduced into (2.1.5) then within $O(\frac{n\theta}{\lambda})$ we obtain for $\kappa_{\alpha\beta}$ the following formula

$$\kappa_{\alpha\beta} = -\frac{1}{2}[\phi_{\alpha|\beta} + \phi_{\beta|\alpha} + b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] -$$
$$- \frac{1}{2} b_{\alpha\beta}\phi^{\lambda}\phi_{\lambda} + \frac{1}{2}(\frac{1}{2}\phi^{\lambda}\phi_{\lambda} + \theta^{\lambda}_{\kappa}\theta^{\kappa}_{\lambda} - \frac{1}{2}\theta^{\kappa}_{\kappa}\theta^{\lambda}_{\lambda})(\phi_{\alpha|\beta} + \phi_{\beta|\alpha}) -$$
$$- \frac{1}{4}\phi^{\lambda}\phi_{\lambda}(b^{\lambda}_{\alpha}\omega_{\lambda\beta} - b^{\lambda}_{\beta}\omega_{\lambda\alpha}) + (\phi^{\lambda} + \phi_{\kappa}\omega^{\kappa\lambda} - \frac{1}{2}\phi^{\lambda}\phi^{2})\theta_{\lambda\alpha\beta} + O(\frac{n\theta}{\lambda}) \tag{5.3.3}$$

$$\theta_{\lambda\alpha\beta} = \theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\alpha\beta|\lambda}$$

Using the relation (4.3.11) we also reduce the total finite rotation vector to the form

$$2\Omega_t = (2\phi_t + \phi_\nu\phi - \phi_\nu\theta_{\nu t} + \phi_t\theta_{\nu\nu})\nu +$$
$$+ [-2\phi_\nu + \phi_t\phi - 3\phi_\nu\theta_{tt} + 3\phi_t\theta_{\nu t} + 2\phi_\nu\phi\theta_{\nu t} + \phi_t\phi(\theta_{tt} - \theta_{\nu\nu}) -$$
$$- \phi_\nu\phi^2]t + [2\phi - 2\theta_{\nu t} - \phi_\nu\phi_t(1 + \theta_{tt}) - \frac{1}{2}(\phi^2_\nu - \phi^2_t)\phi +$$
$$+ \frac{1}{2}(\phi^2_\nu + 3\phi^2_t)\theta_{\nu t}]n + O(\theta^2\sqrt{\theta}) \tag{5.3.4}$$

When introduced into (4.1.16) the strain measures (2.1.4) and (5.3.3) lead to the Lagrangean equilibrium equations $(4.1.18)_1$, where

$$GN^{\beta} = \{1^{\alpha}_{\cdot\lambda}N^{\lambda\beta} - b^{\alpha}_{\lambda}M^{\lambda\beta} + \frac{1}{4}\phi^{\kappa}\phi_{\kappa}(b^{\alpha}_{\lambda}M^{\lambda\beta} - b^{\beta}_{\lambda}M^{\alpha\lambda}) +$$
$$+ (2\theta^{\alpha\beta} - a^{\alpha\beta}\theta^{\kappa}_{\kappa})\phi_{\lambda|\mu}M^{\lambda\mu} + \frac{1}{2}\omega^{\alpha\beta}\phi^{\kappa}\theta_{\kappa\lambda\mu}M^{\lambda\mu} -$$
$$- \frac{1}{2}(\phi^{\alpha}\theta^{\beta}_{\cdot\lambda\mu} - \phi^{\beta}\theta^{\alpha}_{\cdot\lambda\mu})M^{\lambda\mu} - [(1 - \frac{1}{2}\phi^2)(\phi^{\alpha}M^{\lambda\beta} + \phi^{\beta}M^{\alpha\lambda} - \phi^{\lambda}M^{\alpha\beta}) +$$
$$+ \phi_{\kappa}(\omega^{\kappa\alpha}M^{\lambda\beta} + \omega^{\kappa\beta}M^{\alpha\lambda} - \omega^{\kappa\lambda}M^{\alpha\beta})]|_{\lambda}\}a_{\alpha} + \tag{5.3.5}$$
$$+ \{\phi_{\lambda}N^{\lambda\beta} + [(1 - \frac{1}{2}\phi^{\kappa}\phi_{\kappa} + \frac{1}{2}\theta^{\kappa}_{\kappa}\theta^{\mu}_{\mu} - \theta^{\kappa}_{\mu}\theta^{\mu}_{\kappa})M^{\lambda\beta}]|_{\lambda} +$$

