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INSTITUTE OF FLUID-FLOW MACHINERY

WOJCIECH PIETRASZKIEWICZ

FINITE ROTATIONS AND LAGRANGEAN DESCRIPTION IN THE NON-LINEAR THEORY OF SHELLS

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1. Introduction

1.1. Preliminary remarks

In the theory of shells an attempt is made to do the impossible, namely to describe the stress and strain state in a thin three-dimensional body by means of a finite number of quantities defined at the middle surface of the body. Thus the theory of shells is by definition an approximate one and, generally, neither complete nor exact information as to the stress and strain state in a thin three-dimensional body can be provided by this theory. However, the partial information obtained from the two-dimensional shell equations is sufficiently accurate for the majority of practical applications. Simultaneously, it is very difficult or even impossible to obtain this information from three-dimensional equations of continuum mechanics.

The classical first-approximation linear theory of thin elastic shells, which began with works of Aron [1] and Love [2], has been widely developed during the recent hundred years. A number of scientific schools arose, many fundamental works were published. Various sets of basic equations of the classical linear theory of shells were formulated and various methods of solving them were discussed on the basis of vast number of shell problems met in practice. Most variants of the linear shell equations differ from each other only by small terms of the order of η/R in the definition of the tensor of change of curvature; here η is the largest strain of the shell and R is the smallest radius of curvature of its middle surface. Koiter [3] noted, that terms of this order result in negligibly small terms, when the expression for the shell strain energy is calculated. Hence all variants of the classical linear theory of shells, differing by terms of the order of η/R in the expression for the tensor of change of curvature, are equivalent from the strain energy point of view. Budiansky and Sanders [4] proposed to recognize as the best, among many equivalent variants of the classical linear theory of shells, the one developed independently by Sanders [5] and Koiter [3]. Some of its properties are: (a) tensors of internal forces and moments, as well as strain and change of curvature tensors are symmetric, (b) two-dimensional variational theorems, analogous to those of the three-dimensional linear theory of elasticity, are satisfied, (c) the static-geometrical analogy is valid, (d) constitutive equations are not coupled. Since then a significant drop of the number of new variants of the linear theory appearing in the literature of the subject was observed. The above mentioned "best" Sanders-Koiter variant can be regarded as the standard variant of the first-approximation linear theory of thin elastic shells.

As far as the non-linear theory of shells is concerned, the situation is still quite different. Despite of many tens of general studies, the majority of which was published in recent fifteen years, there is still a dispute as to the fundamentals of the theory. Various variants of the general non-linear theory of shells were discussed, among others, in monographs by WOŹNIAK [6], NAGHDI [7], LIBRESCU [8] (see also [82, 90]) and in reviews made by KOITER and SIMMONDS [9], WOŹNIAK [10], KOITER [11] and the present author [12, 89]. In these works the fundamentals of the non-linear theory of shells were discussed and an extensive number of references was quoted. Monographs by MUSHTARI and GALIMOV [13], VOLMIR [14, 15], GRIGOLIUK and KABANOV [16], BRUSH and ALMROTH [17] provide many examples of application of various simple variants of the non-linear shell equations, mainly the equations of geometrically non-linear theory of shallow shells. In these works an extensive reference to other studies of the subject may also be found. Thus it seems unnecessary for the author to review all problems arising during the construction of basic equations of the non-linear theory of shells. This allows for concentrating on selected problems, which up to now have not been explained satisfactorily in the literature of the subject.

The necessity to clearly distinguish between the Eulerian (spatial) description and the Lagrangean (material) description, as in the non-linear continuum mechanics [18], is one of fundamental problems appearing in the non-linear shell theory. Generally speaking, in the Eulerian description all quantities are defined and decomposed in the geometry of the space occupied by the shell in its actual deformed configuration, while in the Lagrangean description they are defined and decomposed in the geometry of the space occupied by the shell in the reference configuration. The later usually is taken as that corresponding to the natural state of the undeformed shell.

In the three-dimensional continuum mechanics the Eulerian and Lagrangean descriptions lead to relations, which are different only from the formal point of view. They are however physically equivalent and interrelated through transformation formulae between the systems of spatial and material coordinates in the three-dimensional Euclidean space. As such is the case, the requirements to be satisfied in each particular problem decide on the use of either the Eulerian or the Lagrangean description in the continuum mechanics.

It is not so when the solution of a non-linear problem for a thin body is sought through the solution of two-dimensional equations of the non-linear theory of shells, which equations are by definition approximate ones. Here the introduction of simplifications of three-dimensional relations in each of the two descriptions independently may lead eventually to physically different sets of two-dimensional shell relations, which in general can not be interrelated with the aid of transformation formulae between the systems of material and spatial surface coordinates. Also the transformation relations in a two-dimensional Riemannian space of the shell middle surface are in themselves more complicated than for three-dimensional problems.

Consider as an example a situation, when Lagrangean constraints of the type "material elements, normal to the middle surface of the shell in the reference configuration, remain straight during the shell deformation" are imposed on a shell deformation. This leads to shell relations, which physically differ from those resulting from the imposition of Eulerian constraints of the type "material elements, normal to the middle surface of the shell in the actual configuration, remain straight during the shell deformation". In the case of statics both types of constraints are dual from the formal point of view [19]. The Lagrangean shell relations, resulting under the Lagrangean constraints, may easily be transformed into the Eulerian form by using two-dimensional transformation relations between material and spatial coordinates. The inverse transformation may be applied to the Eulerian shell relations resulting under the Eulerian constraints mentioned above in order to obtain their Lagrangean form. It is obvious, that in this case both types of constraints would lead to different sets of the Lagrangean or Eulerian shell relations. A physical equivalence of shell equations in both these descriptions is possible in particular cases of constraints only, for instance in the case of assumption of Kirchhoff-Love constraints "material elements, normal to the shell middle surface in one configuration, remain normal to it in any configuration and do not change their lengths during the shell deformation". It is therefore essential, when formulating the basic relations of the non-linear theory of shells, to both distinguish the configuration in which approximations of three-dimensional problem are made (eg. the natural state of undeformed shell, a fixed reference configuration, the variable actual shell configuration, and so on) and distinguish the description (Eulerian or Lagrangean) used for simplified two-dimensional relations.

In many works of general nature, especially when the concept of convected coordinate system is used, no information at all is given as to the kind of description applied. In the convected coordinate system some analogous Eulerian and Langrangean tensors may have numerically identical representations, although these representations are related to entirely different base systems. These bases, however, do not enter the considerations and relations. The notion of convected coordinate system, which has many advantages as far as the mnemonics is taken into account, is widely used in the shell literature (see, for instance, references [7, 9, 11, 12, 20, 21, 22, 23]). More general coordinate systems have been used in few papers only [6, 10, 19, 24]. Which of the two descriptions, the Eulerian description or the Lagrangean one, is used in a paper, in which the convected coordinate system is adopted, can be concluded only indirectly, mainly from the final form of equations of motion and natural boundary conditions.

The equations of motion and natural boundary conditions have in the Eulerian description a relatively simple and clear structure and they are easy to operate with in considerations of general nature. For this reason the Eulerian description, with various initial assumptions, has readily been used for the formation of equations of motion. Here general studies of Woźniak [6], Green and Naghdi [25], Naghdi [7], Sanders [20], Krätzig [26] and Langhaar [23] can be mentioned. There is however a disadvantage of using the Eulerian description, consisting in that the geometry of actual shell configuration is, as a rule, unknown prior to the solution of the problem, so a direct use of simple Eulerian formulae is usually out of question. Each of the Eulerian quantities and operations, defined in an unknown geometry of the actual shell configuration, must be expressed first in terms of the Lagrangean quantities and operations, which are defined in a known geometry of the reference shell configuration. However, such transformations are, as a rule, omitted in general works in which the Eulerian description was adopted. It was only for the non-linear Kirchhoff-Love type theory of shells that formulation of Eulerian relations in terms of the Lagrangean ones was discussed; this concerns the works of Da-

NIELSON [27] and the present author [12, 21], and – with an additional small strain assumption – also those of KOITER [11], WOŹNIAK [6], SANDERS [20], KOITER and SIMMONDS [9] and others. Transformation rules were also discussed by WOŹNIAK [6] within the frames of the small strain theory of shells based on a more general kinematic hypothesis "material elements that are straight in one configuration remain such in any configuration".

In considerations of actual problems of the non-linear theory of shells there is always at least one configuration of a shell with the known geometry, most often that of the natural state of undeformed shell. The known natural state (or another configuration with the known geometry) can be adopted as the reference configuration and simplifying assumptions with respect to this configuration can be introduced. Then it is possible from the beginning to formulate all basic relations of the non-linear theory of shells in the Lagrangean description, expressing them entirely in terms of quantities and operations defined in the known reference configuration of the shell. As a consequence of application of the Lagrangean description components of the shell deformation gradient appear explicitly in the equations of motion and natural boundary conditions. This causes that their structure become more complex than that of relations to which the Eulerian description leads. However, even for a most general variant of the non-linear theory of shells in the Lagrangean description, further transformations of the equations are unnecessary and an immediate use of them is potentially possible. All the relations and operations are here defined completely in the known geometry of the reference shell configuration, which allows for direct programming for calculations made with the aid of a computer.

In the pioneer works of KIRCHHOFF [28] and KÁRMÁN [29], concerning large bending of plates, and also in widely known fundamental works of DONNELL [30], VLASOV [31], MARQUERRE [79] and others, dealing with the simplest geometrically non-linear variants of basic equations for shells, the Lagrangean description is used, even though not stated clearly by the authors. However, this is not the case for the majority of more recent works of general nature which, giving up the use of too strong simplifying assumptions, take advantage of mainly the Eulerian description. In several works the basic equations of the non-linear theory of shells have been formulated from the beginning in the Lagrangean description. Some results of these works will be discussed below.

SANDERS [20] quoted, without deriving, the Lagrangean equations of equilibrium (formulae 58′–60′) within the frames of Kirchhoff-Love type non-linear theory of shells. These relations, when linearized, do not lead however to the "best" variant of the classical linear theory of shells. Another form of the Lagrangean equations of equilibrium, whose linearization results in the "best" variant of the linear theory, was proposed by BUDIANSKY [22]. He used there an unconventional definition of the tensor of change of curvature, which made it possible to express all relations in the form of polynomials in components of the displacement vector. Unfortunately, the final results turned out to be so involved that in [22] the basic relations were presented in the operator form rather than in the final expanded form. The present author proposed in [21] completely defined Lagrangean equations of equilibrium and natural boundary conditions in a symmetric, easily memorized form. The formulae given in [21] were expressed in terms of components of symmetric tensors of internal forces and internal moments, and symmetric measures of strain and change of curvature of the shell middle surface. Linearization of these formulae leads to the "best" variant of the linear theory of shells [4]. Another work [32] of the author contains also the Lagrangean non-linear equations of motion of shells moving in a non-inertial frame of reference; besides, in that work the Lagrangean equations of shell stability were formulated. SHRIVASTAVA and GLOCKNER [33] presented Lagrangean equations of shell equilibrium in terms of unsymmetric quantities.

A more general assumption about the linear distribution of displacements across the shell thickness in the reference configuration, was adopted in works [19, 24] by the present author. Lagrangean equations of motion and natural boundary conditions for a shell were obtained in [19, 24] by direct integration of Lagrangean form of three-dimensional equations of motion of continuum across the thickness of the shell in the reference configuration. Two separate systems of coordinates were used, namely material and spatial coordinates. It was shown [19] that shell relations obtained as a result of introduction formally dual simplifications to continuum equations in, separately, the reference and actual configurations, are physically nonequivalent. Relations based on the assumption of linear distribution of displacements in the reference configuration were also derived by HABIP and EBCIOGLU in [34], and HABIP in [35] using a modified Hellinger-Reissner variational principle and a convected coordinate system. From a comparison of equations of motion obtained in [34, 35] with those derived by the author [19, 24] it is evident, that in [34, 35] terms containing components of moments $M^{\beta 3}$ were for the most part omitted. However, the contribution of $M^{\beta 3}$ to the shell strain energy is, even for small strains, of the same order of magnitude as the contribution due to shear forces or coupling between various strain measures. It seems then that omission of terms with $M^{\beta 3}$ without simultaneous omission of other terms of the same order of magnitude is improper at this level of generality.

Other variants of the Lagrangean equations, whose range of applications is narrower than in [19, 24] but wider than in [21], can be found in works of EBCIOGLU [36], OSHIMA, SEGUCHI and SHINDO [37], and AINOLA [38, 39]. On the other hand, YOKOO and MATSUNAGA [40] did not use any simplified assumptions but expanded the Lagrangean equations of motion of continuum, together with constitutive equations for an elastic material, into Taylor series in the vicinity of the middle shell surface. As a result extremely complex recurrent relations were obtained. More promising seems an approach used by NOVOTNY [41], who applied a method of asymptotic integration [42] to the Lagrangean equations of motion of continuum. As the zeroth approximation to the internal shell problem he obtained the known equations of the non-linear theory of shallow shells.

Explicit appearance of components of the deformation gradient tensor in the Lagrangean shell relations requires that the structure of the shell deformation gradient tensor, and its rotational part in particular, be studied more closely. It is known that appearance of the large rotations of the material elements of a shell during its deformation is the basic kinematic property of any non-linear theory of shells. Large rotations of the material elements appear even if one restricts the considerations to the small strain theory only. Consider as a trivial example the rolling of a sheet of paper into a cylinder, where strains are small but rotations can be even multiplicity of 2π.

For a three-dimensional continuum the description of the rotational part of deformation gradient, expressed by a tensor of rotation **R**, was studied, among others, in monographs by TRUESDELL and TOUPIN [43], TRUESDELL and NOLL [18], and ERINGEN [44].

An equivalent description, expressed by a finite rotation vector $\mathbf{\Omega}$, was used by SHAMINA [45] for investigations of the combatibility conditions in a three-dimensional continuum. Rotation parameters appear explicitly in many three-dimensional relations. They are especially important in the theory of constitutive equations [18] and in the analytic mechanics of rigid-body motion [46, 47].

It was already shown by NOVOZHILOV [48] that, for small strains and when rigid-body motion has been eliminated, only small rotations of material elements may appear in a three-dimensional elastic body. On the other hand, under the same assumptions, large rotations of material elements can occur in this bodies, such as beams, thin-walled rods, plates and shells. Thus there is a substantial qualitative difference between such bodies and those of truly three-dimensional nature. This shows that in the non-linear theory of shells the rotational part of deformation should play much more important role than in the problems of continuum mechanics.

There exist many works concerned with simplified variants of geometrically non-linear Kirchhoff-Love type theory of shells, where such names as a theory of "small rotations", "moderately small rotations", "moderate rotations", "large rotations" etc. have been used. Judging from these names one should conclude that appropriate restrictions have been imposed on components of finite rotations of material elements during the shell deformation. In spite of that a more close analysis of all works concerned with these problems has shown, that such restrictions were imposed on either surface displacement gradients and their aggregates, that do not constitute components of the finite rotation vector, or components of the linearized rotation vector used in the classical linear theory of shells.

SANDERS [20] defined the variant of "moderately small rotation" by setting restrictions on components of the linearized rotation vector to the effect, that squares of these components can be at most of the order of magnitude of strains. KOITER [11] defined the variant of "moderate deflections" by adopting the assumption that the surface displacement gradients are small as compared with unity. The variant of "small finite deflections" was defined by additionally limiting the values of components of the linearized rotation vector to the range where their square will not exceed the order of magnitude of strains. CHIEN [49] and MUSHTARI and GALIMOV [13] postulated various restrictions to be set on strain tensor and tensor of change of curvature. DUSZEK [50] based her simplifications on postulated orders of magnitude for displacements and their derivatives. Restrictions imposed on combinations of displacement gradients or on components of linearized rotation vector were also used by MUSHTARI and GALIMOV [13], HABIP [35], EBCIOGLU [35] and the present author [12, 51, 52].

Obviously, for small rotations the parameters of the finite rotation of the shell material fibres are equivalent, within the accuracy defined by simplified assumptions adopted, to the parameters of the linearized rotation. Note that components of the linearized rotation vector are linear functions of the surface displacement gradients. As the rotations become larger and larger, the numerical values of parameters of the finite rotation differ more and more from the numerical values of the linearized rotation parameters. Thus no substantial reasons can be given for the use of the linearized rotation vector in the non-linear theory of shells. The only motivation for this situation seems to be found in the fact, that a theory of finite rotations for the non-linear two-dimensional problems of the theory of shells has not been developed as yet.

To the best of the author's knowledge there are only three works from the domain of the theory of shells in which the finite rotation vector has been used explicitly, all of them dealing with the non-linear theory of shells of Kirchhoff-Love type. SIMMONDS and DANIELSON [53, 54] employed the finite rotation vector as one of two basic independent variables (the stress function vector or the symmetric tensor of internal forces is the second independent variable) in terms of which all remaining relations of so called intrinsic theory of shells were expressed. NOVOZHILOV and SHAMINA [55] used the notion of the finite rotation vector for the derivation of kinematical and deformational boundary conditions. In each of the papers the finite rotation vector was introduced in a descriptive manner, without reference to other quantities characterizing the shell deformation, such as the displacement field or the deformation gradient tensor.

It seems that several factors are responsible for the lack of works on the theory of finite rotations in shells, despite of their potentially great importance for any thin body. First of all, difficulties arising in connection with the numerical analysis of the non-linear problems have caused that most of the numerical calculations for various non-linear problems of shells has been based on the simplest Donnell-Mushtari-Vlasov variant of the geometrically non-linear theory of shallow shells. In this variant terms containing squares of displacement normal to the middle surface of the shell are the only non-linear terms, and permissible rotations of the material elements do not exceed several degrees. For this variant, which forms the simplest generation of classical equations of the non-linear theory, the difference between finite rotations and linearized rotations is negligibly small indeed. Numerical calculations of the non-linear shell problems involving large or even unlimited rotations became possible only recently, when large and fast computers have been put into operation.

Secondly, as it was already mentioned, most of the general studies on the non-linear theory of shells had employed the convected coordinate system. This causes that the part of deformation that concerns strain is the only one defined explicitly. The rotational part of deformation is automatically eliminated and does not directly enter most of the relations which contain components of tensors with respect to the moving convected system bases. Thirdly, a direct and the most simple method of separation of the strain and the rotational parts of deformation consists in applying the polar decomposition to the deformation gradient tensor [18]. This is associated with employment of some notions of tensor calculus in the absolute notation. This form of tensor calculus has been however used only in some parts of the works [19, 24, 56, 57] concerning shell problems.

1.2. Scope of the work

Formulation of basic relations of the non-linear theory of shells in the Lagrangean description, the theory of finite rotations in shells and a number of associated problems are presented in this study. In this paragraph the contents of respective chapters will be presented and results which are believed to be novel in the literature will be pointed out.

In chapter 2 basic definitions and geometrical relations for a surface and a three-dimensional space are presented, some relations of the tensor algebra and analysis in the absolute notation are given, and relations for a normal coordinate system are derived. The description of tensor fields used in this work, based on the notion of tensor calculus in the absolute notation, is a convenient alternative to the descriptions as proposed by NAGHDI [58] and WOŹNIAK [6].

Chapter 3 contains a description of the shell deformation. The shell deformation function has been expanded into a Taylor series in terms of the shell thickness in a reference configuration, with the only linear part of the series (3.1.4) taken into account. A deformation gradient tensor **G** introduced in this chapter provides an exact and complete information about the state of shell deformation in the neighbourhood of middle surface points. The displacement field is linear across the thickness of the shell, (3.1.7); it is expressed by means of two independent vector parameters, namely a displacement **u** of the shell middle surface and a vector $\boldsymbol{\beta}$ of the change of direction of the normal material fibres during the shell deformation. An analysis of the shell strain measures in the Lagrangean and Eulerian descriptions, as carried out in the chapter, has shown, that in a convected coordinate system the Lagrangean and Eulerian measures take formally the same forms, (3.2.7) and (3.2.15). Essential geometric differences between them become evident when the absolute notation, (3.2.5), (3.2.6) and (3.2.13), (3.2.14), is used. In this chapter the shell deformation under Kirchhoff-Love constraints is also discussed.

An exact theory of finite rotations in shells is presented in chapter 4. A polar decomposition of the deformation gradient tensor **G** has been used to decompose the shell deformation into the pure translation, the pure stretch along the principal directions of strain, followed by the finite rotation of the principal directions. An exact formula (4.1.24) for the rotation tensor **R** is given. This is obtained by using an intermediate basis $\breve{\mathbf{a}}_a$ $(a=1, 2, 3)$ generated from the reference basis $\mathbf{a}_a$ by its pure stretching along the principal directions of the Lagrangean strain measures. An equivalent description of the rotational part of deformation by means of a finite rotation vector $\boldsymbol{\Omega}$ is introduced. The direction and sense of this vector is defined by an unit vector **e** of the principal direction of tensor **R**, while its length is equal to $|\sin\omega|$, where ω is the angle of rotation about **e**. Three equivalent exact relations, (4.2.6), (4.2.8), and (4.2.11), expressing $\boldsymbol{\Omega}$ in terms of **u** and $\boldsymbol{\beta}$, are presented. Also in chapter 4 the geometry of the base $\breve{\mathbf{a}}_a$ is discussed together with the modification of strain measures and differentiation of the vector $\boldsymbol{\Omega}$. As a result many interesting geometrical relations expressed in terms of $\boldsymbol{\Omega}$ are derived.

A thorough analysis of deformation of a shell boundary element is presented. Special attention is paid to the rotational part of deformation, consisting of two subsequent rotations. The first rotation is related to a pure stretch along the principal directions of strain, the second one – to a finite rotation of the principal directions. Exact formulae (4.3.27) and (4.3.28) for a total rotation tensor $\mathbf{R}_t$ and its equivalent – a total finite rotation vector $\boldsymbol{\Omega}_t$ – are obtained for the shell boundary element. Purely geometrical considerations lead to three general forms of geometrical boundary conditions: 1) displacement boundary conditions (4.4.3) expressed in terms of parameters **u** and $\boldsymbol{\beta}$; 2) kinematical boundary conditions (4.4.9) expressed in terms of the vector $\boldsymbol{\Omega}_t$ and physical components γ_{tt}, γ_{3t} and γ_{33} of the Green strain tensor; 3) deformational boundary conditions (4.4.40) expressed in terms of the vector $\mathbf{k}_t$ of the change of curvature of the boundary line and the components γ_{tt}, γ_{3t} and γ_{33}. The components of $\mathbf{k}_t$ are shown to depend only on values of strain measures at the shell boundary.

The contents of chapter 4 is novel in the literature concerning the shell theory. Formulae derived in paragraph 4.1–4.4 are exact at the shell middle surface. Relations valid for various simplified variants of the shell theory can be obtained by making the appropriate simplifications. In particular, various simplified relations for the non-linear theory of shells under Kirchhoff-Love type constraints are given. At the shell boundary some of the simplified results agree with those obtained recently by NOVOZHILOV and SHAMINA [55].

The principle of stationary action has been used in chapter 5 to derive a basic set of six equations of motion of shells in the Lagrangean description, together with corresponding natural boundary conditions. The obtained results are in accordance with the assumption of linear distribution of deformation across the shell thickness. Their symmetric structure enables us to present the equations of motion in a clear and short absolute form in terms of some vectors and tensors. Decomposition of this form with respect to the actual basis $\bar{\mathbf{a}}_a$, the reference basis $\mathbf{a}_a$ and the intermediate basis $\check{\mathbf{a}}_a$ leads to five equivalent sets of shell equations written out entirely in terms of Lagrangean quantities. These relations are new in the literature of the subject. As particular cases, several Lagrangean forms of equations for the Kirchhoff-Love type shell theory are also obtained.

Chapter 6 contains an analysis of possible simplifications of basic shell relations in the case of elastic material and under an assumption of small strains everywhere in the shell. By making use of expansions of each quantity into series with respect to the shell thickness one obtains a two-dimensional elastic strain energy function in the form of infinite series. Introduction of a small parameter ϑ and use of estimates of the stress components and their derivatives given by JOHN [62, 63] makes it possible to prove, that the elastic strain energy function for a shell consists of only seven important terms. Two leading terms have the order of magnitude of $Eh\eta^2$ and five terms $O(Eh\eta^2\vartheta^2)$. The remaining terms are of the higher order. The first two terms, which take into account the main part of the elastic strain energy due to membrane and bending strains, as well as due to change in the shell thickness, correspond to the first-approximation classical theory of thin elastic shells as studied by KOITER [3]. The five additional terms in (6.1.13) constitute a consistent correction to the elastic strain energy of the first-approximation theory. This makes it possible to construct the proper constitutive equations (6.1.14) of the second-approximation theory of thin elastic shells and to obtain a consistently simplified set of six equations for geometrically non-linear bending shell theory of Reissner type. Consistent simplification of some geometric relations as well as kinematical and deformational boundary conditions is also presented. Relations for the geometrically non-linear Kirchhoff-Love type theory of shells are obtained as a particular case.

In chapter 7 a simplification of relations of the small strain theory under additional restrictions imposed on the finite rotation parameters is discussed. Taking the restriction of the finite rotation angle ω as the basis for classification of simplified variants, one can distinguish four cases: theory of small, moderate, large, and finite rotations. For the theory of moderate rotations consistently simplified formulae of the general theory of shells are given, together with discussion of a number of particular cases, such as the Kirchhoff-Love type theory and the classical theory of shallow shells.

The results obtained in this treatise disclosed a number of problems of basic nature still awaiting their solution. These are discussed briefly in chapter 8.

1.3. General scheme of notation

Sets of material particles are denoted by $\mathscr{S}, \mathscr{M}, \mathscr{B}, \ldots$ while their images in a three-dimensional Euclidean point space $\mathscr{E}$ are denoted by $\mathscr{S}_\kappa, \mathscr{M}_\kappa, \mathscr{B}_\kappa$ or $\mathscr{S}_\gamma, \mathscr{M}_\gamma, \mathscr{B}_\gamma$, where the subscripts κ or γ indicate the reference or the actual configurations, respectively.

Points of the space are denoted by $M, P, Q, \ldots \in \mathscr{E}$. Vectors and tensors are denoted in bold-type symbols, e.g. $\mathbf{u}$, $\mathbf{v}$, $\boldsymbol{\beta}$, $\mathbf{N}^\alpha, \mathbf{M}^\alpha \in \mathscr{V}$; $\mathbf{E}, \mathbf{G}, \mathbf{R}, \boldsymbol{\gamma}, \boldsymbol{\kappa}, \mathbf{N} \in \mathscr{T}_2$; $\mathbf{L} \in \mathscr{T}_4$. Components of vectors and tensors in the respective bases are denoted by identical symbols in normal type with the appropriate subscripts or superscripts, e.g. u^α, $N^{\alpha\beta}$, γ_{ab}.

A convected system of curvilinear coordinates ϑ^1, ϑ^2, ϑ^3 is used throughout the work. This system coincides in the reference configuration κ with the normal coordinate system.

Indices i, j, k, l assuming values of 1, 2, 3, and indices $\varphi, \psi, \vartheta, \ldots$ assuming values of 1, 2 are used for the description of components of vectors and tensors in a point belonging to a shell space. Indices a, b, c, d, which take values of 1, 2, 3, and indices $\alpha, \beta, \gamma, \ldots$ assuming values of 1, 2, are used for the description of components of vectors and tensors defined at the middle surface of the shell space. A use of the same index as a subscript and superscript means a summation over this index.

A partial derivative is denoted by means of a coma, a covariant spatial derivative – by means of a semicolon, and a covariant surface derivative is denoted by means of a vertical stroke: $T^{ij}_{\ ,k}$, $N^{ab}_{\ ;c}$, $\gamma_{\alpha\beta|\gamma}$.

Natural bases of the coordinate system ϑ^i are denoted by $\mathbf{g}_i$, $\mathbf{a}_a$, whereas $\mathbf{e}_m$, $\mathbf{i}_m$, $\mathbf{k}_m$ denote orthonormal bases not related to ϑ^i.

Each geometric object and operation defined in the actual configuration, which is related through the shell motion to analogous symbol or operation in the reference configuration κ, is distinguished by an additional bar, e.g. $\bar{P}$, $\bar{\mathbf{g}}_i$, $\bar{\mathbf{a}}_a$, $\bar{G}^c_{ab}$, $\bar{\mathbf{N}}$, $(\ )_{||\alpha}$. Analogous objects related to an intermediate base $\check{\mathbf{a}}_a$ are distinguished by the mark $\check{}$, e.g. $\check{\gamma}_{ab}$, $\check{\boldsymbol{\Omega}}_t$, $\check{\mathbf{v}}$, $\check{\mathbf{N}}$, $\check{\mathbf{a}}_t$.

A scalar product of vectors is denoted by a dot (which is sometimes omitted), a vector product is denoted by a cross $\times$, and a tensor product – by the symbol $\otimes$.

Definitions of all symbols are given directly where they appear for the first time.

2. Introductory relations

2.1. Notation and geometric relations

This work is concerned with a body $\mathscr{S}$ consisting of material particles $X, Y, \ldots \in \mathscr{S}$. The body $\mathscr{S}$ is considered in a reference configuration $\kappa: \mathscr{S} \rightarrow \mathscr{S}_\kappa \subset \mathscr{E}$ and in the actual configuration $\gamma: \mathscr{S} \rightarrow \mathscr{S}_\gamma \subset \mathscr{E}$, where $\mathscr{S}_\kappa$ and $\mathscr{S}_\gamma$ are regions of a three-dimensional Euclidean point space $\mathscr{E}$ that are occupied by the body $\mathscr{S}$ in the configurations κ and γ [18, 19, 24].

The motion of a particle $X \in \mathscr{S}$ is defined by the deformation $\chi = \gamma \circ \kappa^{-1}$ from the reference configuration κ, $\bar{P} = \chi(P, t)$, where $P = \kappa(X)$ and $\bar{P} = \gamma(X, t)$ denote places of the particle $X \in \mathscr{S}$ in the configurations κ and γ, respectively, while the parameter t denotes time.

Characteristic for the body as considered in this work is, that at any time t one dimension of the region $\mathscr{S}_\kappa$ as well as of the family of regions $\mathscr{S}_\gamma$ is always much smaller than the other two. Such a body $\mathscr{S}$ will be called a shell.

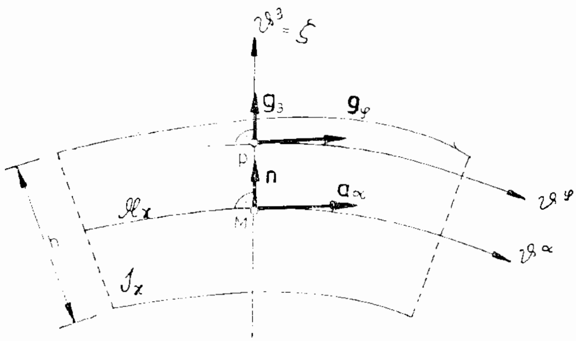

Fig. 1

Let us introduce in the region $\mathscr{S}_\kappa$ a normal curvilinear coordinate system $\{\vartheta^i\}$, $i = 1, 2, 3$, such that a surface $\vartheta^3 = 0$ defines the middle surface $\mathscr{M}_\kappa$ of the region $\mathscr{S}_\kappa$. The coordinate $\vartheta^3 \equiv \zeta$ is the measure of distance between points $P \in \mathscr{S}_\kappa$ and $\mathscr{M}_\kappa$, with $-h/2 \leqslant \zeta \leqslant +h/2$, where h is the thickness of the region $\mathscr{S}_\kappa$ (Fig. 1). The set of particles $Y = \kappa^{-1}(M)$, $M \in \mathscr{M}_\kappa$, forms a material surface $\mathscr{M} \subset \mathscr{S}$ called hereafter the middle surface of the shell.

Geometric quantities of the region $\mathscr{S}_\kappa \subset \mathscr{E}$ are defined by a position vector $\mathbf{p} = \mathbf{p}(P) = \mathbf{p}(\vartheta^i) = P - O$, $O \in \mathscr{E}$, which is used for defining the natural base vectors $\mathbf{g}_i$, dual base vectors $\mathbf{g}^j$, components g_{ij}, δ_i^j, g^{ij} of the metric tensor, components ϵ_{ijk}, ϵ^{ijk} of the Ricci tensor, Christoffel symbols G_{kij}, G_{ij}^k, spatial covariant derivative operation $(\ )_{;i}$ as well as other quantities and operations that have been discussed in detail e.g. in [51, 58, 64, 81]. Some of these relations most often used in the present work have been presented below:

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \vartheta^i} \equiv \mathbf{p}_{,i}, \qquad \mathbf{g}^j \cdot \mathbf{g}_i = \delta_i^j \tag{2.1.2}$$

$$g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \qquad g^{ij} = \mathbf{g}^i \cdot \mathbf{g}^j, \qquad |g_{ij}| = g$$

$$\epsilon_{ijk} = \mathbf{g}_i \cdot (\mathbf{g}_j \times \mathbf{g}_k), \qquad \epsilon^{ijk} = \mathbf{g}^i \cdot (\mathbf{g}^j \times \mathbf{g}^k) \tag{2.1.3}$$

$$\mathbf{g}_i \times \mathbf{g}_j = \epsilon_{ijk} \mathbf{g}^k, \qquad \mathbf{g}^i \times \mathbf{g}^j = \epsilon^{ijk} \mathbf{g}_k$$

$$\mathbf{g}_{i,j} = G_{kij} \mathbf{g}^k = G_{ij}^k \mathbf{g}_k, \qquad \mathbf{g}^i_{,j} = -G_{jk}^i \mathbf{g}^k$$

$$G_{kij} = \tfrac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}) = \mathbf{g}_k \cdot \mathbf{g}_{i,j} \tag{2.1.4}$$

$$G_{ij}^k = g^{kl} G_{lij} = \mathbf{g}^k \cdot \mathbf{g}_{i,j} = -\mathbf{g}_i \cdot \mathbf{g}^k_{,j}$$

$$\mathbf{v}_{,j} = v^i_{;j}\,\mathbf{g}_i = v_{i;\,j}\,\mathbf{g}^i$$

$$v_{i;\,j} = v_{i,\,j} - G^k_{ij}\,v_k\,, \qquad v^i_{;j} = v^i_{,j} + G^i_{kj}\,v^k \tag{2.1.5}$$

$$T^i_{.j;\,k} = T^i_{.j,\,k} + G^i_{kl}\,T^l_{.j} - G^l_{jk}\,T^i_{.l}$$

Geometric quantities describing the surface $\mathcal{M}_\kappa$ are defined by a position vector of the surface points

$$\mathbf{r} = \mathbf{r}(M) = \mathbf{r}(\vartheta^\alpha) = M - O\,, \qquad \alpha = 1\,,2\,. \tag{2.1.6}$$

Defined with the aid of this vector are, among others, the natural base vectors $\mathbf{a}_\alpha$, dual base vectors $\mathbf{a}^\beta$, unit vector $\mathbf{n} \equiv \mathbf{a}_3 \equiv \mathbf{a}^3$ normal to $\mathcal{M}_\kappa$, coefficients $a_{\alpha\beta}$ and $b_{\alpha\beta}$ of the first and second quadratic forms, components $\epsilon_{\alpha\beta}$ and $\epsilon^{\alpha\beta}$ of the surface Ricci tensor, Gauss curvature K and mean curvature H of the surface $\mathcal{M}_\kappa$, Christoffel symbols $\Gamma_{\lambda\alpha\beta}$, $\Gamma^\lambda_{\alpha\beta}$, surface covariant derivative operation $(\ )_{|\alpha}$ and also other quantities and operations as discussed in detail in, for instance, [11, 51, 58, 82, 64]. Some of them, containing the relations most often used in this work, have been listed below:

$$\mathbf{a}_\alpha = \frac{\partial \mathbf{r}}{\partial \vartheta^\alpha} = \mathbf{r}_{,\alpha}\,, \qquad \mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha$$

$$a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta\,, \qquad a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta\,, \qquad \mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta \tag{2.1.7}$$

$$\mathbf{n} = \tfrac{1}{2}\epsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta = \tfrac{1}{2}\epsilon_{\alpha\beta}\,\mathbf{a}^\alpha \times \mathbf{a}^\beta$$

$$\mathbf{a}_\alpha \times \mathbf{a}_\beta = \epsilon_{\alpha\beta}\,\mathbf{n}\,, \qquad \mathbf{n} \times \mathbf{a}_\alpha = \epsilon_{\alpha\beta}\,\mathbf{a}^\beta \tag{2.1.8}$$

$$b_{\alpha\beta} = -\mathbf{n}_{,\alpha} \cdot \mathbf{a}_\beta = \mathbf{a}_{\alpha,\,\beta} \cdot \mathbf{n}$$

$$H = \tfrac{1}{2}b^\alpha_\alpha\,, \qquad K = \tfrac{1}{2}\epsilon^{\alpha\lambda}\epsilon_{\beta\mu}\,b^\beta_\alpha\,b^\mu_\lambda = \frac{b}{a}$$

$$a = |a_{\alpha\beta}|\,, \qquad b = |b_{\alpha\beta}| \tag{2.1.9}$$

$$\Gamma_{\lambda\alpha\beta} = \mathbf{a}_\lambda \cdot \mathbf{a}_{\alpha,\,\beta} = \tfrac{1}{2}(a_{\lambda\alpha,\,\beta} + a_{\lambda\beta,\,\alpha} - a_{\alpha\beta,\,\lambda})$$

$$\Gamma^\mu_{\alpha\beta} = a^{\lambda\mu}\Gamma_{\lambda\alpha\beta} = \mathbf{a}^\mu \cdot \mathbf{a}_{\alpha,\,\beta} = -\mathbf{a}_\alpha \cdot \mathbf{a}^\mu_{,\beta} \tag{2.1.10}$$

$$\mathbf{v} = v^\alpha \mathbf{a}_\alpha + v^3 \mathbf{n}$$

$$\mathbf{v}_{,\beta} = (v^\alpha|_\beta - b^\alpha_\beta\,v^3)\,\mathbf{a}_\alpha + (v^3_{,\beta} + b_{\beta\gamma}\,v^\gamma)\,\mathbf{n} \tag{2.1.11}$$

$$v^\alpha|_\beta = v^\alpha_{,\beta} + \Gamma^\alpha_{\lambda\beta}\,v^\lambda$$

$$T^\alpha_{.\beta|\lambda} = T^\alpha_{.\beta,\,\lambda} + \Gamma^\alpha_{\mu\lambda}\,T^\mu_{.\beta} - \Gamma^\mu_{\beta\lambda}\,T^\alpha_{.\mu}$$

$$b_{\beta\lambda|\mu} = b_{\beta\mu|\lambda}$$

$$b_{\alpha\lambda}\,b_{\beta\mu} - b_{\alpha\mu}\,b_{\beta\lambda} = \epsilon_{\alpha\beta}\,\epsilon_{\lambda\mu}\,K \tag{2.1.12}$$

In what follows we shall often use some concepts and relations of algebra and tensor analysis in the three-dimensional Euclidean point space expressed directly in an absolute notation [65 ÷ 70]. The use of absolute calculus, resulting mainly from the adoption of a number of relations from the continuum mechanics [18], made it possible to substantially shorten and simplify many known relations, and to present their simpler interpretation. The absolute notation has also helped to obtain relations, which are new in the non-linear mechanics of shells.

A tensor product $\otimes$ of two vectors $\mathbf{u}, \mathbf{v} \in \mathscr{V}$ belonging to a three-dimensional Euclidean vector space, gives an Euclidean tensor $\mathbf{u} \otimes \mathbf{v} \in \mathscr{T}_2 = \mathscr{V} \otimes \mathscr{V}$ of the second order, defined by the relation

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) \tag{2.1.13}$$

for each $\mathbf{w} \in \mathscr{V}$. If $\mathbf{g}_i$, $\mathbf{g}^j$, $\mathbf{a}_a$, ... $(i, a = 1, 2, 3)$ are the bases of $\mathscr{V}$ then any set of nine tensors of the type $\mathbf{g}_i \otimes \mathbf{g}_j$, $\mathbf{g}_i \otimes \mathbf{a}_a$, $\mathbf{a}_a \otimes \mathbf{g}^j$, ... $\in \mathscr{T}_2$ is the base of the space $\mathscr{T}_2$. Hence any tensor $\mathbf{T} \in \mathscr{T}_2$ can be represented in the form

$$\mathbf{T} = T^{ij}\mathbf{g}_i \otimes \mathbf{g}_j = T^{ia}\mathbf{g}_i \otimes \mathbf{a}_a = T^{\alpha}_{.j}\, \mathbf{a}_a \otimes \mathbf{g}^j = \ldots \tag{2.1.14}$$

For a tensor $\mathbf{T} \in \mathscr{T}_2$ the following operations will be used: transposition $\mathbf{T}^T$, trace $\operatorname{tr} \mathbf{T}$, determinant $\det \mathbf{T}$, and inverse $\mathbf{T}^{-1}$. If $\mathbf{S} \in \mathscr{T}_2$ then the simple and total dot operations are defined by the relations

$$\begin{aligned} \mathbf{TS} &= T^{ij} S^{kl} g_{jk}\, \mathbf{g}_i \otimes \mathbf{g}_l = T^i_j\, S^{jl} \mathbf{g}_i \otimes \mathbf{g}_l \\ \mathbf{T} \cdot \mathbf{S} &= T^{ij} T^{kl} g_{ik}\, g_{jl} = T^{ij} S_{ij} \end{aligned} \tag{2.1.15}$$

Derivatives of a tensor function $f\colon \mathscr{T}_2 \to \mathscr{T}_p$, $\mathbf{P} = f(\mathbf{T})$ will be denoted by $f_{,\mathbf{A}}, f_{,\mathbf{AA}}, \ldots$ They will be computed according to formulae presented in [65, 66, 68].

2.2. Relations for the normal coordinate system

For the normal coordinate system $\{\vartheta^i\}$ as introduced above we have

$$P = M + \zeta \mathbf{n}, \qquad \mathbf{p} = \mathbf{r} + \zeta \mathbf{n} \tag{2.2.1}$$

The metric tensor $\mathbf{1} \in \mathscr{T}_2$ in the region $\mathscr{S}_\kappa$ can be conveniently represented in three different ways

$$\begin{aligned} \mathbf{1} = \mathbf{1}^T &= g_{ij}\, \mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{g}_i \otimes \mathbf{g}^i \\ &= a_{ab}\, \mathbf{a}^a \otimes \mathbf{a}^b = \mathbf{a}_a \otimes \mathbf{a}^a \\ &= \mu^a_i\, \mathbf{a}_a \otimes \mathbf{g}^i = \mu^j_b\, \mathbf{g}_j \otimes \mathbf{a}^b \end{aligned} \tag{2.2.2}$$

where, for the system adopted

$$\mathbf{a}_a(M) = \delta^i_a\, \mathbf{g}_i(P)\big|_{P=M} \equiv \{\mathbf{a}_\alpha, \mathbf{n}\} \tag{2.2.3}$$

In the above relations we have used different symbols and indices for the components of various quantities with respect to the bases, which are natural in P or M. This differentiation will be important for the further part of the present work. It is assumed in particular, that indices $i, j, k, l = 1, 2, 3$ and $\varphi, \psi, \vartheta = 1, 2$ will indicate components in the base $\mathbf{g}_i$ natural in P, while the indices $a, b, c, d=1,2,3$ together with $\alpha, \beta, \lambda, \mu=1,2$ will be used for components in the base $\mathbf{a}_a$ natural in M.

From (2.2.1) and (2.2.2) it follows that

$$\mathbf{g}_i=\mu_i^a\,\mathbf{a}_a,\quad \mathbf{g}^j=\mu_b^j\,\mathbf{a}^b$$
$$g_{ij}=\mu_i^a\,\mu_j^b\,a_{ab},\quad g^{ij}=\mu_a^i\,\mu_b^j\,a^{ab} \tag{2.2.4}$$

where

$$g_{33}=g^{33}=a_{33}=a^{33}=\mu_3^3=1$$
$$g_{\varphi 3}=g^{\varphi 3}=a_{\alpha 3}=a^{\alpha 3}=\mu_3^\alpha=\mu_\varphi^3=0 \tag{2.2.5}$$
$$\mathbf{g}_3=\mathbf{g}^3=\mathbf{a}_3=\mathbf{a}^3=\mathbf{n}$$

$$\mu_\varphi^\alpha=\mathbf{a}^\alpha\cdot\mathbf{g}_\varphi=\delta_\varphi^\alpha-\zeta\delta_\varphi^\beta\,b_\beta^\alpha$$
$$\mu_\beta^\psi=\mathbf{g}^\psi\cdot\mathbf{a}_\beta=a_{\alpha\beta}\,\mu_\varphi^\alpha\,g^{\varphi\psi}=\delta_\beta^\psi+\zeta\delta_\alpha^\psi\,b_\beta^\alpha+\zeta^2\delta_\alpha^\psi\,b_\lambda^\alpha\,b_\beta^\lambda+\dots \tag{2.2.6}$$
$$\mu_\varphi^\alpha\,\mu_\beta^\varphi=\delta_\beta^\alpha,\quad \mu_\varphi^\alpha\,\mu_\alpha^\psi=\delta_\varphi^\psi$$

Symbols of the type μ_i^a, μ_b^j are often called "translators" [6, 58]. It is evident from the absolute notation in (2.2.2) that they are components of the metric tensor $\mathbf{1}\in\mathscr{T}_2$ in a mixed basis.

For the subsequent transformations we also will use a symmetric translation tensor $\mathbf{g}\in\mathscr{T}_2$, which will be defined in the normal coordinate system as follows

$$\mathbf{g}=\delta_a^i\,\mathbf{g}_i\otimes\mathbf{a}^a,\quad \mathbf{g}^{-1}=\delta_j^b\,\mathbf{a}_b\otimes\mathbf{g}^j \tag{2.2.7}$$

Let us introduce a metric tensor $\mathbf{a}$ and a curvature tensor $\mathbf{b}$ of the surface $\mathscr{M}_\kappa$

$$\mathbf{a}=\mathbf{a}(M)=a_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta=\mathbf{a}_\alpha\otimes\mathbf{a}^\alpha \tag{2.2.8}$$
$$\mathbf{b}=\mathbf{b}(M)=b_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta=-\mathbf{n}_{,\alpha}\otimes\mathbf{a}^\alpha$$

Then we have the following formulae

$$\mathbf{g}=\mathbf{1}-\zeta\mathbf{b},\quad \mathbf{g}^2=\mathbf{1}-2\zeta\mathbf{b}+\zeta^2\mathbf{b}^2 \tag{2.2.9}$$

$$\mathbf{g}_i=\delta_i^a\,\mathbf{g}\mathbf{a}_a,\quad \mathbf{g}^j=\delta_b^j\,\mathbf{g}^{-1}\mathbf{a}^b$$
$$\mathbf{a}_a=\delta_a^i\,\mathbf{g}^{-1}\mathbf{g}_i,\quad \mathbf{a}^b=\delta_j^b\,\mathbf{g}\mathbf{g}^j \tag{2.2.10}$$

$$g_{\varphi\psi}=\mu_\varphi^\alpha\,\mu_\psi^\beta\,a_{\alpha\beta}=\delta_\varphi^\alpha\,\delta_\psi^\beta(a_{\alpha\beta}-2\zeta b_{\alpha\beta}+\zeta^2 b_\alpha^\lambda\,b_{\lambda\beta})$$
$$g^{\varphi\psi}=\mu_\alpha^\varphi\,\mu_\beta^\psi\,a^{\alpha\beta}=\delta_\alpha^\varphi\,\delta_\beta^\psi(a^{\alpha\beta}+2\zeta b^{\alpha\beta}+3\zeta^2 b_\lambda^\alpha\,b^{\lambda\beta}+\dots) \tag{2.2.11}$$

$$\mu=\sqrt{\frac{g}{a}}=1-2\zeta H+\zeta^2 K$$
$$\mu^{-1}=\sqrt{\frac{a}{g}}=1+2\zeta H+\zeta^2(4H^2-K)+\dots \tag{2.2.12}$$

For other relations see [19]. They can also be derived by an appropriate adaptation of formulae presented in [6, 58].