$$+ [\phi^{\beta}(- b_{\lambda\mu} + \phi_{\lambda|\mu} - \underline{b^{\kappa}_{\lambda}\omega_{\kappa\mu}}) +$$

$$+ (1 - \underline{\tfrac{1}{2}\phi^2})\theta^{\beta}_{\cdot\lambda\mu} + \omega^{\beta\kappa}\theta_{\kappa\lambda\mu}]M^{\lambda\mu}\}\underset{\sim}{n}$$

It is obvious how to obtain the component form of $(4.1.18)_1$ with (5.3.5), although it is quite complex.

At the expence of a possible loss in accuracy we may neglect in (5.3.3) some terms $O(\frac{\eta\sqrt{\theta}}{\lambda})$ underlined by dots. As a result, terms underlined by dots will not appear in (5.3.5).

In some shell structures only rotations about a tangent may be large while rotations about a normal are at most moderate. In such a case, we assume $\underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\theta)$ and $\phi = O(\theta)$ in (5.3.1) and relations (5.3.3), (5.3.4) and (5.3.5) can be simplified, without any loss in accuracy, by omitting there all terms underlined by a solid line.

In many engineering shell structures the rotations about a normal are always small even if rotations about a tangent are allowed to be large. In such a case

$$|\underset{\sim}{\Omega}| = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{a}_{\alpha} = O(\sqrt{\theta}) \quad , \quad \underset{\sim}{\Omega}\cdot\underset{\sim}{n} = O(\theta^2)$$

$$\phi_{\alpha} = O(\sqrt{\theta}) \quad , \quad \phi = O(\theta^2) \quad , \quad \theta_{\alpha\beta} = O(\theta) \tag{5.3.6}$$

$$\underset{\sim}{\Omega} = \epsilon^{\beta\alpha}[\phi_{\alpha}(1 + \tfrac{1}{2}\theta^{\kappa}_{\kappa}) - \tfrac{1}{2}\phi^{\lambda}\theta_{\lambda\alpha}]\underset{\sim}{a}_{\beta} + \phi\underset{\sim}{n} + O(\eta\sqrt{\theta})$$

The shell strain measures can now be approximated by

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \tfrac{1}{2}\phi_{\alpha}\phi_{\beta} + \tfrac{1}{2}\theta^{\lambda}_{\alpha}\theta_{\lambda\beta} - \tfrac{1}{2}(\theta^{\lambda}_{\alpha}\omega_{\lambda\beta} + \theta^{\lambda}_{\beta}\omega_{\lambda\alpha}) + O(\eta\theta^2)$$

$$\begin{aligned}\kappa_{\alpha\beta} = &- \tfrac{1}{2}[\phi_{\alpha|\beta} + \phi_{\beta|\alpha} + b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] - \\ &- \tfrac{1}{2}b_{\alpha\beta}\phi^{\lambda}\phi_{\lambda} + \tfrac{1}{2}[\tfrac{1}{2}\phi^{\lambda}\phi_{\lambda} + \underline{\theta^{\lambda}_{\kappa}\theta^{\kappa}_{\lambda} - \tfrac{1}{2}\theta^{\kappa}_{\kappa}\theta^{\lambda}_{\lambda}}](\phi_{\alpha|\beta} + \phi_{\beta|\alpha}) + \\ &+ \phi^{\lambda}\theta_{\lambda\alpha\beta} + O(\tfrac{\eta\theta}{\lambda})\end{aligned} \tag{5.3.7}$$