2.3. Tensor fields in a shell

The description of tensor fields and their derivatives in a shell region has been discussed in [58] using the normal coordinate system, and in [6] using a skew coordinate system. In both works all tensor fields have been described entirely be means of their components.

Another description of tensor fields in the region $\mathscr{S}_\kappa$, as proposed by the author in [19, 24], will be used here. It is based directly on concepts of the absolute tensor analysis. This approach leads to a remarkable simplification of transformations, showing at the same time the geometric meaning of the performed transformations and of the obtained relations.

Values of any vector field $\mathbf{v}=\mathbf{v}(P)\in\mathscr{V}$ or tensor field $\mathbf{T}=\mathbf{T}(P)\in\mathscr{T}_2$ are elements of linear spaces. Thus they can be represented by their components either in the basis $\mathbf{g}_i$ or in the basis $\mathbf{a}_a$ as follows

$$\begin{aligned} \mathbf{v}&=v^i\mathbf{g}_i=v_j\,\mathbf{g}^j=v^a\mathbf{a}_a=v_b\,\mathbf{a}^b \\ \mathbf{T}&=T^{ij}\mathbf{g}_i\otimes\mathbf{g}_j=T_{ij}\,\mathbf{g}^i\otimes\mathbf{g}^j \\ &=T^{ab}\mathbf{a}_a\otimes\mathbf{a}_b=T_{ab}\,\mathbf{a}^a\otimes\mathbf{a}^b \end{aligned} \tag{2.3.1}$$

In view of (2.2.4) or (2.2.10) these different components of the same values are related to each other in the following manner

$$\begin{aligned} v^i&=\mu^i_a v^a, \quad v_j=\mu^b_j v_b \\ v^a&=\mu^a_i v^i, \quad v_b=\mu^j_b v_j \end{aligned} \tag{2.3.2}$$

$$\begin{aligned} T^{ij}&=\mu^i_a\mu^j_b T^{ab}, \quad T_{ij}=\mu^a_i\mu^b_j T_{ab} \\ T^{ab}&=\mu^a_i\mu^b_j T^{ij}, \quad T_{ab}=\mu^i_a\mu^j_b T_{ij} \end{aligned} \tag{2.3.3}$$

Applying spatial operations of gradient and divergence to the vector fields $\mathbf{g}_i(P)$, $\mathbf{v}(P)$ and the tensor field $\mathbf{T}(P)$ we obtain the new tensor fields in the region $\mathscr{S}_\kappa$. The values of these new fields at $P\in\mathscr{S}_\kappa$ can also be represented by their components either in the $\mathbf{g}_i$ or $\mathbf{a}_a$ bases. By taking into account (2.2.4) and making some simple transformations one arrives at the following formulae for the new fields [19, 24].

$$\begin{aligned} \operatorname{grad}\mathbf{g}_i(P)&=G^k_{ij}\,\mathbf{g}_k\otimes\mathbf{g}^j=\mathbf{g}_{i,\,j}\otimes\mathbf{g}^j \\ &=\mathbf{g}_k\otimes\mu^k_c\left[\delta^\beta_\psi(\mu^c_{i,\,\beta}+G^c_{\alpha\beta}\,\mu^\alpha_i)\,\mathbf{g}^\psi+\mu^c_{i,\,3}\,\mathbf{g}^3\right] \end{aligned} \tag{2.3.4}$$

$$\begin{aligned} \operatorname{grad}\mathbf{v}(P)&=v_{i;\,j}\,\mathbf{g}^i\otimes\mathbf{g}^j=\mathbf{v}_{,j}\otimes\mathbf{g}^j \\ &=\mathbf{g}^i\otimes\mu^a_i(\delta^\beta_\psi\,v_{a;\,\beta}\,\mathbf{g}^\psi+v_{a,\,3}\,\mathbf{g}^3) \end{aligned} \tag{2.3.5}$$

$$\begin{aligned} \operatorname{grad}\mathbf{T}(P)&=T^{ij}{}_{;k}\,\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mathbf{g}^k=\mathbf{T}_{,k}\otimes\mathbf{g}^k \\ &=\mathbf{g}_i\otimes\mathbf{g}_j\otimes\mu^i_a\,\mu^j_b(\delta^\gamma_\vartheta\,T^{ab}{}_{;\gamma}\,\mathbf{g}^\vartheta+T^{ab}{}_{,3}\,\mathbf{g}_3) \end{aligned} \tag{2.3.6}$$

$$\begin{aligned} \operatorname{div}\mathbf{T}(P)&=T^{ij}{}_{;j}\,\mathbf{g}_i \\ &=\mu^i_a(\delta^\gamma_\vartheta\,\mu^\vartheta_\beta\,T^{a\beta}{}_{;\gamma}+T^{a3}{}_{,3})\,\mathbf{g}_i \end{aligned} \tag{2.3.7}$$

where $(\ )_{;\gamma}$ denotes the spatial covariant derivative at $P \in \mathscr{S}_\kappa$, which however has been computed with the aid of Christoffel symbols of the basis $\mathbf{a}_a(M)$,

$$G^c_{a\gamma} = \mathbf{a}^c \cdot \mathbf{a}_{a,\gamma} \tag{2.3.8}$$

The formulae (2.3.4)÷(2.3.7), written in terms of components with respect to different bases, lead to a number of relations for the components of tensor fields with respect to the basis $\mathbf{g}_i(P)$ expressed in terms of appropriate components with respect to the basis $\mathbf{a}_a(M)$. These relations have been presented in detail in [19].

In particular, it follows from (2.3.4) that

$$\begin{aligned} G^\vartheta_{\varphi\psi} &= \delta^\alpha_\varphi \delta^\beta_\psi (\delta^\vartheta_\lambda \Gamma^\lambda_{\alpha\beta} - \zeta \mu^\vartheta_\lambda b^\lambda_{\alpha|\beta}) \\ G^\vartheta_{\varphi 3} &= -\mu^\vartheta_\lambda \delta^\alpha_\varphi b^\lambda_\alpha \\ G^3_{\varphi\psi} &= \mu^\alpha_\varphi \delta^\beta_\psi b_{\alpha\beta} \end{aligned} \tag{2.3.9}$$

while from (2.3.5) we obtain

$$\begin{aligned} v_{\varphi;\psi} &= \mu^\alpha_\varphi \delta^\beta_\psi (v_{\alpha|\beta} - b_{\alpha\beta} v_3), \qquad v_{\varphi;3} = \mu^\alpha_\varphi v_{\alpha,3} \\ v_{3;\psi} &= \delta^\beta_\psi (v_{3,\beta} + b^\alpha_\beta v_\alpha), \qquad v_{3;3} = v_{3,3} \end{aligned} \tag{2.3.10}$$

The relations for the components of (2.3.6) are the following:

$$\begin{aligned} T^{\varphi\psi}{}_{;\vartheta} &= \mu^\varphi_\alpha \mu^\psi_\beta \delta^\lambda_\vartheta (T^{\alpha\beta}|_\lambda - b^\alpha_\lambda T^{3\beta} - b^\beta_\lambda T^{\alpha 3}) \\ T^{\varphi 3}{}_{;\vartheta} &= \mu^\varphi_\alpha \delta^\lambda_\vartheta (T^{\alpha 3}|_\lambda - b^\alpha_\lambda T^{33} + b_{\beta\lambda} T^{\alpha\beta}) \\ T^{3\psi}{}_{;\vartheta} &= \mu^\psi_\beta \delta^\lambda_\vartheta (T^{3\beta}|_\lambda + b_{\alpha\lambda} T^{\alpha\beta} - b^\beta_\lambda T^{33}) \\ T^{33}{}_{;\vartheta} &= \delta^\lambda_\vartheta (T^{33}{}_{,\lambda} + b_{\alpha\lambda} T^{\alpha 3} + b_{\beta\lambda} T^{3\beta}) \end{aligned} \tag{2.3.11}$$

$$\begin{aligned} T^{\varphi\psi}{}_{;3} &= \mu^\varphi_\alpha \mu^\psi_\beta T^{\alpha\beta}{}_{,3}, \qquad T^{\varphi 3}{}_{;3} = \mu^\varphi_\alpha T^{\alpha 3}{}_{,3} \\ T^{3\psi}{}_{;3} &= \mu^\psi_\beta T^{3\beta}{}_{,3}, \qquad T^{33}{}_{;3} = T^{33}{}_{,3} \end{aligned} \tag{2.3.12}$$

Finally, for the components of div $\mathbf{T}(P)$ we obtain from (2.3.7)

$$\begin{aligned} \mu\mu^\alpha_\varphi T^{\varphi\psi}{}_{;\psi} &= (\mu\mu^\alpha_\varphi T^{\varphi\psi} \delta^\beta_\psi)_{|\beta} - \mu b^\alpha_\beta \delta^\beta_\psi T^{3\psi} + \mu_{,3} \mu^\alpha_\varphi T^{\varphi 3} \\ \mu T^{3\psi}{}_{;\psi} &= (\mu T^{3\psi} \delta^\beta_\psi)_{|\beta} + b_{\alpha\beta} \mu\mu^\alpha_\varphi \delta^\beta_\psi T^{\varphi\psi} + \mu_{,3} T^{33} \\ \mu^\alpha_\varphi T^{\varphi 3}{}_{;3} &= (\mu^\alpha_\varphi T^{\varphi 3})_{,3} \\ T^{33}{}_{;3} &= T^{33}{}_{,3} \end{aligned} \tag{2.3.13}$$

3. Deformation

3.1. Deformation of a shell region

Consider a deformation χ of a shell $\mathscr{S}$ from the reference configuration κ to the actual configuration γ. Assume that the normal coordinate system $\{\vartheta^i\}$ adopted in the region $\mathscr{S}_\kappa$ is a convected one, that is that the coordinates of the places P and $\bar{P} = \chi(P, t)$ for the same particle $X \in \mathscr{S}$ are identical during the shell motion. Thus the coordinate system $\{\vartheta^i\}$ adopted in κ defines also each geometric quantity of the region $\mathscr{S}_\gamma$, these quantities being analogous to those defined in $\mathscr{S}_\kappa$. In order to avoid the introduction of new symbols we shall distinguish these geometric quantities in $\bar{P} \in \mathscr{S}_\gamma$ putting an additional bar: $\bar{\mathbf{p}}$, $\bar{\mathbf{g}}_i$, $\bar{\mathbf{g}}^j$, $\bar{\mathbf{g}}_{ij}$, $\bar{\mathbf{g}}^{ij}$, $\bar{\epsilon}_{ijk}$, $\bar{\epsilon}^{ijk}$, $\bar{G}_{kij}$, $\bar{G}^k_{ij}$, $(\,)_{;|i}$, $\bar{\mathbf{1}}$ etc. For these quantities formulae analogous to (2.1.2) $\div$ (2.1.5) hold true. Similarly, geometric quantities for the surface $\mathscr{M}_\gamma$, that are analogous to those defined for $\mathscr{M}_\kappa$, will also be distinguished by an additional bar: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{\epsilon}_{\alpha\beta}$, $\bar{\epsilon}^{\alpha\beta}$, $\bar{K}$, $\bar{H}$, $\bar{\Gamma}_{\lambda\alpha\beta}$, $\bar{\Gamma}^\lambda_{\alpha\beta}$, $(\;)_{||\alpha}$ etc. Again relations analogous to (2.1.7) $\div$ (2.1.12) hold true.

In the Lagrangean description the displacement of a particle X from κ to γ is described by a displacement vector $\mathbf{v}(P, t)$, defined by components with respect to the geometry of the region $\mathscr{S}_\kappa$

$$\mathbf{v} = \mathbf{v}(P, t) = \chi(P, t) - P = \bar{\mathbf{p}} - \mathbf{p} = v^i \mathbf{g}_i = v_j \mathbf{g}^j \tag{3.1.1}$$

Deformation of the neighbourhood of the particle X from κ to γ is defined by the deformation gradient tensor

$$\mathbf{F} = \mathbf{F}(P, t) = \nabla\chi(P, t) = \mathbf{1} + \operatorname{grad}\mathbf{v} = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i \tag{3.1.2}$$

while the strain is expressed by the Green strain tensor

$$\begin{aligned} \mathbf{E} &= \mathbf{E}(P, t) = E_{ij}\,\mathbf{g}^i \otimes \mathbf{g}^j = \tfrac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) \\ E_{ij} &= \tfrac{1}{2}(v_{i;\,j} + v_{j;\,i} + g^{kl} v_{k;\,i} v_{l;\,j}) = \tfrac{1}{2}(\bar{g}_{ij} - g_{ij}) \end{aligned} \tag{3.1.3}$$

Let us expand the deformation function $\chi(P, t)$ into a Taylor series in the vicinity of a point $M \in \mathscr{M}_\kappa$, restricting the expansion to the linear part of the series only, on account of thinness of the shell. Bearing in mind (2.2.1) we obtain

$$\bar{P} = \chi(P, t) = \chi(M, t) + \zeta\,\mathbf{G}\mathbf{n} + \ldots \tag{3.1.4}$$

where

$$\mathbf{G} = \nabla\chi(P, t)_{|P=M} = \mathbf{F}(M, t) = \bar{\mathbf{a}}_a \otimes \mathbf{a}^a \tag{3.1.5}$$

$$\bar{\mathbf{a}}_a = \delta^i_a\,\bar{\mathbf{g}}_i\big|_{P=M} \tag{3.1.6}$$

The restriction to the linear part in (3.1.4) means physically, that the shell deformation is described completely by the deformation of the neighbourhood of the shell middle surface points, while the higher-order deformation changes across the shell thickness are omitted. In particular, within the frame of this approximation material fibres orthogonal to $\mathscr{M}_\kappa$ remain straight after the shell deformation, becoming however elongated and, in general, not orthogonal to $\mathscr{M}_\gamma$. The tensor $\mathbf{G}$ in (3.1.5) is a spatial deformation gradient tensor written with respect to the convected coordinate system. It contains the complete information about the deformation of the neighbourhood of the middle surface points and, in accordance with (3.1.4), defines deformation of the region $\mathscr{S}_\kappa$ completely as well.

The spatial basis $\bar{\mathbf{a}}_a \equiv \{\bar{\mathbf{a}}_\alpha, \bar{\mathbf{a}}_3\}$ introduced in (3.1.6) is, in general, a skew one in the region $\mathscr{S}_\gamma$. It should be distinguished from the basis $\{\bar{\mathbf{a}}_\alpha, \bar{\mathbf{n}}\}$ of a normal coordinate system in $\mathscr{S}_\gamma$, which results from the basis $\bar{\mathbf{a}}_a$ for the shell deformation with Kirchhoff-Love type constraints. It is worth to point out that the vector $\bar{\mathbf{a}}_3$ is, in general, neither a unit vector nor an orthogonal to $\mathscr{M}_\gamma$, $\bar{\mathbf{a}}_3 \neq \bar{\mathbf{n}}$.

Using (3.1.5) and (3.1.2) one obtains an expansion of the displacement field

$$\mathbf{v}=\mathbf{v}(P,t)=\mathbf{u}+\zeta\boldsymbol{\beta}+\ldots \tag{3.1.7}$$

where

$$\begin{aligned} &\mathbf{u}=\bar{M}-M=\bar{\mathbf{r}}-\mathbf{r}=u^\alpha\mathbf{a}_\alpha+w\mathbf{n}\\ &\boldsymbol{\beta}=(\mathbf{G}-\mathbf{1})\mathbf{n}=\bar{\mathbf{a}}_3-\mathbf{n}=\beta^\alpha\mathbf{a}_\alpha+\beta\mathbf{n}\end{aligned} \tag{3.1.8}$$

Furthermore, according to $(2.1.11)_2$ there is

$$\mathbf{u}_{,\alpha}=\varphi^{\lambda}_{.\alpha}\mathbf{a}_\lambda+\varphi_\alpha\mathbf{n}, \qquad \boldsymbol{\beta}_{,\alpha}=\psi^{\lambda}_{.\alpha}\mathbf{a}_\lambda+\psi_\alpha\mathbf{n} \tag{3.1.9}$$

where

$$\begin{aligned} &\varphi^{\lambda}_{.\alpha}=u^\lambda|_\alpha-b^\lambda_\alpha w, \qquad \varphi_\alpha=w_{,\alpha}+b^\lambda_\alpha u_\lambda\\ &\psi^{\lambda}_{.\alpha}=\beta^\lambda|_\alpha-b^\lambda_\alpha\beta, \qquad \psi_\alpha=\beta_{,\alpha}+b^\lambda_\alpha\beta_\lambda\end{aligned} \tag{3.1.10}$$

Under (3.1.4) the displacement field (3.1.7) in the shell is described by the displacement vector $\mathbf{u}$ of the middle surface points together with a vector $\boldsymbol{\beta}$ of the change of the normal to the middle surface, or by six independent components of these vectors.

It follows from (3.1.1) and $(2.2.6)_1$ that under (3.1.7)

$$\bar{\mathbf{g}}_\varphi=\delta^\alpha_\varphi(\bar{\mathbf{a}}_\alpha+\zeta\bar{\mathbf{a}}_{3,\alpha}+\ldots), \ \bar{\mathbf{g}}_3=\bar{\mathbf{a}}_3+\ldots \tag{3.1.11}$$

where

$$\begin{aligned} &\bar{\mathbf{a}}_\alpha=\mathbf{a}_\alpha+\mathbf{u}_{,\alpha}=(\delta^\lambda_\alpha+\varphi^{\lambda}_{.\alpha})\mathbf{a}_\lambda+\varphi_\alpha\mathbf{n}\\ &\bar{\mathbf{a}}_3=\mathbf{n}+\boldsymbol{\beta}=\beta^\lambda\mathbf{a}_\lambda+(1+\beta)\mathbf{n}\\ &\bar{\mathbf{a}}_{3,\alpha}=\mathbf{n}_{,\alpha}+\boldsymbol{\beta}_{,\alpha}=(\psi^{\lambda}_{.\alpha}-b^\lambda_\alpha)\mathbf{a}_\lambda+\psi_\alpha\mathbf{n}\end{aligned} \tag{3.1.12}$$

Relations $(2.3.3)_1$ and (3.1.11) yield the following formulae for the components of the Green strain tensor

$$\begin{aligned} &E_{\varphi\psi}=\tfrac{1}{2}(\bar{\mathbf{g}}_\varphi\cdot\bar{\mathbf{g}}_\psi-\mathbf{g}_\varphi\cdot\mathbf{g}_\psi)=\delta^\alpha_\varphi\delta^\beta_\psi[\gamma_{\alpha\beta}+\zeta\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})+\zeta^2\mu_{\alpha\beta}+\ldots]\\ &E_{3\psi}=\tfrac{1}{2}(\bar{\mathbf{g}}_3\cdot\bar{\mathbf{g}}_\psi-\mathbf{g}_3\cdot\mathbf{g}_\psi)=\delta^\beta_\psi(\gamma_{3\beta}+\zeta\tfrac{1}{2}\kappa_{3\beta}+\ldots)\\ &E_{33}=\tfrac{1}{2}(\bar{\mathbf{g}}_3\cdot\bar{\mathbf{g}}_3-\mathbf{g}_3\cdot\mathbf{g}_3)=\gamma_{33}+\ldots\end{aligned} \tag{3.1.13}$$

where

$$\begin{aligned} 2\gamma_{\alpha\beta}&=\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_\beta-\mathbf{a}_\alpha\cdot\mathbf{a}_\beta\\ \kappa_{\alpha\beta}+\kappa_{\beta\alpha}&=\bar{\mathbf{a}}_\alpha\cdot\bar{\mathbf{a}}_{3,\beta}+\bar{\mathbf{a}}_\beta\cdot\bar{\mathbf{a}}_{3,\alpha}-2\mathbf{a}_\alpha\cdot\mathbf{n}_{,\beta}\\ 2\mu_{\alpha\beta}&=\bar{\mathbf{a}}_{3,\alpha}\cdot\bar{\mathbf{a}}_{3,\beta}-\mathbf{n}_{,\alpha}\cdot\mathbf{n}_{,\beta}\\ 2\gamma_{3\beta}&=\bar{\mathbf{a}}_3\cdot\bar{\mathbf{a}}_\beta-\mathbf{n}\cdot\mathbf{a}_\beta\\ \kappa_{3\beta}&=\bar{\mathbf{a}}_3\cdot\bar{\mathbf{a}}_{3,\beta}-\mathbf{n}\cdot\mathbf{n}_{,\beta}\\ 2\gamma_{33}&=\bar{\mathbf{a}}_3\cdot\bar{\mathbf{a}}_3-\mathbf{n}\cdot\mathbf{n}\end{aligned} \tag{3.1.14}$$

Making use of (3.1.12) we obtain, after some transformations,

$$
\begin{aligned}
2\gamma_{\alpha\beta} &= \varphi_{\alpha\beta} + \varphi_{\beta\alpha} + a^{\lambda\mu}\varphi_{\lambda\alpha}\,\varphi_{\mu\beta} + \varphi_\alpha\,\varphi_\beta \\
2\kappa_{(\alpha\beta)} &= \kappa_{\alpha\beta} + \kappa_{\beta\alpha} = \psi_{\alpha\beta} + \psi_{\beta\alpha} - b^\lambda_\alpha\,\varphi_{\lambda\beta} - b^\lambda_\beta\,\varphi_{\lambda\alpha} + \\
&\qquad + a^{\lambda\mu}(\varphi_{\lambda\alpha}\psi_{\mu\beta} + \varphi_{\lambda\beta}\psi_{\mu\alpha}) + \varphi_\alpha\psi_\beta + \varphi_\beta\psi_\alpha \\
2\mu_{\alpha\beta} &= a^{\lambda\mu}\psi_{\lambda\alpha}\psi_{\mu\beta} - b^\lambda_\alpha\psi_{\lambda\beta} - b^\lambda_\beta\psi_{\lambda\alpha} + \psi_\alpha\psi_\beta \\
2\gamma_{3\alpha} &= \varphi_\alpha + \beta_\alpha + a^{\lambda\mu}\varphi_{\lambda\alpha}\beta_\mu + \varphi_\alpha\beta \\
\kappa_{3\alpha} &= \psi_\alpha - b^\lambda_\alpha\beta_\lambda + a^{\lambda\mu}\psi_{\lambda\alpha}\beta_\mu + \psi_\alpha\beta \\
2\gamma_{33} &= 2\beta + a^{\lambda\mu}\beta_\lambda\beta_\mu + \beta^2
\end{aligned}
\tag{3.1.15}
$$

With the help of (3.1.10) the relations (3.1.15) can be written directly in terms of six components of the displacement state. This will lead to the formulae presented in the previous works of the author [19, 24] and in the paper by HABIP [35].

3.2. Lagrangean and Eulerian description

Within the frames of approximation (3.1.4) adopted in this work, a convected coordinate system normal in $\mathscr{S}_\kappa$ changes after deformation χ into a coordinate system that is skew in $\mathscr{S}_\gamma$. This skew system is defined by the spatial basis $\bar{\mathbf{a}}_a$ natural in $\bar{M} \in \mathscr{M}_\gamma$.

According to relations (2.1.2)÷(2.1.5) for the basis we have

$$
\begin{aligned}
&\bar{a}_{ab} = \bar{\mathbf{a}}_a \cdot \bar{\mathbf{a}}_b\,, \qquad \bar{a} = |\bar{a}_{ab}| = \bar{g}|_{\zeta=0} \\
&\bar{a}^{ab} = \frac{\min(\bar{a}_{ab})}{\bar{a}}\,, \quad \bar{\mathbf{a}}^a = \bar{a}^{ab}\bar{\mathbf{a}}_b\,, \quad \bar{a}^{ab}\bar{a}_{ac} = \delta^b_c \\
&\bar{\epsilon}_{abc} = \bar{\mathbf{a}}_a \cdot (\bar{\mathbf{a}}_b \times \bar{\mathbf{a}}_c)\,, \qquad \bar{\epsilon}^{abc} = \bar{\mathbf{a}}^a \cdot (\bar{\mathbf{a}}^b \times \bar{\mathbf{a}}^c)
\end{aligned}
\tag{3.2.1}
$$

Let $\bar{\mathbf{g}}$ be a translation tensor in $\mathscr{S}_\gamma$ for a skew parametrization satisfying (3.1.4). Then we can formulate in $\mathscr{S}_\gamma$ the relations analogous to those for the normal system

$$
\begin{aligned}
&\bar{\mathbf{g}} = \delta^i_a\,\bar{\mathbf{g}}_i \otimes \bar{\mathbf{a}}^a\,, \qquad \bar{\mathbf{g}}^{-1} = \delta^a_i\,\bar{\mathbf{a}}_a \otimes \bar{\mathbf{g}}^i \\
&\bar{\boldsymbol{\lambda}} = -\bar{\mathbf{a}}_{3,\beta} \otimes \bar{\mathbf{a}}^\beta = \bar{\lambda}_{\alpha\beta}\,\bar{\mathbf{a}}^\alpha \otimes \bar{\mathbf{a}}^\beta \\
&\bar{\mathbf{g}} = \bar{\mathbf{1}} - \zeta\bar{\boldsymbol{\lambda}} + \ldots\,, \quad \bar{\mathbf{g}}^T\bar{\mathbf{g}} = \bar{\mathbf{1}} - \zeta(\bar{\boldsymbol{\lambda}} + \bar{\boldsymbol{\lambda}}^T) + \zeta^2\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}} + \ldots \\
&\bar{\mathbf{g}}_i = \delta^a_i\,\bar{\mathbf{g}}\bar{\mathbf{a}}_a\,, \quad \bar{\mathbf{g}}^j = \delta^j_b\,\bar{\mathbf{g}}^{-1}\bar{\mathbf{a}}^b
\end{aligned}
\tag{3.2.2}
$$

Now the deformation gradient tensor of any point of the shell region can be expressed as follows

$$
\mathbf{F} = \bar{\mathbf{g}}\mathbf{G}\mathbf{g}^{-1} = (\mathbf{G} - \zeta\bar{\boldsymbol{\lambda}}\mathbf{G} + \ldots)\,\mathbf{g}^{-1} \tag{3.2.3}
$$

Using the Lagrangean description we obtain the following relation for the Green strain tensor

$$2\mathbf{E}=\mathbf{g}^{-1}\{(\mathbf{G}^T\mathbf{G}-\mathbf{1})-\zeta[\mathbf{G}^T(\bar{\boldsymbol{\lambda}}+\bar{\boldsymbol{\lambda}}^T)\mathbf{G}-2\mathbf{b}]+\zeta^2(\mathbf{G}^T\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}}\mathbf{G}-\mathbf{b}^2)+\dots\}\mathbf{g}^{-1} \quad (3.2.4)$$

Hence the tensors defined at the surface $\mathscr{M}_\kappa$

$$\boldsymbol{\gamma}=\tfrac{1}{2}(\mathbf{G}^T\mathbf{G}-\mathbf{1}), \quad \boldsymbol{\kappa}=-(\mathbf{G}^T\bar{\boldsymbol{\lambda}}\mathbf{G}-\mathbf{b}) \quad (3.2.5)$$

determine completely the shell strain state during deformation from the configuration κ to γ, whereas the tensor

$$\boldsymbol{\mu}=\tfrac{1}{2}(\mathbf{G}^T\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}}\mathbf{G}-\mathbf{b}^2)=\tfrac{1}{2}[(\mathbf{b}-\boldsymbol{\kappa}^T)(\mathbf{1}+2\boldsymbol{\gamma})^{-1}(\mathbf{b}-\boldsymbol{\kappa})-\mathbf{b}^2] \quad (3.2.6)$$

is a quantity dependent on $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ as well as on the geometry of $\mathscr{M}_\kappa$.

Tensors (3.2.5) and (3.2.6) expressed by their components in the basis $\mathbf{a}^a\otimes\mathbf{a}^b$ take the form

$$\begin{gathered} 2\gamma_{ab}=\bar{a}_{ab}-a_{ab}, \quad \kappa_{\alpha\beta}=-(\bar{\lambda}_{\alpha\beta}-b_{\alpha\beta}) \\ 2\mu_{\alpha\beta}=\bar{\lambda}^{c}_{.\alpha}\bar{\lambda}_{c\beta}-b^{\lambda}_{\alpha}b_{\lambda\beta} \end{gathered} \quad (3.2.7)$$

In the Eulerian description the displacement of a particle X from the configuration κ to γ is described by a displacement vector $\bar{\mathbf{v}}(\bar{P},t)$ expressed by the components in geometry of the region $\mathscr{S}_\gamma$

$$\bar{\mathbf{v}}=\bar{\mathbf{v}}(\bar{P},t)=\bar{P}-\chi^{-1}(\bar{P},t)=\bar{\mathbf{p}}-\mathbf{p}=\bar{v}^i\bar{\mathbf{g}}_i=\bar{v}_j\bar{\mathbf{g}}^j \quad (3.2.8)$$

The deformation of the neighbourhood of the particle X is defined by the inverse of the deformation gradient tensor

$$\mathbf{F}^{-1}=\mathbf{F}^{-1}(\bar{P},t)=\nabla\chi^{-1}(\bar{P},t)=\bar{\mathbf{1}}-\operatorname{grad}\bar{\mathbf{v}}=\mathbf{g}_i\otimes\bar{\mathbf{g}}^i \quad (3.2.9)$$

Making use of (3.2.2) and (3.2.3) for a deformation compatible with linear deformation field (3.1.4) we obtain

$$\mathbf{F}^{-1}=(\bar{\mathbf{g}}\mathbf{G}\mathbf{g}^{-1})^{-1}=(\mathbf{G}^{-1}-\zeta\mathbf{b}\mathbf{G}^{-1})\bar{\mathbf{g}}^{-1} \quad (3.2.10)$$

In the Eulerian description the strain state is described by the Almansi strain tensor, [18], defined by

$$\bar{\mathbf{E}}=\bar{\mathbf{E}}(\bar{P},t)=\tfrac{1}{2}[\bar{\mathbf{1}}-(\mathbf{F}^{-1})^T\mathbf{F}^{-1}] \quad (3.2.11)$$

In view of (3.2.10) the tensor takes the form

$$\begin{aligned} 2\bar{\mathbf{E}}=(\bar{\mathbf{g}}^{-1})^T\{[\bar{\mathbf{1}}-(\mathbf{G}^{-1})^T\mathbf{G}^{-1}]-\zeta[\bar{\boldsymbol{\lambda}}+\bar{\boldsymbol{\lambda}}^T-(\mathbf{G}^{-1})^T2\mathbf{b}\mathbf{G}^{-1}]+\\ +\zeta^2[\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}}-(\mathbf{G}^{-1})^T\mathbf{b}^2\mathbf{G}^{-1}]\}\bar{\mathbf{g}}^{-1} \end{aligned} \quad (3.2.12)$$

Hence it is evident that tensors analogous to (3.2.5) and defined over the surface $\mathscr{M}_\gamma$ by

$$\bar{\boldsymbol{\gamma}}=\tfrac{1}{2}[\bar{\mathbf{1}}-(\mathbf{G}^{-1})^T\mathbf{G}^{-1}], \quad \bar{\boldsymbol{\kappa}}=-[\bar{\boldsymbol{\lambda}}-(\mathbf{G}^{-1})^T\mathbf{b}\mathbf{G}^{-1}] \quad (3.2.13)$$

describe completely the shell strain state during the deformation from κ to γ, while the tensor

$$\bar{\boldsymbol{\mu}}=\tfrac{1}{2}[\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}}-(\mathbf{G}^{-1})^T\mathbf{b}^2\mathbf{G}^{-1}]=\tfrac{1}{2}[\bar{\boldsymbol{\lambda}}^T\bar{\boldsymbol{\lambda}}-(\bar{\boldsymbol{\kappa}}+\bar{\boldsymbol{\lambda}})(\bar{\mathbf{1}}-2\bar{\boldsymbol{\gamma}})^{-1}(\bar{\boldsymbol{\kappa}}+\bar{\boldsymbol{\lambda}})] \quad (3.2.14)$$

is a quantity dependent on $\bar{\gamma}$ and $\bar{\kappa}$ and on the geometry of $\mathscr{M}_\gamma$ expressed in the skew coordinate system $\bar{\mathbf{a}}_a$.

Formulae (3.2.13) and (3.2.14) decomposed into components in the basis $\bar{\mathbf{a}}^a \otimes \bar{\mathbf{a}}^b$ take the form

$$2\bar{\gamma}_{ab} = \bar{a}_{ab} - a_{ab}, \qquad \bar{\kappa}_{\alpha\beta} = -(\bar{\lambda}_{\alpha\beta} - b_{\alpha\beta})$$
$$2\bar{\mu}_{\alpha\beta} = \bar{\lambda}^{c}_{.\alpha}\bar{\lambda}_{c\beta} - b^{\lambda}_{\alpha} b_{\lambda\beta} \tag{3.2.15}$$

From (3.2.5), (3.2.6) and (3.2.13), (3.2.14) it is easy to note the following transformation rules between Lagrangean and Eulerian strain measures

$$\boldsymbol{\gamma} = \mathbf{G}^T \bar{\boldsymbol{\gamma}} \mathbf{G}, \qquad \bar{\boldsymbol{\gamma}} = (\mathbf{G}^{-1})^T \boldsymbol{\gamma} \mathbf{G}^{-1}$$
$$\boldsymbol{\kappa} = \mathbf{G}^T \bar{\boldsymbol{\kappa}} \mathbf{G}, \qquad \bar{\boldsymbol{\kappa}} = (\mathbf{G}^{-1})^T \boldsymbol{\kappa} \mathbf{G}^{-1} \tag{3.2.16}$$
$$\boldsymbol{\mu} = \mathbf{G}^T \bar{\boldsymbol{\mu}} \mathbf{G}, \qquad \bar{\boldsymbol{\mu}} = (\mathbf{G}^{-1})^T \boldsymbol{\mu} \mathbf{G}^{-1}$$

It can easily be seen that in the convected coordinate system as used here the both groups of the shell strain measures — the Lagrangean tensors (3.2.5) and the Eulerian ones (3.2.13) — have the components (3.2.7) and (3.2.15) identically defined, although they are related to entirely different bases. If the component notation is used exclusively and simultaneously the notion of convected coordinate system is adopted, then the difference between the Lagrangean and Eulerian descriptions becomes barely noticeable. Sometime this may lead to misunderstandings. The absolute notation will often be used in the present work in order to emphasize the principal geometric difference between the Lagrangean description and the analogous Eulerian description.

3.3. Deformation of the Kirchhoff-Love type

Let us consider possible simplifications of the relations describing deformation of the region $\mathscr{S}_\kappa$, which result from imposing on the deformation the Kirchhoff-Love type constraints. The introduction of Kirchhoff-Love constraints is justified only for the description of strains and rotations within the first — approximation geometrically non-linear theory of shells, which is the ultimate goal of the simplified relations.

According to the Kirchhoff-Love constraints material fibres of the shell, which are rectilinear and orthogonal to $\mathscr{M}_\kappa$, remain after an arbitrary deformation χ, rectilinear and orthogonal to $\mathscr{M}_\gamma$, and do not change their length. This means in particular that the shell deformation depends entirely on the deformation of its middle surface, which is described by the displacement vector $\mathbf{u}$ only.

Thus for the Kirchhoff-Love type deformation there is $\bar{\mathbf{a}}_3 \equiv \bar{\mathbf{n}}$, hence

$$\boldsymbol{\beta} = \bar{\mathbf{n}} - \mathbf{n} = n^\alpha \mathbf{a}_\alpha + (n-1)\mathbf{n}$$
$$\mathbf{G} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} \tag{3.3.1}$$

where [11, 21, 51] in the Lagrangean description the following relations hold true

$$\bar{\mathbf{a}}_\alpha = l^\lambda_{.\alpha}\mathbf{a}_\lambda + \varphi_\alpha \mathbf{n}, \quad \bar{\mathbf{n}} = n^\lambda \mathbf{a}_\lambda + n\mathbf{n} \tag{3.3.2}$$

$$\begin{gathered} l^\lambda_{.\alpha} = \delta^\lambda_\alpha + \varphi^\lambda_{.\alpha} = \delta^\lambda_\alpha + u^\lambda|_\alpha - b^\lambda_\alpha w \\ n^\mu = \sqrt{\frac{a}{\bar{a}}}\, \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\varphi_\alpha l_{\lambda\beta}, \quad n = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\, \epsilon^{\alpha\beta}\epsilon^{\lambda\mu} l_{\lambda\alpha} l_{u\beta} \end{gathered} \tag{3.3.3}$$

while for components (3.1.12), (3.1.13) of the Green strain tensor we have

$$\begin{gathered} \gamma_{3\alpha} = \kappa_{3\alpha} = \gamma_{33} = 0 \\ \gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \tfrac{1}{2}(l^\lambda_{.\alpha} l_{\lambda\beta} + \varphi_\alpha \varphi_\beta - a_{\alpha\beta}) \\ \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = -\left[n(\varphi_{\alpha|\beta} + b^\lambda_\beta l_{\lambda\alpha}) + n_\lambda(l^\lambda_{.\alpha|\beta} - b^\lambda_\beta \varphi_\alpha) - b_{\alpha\beta}\right] \end{gathered} \tag{3.3.4}$$

Moreover

$$\mu_{\alpha\beta} = \tfrac{1}{2}(\bar{b}^\lambda_\alpha \bar{b}_{\lambda\beta} - b^\lambda_\alpha b_{\lambda\beta}) \tag{3.3.5}$$

can be expressed in terms of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. By application of the Cayley-Hamilton theorem to the curvature tensor $\mathbf{b}$ one obtains a relation that does not contain an inverse operation of tensors. Thus

$$\mathbf{b}^2 - (\operatorname{tr}\mathbf{b})\,\mathbf{b} + (\det\mathbf{b})\,\mathbf{a} = \mathbf{0} \tag{3.3.6}$$

where

$$\operatorname{tr}\mathbf{b} = b^\alpha_\alpha = 2H, \quad \det\mathbf{b} = |b^\alpha_\beta| = K \tag{3.3.7}$$

Now (3.3.5) becomes

$$\mu_{\alpha\beta} = -K\gamma_{\alpha\beta} - H\kappa_{\alpha\beta} - \tfrac{1}{2}(\bar{K} - K)\,a_{\alpha\beta} + (\bar{H} - H)\,b_{\alpha\beta} - (\bar{K} - K)\,\gamma_{\alpha\beta} - (\bar{H} - H)\,\kappa_{\alpha\beta} \tag{3.3.8}$$

where invariants $\bar{H} - H$ and $\bar{K} - K$ are given in terms of invariants of $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ as well as the geometry of $\mathscr{M}_\kappa$ by the relations

$$\begin{gathered} \bar{H} - H = \tfrac{1}{2}\frac{a}{\bar{a}}\left[-\kappa^\alpha_\alpha - 4H(\gamma^\alpha_\alpha + \hat{\gamma}^{\alpha\beta}\gamma_{\alpha\beta}) + 2\hat{\gamma}^{\alpha\beta}(b_{\alpha\beta} - \kappa_{\alpha\beta})\right] \\ \bar{K} - K = \tfrac{1}{2}\frac{a}{\bar{a}}\left[-\hat{\kappa}^{\alpha\beta} b_{\alpha\beta} - 4K(\gamma^\alpha_\alpha + \hat{\gamma}^{\alpha\beta}\gamma_{\alpha\beta}) - \hat{\kappa}^{\alpha\beta}(b_{\alpha\beta} - \kappa_{\alpha\beta})\right] \end{gathered} \tag{3.3.9}$$

This result has been obtained with the aid of relations [21, 51]

$$\begin{gathered} \frac{\bar{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2\hat{\gamma}^{\alpha\beta}\gamma_{\alpha\beta} \\ \bar{a}^{\alpha\beta} = \bar{\epsilon}^{\alpha\lambda}\bar{\epsilon}^{\beta\mu}\bar{a}_{\lambda\mu} = \frac{a}{\bar{a}}(a^{\alpha\beta} + 2\hat{\gamma}^{\alpha\beta}) \end{gathered} \tag{3.3.10}$$

with the following denotations

$$\begin{gathered} \hat{\gamma}^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\gamma_{\lambda\mu} = a^{\alpha\beta}\gamma^\lambda_\lambda - \gamma^{\alpha\beta} \\ \hat{\kappa}^{\alpha\beta} = \epsilon^{\alpha\lambda}\epsilon^{\beta\mu}\kappa_{\lambda\mu} = a^{\alpha\beta}\kappa^\lambda_\lambda - \kappa^{\alpha\beta} \end{gathered} \tag{3.3.11}$$

It is worth mentioning that NAGHDI and NORDGREN [59] have stated that $\mu_{\alpha\beta}$ can not, in general, be expressed in terms of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ only. This statement has been repeated recently without proof by JEFFERS and BRULL [80]. General formulae (3.3.8) and (3.3.9) derived in this work show explicitly how to calculate $\mu_{\alpha\beta}$ by means of $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. Moreover, this can be done without necessity to invert the tensors.

Now let us specify briefly some relations between the Lagrangean and Eulerian descriptions in the particular case of the Kirchhoff-Love type deformation. They result from general formulae (3.2.4) and (3.2.12) after taking into account (3.3.1) and substituting $\bar{\mathbf{b}}$ for $\bar{\boldsymbol{\lambda}}$.

The Green strain tensor takes the form

$$\mathbf{E}=\mathbf{g}^{-1}(\boldsymbol{\gamma}+\zeta\boldsymbol{\kappa}+\zeta^2\boldsymbol{\mu})\,\mathbf{g}^{-1} \tag{3.3.12}$$

where

$$\begin{aligned}
\boldsymbol{\gamma}&=\tfrac{1}{2}(\mathbf{G}^T\mathbf{G}-\mathbf{1})=\tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta})\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta\\
\boldsymbol{\kappa}&=-(\mathbf{G}^T\bar{\mathbf{b}}\mathbf{G}-\mathbf{b})=-(\bar{b}_{\alpha\beta}-b_{\alpha\beta})\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta\\
\boldsymbol{\mu}&=\tfrac{1}{2}(\mathbf{G}^T\bar{\mathbf{b}}^2\mathbf{G}-\mathbf{b}^2)=\tfrac{1}{2}(\bar{b}_\alpha^\lambda\,\bar{b}_{\lambda\beta}-b_\alpha^\lambda\,b_{\lambda\beta})\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta
\end{aligned} \tag{3.3.13}$$

The Almansi strain tensor is found to take the form

$$\bar{\mathbf{E}}=\bar{\mathbf{g}}^{-1}(\bar{\boldsymbol{\gamma}}+\zeta\bar{\boldsymbol{\kappa}}+\zeta^2\bar{\boldsymbol{\mu}})\,\bar{\mathbf{g}}^{-1} \tag{3.3.14}$$

where

$$\begin{aligned}
\bar{\boldsymbol{\gamma}}&=\tfrac{1}{2}[\bar{\mathbf{1}}-(\mathbf{G}^{-1})^T\mathbf{G}^{-1}]=\tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta})\,\bar{\mathbf{a}}^\alpha\otimes\bar{\mathbf{a}}^\beta\\
\bar{\boldsymbol{\kappa}}&=-[\bar{\mathbf{b}}-(\mathbf{G}^{-1})^T\mathbf{b}\mathbf{G}^{-1}]=-(\bar{b}_{\alpha\beta}-b_{\alpha\beta})\,\bar{\mathbf{a}}^\alpha\otimes\bar{\mathbf{a}}^\beta\\
\bar{\boldsymbol{\mu}}&=\tfrac{1}{2}[\bar{\mathbf{b}}^2-(\mathbf{G}^{-1})^T\mathbf{b}^2\mathbf{G}^{-1}]=\tfrac{1}{2}(\bar{b}_\alpha^\lambda\,\bar{b}_{\lambda\beta}-b_\alpha^\lambda\,b_{\lambda\beta})\,\bar{\mathbf{a}}^\alpha\otimes\bar{\mathbf{a}}^\beta
\end{aligned} \tag{3.3.15}$$

Comparison of (3.3.13) with (3.3.15) shows clearly the essential difference between the definitions of the both tensors, despite the similarity of formulae for their components.

A deformation of the Kirchhoff-Love type causes transition of a coordinate system normal in $\mathscr{S}_\kappa$ into the one normal in $\mathscr{S}_\gamma$. Hence variants of the theory of shells developed independently in the Lagrangean and Eulerian descriptions are equivalent under Kirchhoff-Love constraints, and can be related to each other with the help of transformation formulae between the surface coordinate systems at $\mathscr{M}_\kappa$ and $\mathscr{M}_\gamma$. Appropriate transformation relations have been given by the author in [21].

3.4. Compatibility conditions under Kirchhoff-Love constraints

During a deformation of the shell middle surface in the three-dimensional Euclidean space, the Codazzi-Gauss equations (2.1.12) should be satisfied in each configuration. If $\mathbf{u}$ is taken as independent variable, these equations will become identities. When the deformation is described in terms of strains $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$, the formulae (2.1.12) provide additional compatibility conditions for strains. Their form for the Lagrangean description will be as follows [11, 82]

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu}+\bar{a}^{\kappa\nu}(b_{\kappa\lambda}-\kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}]=0$$
$$K\gamma^{\lambda}_{\lambda}+\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda}-b_{\alpha\mu}\kappa_{\beta\lambda}+\tfrac{1}{2}(\kappa_{\alpha\mu}\kappa_{\beta\lambda}+\bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda})]=0 \tag{3.4.1}$$

with

$$\gamma_{\lambda\alpha\beta}=\gamma_{\lambda\alpha|\beta}+\gamma_{\lambda\beta|\alpha}-\gamma_{\alpha\beta|\lambda} \tag{3.4.2}$$

Formulae (3.4.1) constitute a nonlinear system of three equations for $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$. This is a very complex system, mainly due to the presence of a function $\sqrt{a/\bar{a}}$, irrational with respect to $\gamma_{\alpha\beta}$, in (3.3.10) and (3.3.3). For this reason in particular applications the analysis is usually based on simplified equations, obtained from (3.4.1) by omission of some nonlinear terms estimated as small for the class of problems under consideration. As a result, a displacement vector $\mathbf{u}_1$, that exactly satisfies these simplified compatibility conditions, does not transform exactly the surface $\mathscr{M}_\kappa$ into the surface $\mathscr{M}_\gamma$. The largest error is expected for the extreme case of linearization of the equations (3.4.1) as used in the linear theory of shells.

However, compatibility conditions appropriate for the linear theory of shells can be obtained directly as integrability conditions of the linear relations for strain measures. This does not impose any restrictions on displacements or their derivatives.