These strain measures lead to the vector equilibrium equation (4.1.18)$_1$, where now

$$\underset{\sim}{GN}^{\beta} = [N^{\alpha\beta} + \frac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} + \theta^{\beta\lambda}N^{\alpha}_{\lambda}) + \frac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} - \theta^{\beta\lambda}N^{\alpha}_{\lambda}) -$$

$$- \frac{1}{2}(\omega^{\alpha\lambda}N^{\beta}_{\lambda} + \omega^{\beta\lambda}N^{\alpha}_{\lambda}) - b^{\alpha}_{\lambda}M^{\lambda\beta} + \underset{\cdots\cdots\cdots}{(2\theta^{\alpha\beta} - a^{\alpha\beta}\theta^{\kappa}_{\kappa})\phi_{\lambda|\mu}M^{\lambda\mu}} -$$

$$- (\phi^{\alpha}M^{\lambda\beta} + \phi^{\beta}M^{\alpha\lambda} - \phi^{\lambda}M^{\alpha\beta})_{|\lambda}]\underset{\sim}{a}_{\alpha} + \tag{5.3.8}$$

$$+ \{\phi_{\lambda}N^{\lambda\beta} + [(1 - \frac{1}{2}\phi^{\kappa}\phi_{\kappa} + \underset{\cdots\cdots\cdots}{\frac{1}{2}\theta^{\kappa}_{\kappa}\theta^{\mu}_{\mu} - \theta^{\kappa}_{\mu}\theta^{\mu}_{\kappa}})M^{\lambda\beta}]_{|\lambda} +$$

$$+ [\phi^{\beta}(-b_{\lambda\mu} + \phi_{\lambda|\mu}) + \theta^{\beta}_{\cdot\lambda\mu}]M^{\lambda\mu}\}\underset{\sim}{n}$$

Again, at the expence of a possible loss in accuracy we may neglect in (5.3.7) terms underlined by dots. As a result, terms underlined by dots will not appear in (5.3.8). Such approximation introduces an additional error $O(Eh\eta^2\theta\sqrt{\theta})$ into the strain energy (4.3.5).

At the expence of a greater error $O(Eh\eta^2\theta)$ in (4.3.5) the strain measures may be approximated by the following formulae

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\phi_{\alpha}\phi_{\beta} + \frac{1}{2}\theta^{\lambda}_{\alpha}\theta_{\lambda\beta} + O(\eta\theta)$$
$$\kappa_{\alpha\beta} = -\frac{1}{2}(1 - \frac{1}{2}\phi^{\lambda}\phi_{\lambda})(\phi_{\alpha|\beta} + \phi_{\beta|\alpha}) + \theta^{\lambda}\theta_{\lambda\alpha\beta} + O(\frac{\eta}{\lambda}) \tag{5.3.9}$$

which result in equilibrium equation (4.1.18)$_1$ with

$$\underset{\sim}{GN}^{\beta} = [N^{\alpha\beta} + \frac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} + \theta^{\beta\lambda}N^{\alpha}_{\lambda}) - (\phi^{\alpha}M^{\lambda\beta} + \phi^{\beta}M^{\alpha\lambda} - \phi^{\lambda}M^{\alpha\beta})_{|\lambda}]\underset{\sim}{a}_{\alpha} +$$
$$+ \{\phi_{\lambda}N^{\lambda\beta} + [(1 - \frac{1}{2}\phi^{\kappa}\phi_{\kappa})M^{\lambda\beta}]_{|\lambda} + (\phi^{\beta}\phi_{\lambda|\mu} + \theta^{\beta}_{\cdot\lambda\mu})M^{\lambda\mu}\}\underset{\sim}{n} \tag{5.3.10}$$

When expressed in the reference basis (4.1.18)$_1$ with (5.3.8) or (5.3.10) leads to appropriate component forms of equilibrium equations.

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