Similar problem is encountered also in the case of a nonlinear deformation of continuum, where components of the Green strain tensor satisfy the following compatibility conditions

$$\epsilon^{rij}\epsilon^{skl}(2E_{il;\,jk}+\bar{g}^{mn}E_{mil}E_{njk})=0 \tag{3.4.3}$$

with

$$E_{mil}=E_{mi;\,l}+E_{ml;\,i}-E_{il;\,m}$$
$$\bar{g}^{mn}=\frac{1}{2}\frac{g}{\bar{g}}\,\epsilon^{mij}\epsilon^{nkl}(g_{ik}+2E_{ik})(g_{jl}+2E_{jl}) \tag{3.4.4}$$
$$\frac{\bar{g}}{g}=\frac{1}{6}\,\epsilon^{ijk}\epsilon^{rst}(g_{ir}+2E_{ir})(g_{js}+2E_{js})(g_{kt}+2E_{kt})$$

Making use of decomposition of grad $\mathbf{v}$ into symmetric and antisymmetric parts

$$\operatorname{grad}\mathbf{v}=v_{i;\,j}\,\mathbf{g}^i\otimes\mathbf{g}^j=(\varepsilon_{ij}-\omega_{ij})\,\mathbf{g}^i\otimes\mathbf{g}^j$$
$$\varepsilon_{ij}=\tfrac{1}{2}(v_{i;\,j}+v_{j;\,i}),\qquad \omega_{ij}=\tfrac{1}{2}(v_{j;\,i}-v_{i;\,j}) \tag{3.4.5}$$

one arrives at the following relation for the components of the tensor $\mathbf{E}$

$$E_{il}=\varepsilon_{il}+\tfrac{1}{2}\,g^{mn}v_{m;\,i}v_{n;\,l}$$
$$E_{mil}=F^{p}_{\cdot\,m}v_{p;\,il} \tag{3.4.6}$$

Upon substitution of (3.4.6) and (3.4.5) into (3.4.3) it turns out that all the non-linear terms cancel each other. Thus (3.4.3) reduces in an exact way to the form that is known from the classical linear theory of elasticity

$$\epsilon^{rij}\epsilon^{skl}\varepsilon_{il;\,jk}=0 \tag{3.4.7}$$

The method of exact reduction of (3.4.3) to the formula (3.4.7), as presented above, has been developed by the author in [60,83] for arbitrary curvilinear coordinate system. An analogous reduction in the Cartesian coordinate system has been presented in [61].

The result obtained above implies the possibility of achieving similar exact reduction also for compatibility conditions of the surface strain tensors (3.4.1). In fact, these conditions can be obtained directly from (3.4.3) by imposition of the Kirchhoff-Love constraints on the three-dimensional shell deformation and substitution $\zeta=0$. In particular, for $r=3$, $s=3$ equation $(3.4.1)_2$ is obtained, the case of $r=\varphi$, $s=3$ leads to two equations $(3.4.1)_1$, and the remaining sets of r, s yield the same equations or identities. In view of that, let us take directly the reduced form (3.4.7) and write its representation for $\zeta=0$. Here we will make use of $(3.1.11)_2$, (2.3.9) and

$$\begin{aligned} \mathbf{u}_{,\alpha} &= (\vartheta_{\lambda\alpha}-\omega_{\lambda\alpha})\,\mathbf{a}^{\lambda}+\varphi_\alpha\,\mathbf{n} \\ \boldsymbol{\beta}_{,\alpha} &= [n_{\lambda|\alpha}-b_{\lambda\alpha}(n-1)]\,\mathbf{a}^{\lambda}+(n_{,\alpha}+b_\alpha^\lambda n_\lambda)\,\mathbf{n} \end{aligned} \tag{3.4.8}$$

where

$$\begin{aligned} \vartheta_{\alpha\beta} &= \tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}\,w \\ \omega_{\alpha\beta} &= \tfrac{1}{2}(u_{\beta|\alpha}-u_{\alpha|\beta})=\epsilon_{\alpha\beta}\,\varphi \end{aligned} \tag{3.4.9}$$

Components of the tensor $\boldsymbol{\varepsilon}(P,t)$ appearing in (3.4.7) may be related to the parameters of deformation of the middle surface. To this end the formulae [60] shall be used

$$\begin{aligned} \delta_\alpha^\varphi\delta_\beta^\psi\,\varepsilon_{\varphi\psi} &= \vartheta_{\alpha\beta}+\zeta\,[\tfrac{1}{2}(n_{\alpha|\beta}+n_{\beta|\alpha})-b_{\alpha\beta}(n-1)-\tfrac{1}{2}\,b_\alpha^\lambda(\vartheta_{\lambda\beta}-\omega_{\lambda\beta})- \\ &\quad -\tfrac{1}{2}\,b_\beta^\lambda(\vartheta_{\lambda\alpha}-\omega_{\lambda\alpha})]+\zeta^2[b_\alpha^\lambda\,b_{\lambda\beta}(n-1)-\tfrac{1}{2}(b_\alpha^\lambda\,n_{\lambda|\beta}+b_\beta^\lambda\,n_{\lambda|\alpha})] \\ \delta_\alpha^\varphi\varepsilon_{\varphi3} &= \tfrac{1}{2}(\varphi_\alpha+n_\alpha)+\zeta\tfrac{1}{2}\,n_{,\alpha} \\ \varepsilon_{33} &= n-1 \end{aligned} \tag{3.4.10}$$

Now we transform the equation (3.4.7) bearing in mind, that symbols $G_{ij}^k(\zeta)$ are used for computation of the covariant derivative. Next we make use of the relations (2.3.9) and (3.4.10) and substitute $\zeta=0$ in the final formulae. For the detailed transformation procedure see [60, 83]. It appears that for $r=3$, $s=3$ and $r=\varphi$, $s=3$ all terms that are nonlinear with respect to displacements and strains and contain n_α, n and their derivatives cancel each other, while the remaining linear terms can be transformed into

$$\begin{aligned} \epsilon^{\alpha\beta}\,\epsilon^{\lambda\mu}[(\rho_{\beta\lambda}-\tfrac{1}{2}\,b_\beta^\kappa\,\vartheta_{\kappa\lambda}-\tfrac{1}{2}\,b_\lambda^\kappa\,\vartheta_{\kappa\beta})_{|\mu}-b_\lambda^\nu(\vartheta_{\nu\beta|\mu}+\vartheta_{\nu\mu|\beta}-\vartheta_{\beta\mu|\nu})] &= 0 \\ \epsilon^{\alpha\beta}\,\epsilon^{\lambda\mu}[\vartheta_{\alpha\mu|\beta\lambda}-b_{\alpha\mu}\,\rho_{\beta\lambda}] &= 0 \end{aligned} \tag{3.4.11}$$

where

$$\rho_{\alpha\beta}=-\tfrac{1}{2}(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha}+b_\alpha^\lambda\,\omega_{\beta\lambda}+b_\beta^\lambda\,\omega_{\alpha\lambda}) \tag{3.4.12}$$

Equations (3.4.11) constitute the compatibility conditions well known in the linear theory of shells, expressed here in terms of strain measures $\vartheta_{\alpha\beta}$ and $\rho_{\alpha\beta}$ appearing in the "best" variant of the linear shell theory [4, 11].

4. Theory of finite rotations

4.1. Polar decomposition of the shell deformation gradient

The occurence of large rotations of the neighbourhood of the shell middle surface points is the fundamental feature of any non-linear shell deformation. This phenomenon appears even in the particular case of negligibly small strains. In a number of fundamental papers on the non-linear theory of shells different approximate variants of the geometrically non-linear theory of shells for "small", "medium", "large" etc. rotations are discussed. The basis for the approximate variants are limitations imposed upon the parameters φ_α and $\omega_{\alpha\beta}$, which are components of the linearized rotation vector. This approach, useful from the practical point of view for relatively small rotations, caeses to model correctly the actual deformation process for increasing rotations. In order to describe correctly the deformation within the unlimited rotation range let us consider then the exact description of the rotational part of shell deformation.

Complete information about the shell deformation, following the linear distribution of the displacement field (3.1.7), is provided by a deformation gradient tensor

$$\mathbf{G}=\bar{\mathbf{a}}_a\otimes\mathbf{a}^a, \qquad \mathbf{G}^{-1}=\mathbf{a}_a\otimes\bar{\mathbf{a}}^a \tag{4.1.1}$$

According to the polar decomposition theorem [18] the tensors $\mathbf{G}$ and $\mathbf{G}^{-1}$ can be represented in two uniquely defined forms

$$\mathbf{G}=\mathbf{RU}=\mathbf{VR}, \qquad \mathbf{G}^{-1}=\mathbf{U}^{-1}\mathbf{R}^T=\mathbf{R}^T\mathbf{V}^{-1} \tag{4.1.2}$$

where $\mathbf{U}$ and $\mathbf{V}$ are right and left stretch tensors, respectively, whereas $\mathbf{R}$ is a finite rotation tensor. The tensors $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite, and $\mathbf{R}$ is a proper orthogonal tensor. The following relations are valid

$$\begin{gathered}\mathbf{U}=\sqrt{\mathbf{G}^T\mathbf{G}}, \qquad \mathbf{V}=\sqrt{\mathbf{G}\mathbf{G}^T}\\ \mathbf{U}=\mathbf{R}^T\mathbf{V}\mathbf{R}, \qquad \mathbf{R}^T=\mathbf{R}^{-1}, \qquad \det\mathbf{R}=+1\end{gathered} \tag{4.1.3}$$

The tensor $\mathbf{U}$ is an appropriate strain measure in the Lagrangean description, while in the Eulerian description the tensor $\mathbf{V}$ is such a measure.

Relations (4.1.1) and (4.1.2) lead to the following formulae for the deformed basis

$$\begin{gathered}\bar{\mathbf{a}}_a=\mathbf{G}\mathbf{a}_a=\mathbf{R}\check{\mathbf{a}}_a=\mathbf{V}\overset{*}{\mathbf{a}}_a\\ \bar{\mathbf{a}}^a=(\mathbf{G}^{-1})^T\mathbf{a}^a=\mathbf{R}\check{\mathbf{a}}^a=\mathbf{V}^{-1}\overset{*}{\mathbf{a}}^a\end{gathered} \tag{4.1.4}$$

Here two additional intermediate bases have been introduced: a basis $\check{\mathbf{a}}_a$, generated from $\mathbf{a}_a$ after the pure strain due to the tensor $\mathbf{U}$, and a basis $\overset{*}{\mathbf{a}}_a$, generated from $\mathbf{a}_a$ as a result of the rigid-body rotation due to the tensor $\mathbf{R}$. In the Lagrangean description these bases are defined as follows

$$\begin{gathered}\check{\mathbf{a}}_a=\mathbf{U}\mathbf{a}_a, \qquad \check{\mathbf{a}}^a=\mathbf{U}^{-1}\mathbf{a}^a\\ \overset{*}{\mathbf{a}}_a=\mathbf{R}\mathbf{a}_a, \qquad \overset{*}{\mathbf{a}}^a=\mathbf{R}\mathbf{a}^a\end{gathered} \tag{4.1.5}$$

Relations inverse to (4.1.4) have the following form

$$\mathbf{a}_a = \mathbf{G}^{-1}\bar{\mathbf{a}}_a = \mathbf{U}^{-1}\check{\mathbf{a}}_a = \mathbf{R}^T\overset{*}{\mathbf{a}}_a$$
$$\mathbf{a}^a = \mathbf{G}^T\bar{\mathbf{a}}^a = \mathbf{U}\check{\mathbf{a}}^a = \mathbf{R}^T\overset{*}{\mathbf{a}}^a \tag{4.1.6}$$

Making use of (4.1.1) we can now represent the tensor $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ in the absolute notation

$$\mathbf{U} = \check{\mathbf{a}}_a \otimes \mathbf{a}^a, \qquad \mathbf{U}^{-1} = \mathbf{a}_a \otimes \check{\mathbf{a}}^a$$
$$\mathbf{V} = \bar{\mathbf{a}}_a \otimes \overset{*}{\mathbf{a}}^a, \qquad \mathbf{V}^{-1} = \overset{*}{\mathbf{a}}_a \otimes \bar{\mathbf{a}}^a \tag{4.1.7}$$
$$\mathbf{R} = \bar{\mathbf{a}}_a \otimes \check{\mathbf{a}}^a = \overset{*}{\mathbf{a}}_a \otimes \mathbf{a}^a$$

The stretch tensor $\mathbf{U}$ and the rotation tensor $\mathbf{R}$ can be expressed in the Lagrangean description in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$. It should only be remembered that

$$\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}, \qquad \bar{\mathbf{a}}_3 = \mathbf{n} + \boldsymbol{\beta} \tag{4.1.8}$$

and, by analogy to (3.4.4),

$$\bar{a}^{ad} = \frac{1}{2}\frac{a}{\bar{a}}\,\epsilon^{abc}\epsilon^{def}(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf})$$
$$\frac{\bar{a}}{a} = \frac{1}{6}\,\epsilon^{abc}\epsilon^{def}(a_{ad} + 2\gamma_{ad})(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}) \tag{4.1.9}$$
$$2\gamma_{ab} = \bar{\mathbf{a}}_a \cdot \bar{\mathbf{a}}_b - \mathbf{a}_a \cdot \mathbf{a}_b$$

Besides, according to (3.2.5) and (4.1.3) we have

$$\mathbf{U} = \sqrt{\mathbf{1} + 2\boldsymbol{\gamma}} \tag{4.1.10}$$

which, together with $(4.1.9)_3$, gives the expression of $\mathbf{U}$ in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$.

Consider more closely the structure of tensor $\mathbf{U}$ and other Lagrangean shell strain measures that will be used further on.

The tensor $\mathbf{U} = \mathbf{U}(M, t)$ is a non-singular, symmetric and positive definite tensor. It has therefore three real and positive eigenvalues U_M, $M = 1, 2, 3$, in three orthogonal principal directions defined by unit vectors $\mathbf{k}_M$, which satisfy the equations

$$\mathbf{U}\mathbf{k}_M = U_M\mathbf{k}_M, \qquad \mathbf{k}_M \cdot \mathbf{k}_N = \delta_{MN} \tag{4.1.11}$$

The tensor $\mathbf{U}$ has in the basis $\mathbf{k}_M$ the diagonal form

$$\mathbf{U} = \sum_M U_M \mathbf{k}_M \otimes \mathbf{k}_M \tag{4.1.12}$$

The tensor $\boldsymbol{\gamma} = \boldsymbol{\gamma}(M, t)$, which is a symmetric tensor, has three real, however not necessarily positive, eigenvalues γ_M. Its principal directions coincide, according to (4.1.10), with the principal directions of $\mathbf{U}$ and are defined by the same unit vectors $\mathbf{k}_M$. Hence $\boldsymbol{\gamma}$ can also be put in a diagonal form similar to (4.1.12)

$$\boldsymbol{\gamma} = \sum_M \gamma_M \mathbf{k}_M \otimes \mathbf{k}_M \tag{4.1.13}$$

with the eigenvalues of the both tensors related to each other through

$$\gamma_M = \tfrac{1}{2}(U_M^2 - 1), \quad U_M = +\sqrt{1+2\gamma_M} \tag{4.1.14}$$

Let

$$\mathbf{k}_M = k_M^a \mathbf{a}_a, \quad k_M^a = \mathbf{a}^a \cdot \mathbf{k}_M \tag{4.1.15}$$

Then

$$\mathbf{a}^a = \sum_M k_M^a \mathbf{k}_M, \quad \mathbf{a}_b = \sum_M k_M^a a_{ab} \mathbf{k}_M$$
$$a_{ab} = \sum_M k_M^c k_M^d a_{ac} a_{bd} \tag{4.1.16}$$

whence (4.1.5) yields

$$\breve{\mathbf{a}}_a = \mathbf{U}\mathbf{a}_a = (\mathbf{1} + \boldsymbol{\gamma} + \boldsymbol{\Lambda})\mathbf{a}_a \tag{4.1.17}$$

Here the tensor

$$\boldsymbol{\Lambda} = \sqrt{\mathbf{1} + 2\boldsymbol{\gamma}} - \mathbf{1} - \boldsymbol{\gamma} = \Lambda_{ab}\, \mathbf{a}^a \otimes \mathbf{a}^b \tag{4.1.18}$$

introduced in [45] is a symmetric tensor, in which $\boldsymbol{\gamma}$ appears raised to a power higher than one. This results from the expansion of the square root function into a Taylor series in the vicinity of $\boldsymbol{\gamma} = \mathbf{0}$. The tensor $\boldsymbol{\Lambda}$ has also a diagonal form

$$\boldsymbol{\Lambda} = \sum_M (\sqrt{1+2\gamma_M} - 1 - \gamma_M)\, \mathbf{k}_M \otimes \mathbf{k}_M \tag{4.1.19}$$

Its components are

$$\Lambda_{ab} = \sum_M (\sqrt{1+2\gamma_M} - 1 - \gamma_M)\, k_M^c k_M^d a_{ac} a_{bd} \tag{4.1.20}$$

Formula (4.1.20) seems to be simpler than the corresponding formula obtained in [45].

In what follows we will also use a strain tensor $\breve{\boldsymbol{\gamma}}$ defined by

$$\breve{\boldsymbol{\gamma}} = \sqrt{\mathbf{1} + 2\boldsymbol{\gamma}} - \mathbf{1} = \boldsymbol{\gamma} + \boldsymbol{\Lambda} = \breve{\gamma}_{ab}\, \mathbf{a}^a \otimes \mathbf{a}^b \tag{4.1.21}$$

$$\breve{\boldsymbol{\gamma}} = \sum_M (\sqrt{1+2\gamma_M} - 1)\, \mathbf{k}_M \otimes \mathbf{k}_M \tag{4.1.22}$$

$$\breve{\gamma}_{ab} = \sum_M (\sqrt{1+2\gamma_M} - 1)\, k_M^c k_M^d a_{ac} a_{bd} \tag{4.1.23}$$

Characteristic for the tensor $\breve{\boldsymbol{\gamma}}$ is that many relations, that are non-rational in terms of $\boldsymbol{\gamma}$, become polynomials or rational functions when expressed in terms of $\breve{\boldsymbol{\gamma}}$. This makes it possible to perform transformations and has led to a number of new relations, presented in p. 4.2.

Let us emphasize that during the pure strain of the shell caused by, for instance, the tensor $\mathbf{U}$, only material fibres coinciding with the principal directions $\mathbf{k}_M$ are subject to the pure stretch alone. Any other material fibre, with in general different eigenvalues U_M, is subject to stretching together with rotation. This is essential from the point of view of studies of a boundary element deformation, as discussed in detail in p. 4.3.

The rotation tensor $\mathbf{R}$ expressed in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$ takes, in view of (4.1.7) together with (4.1.9), and (4.1.21), the form

$$\mathbf{R}=[(\mathbf{a}_\alpha+\mathbf{u}_{,\alpha})\bar{a}^{\alpha b}+(\mathbf{n}+\boldsymbol{\beta})\bar{a}^{3b}]\otimes(\mathbf{a}_b+\check{\gamma}_{bc}\mathbf{a}^c) \tag{4.1.24}$$

One can obtain other equivalent forms by putting in (4.1.24) $\mathrm{U}, \gamma, \Lambda$ or U^{-1} instead of $\check{\gamma}$.

Any rotation tensor $\mathbf{R}$ has one real eigenvalue equal to $+1$. Let $\mathbf{e}_1$ be a unit vector of the principal direction of $\mathbf{R}$ corresponding to this eigenvalue. If we take an arbitrary right-handed orthonormal basis $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ it is possible to write the tensor $\mathbf{R}$ in the form [66, 70]

$$\mathbf{R}=\mathbf{e}_1\otimes\mathbf{e}_1+\cos\omega(\mathbf{e}_2\otimes\mathbf{e}_2+\mathbf{e}_3\otimes\mathbf{e}_3)-\sin\omega(\mathbf{e}_2\otimes\mathbf{e}_3-\mathbf{e}_3\otimes\mathbf{e}_2) \tag{4.1.25}$$

The direction defined by the unit vector $\mathbf{e}_1$ is called the axis of rotation of tensor $\mathbf{R}$, and the angle ω is called the angle of rotation of the tensor $\mathbf{R}$ about the axis of rotation.

The action of tensor $\mathbf{R}$ on a vector $\mathbf{v}$ can be interpreted geometrically as follows. Any vector $\mathbf{v}$ may be uniquely represented as a sum of a vector $\mathbf{v}_1$ directed along $\mathbf{e}_1$ and a vector $\mathbf{v}_p\perp\mathbf{e}_1$. Let α is the angle between $\mathbf{v}_p$ and $\mathbf{e}_2$. Then

$$\mathbf{v}=\mathbf{v}_1+\mathbf{v}_p=v_1\,\mathbf{e}_1+v_p(\cos\alpha\mathbf{e}_2+\sin\alpha\mathbf{e}_3) \tag{4.1.26}$$

The action of tensor $\mathbf{R}$ on the vector $\mathbf{v}$ results in a new vector $\overset{*}{\mathbf{v}}$ which takes, according to (4.1.25) and (4.1.26), the form

$$\overset{*}{\mathbf{v}}\equiv\mathbf{R}\mathbf{v}=v_1\,\mathbf{e}_1+v_p[\cos(\alpha+\omega)\,\mathbf{e}_2+\sin(\alpha+\omega)\,\mathbf{e}_3] \tag{4.1.27}$$

Hence it is evident that the tensor $\mathbf{R}$ rotates the vector $\mathbf{v}$ through the angle ω about the axis of rotation defined by $\mathbf{e}_1$.

4.2. Finite rotation vector

Within the analytical mechanics an operation analogous to (4.1.27) is formulated by means of a finite rotation vector $\boldsymbol{\Omega}$, which has direction and sense defined by the unit vector $\mathbf{e}_1\equiv\mathbf{e}$, and the length as required; for instance, in reference [46] this length was taken as equal to $|\mathrm{tg}\,\omega/2|$ while in [47] the magnitude $|2\,\mathrm{tg}\,\omega/2|$ was adopted. Hereafter we will assume, for the sake of simplicity in further applications, the length to be equal to $|\sin\omega|$, that is for $|\omega|<\pi$

$$\boldsymbol{\Omega}\equiv\sin\omega\mathbf{e} \tag{4.2.1}$$

Using formulae presented in [46, 47] and adopting them to the magnitude of $\boldsymbol{\Omega}$ as given by (4.2.1) we obtain the following formula for the action of the finite rotation vector $\boldsymbol{\Omega}$ on a vector $\mathbf{v}$

$$\begin{aligned}\overset{*}{\mathbf{v}}&=\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{v}+\frac{1}{2\cos^2\omega/2}\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{v})=\\&=\cos\omega\mathbf{v}+\boldsymbol{\Omega}\times\mathbf{v}+\frac{1}{2\cos^2\omega/2}\,(\boldsymbol{\Omega}\cdot\mathbf{v})\,\boldsymbol{\Omega}\end{aligned} \tag{4.2.2}$$

In view of $(4.1.4)_1$ we find, in particular, that the basis $\bar{\mathbf{a}}_a$ is obtained from the intermediate basis $\breve{\mathbf{a}}_a$ with the aid of a formula

$$\bar{\mathbf{a}}_a = \breve{\mathbf{a}}_a + \boldsymbol{\Omega} \times \breve{\mathbf{a}}_a + \frac{1}{2\cos^2\omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \breve{\mathbf{a}}_a) \tag{4.2.3}$$

where

$$\breve{\mathbf{a}}_a = \sqrt{1+2\gamma}\,\mathbf{a}_a \tag{4.2.4}$$

The vector $\boldsymbol{\Omega}$ is defined uniquely by the tensor $\mathbf{R}$. Appropriate relations in terms of the components of the tensor $\mathbf{R}$ with respect to an orthonormal basis have been presented in [71]. Such a base can be constructed of, for instance, unit vectors $\mathbf{i}_k$ of a Cartesian coordinate system fixed to O. Then relation (2.1.6) together with (2.1.8) yields

$$\begin{aligned} \mathbf{r} &= \mathbf{r}(M) = x^k(\vartheta^\alpha)\,\mathbf{i}_k \\ \mathbf{a}_\alpha &= x^k_{,\alpha}\,\mathbf{i}_k \\ \mathbf{n} &= \tfrac{1}{2}\,\epsilon^{\alpha\beta} x^k_{,\alpha} x^l_{,\beta}\, e_{klm}\,\mathbf{i}^m \end{aligned} \tag{4.2.5}$$

where e_{klm} is a permutation symbol. From (4.1.24) in conjunction with (4.2.5) it follows that the tensor $\mathbf{R}$ can be written as $R_{kl}\mathbf{i}^k \otimes \mathbf{i}^l$, where $\mathbf{i}^k \otimes \mathbf{i}^l$ is the orthonormal tensor basis, while R_{kl} depend only on the geometry of the surface $\mathscr{M}_\kappa$ and the displacement parameters $\mathbf{u}$ and $\boldsymbol{\beta}$. Formulae for R_{kl} result directly from substitution of (4.2.5) into (4.1.24). Then parameters of the vector $\boldsymbol{\Omega}$ can be computed using the relations [71]

$$\begin{aligned} \mathbf{e} &= -\frac{e^{klm}R_{kl}}{2\sin\omega}\,\mathbf{i}_m \\ \cos\omega &= \tfrac{1}{2}(\operatorname{tr}\mathbf{R}-1) = \tfrac{1}{2}(R_{11}+R_{22}+R_{33}-1) \\ \cos^2\omega/2 &= \tfrac{1}{2}(1+\cos\omega), \qquad \sin\omega = \sqrt{1-\cos^2\omega} \end{aligned} \tag{4.2.6}$$

The obtained in such a manner general relation for the finite rotation vector $\boldsymbol{\Omega}$, expressing it in terms of the displacement parameters $\mathbf{u}$ and $\boldsymbol{\beta}$, is too complicated for our future purposes. There exist other ways of expressing $\boldsymbol{\Omega}$ directly in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$, without reference to components of the tensor $\mathbf{R}$. Two of them are presented below.

First let us discuss determination of $\boldsymbol{\Omega}$ by way of determination of its components in the basis $\breve{\mathbf{a}}_a$. To this end, we will use representation $(4.2.2)_2$ of (4.2.3), taking next the scalar product of $\bar{\mathbf{a}}_a$ and $\breve{\epsilon}^{abc}\breve{\mathbf{a}}_b$. Making use of the symmetry we obtain

$$\breve{\epsilon}^{abc}\bar{\mathbf{a}}_a \cdot \breve{\mathbf{a}}_b = \breve{\epsilon}^{abc}\boldsymbol{\Omega}\cdot(\breve{\mathbf{a}}_a \times \breve{\mathbf{a}}_b) = 2\boldsymbol{\Omega}\cdot\breve{\mathbf{a}}^c \tag{4.2.7}$$

hence

$$\boldsymbol{\Omega} = \tfrac{1}{2}\,\breve{\epsilon}^{abc}(\bar{\mathbf{a}}_a \cdot \breve{\mathbf{a}}_b)\,\breve{\mathbf{a}}_c = \breve{\Omega}^\alpha \breve{\mathbf{a}}_\alpha + \breve{\Omega}^3\,\breve{\mathbf{a}}_3 \tag{4.2.8}$$

or, in terms of components

$$\begin{aligned} \breve{\Omega}^\alpha &= \tfrac{1}{2}\,\breve{\epsilon}^{3\beta\alpha}[\boldsymbol{\beta}\cdot\breve{\mathbf{a}}_\beta - (\mathbf{a}_\beta + \mathbf{u}_{,\beta})\cdot\breve{\mathbf{a}}_3] \\ \breve{\Omega}^3 &= \tfrac{1}{2}\,\breve{\epsilon}^{3\alpha\beta}(\mathbf{u}_{,\alpha}\cdot\breve{\mathbf{a}}_\beta) \end{aligned} \tag{4.2.9}$$

where $\breve{\mathbf{a}}_a$ is given by the formula (4.2.4).

Another direct method of expressing $\mathbf{\Omega}$ takes advantage of the fact, that in case of rotation of an orthonormal basis $\mathbf{k}_i$ about the axis of rotation defined by $\mathbf{e}$ through the angle ω to the position $\overset{*}{\mathbf{k}}_i$ the following holds true

$$2\mathbf{\Omega} = \mathbf{k}_1 \times \mathbf{k}_1^* + \mathbf{k}_2 \times \mathbf{k}_2^* + \mathbf{k}_3 \times \mathbf{k}_3^* \tag{4.2.10}$$

Thus rotation of any skew basis, for instance $\check{\mathbf{a}}_a$ in the case under consideration, leads to generalization of formula (4.2.10)

$$\mathbf{\Omega} = \tfrac{1}{2} \check{\mathbf{a}}_a \times \bar{\mathbf{a}}^a \tag{4.2.11}$$

One does not see at once that (4.2.11) and (4.2.8) really describe the same rotation vector. Therefore, prior to an application of (4.2.11), consider more closely geometric relations of the deformed configuration, and how they are expressed in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$ as well as geometric quantities of the reference configuration. Bearing in mind that

$$\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}, \qquad \bar{\mathbf{a}}_3 = \mathbf{n} + \boldsymbol{\beta} \tag{4.2.12}$$

and using formulae (3.4.4) we find that quantities

$$\begin{aligned} \bar{a}_{ab} &= a_{ab} + 2\gamma_{ab}, \qquad 2\gamma_{ab} = \bar{\mathbf{a}}_a \cdot \bar{\mathbf{a}}_b - \mathbf{a}_a \cdot \mathbf{a}_b \\ \bar{a}^{ad} &= \frac{1}{2}\frac{a}{\bar{a}}\, \epsilon^{abc} \epsilon^{def} (a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}) \\ \frac{\bar{a}}{a} &= \frac{1}{6}\, \epsilon^{abc} \epsilon^{def} (a_{ad} + 2\gamma_{ad})(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}) \end{aligned} \tag{4.2.13}$$

can also be expressed in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$ and the geometry of $\mathcal{M}_\kappa$. There are however many formulae related to the basis $\bar{\mathbf{a}}_a$ that contain square-root functions, as for instance (4.2.4), or the invariant $\sqrt{\bar{a}/a}$, which is a non-rational function of the tensor $\boldsymbol{\gamma}$. It is therefore convenient to express subsequent relations in terms of a symmetric strain tensor $\check{\boldsymbol{\gamma}}$ rather than the Green strain tensor $\boldsymbol{\gamma}$, the former having also been defined in the κ configuration geometry by the formula (4.1.21). Now in terms of $\check{\boldsymbol{\gamma}}$ we obtain the relations

$$\begin{aligned} \check{\mathbf{a}}_a &= \mathbf{a}_a + \check{\gamma}_{ab} \mathbf{a}^b = (\delta_a^c + \check{\gamma}_a^{\,c}) \mathbf{a}_c \\ \check{\mathbf{a}}_a \cdot \check{\mathbf{a}}_b &= \bar{\mathbf{a}}_a \cdot \bar{\mathbf{a}}_b = \bar{a}_{ab} = (\delta_a^c + \check{\gamma}_a^c)(\delta_b^d + \check{\gamma}_b^d)\, a_{cd} \\ 2\gamma_{ab} &= 2\check{\gamma}_{ab} + \check{\gamma}_a^{\,c} \check{\gamma}_{cb} \end{aligned} \tag{4.2.14}$$

where for the operations of raising and lowering an index the tensors a_{ab} and a^{ab} have been used. Besides, from (4.2.14) and (3.2.1) it follows that

$$\begin{aligned} \bar{\epsilon}_{abc} &= \sqrt{\frac{\bar{a}}{a}}\, \epsilon_{abc} = (\delta_a^d + \check{\gamma}_a^d)(\delta_b^e + \check{\gamma}_b^e)(\delta_c^f + \check{\gamma}_c^f)\, \epsilon_{def} \\ \sqrt{\frac{\bar{a}}{a}} &= \frac{1}{6}\, \epsilon^{abc} \epsilon_{def} (\delta_a^d + \check{\gamma}_a^{\,d})(\delta_b^e + \check{\gamma}_b^{\,e})(\delta_c^f + \check{\gamma}_c^{\,f}) \\ \bar{\epsilon}^{def} &= \sqrt{\frac{a}{\bar{a}}}\, \epsilon^{def} = \frac{a}{\bar{a}} (\delta_a^d + \check{\gamma}_a^d)(\delta_b^e + \check{\gamma}_b^e)(\delta_c^f + \check{\gamma}_c^f)\, \epsilon^{abc} \end{aligned} \tag{4.2.15}$$

which shows, among others, that the invariant $\sqrt{\bar{a}/a}$ is a polynomial of the third order in $\check{\gamma}$. Formulae (4.2.15) give, after some transformations, also the relations

$$\mathbf{a}_d = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{abc}\epsilon_{def}(\delta^e_b + \check{\gamma}^e_b)(\delta^f_c + \check{\gamma}^f_c)\check{\mathbf{a}}_a$$

$$\bar{a}^{ad} = \frac{1}{2}\frac{a}{\bar{a}}\,\epsilon^{abc}\epsilon^{def}(\delta^g_b + \check{\gamma}^g_b)(\delta^h_e + \check{\gamma}^h_e)(\delta^p_c + \check{\gamma}^p_c)(\delta^q_f + \check{\gamma}^q_f)\,a_{gh}\,a_{pq} \tag{4.2.16}$$

$$\check{\mathbf{a}}^a = \bar{a}^{ab}\check{\mathbf{a}}_b$$

It follows then that all geometric relations are rational functions of the tensor $\check{\gamma}$.

Now rewrite formulae (4.2.8) and (4.2.11) in the common basis $\mathbf{a}_a$, in order to prove that both of them define really the same finite rotation vector. Denoting

$$G_{da} = \mathbf{a}_d\,\mathbf{G}\mathbf{a}_a = \mathbf{a}_d \cdot \bar{\mathbf{a}}_a \tag{4.2.17}$$

we obtain from (4.2.8) together with (4.2.14)

$$2\boldsymbol{\Omega} = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{abc}(\delta^e_b + \check{\gamma}^e_b)(\delta^f_c + \check{\gamma}^f_c)\,G_{ea}\,\mathbf{a}_f \tag{4.2.18}$$

while (4.2.11), together with (4.2.14) and (2.1.13) yields

$$2\boldsymbol{\Omega} = \epsilon_{dpq}(\delta^d_a + \check{\gamma}^d_a)\bar{a}^{ab}G^p_{.b}\,\mathbf{a}^q \tag{4.2.19}$$

Next we transform (4.2.19) to the form of (4.2.18). Subsequent stages of the transformation procedure are given below, the underlined terms being those currently transformed. In view of $(4.2.16)_2$ we have

$$2\boldsymbol{\Omega} = \epsilon_{dpq}\underline{(\delta^d_a + \check{\gamma}^d_a)}\,\frac{1}{2}\underline{\frac{a}{\bar{a}}}\,\epsilon^{aef}\epsilon^{brs}\underline{(\delta^g_e + \check{\gamma}^g_e)}\times$$

$$\times(\delta^h_r + \check{\gamma}^h_r)\underline{(\delta^m_f + \check{\gamma}^m_f)}(\delta^n_s + \check{\gamma}^n_s)\,a_{gh}\,a_{mn}\,G^p_{.b}\,\mathbf{a}^q =$$

$$= \underline{\epsilon_{dpq}}\,\frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\underline{\epsilon^{dgm}}\epsilon^{brs}\underline{a_{gh}\,a_{mn}}(\delta^h_r + \check{\gamma}^h_r)(\delta^n_s + \check{\gamma}^n_s)\,G^p_{.b}\,\mathbf{a}^q =$$

$$= \underline{(a_{ph}\,a_{qn} - a_{qh}\,a_{pn})}\,\frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{brs}(\delta^h_r + \check{\gamma}^h_r)(\delta^n_s + \check{\gamma}^n_s)\underline{G^p_{.b}\,\mathbf{a}^q} =$$

$$= \underline{\frac{1}{2}}\sqrt{\frac{a}{\bar{a}}}\,\underline{\epsilon^{brs}}(\delta^h_r + \check{\gamma}^h_r)(\delta^n_s + \check{\gamma}^n_s)\underline{(G_{hb}\,\mathbf{a}_n - G_{nb}\,\mathbf{a}_h)} =$$

$$= \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{brs}(\delta^h_r + \check{\gamma}^h_r)(\delta^n_s + \check{\gamma}^n_s)\,G_{hb}\,\mathbf{a}_n \tag{4.2.20}$$

which, except for the index denotation, is exactly the formula (4.2.18).

The general formula (4.2.19) can also be transformed in the expanded form by means of displacement parameters $\mathbf{u}$ and $\boldsymbol{\beta}$ and the geometry of the surface $\mathscr{M}_\kappa$. Making use of

(4.2.14), (4.2.12) and (2.1.8), the following relation is arrived at after some transformations

$$2\mathbf{\Omega}=\epsilon_{\lambda\mu}\{(\delta_a^3+\check{\gamma}_a^3)[\bar{a}^{a\beta}(\delta_\beta^\lambda+\varphi_{.\beta}^\lambda)+\bar{a}^{a3}\beta^\lambda]-$$
$$-(\delta_a^\lambda+\check{\gamma}_a^\lambda)[\bar{a}^{a\beta}\varphi_\beta+\bar{a}^{a3}(1+\beta)]\}\,\mathbf{a}^\mu+$$
$$+\epsilon_{\lambda\mu}(\delta_a^\lambda+\check{\gamma}_a^\lambda)[\bar{a}^{a\beta}(\delta_\beta^\mu+\varphi_{.\beta}^\mu)+\bar{a}^{a3}\beta^\mu]\,\mathbf{n} \qquad (4.2.21)$$

Another equivalent form of (4.2.21) that does not contain components $\bar{a}^{ab}$ can be obtained by developing (4.2.18) or by direct substitution of (4.2.16) into (4.2.21).

Let $\mathbf{k}_\beta$ be the vector of change of curvature of the coordinate line ϑ^β at the middle surface of a shell when deformed from $\mathscr{M}_\kappa$ to $\mathscr{M}_\gamma$. The vector $\mathbf{k}_\beta$ can be computed from the formula

$$2\mathbf{k}_\beta=\bar{\epsilon}^{aef}[(\bar{G}_{a\beta}^c-G_{a\beta}^c)\bar{a}_{ce}-\check{\mathbf{a}}_{a;\,\beta}\cdot\check{\mathbf{a}}_e]\,\check{\mathbf{a}}_f \qquad (4.2.22)$$

according to relation derived for three-dimensional problems [45]. This can be transformed to

$$2\mathbf{k}_\beta=\bar{\epsilon}^{aef}[2\gamma_{e\beta;\,a}-a^{pq}\check{\gamma}_{ep}\check{\gamma}_{aq;\,\beta}]\,\check{\mathbf{a}}_f \qquad (4.2.23)$$

where the spatial covariant derivative $(\ )_{;\beta}$ at the surface $\mathscr{M}_\kappa$ is computed as shown in (2.1.5). The vector $\mathbf{k}_\beta$ can serve for the formulation of covariant surface differentiation of the intermediate basis $\check{\mathbf{a}}_a$. For the partial derivative there is

$$\check{\mathbf{a}}_{a,\,\beta}=\bar{G}_{a\beta}^c\,\check{\mathbf{a}}_c-\mathbf{k}_\beta\times\check{\mathbf{a}}_a \qquad (4.2.24)$$

Hence

$$\check{\mathbf{a}}_{a|\beta}=\check{\mathbf{a}}_{a,\,\beta}-\Gamma_{a\beta}^\lambda\,\check{\mathbf{a}}_\lambda=(\bar{G}_{a\beta}^\lambda-\Gamma_{a\beta}^\lambda)\,\check{\mathbf{a}}_\lambda+\bar{G}_{a\beta}^3\,\check{\mathbf{a}}_3-\mathbf{k}_\beta\times\check{\mathbf{a}}_a \qquad (4.2.25)$$

or, after rewriting

$$\check{\mathbf{a}}_{\alpha|\beta}=b_{\alpha\beta}\,\check{\mathbf{a}}_3+\bar{a}^{cd}\gamma_{d\alpha\beta}\,\check{\mathbf{a}}_c-\mathbf{k}_\beta\times\check{\mathbf{a}}_\alpha$$
$$\check{\mathbf{a}}_{3|\beta}=-b_\beta^\lambda\check{\mathbf{a}}_\lambda+\bar{a}^{cd}\gamma_{d3\beta}\,\check{\mathbf{a}}_c-\mathbf{k}_\beta\times\check{\mathbf{a}}_3 \qquad (4.2.26)$$

The derivative of the finite rotation vector $\mathbf{\Omega}$ can also be expressed in terms of $\mathbf{k}_\beta$

$$\frac{d\mathbf{\Omega}}{d\vartheta^\beta}=\cos\omega\mathbf{k}_\beta+\frac{1}{2}\,\mathbf{\Omega}\times\mathbf{k}_\beta-\frac{1}{4\cos^2\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{k}_\beta) \qquad (4.2.27)$$

The integrability condition of the set of equations (4.2.27) can be expressed in terms of $\mathbf{k}_\beta$ by

$$\epsilon^{\alpha\beta}(\mathbf{k}_{\beta|\alpha}+\tfrac{1}{2}\mathbf{k}_\alpha\times\mathbf{k}_\beta)=\mathbf{0} \qquad (4.2.28)$$

where

$$\mathbf{k}_\beta=\frac{d\mathbf{\Omega}}{d\vartheta^\beta}+\frac{1}{2\cos^2\omega/2}\,\frac{d\mathbf{\Omega}}{d\vartheta^\beta}\times\mathbf{\Omega}+\frac{d\omega}{d\vartheta^\beta}\ \mathrm{tg}\,\omega/2\,\mathbf{\Omega}$$
$$\frac{d\omega}{d\vartheta^\beta}=\mathbf{k}_\beta\cdot\mathbf{e} \qquad (4.2.29)$$

The vector $\mathbf{\Omega}$ can be used for derivation of a variety of geometric relations and formulae describing strain parameters.

From (3.1.7) and (4.2.3) it follows

$$\mathbf{u}_{,\beta}=\breve{\gamma}_{a\beta}\,\mathbf{a}^{a}+\mathbf{\Omega}\times\breve{\mathbf{a}}_{\beta}+\frac{1}{2\cos^{2}\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\breve{\mathbf{a}}_{\beta})$$
$$\boldsymbol{\beta}=\breve{\gamma}_{a3}\,\mathbf{a}^{a}+\mathbf{\Omega}\times\breve{\mathbf{a}}_{3}+\frac{1}{2\cos^{2}\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\breve{\mathbf{a}}_{3}) \qquad (4.2.30)$$

whence, after transformations

$$\breve{\gamma}_{a\beta}=\mathbf{a}_{a}\cdot\mathbf{u}_{,\beta}+(\delta^{c}_{\beta}+\breve{\gamma}^{c}_{\beta})\left[\epsilon_{acd}(\mathbf{\Omega}\cdot\mathbf{a}^{d})+\frac{1}{2\cos^{2}\omega/2}(\mathbf{\Omega}\times\mathbf{a}_{a})\cdot(\mathbf{\Omega}\times\mathbf{a}_{c})\right]$$

$$\breve{\gamma}_{a3}=\mathbf{a}_{a}\cdot\boldsymbol{\beta}+(\delta^{c}_{3}+\breve{\gamma}^{c}_{3})\left[\epsilon_{acd}(\mathbf{\Omega}\cdot\mathbf{a}^{d})+\frac{1}{2\cos^{2}\omega/2}(\mathbf{\Omega}\times\mathbf{a}_{a})\cdot(\mathbf{\Omega}\times\mathbf{a}_{c})\right]$$

$$\mathbf{a}_{a}\times\mathbf{u}_{,\beta}=\epsilon_{acd}\,\breve{\gamma}^{c}_{\beta}\,\mathbf{a}^{d}+(\delta^{c}_{\beta}+\breve{\gamma}^{c}_{\beta})\left\{\left[a_{ac}-\frac{1}{2\cos^{2}\omega/2}\,\epsilon_{acd}(\mathbf{\Omega}\cdot\mathbf{a}^{d})\right]\mathbf{\Omega}-\right.$$
$$\left.-(\mathbf{\Omega}\cdot\mathbf{a}_{a})\left[\mathbf{a}_{c}+\frac{1}{2\cos^{2}\omega/2}\,\mathbf{\Omega}\times\mathbf{a}_{c}\right]\right\} \qquad (4.3.31)$$

$$\mathbf{a}_{a}\times\boldsymbol{\beta}=\epsilon_{acd}\,\breve{\gamma}^{c}_{3}\,\mathbf{a}^{d}+(\delta^{c}_{3}+\breve{\gamma}^{c}_{3})\left\{\left[a_{ac}-\frac{1}{2\cos^{2}\omega/2}\,\epsilon_{acd}(\mathbf{\Omega}\cdot\mathbf{a}^{d})\right]\mathbf{\Omega}-\right.$$
$$\left.-(\mathbf{\Omega}\cdot\mathbf{a}_{a})\left(\mathbf{a}_{c}+\frac{1}{2\cos^{2}\omega/2}\,\mathbf{\Omega}\times\mathbf{a}_{c}\right)\right\}$$

In view of (4.2.17) the components of the deformation gradient tensor take the form

$$G_{ab}=\mathbf{a}_{a}\,\mathbf{G}\mathbf{a}_{b}=\mathbf{a}_{a}\cdot\bar{\mathbf{a}}_{b}=\begin{bmatrix} l_{\alpha\beta}, & \beta_{\alpha} \\ \varphi_{\beta}, & 1+\beta \end{bmatrix} \qquad (4.2.32)$$

Now from (4.2.3) and (4.2.14) we obtain

$$G_{ab}=(\delta^{c}_{b}+\breve{\gamma}^{c}_{b})\left[a_{ca}+\epsilon_{cad}(\mathbf{\Omega}\cdot\mathbf{a}^{d})-\frac{1}{2\cos^{2}\omega/2}(\mathbf{\Omega}\times\mathbf{a}_{c})(\mathbf{\Omega}\times\mathbf{a}_{a})\right] \qquad (4.2.33)$$

This leads easily to relations for $l_{\alpha\beta}$, φ_{α}, β_{α} and β.

4.3. Deformation of the boundary element

Consider a boundary curve $\mathscr{C}_{\kappa}$ on the middle surface of a shell in the κ configuration, described as follows

$$\vartheta^{1}=\vartheta^{1}(s),\qquad \vartheta^{2}=\vartheta^{2}(s) \qquad (4.3.1)$$

where s is the arc length of $\mathscr{C}_\kappa$. The curve $\mathscr{C}_\kappa$ creates in $\mathscr{S}_\kappa$ a boundary surface $\mathscr{B}_\kappa$, which is obtained by providing $\mathscr{C}_\kappa$ with vectors $\mathbf{n}$ normal to $\mathscr{M}_\kappa$. The geometry of this boundary surface is described by an orthogonal coordinate system defined by an orthonormal triad of vectors related to $\mathscr{C}_\kappa$. These are: $\mathbf{t}$, a vector tangent to $\mathscr{C}_\kappa$, $\mathbf{n}$, a vector normal to $\mathscr{M}_\kappa$, and $\mathbf{v} \equiv \mathbf{t} \times \mathbf{n}$, an outward normal to $\mathscr{C}_\kappa$ in a plane tangent to $\mathscr{M}_\kappa$; see Fig. 2.

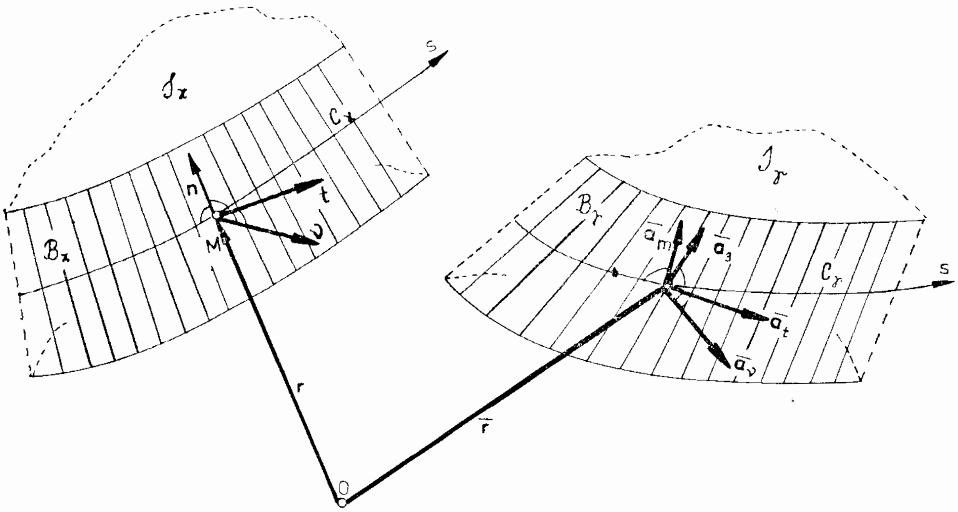

Fig. 2

Then there is over $\mathscr{C}_\kappa$ [64,82]

$$\mathbf{r}=\mathbf{r}(s), \quad \mathbf{n}=\mathbf{n}(s), \quad \mathbf{v}=\mathbf{t}\times\mathbf{n}=\mathbf{v}(s)$$

$$\mathbf{t}=\frac{d\mathbf{r}}{ds}=t^\alpha \mathbf{a}_\alpha=\epsilon^{\beta\alpha}\nu_\beta\,\mathbf{a}_\alpha\,, \quad \mathbf{v}=\nu^\alpha\mathbf{a}_\alpha=\epsilon^{\alpha\beta}t_\beta\,\mathbf{a}_\alpha \tag{4.3.2}$$

$$t^\alpha=\frac{d\vartheta^\alpha}{ds}, \quad t_\alpha=\frac{ds}{d\vartheta^\alpha}, \quad \mathbf{a}_\alpha=\nu_\alpha\mathbf{v}+t_\alpha\mathbf{t}$$

The deformation χ satisfying (3.1.4) causes the surface $\mathscr{B}_\kappa$ to turn approximately into a surface $\mathscr{B}_\gamma$, which is obtained from a boundary curve $\mathscr{C}_\gamma$ by providing the latter with vectors $\bar{\mathbf{a}}_3$. In general, the surface $\mathscr{B}_\gamma$ is not orthogonal to the surface $\mathscr{M}_\gamma$, although it still consists of approximately rectilinear elements. The geometry of this surface is defined by the convected coordinate system (s, ζ) introduced in $\mathscr{B}_\kappa$, which generates a skew coordinate system in $\mathscr{B}_\gamma$. Vectors $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$, not orthonormal in general, form the basis along the curve $\mathscr{C}_\gamma$. They also define a vector $\bar{\mathbf{a}}_\nu$, normal to $\mathscr{B}_\gamma$, which in general is not tangent to $\mathscr{M}_\gamma$ (Fig. 2).

For $\mathscr{C}_\gamma$ we find that

$$\bar{\mathbf{a}}_t=\bar{\mathbf{a}}_t(s)=\frac{d\bar{\mathbf{r}}}{ds}=t^\alpha\bar{\mathbf{a}}_\alpha=\mathbf{t}+\frac{d\mathbf{u}}{ds} \tag{4.3.3}$$

$$\bar{\mathbf{a}}_3=\bar{\mathbf{a}}_3(s)=\mathbf{n}(s)+\boldsymbol{\beta}(s) \tag{4.3.4}$$

$$\bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_\nu(s) = \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 = t^\alpha \bar{\epsilon}_{\alpha 3\beta}\, \bar{\mathbf{a}}^\beta = \sqrt{\frac{\bar{a}}{a}}\, \bar{\mathbf{a}}^\alpha \nu_\alpha = \left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right) \times (\mathbf{n} + \boldsymbol{\beta}) \tag{4.3.5}$$

$$\begin{aligned} \bar{\mathbf{a}}_m = \bar{\mathbf{a}}_\nu \times \bar{\mathbf{a}}_t = \sqrt{\frac{\bar{a}}{a}}\, \nu_\alpha\, \bar{\mathbf{a}}^\alpha \times \bar{\mathbf{a}}^b \bar{a}_{b\gamma}\, t^\gamma = \nu_\alpha\, \epsilon^{\alpha\beta} (\bar{a}_{\beta\gamma} \bar{\mathbf{a}}_3 - \bar{\mathbf{a}}_\beta \bar{a}_{3\gamma})\, t^\gamma = \\ = (\mathbf{n}+\boldsymbol{\beta}) \left[\left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right)\left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right)\right] - \left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right)\left[(\mathbf{n}+\boldsymbol{\beta})\left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right)\right] = (1+2\gamma_{tt})\,\bar{\mathbf{a}}_3 - 2\gamma_{3t}\,\bar{\mathbf{a}}_t \end{aligned} \tag{4.3.6}$$

Lengths of these vectors are given by the formulae

$$\bar{\mathbf{a}}_t \cdot \bar{\mathbf{a}}_t = (a_{\alpha\beta} + 2\gamma_{\alpha\beta})\, t^\alpha t^\beta = 1 + 2\gamma_{tt}, \quad |\bar{\mathbf{a}}_t| = \bar{a}_t = \sqrt{1+2\gamma_{tt}} \tag{4.3.7}$$

$$2\gamma_{tt} = 2\gamma_{\alpha\beta}\, t^\alpha t^\beta = 2\mathbf{t} \cdot \frac{d\mathbf{u}}{ds} + \frac{d\mathbf{u}}{ds} \cdot \frac{d\mathbf{u}}{ds}$$

$$\bar{\mathbf{a}}_3 \cdot \bar{\mathbf{a}}_3 = 1 + 2\gamma_{33}, \qquad |\bar{\mathbf{a}}_3| = \bar{a}_3 = \sqrt{1+2\gamma_{33}} \tag{4.3.8}$$

$$2\gamma_{33} = 2\mathbf{n} \cdot \boldsymbol{\beta} + \boldsymbol{\beta} \cdot \boldsymbol{\beta}$$

$$\bar{\mathbf{a}}_\nu \cdot \bar{\mathbf{a}}_\nu = (1+2\gamma_{tt})(1+2\gamma_{33}) - (2\gamma_{3t})^2$$

$$2\gamma_{3t} = 2\gamma_{3\alpha}\, t^\alpha = \mathbf{n} \cdot \frac{d\mathbf{u}}{ds} + \boldsymbol{\beta} \cdot \left(\mathbf{t} + \frac{d\mathbf{u}}{ds}\right) \tag{4.3.9}$$

$$|\bar{\mathbf{a}}_\nu| = \bar{a}_\nu = \sqrt{(1+2\gamma_{tt})(1+2\gamma_{33}) - (2\gamma_{3t})^2}$$

$$\begin{aligned} \bar{\mathbf{a}}_m \cdot \bar{\mathbf{a}}_m = (1+2\gamma_{tt})\left[(1+2\gamma_{tt})(1+2\gamma_{33}) - (2\gamma_{3t})^2\right] \\ |\bar{\mathbf{a}}_m| = \bar{a}_m = \bar{a}_\nu\, \bar{a}_t \end{aligned} \tag{4.3.10}$$

According to the polar decomposition (4.1.2), deformation of the boundary element can also be decomposed into a pure strain due to the tensor $\mathbf{U}$ and a rigid-body rotation done with the aid of the tensor $\mathbf{R}$ or the vector $\boldsymbol{\Omega}$. Thus there exist such vectors $\breve{\mathbf{a}}_t$, $\breve{\mathbf{a}}_3$, $\breve{\mathbf{a}}_\nu$ and $\breve{\mathbf{a}}_m$, that for each of them formula of the type as given by (4.2.3) holds, e.g.

$$\bar{\mathbf{a}}_t = \breve{\mathbf{a}}_t + \boldsymbol{\Omega} \times \breve{\mathbf{a}}_t + \frac{1}{2\cos^2 \omega/2}\, \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \breve{\mathbf{a}}_t) \tag{4.3.11}$$

We compute these vectors using $(4.2.14)_1$

$$\breve{\mathbf{a}}_t = \breve{\mathbf{a}}_\alpha\, t^\alpha = (a_{\alpha\beta} + \breve{\gamma}_{\alpha\beta})\, t^\alpha \mathbf{a}^\beta + \breve{\gamma}_{3\alpha}\, t^\alpha \mathbf{n} = \breve{\gamma}_{\nu t}\, \boldsymbol{\nu} + (1+\breve{\gamma}_{tt})\, \mathbf{t} + \breve{\gamma}_{3t}\, \mathbf{n} \tag{4.3.12}$$

$$\breve{\mathbf{a}}_3 = \mathbf{n} + \breve{\gamma}_{3\beta}\, \mathbf{a}^\beta + \breve{\gamma}_{33}\, \mathbf{n} = \breve{\gamma}_{3\nu}\, \boldsymbol{\nu} + \breve{\gamma}_{3t}\, \mathbf{t} + (1+\breve{\gamma}_{33})\, \mathbf{n} \tag{4.3.13}$$

$$\begin{aligned} \breve{\mathbf{a}}_\nu = \breve{\mathbf{a}}_t \times \breve{\mathbf{a}}_3 = \left[(1+\breve{\gamma}_{tt})(1+\breve{\gamma}_{33}) - (\breve{\gamma}_{3t})^2\right] \boldsymbol{\nu} + \left[\breve{\gamma}_{3\nu}\breve{\gamma}_{3t} - \breve{\gamma}_{\nu t}(1+\breve{\gamma}_{33})\right] \mathbf{t} + \\ + \left[\breve{\gamma}_{\nu t}\breve{\gamma}_{3t} - \breve{\gamma}_{3\nu}(1+\breve{\gamma}_{tt})\right] \mathbf{n} \end{aligned} \tag{4.3.14}$$

$$\begin{aligned} \breve{\mathbf{a}}_m = \breve{\mathbf{a}}_\nu \times \breve{\mathbf{a}}_t = (\breve{\mathbf{a}}_t \times \breve{\mathbf{a}}_3) \times \breve{\mathbf{a}}_t = (1+2\gamma_{tt})\, \breve{\mathbf{a}}_3 - (2\gamma_{3t})\, \breve{\mathbf{a}}_t = \\ = \left[(1+2\gamma_{tt})\breve{\gamma}_{3\nu} - 2\gamma_{3t}\breve{\gamma}_{\nu t}\right] \boldsymbol{\nu} + \left[(1+2\gamma_{tt})\breve{\gamma}_{3t} - 2\gamma_{3t}(1+\breve{\gamma}_{tt})\right] \mathbf{t} + \\ + \left[(1+2\gamma_{tt})(1+\breve{\gamma}_{33}) - 2\gamma_{3t}\breve{\gamma}_{3t}\right] \mathbf{n} \end{aligned} \tag{4.3.15}$$

Lengths of these vectors are equal to those of analogous rotated vectors given by (4.3.7) ÷ (4.3.10), which is obvious on account of their geometric interpretation.

In order to prove analytically this equality, relations between the components of $\boldsymbol{\gamma}$ and $\check{\boldsymbol{\gamma}}$ in the basis **v**, **t**, **n** ought to be applied. It follows from $(4.2.14)_3$ that

$$
\begin{aligned}
2\gamma_{\alpha\beta} &= 2\check{\gamma}_{\alpha\beta} + \check{\gamma}_{\alpha}^{\gamma}\,\check{\gamma}_{\gamma\beta} + \check{\gamma}_{\alpha}^{3}\,\check{\gamma}_{3\beta} \\
2\gamma_{3\beta} &= 2\check{\gamma}_{3\beta} + \check{\gamma}_{3}^{\gamma}\,\check{\gamma}_{\gamma\beta} + \check{\gamma}_{3}^{3}\,\check{\gamma}_{3\beta} \\
2\gamma_{33} &= 2\check{\gamma}_{33} + \check{\gamma}_{3}^{\alpha}\,\check{\gamma}_{\alpha 3} + \check{\gamma}_{3}^{3}\,\check{\gamma}_{33}
\end{aligned}
\tag{4.3.16}
$$

which leads to

$$
\begin{aligned}
1 + 2\gamma_{tt} &= (\check{\gamma}_{vt})^2 + (1 + \check{\gamma}_{tt})^2 + (\check{\gamma}_{3t})^2 \\
2\gamma_{3t} &= \check{\gamma}_{3v}\,\check{\gamma}_{vt} + \check{\gamma}_{3t}(1 + \check{\gamma}_{tt}) + \check{\gamma}_{3t}(1 + \check{\gamma}_{33}) \\
1 + 2\gamma_{33} &= (\check{\gamma}_{3v})^2 + (\check{\gamma}_{3t})^2 + (1 + \check{\gamma}_{33})^2
\end{aligned}
\tag{4.3.17}
$$

Computation of scalar products of vectors (4.3.12) ÷ (4.3.15) and reference of formulae (4.3.17) results in relations (4.3.7) ÷ (4.3.10)

Hence we obtain

$$
\begin{aligned}
&\check{a}_t = \sqrt{1 + 2\gamma_{tt}}, \qquad \check{a}_3 = \sqrt{1 + 2\gamma_{33}} \\
&\check{a}_v = \sqrt{(1 + 2\gamma_{tt})(1 + 2\gamma_{33}) - (2\gamma_{3t})^2}, \quad \check{a}_m = \check{a}_v\,\check{a}_t
\end{aligned}
\tag{4.3.18}
$$

During the pure strain caused by the tensor **U** the material fibres of the shell boundary $\mathscr{B}_\kappa$, which are defined along $\mathscr{C}_\kappa$ by vectors **t** and **n**, are being stretched and rotated as well. Stretching of the fibres is described by γ_{tt} and γ_{33} while the change of angles is given by γ_{3t}.

Let $\check{\boldsymbol{\Omega}}_t(s)$ be a finite rotation vector and $\check{\mathbf{R}}_t(s)$ a rotation tensor transforming **v**, **t**, **n** into $\check{\mathbf{v}}$, $\check{\mathbf{t}}$ $\check{\mathbf{m}}$, where

$$
\check{\mathbf{v}} = \frac{\check{\mathbf{a}}_v}{\check{a}_v}, \qquad \check{\mathbf{t}} = \frac{\check{\mathbf{a}}_t}{\check{a}_t}, \qquad \check{\mathbf{m}} = \frac{\check{\mathbf{a}}_m}{\check{a}_m}
\tag{4.3.19}
$$

Then according to (4.2.10) or $(4.1.7)_3$ there is

$$
2\check{\boldsymbol{\Omega}}_t = \mathbf{v} \times \check{\mathbf{v}} + \mathbf{t} \times \check{\mathbf{t}} + \mathbf{n} \times \check{\mathbf{m}}
\tag{4.3.20}
$$

$$
\check{\mathbf{R}}_t = \check{\mathbf{v}} \otimes \mathbf{v} + \check{\mathbf{t}} \otimes \mathbf{t} + \check{\mathbf{m}} \otimes \mathbf{n}
\tag{4.3.21}
$$

Formula (4.3.21) can be written out in a component form with respect to an orthonormal basis of the vectors **v**, **t** and **n**

$$
\check{\mathbf{R}}_t = (\check{R}_{11}\,\mathbf{v} + \check{R}_{21}\,\mathbf{t} + \check{R}_{31}\,\mathbf{n}) \otimes \mathbf{v} + (\check{R}_{12}\,\mathbf{v} + \check{R}_{22}\,\mathbf{t} + \check{R}_{32}\,\mathbf{n}) \otimes \mathbf{t} + (\check{R}_{13}\,\mathbf{v} + \check{R}_{23}\,\mathbf{t} + \check{R}_{33}\,\mathbf{n}) \otimes \mathbf{n}
\tag{4.3.22}
$$

where definitions of coefficients $\check{R}_{kl}$ result directly from (4.3.19), (4.3.18) together with (4.3.12), (4.3.14) and (4.3.15).

Computations yield them to be

$$\check{R}_{11}=\frac{(1+\check{\gamma}_{tt})(1+\check{\gamma}_{33})-(\check{\gamma}_{3t})^2}{|\check{\mathbf{a}}_v|}, \quad \check{R}_{21}=\frac{\check{\gamma}_{3v}\check{\gamma}_{3t}-\check{\gamma}_{vt}(1+\check{\gamma}_{33})}{|\check{\mathbf{a}}_v|}$$

$$\check{R}_{31}=\frac{\check{\gamma}_{vt}\check{\gamma}_{3t}-\check{\gamma}_{3v}(1+\check{\gamma}_{tt})}{|\check{\mathbf{a}}_v|}$$

$$\check{R}_{12}=\frac{\check{\gamma}_{vt}}{|\check{\mathbf{a}}_t|}, \qquad \check{R}_{22}=\frac{1+\check{\gamma}_{tt}}{|\check{\mathbf{a}}_t|}, \qquad \check{R}_{32}=\frac{\check{\gamma}_{3t}}{|\check{\mathbf{a}}_t|} \tag{4.3.23}$$

$$\check{R}_{13}=\frac{(1+2\gamma_{tt})\check{\gamma}_{3v}-2\gamma_{3t}\cdot\check{\gamma}_{vt}}{|\check{\mathbf{a}}_m|}, \quad \check{R}_{23}=\frac{(1+2\gamma_{tt})\check{\gamma}_{3t}-2\gamma_{3t}(1+\check{\gamma}_{tt})}{|\check{\mathbf{a}}_m|}$$

$$\check{R}_{33}=\frac{(1+2\gamma_{tt})(1+\check{\gamma}_{33})-2\gamma_{3t}\check{\gamma}_{3t}}{|\check{\mathbf{a}}_m|}$$

Direction cosines for the rotation axis unit vector $\check{\mathbf{e}}_t$ in the system $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$, as well as the angle $\check{\omega}_t$ through which the rotation about that axis is accomplished, are computed easily from the relations [71]

$$\check{\mathbf{e}}_t=c_1\,\mathbf{v}+c_2\,\mathbf{t}+c_3\,\mathbf{n}$$

$$\cos\check{\omega}_t=\tfrac{1}{2}(\operatorname{tr}\check{\mathbf{R}}_t-1)=\tfrac{1}{2}(\check{R}_{11}+\check{R}_{22}+\check{R}_{33}-1) \tag{4.3.24}$$

$$c_1=\frac{\check{R}_{32}-\check{R}_{23}}{2\sin\check{\omega}_t}, \qquad c_2=\frac{\check{R}_{13}-\check{R}_{31}}{2\sin\check{\omega}_t}, \qquad c_3=\frac{\check{R}_{21}-\check{R}_{12}}{2\sin\check{\omega}_t}$$

$$\sin\check{\omega}_t=\sqrt{1-\cos^2\check{\omega}_t}, \qquad \cos^2\check{\omega}_t/2=\tfrac{1}{2}(1+\cos\check{\omega}_t)$$

The resulting general form of the rotation vector is $\check{\mathbf{\Omega}}_t=\check{\mathbf{e}}_t\sin\check{\omega}_t$.

Transformation of the vectors $\mathbf{v}$ and $\mathbf{t}$ into vectors $\bar{\mathbf{a}}_v$ and $\bar{\mathbf{a}}_t$ is done in two stages. First, the stretching is realized with the help of $\mathbf{U}$, followed by the rotation realized with the help of $\check{\mathbf{R}}_t$ or $\check{\mathbf{\Omega}}_t$. These yield

$$\check{\mathbf{a}}_v=\bar{a}_v\check{\mathbf{R}}_t\,\mathbf{v}=\bar{a}_v\left[\mathbf{v}+\check{\mathbf{\Omega}}_t\times\mathbf{v}+\frac{1}{2\cos^2\check{\omega}_t/2}\check{\mathbf{\Omega}}_t\times(\check{\mathbf{\Omega}}_t\times\mathbf{v})\right] \tag{4.3.25}$$

$$\check{\mathbf{a}}_t=\bar{a}_t\check{\mathbf{R}}_t\,\mathbf{t}=\bar{a}_t\left[\mathbf{t}+\check{\mathbf{\Omega}}_t\times\mathbf{t}+\frac{1}{2\cos^2\check{\omega}_t/2}\check{\mathbf{\Omega}}_t\times(\check{\mathbf{\Omega}}_t\times\mathbf{t})\right] \tag{4.3.26}$$

Next, the vectors $\check{\mathbf{a}}_v$ or $\check{\mathbf{a}}_t$ are rotated to the final position. This is done with the help of $\mathbf{R}$ or $\mathbf{\Omega}$, using formula of the type (4.1.4) or (4.2.3).

Let $\mathbf{R}_t$ be a total rotation tensor and $\mathbf{\Omega}_t$ be a total finite rotation vector of the boundary element, resulting from a composition of two subsequent rotations caused by $\check{\mathbf{R}}_t$ or $\check{\mathbf{\Omega}}_t$, and $\mathbf{R}$ or $\mathbf{\Omega}$. After using the rules of composition of rotation tensors [18] and finite rotation vectors [47], and allowing for differences in definitions of lengths of finite rotation vectors we obtain

$$\mathbf{R}_t=\mathbf{R}\check{\mathbf{R}}_t$$

$$\mathbf{\Omega}_t=\left(1-\frac{\check{\mathbf{\Omega}}_t\cdot\mathbf{\Omega}}{4\cos^2\check{\omega}_t/2\,\cos^2\omega/2}\right)\left[\cos^2\omega/2\;\check{\mathbf{\Omega}}_t+\cos^2\check{\omega}_t/2\,\mathbf{\Omega}+\tfrac{1}{2}\mathbf{\Omega}\times\check{\mathbf{\Omega}}_t\right] \tag{4.3.27}$$

or analogously to (4.3.20) and (4.3.21)

$$\mathbf{R}_t = \frac{\bar{\mathbf{a}}_v}{\bar{a}_v} \otimes \mathbf{v} + \frac{\bar{\mathbf{a}}_t}{\bar{a}_t} \otimes \mathbf{t} + \frac{\bar{\mathbf{a}}_m}{\bar{a}_m} \otimes \mathbf{n}$$

$$2\boldsymbol{\Omega}_t = \mathbf{v} \times \frac{\bar{\mathbf{a}}_v}{\bar{a}_v} + \mathbf{t} \times \frac{\bar{\mathbf{a}}_t}{\bar{a}_t} + \mathbf{n} \times \frac{\bar{\mathbf{a}}_m}{\bar{a}_m} \tag{4.3.28}$$

Thus the final relations for $\bar{\mathbf{a}}_v$ and $\bar{\mathbf{a}}_t$ take the form

$$\bar{\mathbf{a}}_v = \bar{a}_v \mathbf{R}_t \mathbf{v} = \bar{a}_v \left[\mathbf{v} + \boldsymbol{\Omega}_t \times \mathbf{v} + \frac{1}{2\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{v}) \right] \tag{4.3.29}$$

$$\bar{\mathbf{a}}_t = \bar{a}_t \mathbf{R}_t \mathbf{t} = \bar{a}_t \left[\mathbf{t} + \boldsymbol{\Omega}_t \times \mathbf{t} + \frac{1}{2\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{t}) \right] \tag{4.3.30}$$

The transformation of $\mathbf{n}$ into $\bar{\mathbf{a}}_m$ has a similar form

$$\bar{\mathbf{a}}_m = \bar{a}_m \mathbf{R}_t \mathbf{n} = \bar{a}_m \left[\mathbf{n} + \boldsymbol{\Omega}_t \times \mathbf{n} + \frac{1}{2\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{n}) \right] \tag{4.3.31}$$

Using (4.3.6) we arrive at the following formula for $\bar{\mathbf{a}}_3$ in terms of $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_m$

$$\bar{\mathbf{a}}_3 = \frac{2\gamma_{3t}}{1 + 2\gamma_{tt}} \bar{\mathbf{a}}_t + \frac{1}{1 + 2\gamma_{tt}} \bar{\mathbf{a}}_m \tag{4.3.32}$$

which together with (4.3.30) and (4.3.31) gives a general relation for $\bar{\mathbf{a}}_3$.

4.4. Geometric boundary conditions

Problems of the non-linear theory of shells are formulated most frequently in terms of components of the displacement parameters. Hence in the basic variant of boundary conditions it is required to set values of parameters of the displacement state along the shell boundary. Unfortunately, the basic relations in the non-linear theory of shells, expressed in terms of displacements, lead as a rule to extremely complex formulae. It is mainly for this reason that effective solutions are based most often on relations valid within the frames of the simplest version of non-linear theory of thin shallow shells.

Formulation of the solution directly in terms of the displacement components is however not always appropriate. One is often more interested in the stress state in the shell, which can be determined from the known strain state with the aid of constitutive equations. Formulation of the problem from the beginning in terms of strain components generally leads to the relations that are considerably simpler and easier to be solved. The displacement parameters can be determined, if necessary, by integration of kinematic relations in the second stage of the solution. For such a procedure it is however required to formulate geometric boundary conditions in terms of components of the strain measures.

Recently Novozhilov and Shamina [55] proposed for the Kirchhoff-Love non-linear theory of shells two other kinds of boundary conditions, namely kinematical boundary conditions, given in terms of components of the total finite rotation vector $\mathbf{\Omega}_t$ together with strains γ_{tt}, and deformational boundary conditions, expressed with the aid of four parameters depending only on values of strain measures along the boundary. The two kinds of boundary conditions constitute generalizations, for the non-linear problems, of analogous variants of boundary conditions proposed by CHERNYKH [64] for the classical linear theory of thin shells.

SHAMINA presented in [72] the form of kinematical and deformational boundary conditions for the linear theory of shells of the Reissner type. A derivation of kinematical and deformational boundary conditions for a general variant of the non-linear theory of shells, based on a linear distribution of deformation across the shell thickness (3.1.4), has been given below. The general relations obtained in this way can be reduced to particular cases discussed in [55, 64, 72].

If the linear distribution of displacements (3.1.7) is assumed, the position vector $\bar{\mathbf{p}}$ of points $\bar{P}$ of the surface $\mathscr{B}_\gamma$ is given by

$$\bar{\mathbf{p}}(s,\zeta)=\bar{\mathbf{r}}(s)+\zeta\bar{\mathbf{a}}_3(s) \tag{4.4.1}$$

where

$$\bar{\mathbf{r}}(s)=\mathbf{r}(s)+\mathbf{u}(s), \quad \bar{\mathbf{a}}_3(s)=\mathbf{n}(s)+\boldsymbol{\beta}(s) \tag{4.4.2}$$

In order to uniquely define the boundary surface $\mathscr{B}_\gamma$ it is sufficient to set along $\mathscr{C}_\kappa$ two vector functions

$$\mathbf{u}(s)=\mathbf{A}(s), \quad \boldsymbol{\beta}(s)=\mathbf{B}(s) \tag{4.4.3}$$

Such form of the boundary conditions will be called displacement boundary conditions.

The same boundary surface can also be described implicitly with the aid of differential relations of the form

$$\frac{\partial\mathbf{p}}{\partial s}=\bar{\mathbf{a}}_t+\zeta\frac{d}{ds}\bar{\mathbf{a}}_3, \quad \frac{\partial\mathbf{p}}{\partial\zeta}=\bar{\mathbf{a}}_3 \tag{4.4.4}$$

$$\frac{\partial^2\mathbf{p}}{\partial s^2}=\frac{d}{ds}\bar{\mathbf{a}}_t+\zeta\frac{d^2}{ds^2}\bar{\mathbf{a}}_3, \quad \frac{\partial^2\mathbf{p}}{\partial s\partial\zeta}=\frac{d}{ds}\bar{\mathbf{a}}_3 \tag{4.4.5}$$

The differential equations define $\mathscr{B}_\gamma$ with accuracy up to a rigid-body translation or a rigid-body motion in space, respectively. These differential equations are uniquely defined if along the boundary either

$$\bar{\mathbf{a}}_t(s)=\mathbf{M}(s), \quad \bar{\mathbf{a}}_3(s)=\mathbf{L}(s) \tag{4.4.6}$$

or

$$\frac{d}{ds}\bar{\mathbf{a}}_t(s)=\mathbf{P}(s), \quad \frac{d}{ds}\bar{\mathbf{a}}_3(s)=\mathbf{Q}(s) \tag{4.4.7}$$

are given. The quantities under consideration are related to each other, namely there is

$$\mathbf{M}=\mathbf{t}+\frac{d\mathbf{A}}{ds}, \quad \mathbf{L}=\mathbf{n}+\mathbf{B}$$

$$\mathbf{P}=\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}, \quad \mathbf{Q}=\frac{d\mathbf{n}}{ds}+\frac{d\mathbf{B}}{ds} \tag{4.4.8}$$

In the formula (4.4.4) $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ are the principal quantities. According to formulae (4.3.30) and (4.3.32) these quantities can be expressed in terms of the total finite rotation vector $\mathbf{\Omega}_t$ and physical components γ_{tt}, γ_{3t} and γ_{33} of the strain tensor, which form together a set of independent variables. The other components $\breve{\gamma}_{tt}$, $\breve{\gamma}_{3t}$ and $\breve{\gamma}_{33}$ appearing in the intermediate formulae can be regarded as auxiliary ones. They can be expressed in terms of γ_{tt}, γ_{3t} and γ_{33} with the aid of relations similar to (4.2.14) with $\breve{\gamma}_{vt}$, $\breve{\gamma}_{3v}$ given then with the help of (4.3.17). Hence, quantities $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ are uniquely defined by setting at the boundary one vector function and three scalar functions

$$\mathbf{\Omega}_t(s)=\mathbf{m}(s),$$
$$\gamma_{tt}(s)=l(s), \quad \gamma_{3t}(s)=m(s), \; \gamma_{33}(s)=n(s) \tag{4.4.9}$$

The so defined boundary conditions will be called kinematical boundary conditions. Using (4.3.7), (4.3.8), (4.3.9) and the relation

$$2\mathbf{\Omega}_t=\mathbf{v}\times\frac{\bar{\mathbf{a}}_v}{\bar{a}_v}+\mathbf{t}\times\frac{\bar{\mathbf{a}}_t}{\bar{a}_t}+\mathbf{n}\times\frac{\bar{\mathbf{a}}_m}{\bar{a}_m} \tag{4.4.10}$$

that results from (4.2.10) we can compute values of functions which appear in (4.4.9). For given $\mathbf{A}(s)$ and $\mathbf{B}(s)$ we have

$$2\mathbf{m}=\frac{\mathbf{v}\times\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]}{\sqrt{(1+2l)(1+2n)-(2m)^2}}+\frac{\mathbf{t}\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)}{\sqrt{1+2l}}+$$
$$+\frac{\mathbf{n}\times\left\{\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\right\}}{\sqrt{1+2l}\sqrt{(1+2l)(1+2n)-(2m)^2}} \tag{4.4.11}$$

$$2l=2\mathbf{t}\cdot\frac{d\mathbf{A}}{ds}+\frac{d\mathbf{A}}{ds}\cdot\frac{d\mathbf{A}}{ds}$$
$$2m=\mathbf{n}\cdot\frac{d\mathbf{A}}{ds}+\mathbf{B}\cdot\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right) \tag{4.4.12}$$
$$2n=2\mathbf{n}\cdot\mathbf{B}+\mathbf{B}\cdot\mathbf{B}$$

A set of variables $\mathbf{\Omega}_t$, γ_{tt}, γ_{3t} and γ_{33} specified at the shell boundary describes the boundary surface $\mathscr{B}_y$ with accuracy up to any rigid-body translation of this surface in space.

In formula (4.4.5) derivatives of vectors $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ with respect to the arc length s are used. For this reason it is necessary to consider more closely differentiation of various vectors along the boundary curve.

Upon differentiation of the unit vectors defined at $\mathscr{C}_\kappa$ the following formulae are obtained

$$\frac{d\mathbf{v}}{ds}=\boldsymbol{\omega}_t\times\mathbf{v},\qquad \frac{d\mathbf{t}}{ds}=\boldsymbol{\omega}_t\times\mathbf{t},\qquad \frac{d\mathbf{n}}{ds}=\boldsymbol{\omega}_t\times\mathbf{n} \tag{4.4.13}$$

where

$$\boldsymbol{\omega}_t=\sigma_t\mathbf{v}+\tau_t\mathbf{t}+\kappa_t\mathbf{n} \tag{4.4.14}$$

$$\begin{gathered}\sigma_t=\mathbf{n}\cdot\frac{d\mathbf{t}}{ds}=-\mathbf{t}\cdot\frac{d\mathbf{n}}{ds},\qquad \tau_t=\mathbf{v}\cdot\frac{d\mathbf{n}}{ds}=-\mathbf{n}\cdot\frac{d\mathbf{v}}{ds}\\ \kappa_t=\mathbf{t}\cdot\frac{d\mathbf{v}}{ds}=-\mathbf{v}\cdot\frac{d\mathbf{t}}{ds}\end{gathered} \tag{4.4.15}$$

Formulae (4.4.15) define the normal curvature σ_t, the geodesic torsion τ_t and the geodesic curvature κ_t of the boundary curve $\mathscr{C}_\kappa$ over the surface $\mathscr{M}_\kappa$. These parameters as well as some other can be expressed in terms of geometric quantities describing the surface $\mathscr{M}_\kappa$. For this purpose the following relations [64] are used

$$\begin{gathered}\sigma_t=b_{\alpha\beta}t^\alpha t^\beta,\qquad \tau_t=-b_{\alpha\beta}v^\alpha t^\beta,\\ \kappa_t=-v_\alpha t^\alpha\big|_\beta t^\beta=t_\alpha v^\alpha\big|_\beta t^\beta\\ \sigma_v=b_{\alpha\beta}v^\alpha v^\beta,\qquad \kappa_v=-t_\alpha v^\alpha\big|_\beta v^\beta=+v_\alpha t^\alpha\big|_\beta v^\beta\end{gathered} \tag{4.4.16}$$

Let us denote the unit vectors of the deformed boundary, defined at $\mathscr{C}_\gamma$, as follows

$$\bar{\mathbf{v}}(s)=\frac{\bar{\mathbf{a}}_v}{\bar{a}_v},\qquad \bar{\mathbf{t}}(s)=\frac{\bar{\mathbf{a}}_t}{\bar{a}_t},\qquad \bar{\mathbf{m}}(s)=\frac{\bar{\mathbf{a}}_m}{\bar{a}_m} \tag{4.4.17}$$

Since $d\bar{s}=\bar{a}_t ds$, where $\bar{s}$ is the length parameter along $\mathscr{C}_\gamma$, let us assume, similarly to (4.4.13), that

$$\frac{d\bar{\mathbf{v}}}{ds}=\bar{\boldsymbol{\omega}}_t\times\bar{\mathbf{v}},\qquad \frac{d\bar{\mathbf{t}}}{ds}=\bar{\boldsymbol{\omega}}_t\times\bar{\mathbf{t}},\qquad \frac{d\bar{\mathbf{m}}}{ds}=\bar{\boldsymbol{\omega}}_t\times\bar{\mathbf{m}} \tag{4.4.18}$$

where

$$\bar{\boldsymbol{\omega}}_t=\bar{a}_t(\bar{\sigma}_t\bar{\mathbf{v}}+\bar{\tau}_t\bar{\mathbf{t}}+\bar{\kappa}_t\bar{\mathbf{m}}) \tag{4.4.19}$$

$$\begin{gathered}\bar{a}_t\bar{\sigma}_t=\bar{\mathbf{m}}\cdot\frac{d\bar{\mathbf{t}}}{ds}=-\bar{\mathbf{t}}\cdot\frac{d\bar{\mathbf{m}}}{ds},\qquad \bar{a}_t\bar{\tau}_t=\bar{\mathbf{v}}\cdot\frac{d\bar{\mathbf{m}}}{ds}=-\bar{\mathbf{m}}\cdot\frac{d\bar{\mathbf{v}}}{ds}\\ \bar{a}_t\bar{\kappa}_t=\bar{\mathbf{t}}\cdot\frac{d\bar{\mathbf{v}}}{ds}=-\bar{\mathbf{v}}\cdot\frac{d\bar{\mathbf{t}}}{ds}\end{gathered} \tag{4.4.20}$$

It is important to note that the unit vector $\bar{\mathbf{m}}$ as used here is not orthogonal to the surface $\mathscr{M}_\gamma$, and that $\bar{\mathbf{v}}$ does not rest on the plane tangent to $\mathscr{M}_\gamma$. Thus parameters $\bar{\sigma}_t$, $\bar{\tau}_t$ and $\bar{\kappa}_t$ are not related directly to the geometry of the surface $\mathscr{M}_\gamma$, although they do describe the curvature properties of the boundary curve $\mathscr{C}_\gamma$. This makes a significant difference when compared to the shell theory of the Kirchhoff-Love type, for which analogous relations have been discussed in [55].

Typical formulae resulting from (4.3.29) ÷ (4.3.31) can be used for the representation of the unit vectors $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$, $\bar{\mathbf{m}}$ in terms of the unit vectors $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$ and the finite rotation vector $\mathbf{\Omega}_t$. It can be noted that

$$\bar{\mathbf{m}} \cdot \mathbf{v} - \bar{\mathbf{v}} \cdot \mathbf{n} = 2\mathbf{\Omega}_t \cdot \mathbf{t} \tag{4.4.21}$$

Differentiation with respect to the are length s yields

$$2\frac{d\mathbf{\Omega}_t}{ds} \cdot \mathbf{t} = \bar{\boldsymbol{\omega}}_t \cdot (\bar{\mathbf{m}} \times \mathbf{v} - \bar{\mathbf{v}} \times \mathbf{n}) - \boldsymbol{\omega}_t \cdot (\mathbf{n} \times \bar{\mathbf{v}} - \mathbf{v} \times \bar{\mathbf{m}}) - 2(\mathbf{\Omega}_t \times \boldsymbol{\omega}_t) \cdot \mathbf{t} \tag{4.4.22}$$

After some transformations we arrive at

$$\bar{\mathbf{m}} \times \mathbf{v} - \bar{\mathbf{v}} \times \mathbf{n} = 2\cos\omega_t \mathbf{t} + \mathbf{\Omega}_t \times \mathbf{t} - \frac{1}{2\cos^2\omega_t/2} \mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \mathbf{t}) \tag{4.4.23}$$

Bearing in mind that

$$\begin{gathered} \bar{\boldsymbol{\omega}}_t = \check{\boldsymbol{\omega}}_t + \mathbf{\Omega}_t \times \check{\boldsymbol{\omega}}_t + \frac{1}{2\cos^2\omega_t/2} \mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \check{\boldsymbol{\omega}}_t) \\ \check{\boldsymbol{\omega}}_t = \bar{a}_t(\bar{\sigma}_t \mathbf{v} + \bar{\tau}_t \mathbf{t} + \bar{\kappa}_t \mathbf{n}) \end{gathered} \tag{4.4.24}$$

for various terms in (4.4.22), after rather lengthy and tedious transformations in which vector and trigonometric identities are utilized, we obtain

$$\bar{\boldsymbol{\omega}}_t \cdot (\bar{\mathbf{m}} \times \mathbf{v} - \bar{\mathbf{v}} \times \mathbf{n}) = \left[2\cos\omega_t \check{\boldsymbol{\omega}}_t + \mathbf{\Omega}_t \times \check{\boldsymbol{\omega}}_t - \frac{1}{2\cos^2\omega_t/2} \mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \check{\boldsymbol{\omega}}_t)\right] \cdot \mathbf{t} \tag{4.4.25}$$

$$-\boldsymbol{\omega}_t \cdot (\mathbf{n} \times \bar{\mathbf{v}} - \mathbf{v} \times \bar{\mathbf{m}}) - 2(\mathbf{\Omega}_t \times \boldsymbol{\omega}_t) \cdot \mathbf{t} =$$

$$= -\left[2\cos\omega_t \boldsymbol{\omega}_t + \mathbf{\Omega}_t \times \boldsymbol{\omega}_t - \frac{1}{2\cos^2\omega_t/2} \mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \boldsymbol{\omega}_t)\right] \cdot \mathbf{t} \tag{4.4.26}$$

Let $\mathbf{k}_t$ be the vector of change of curvature of the boundary contour during the shell deformation, defined by

$$\mathbf{k}_t = \check{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t \tag{4.4.27}$$

It follows from (4.4.22), (4.4.25) and (4.4.26) that

$$\frac{d\mathbf{\Omega}_t}{ds} = \cos\omega_t \mathbf{k}_t + \frac{1}{2} \mathbf{\Omega}_t \times \mathbf{k}_t - \frac{1}{4\cos^2\omega_t/2} \mathbf{\Omega}_t \times (\mathbf{\Omega}_t \times \mathbf{k}_t) \tag{4.4.28}$$

In view of (4.4.18), (4.4.17) and (4.3.7) ÷ (4.3.10) it is evident that derivatives of vectors $\bar{\mathbf{a}}_v$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_m$ along the boundary are defined by $\bar{\boldsymbol{\omega}}_t$ together with γ_{tt}, γ_{3t} and γ_{33}. Since (4.4.28) is equivalent to formulae (4.4.18), the above parameters can be substituted with equivalent quantities $\mathbf{k}_t$ and γ_{tt}, γ_{3t} and γ_{33}. Let us compute the components of $\mathbf{k}_t$ with respect to **v**, **t**, **n**. Let

$$\mathbf{k}_t = -k_{tt}\,\mathbf{v} + k_{vt}\,\mathbf{t} - k_{nt}\,\mathbf{n} \tag{4.4.29}$$

where

$$-k_{tt} = \bar{a}_t\,\bar{\sigma}_t - \sigma_t\,, \qquad k_{vt} = \bar{a}_t\,\bar{\tau}_t - \tau_t\,, \qquad -k_{nt} = \bar{a}_t\,\bar{\kappa}_t - \kappa_t \tag{4.4.30}$$

Compute derivatives with respect to underformed arc length of vectors of the deformed boundary. For vectors $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ we obtain

$$\frac{d}{ds}\,\bar{\mathbf{a}}_t = \bar{\mathbf{a}}_{\alpha,\,\beta}\,t^\alpha t^\beta + \bar{\mathbf{a}}_\alpha\,t^\alpha_{,\,\beta}\,t^\beta = t^\alpha\big|_\beta\,t^\beta\bar{\mathbf{a}}_\alpha + \sigma_t\,\bar{\mathbf{a}}_3 + \bar{a}^{dc}\gamma_{c\alpha\beta}\,t^\alpha t^\beta\bar{\mathbf{a}}_d \tag{4.4.31}$$

$$\frac{d}{ds}\,\bar{\mathbf{a}}_3 = \bar{\mathbf{a}}_{3,\,\beta}\,t^\beta = -\bar{\lambda}_{c\beta}\,t^\beta\bar{\mathbf{a}}^c = -\,b^\mu_\beta\,t^\beta\bar{\mathbf{a}}_\mu + \gamma_{c3\beta}\,t^\beta\bar{\mathbf{a}}^c \tag{4.4.32}$$

It follows from (4.3.32) that

$$\bar{\mathbf{a}}_m = (1+2\gamma_{tt})\,\bar{\mathbf{a}}_3 - 2\gamma_{3t}\,\bar{\mathbf{a}}_t \tag{4.4.33}$$

from which

$$\frac{d}{ds}\,\bar{\mathbf{a}}_m = 2\,\frac{d\gamma_{tt}}{ds}\,\bar{\mathbf{a}}_3 + (1+2\gamma_{tt})\,\frac{d}{ds}\,\bar{\mathbf{a}}_3 - 2\,\frac{d\gamma_{3t}}{ds}\,\bar{\mathbf{a}}_t - 2\gamma_{3t}\,\frac{d}{ds}\,\bar{\mathbf{a}}_t \tag{4.4.34}$$

The derivative of the vector $\bar{\mathbf{a}}_v$ obtained from (4.3.5), (4.4.31) and (4.4.32) is equal to

$$\frac{d}{ds}\,\bar{\mathbf{a}}_v = \frac{d}{ds}\,\bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 + \bar{\mathbf{a}}_t \times \frac{d}{ds}\,\bar{\mathbf{a}}_3 = -\sqrt{\frac{\bar{a}}{a}}\;\epsilon_{\lambda\mu}\,(t^\lambda\big|_\beta + \bar{a}^{\lambda c}\gamma_{c\alpha\beta}\,t^\alpha)\,t^\beta\bar{\mathbf{a}}^\mu -$$

$$-\sqrt{\frac{\bar{a}}{a}}\;\tau_t\,\bar{\mathbf{a}}^3 + \sqrt{\frac{\bar{a}}{a}}\;\nu_\lambda\,(\bar{a}^{3c}\bar{\mathbf{a}}^\lambda - \bar{a}^{\lambda c}\bar{\mathbf{a}}^3)\,\gamma_{c3\beta}\,t^\beta \tag{4.4.35}$$

Two different forms of formula $(4.4.20)_1$ may serve as a basis for the determination of $\bar{\sigma}_t$. The first form yields

$$\bar{a}_m\,\bar{a}_t^2\bar{\sigma}_t = \bar{\mathbf{a}}_m\cdot\frac{d}{ds}\,\bar{\mathbf{a}}_t = (1+2\gamma_{tt})\,(\bar{a}_{3\alpha}\,t^\alpha\big|_\beta\,t^\beta + \bar{a}_{33}\,\sigma_t + \gamma_{3\alpha\beta}\,t^\alpha t^\beta) -$$

$$-2\gamma_{3t}\,t^\lambda[(\bar{a}_{\lambda\alpha}\,t^\alpha\big|_\beta + \gamma_{\lambda\alpha\beta}\,t^\alpha)\,t^\beta + \sigma_t\,\bar{a}_{\lambda 3}] \tag{4.4.36}$$

The second one leads to the relation

$$\bar{a}_m\,\bar{a}_t^2\bar{\sigma}_t = -\bar{\mathbf{a}}_t\cdot\frac{d}{ds}\bar{\mathbf{a}}_m = -2\,\frac{d\gamma_{tt}}{ds}\,\bar{a}_{3\alpha}\,t^\alpha + (1+2\gamma_{tt})\,(b^\mu_\beta\,\bar{a}_{\alpha\mu} - \gamma_{\alpha 3\beta})\,t^\alpha t^\beta + \tag{4.4.37}$$

$$+2\,\frac{d\gamma_{3t}}{ds}\,\bar{a}_{\alpha\beta}\,t^\alpha t^\beta + 2\gamma_{3t}\,t^\lambda(\bar{a}_{\lambda\alpha}\,t^\alpha\big|_\beta\,t^\beta + \sigma_t\,\bar{a}_{3\lambda} + \gamma_{\lambda\alpha\beta}\,t^\alpha t^\beta)$$

which is equivalent to (4.4.36).

Similarly, using $(4.4.20)_{2,3}$ and (4.4.35) for $\bar{\tau}_t$ and $\bar{\kappa}_t$, we have

$$\bar{a}_\nu \bar{a}_t \bar{a}_m \bar{\tau}_t = -\bar{\mathbf{a}}_m \cdot \frac{d}{ds}\bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_\nu \cdot \frac{d}{ds}\bar{\mathbf{a}}_m = -\sqrt{\frac{\bar{a}}{a}}\,\nu_\lambda\left[(1+2\gamma_{tt})(b^\lambda_\beta - \bar{a}^{\lambda c}\gamma_{c3\beta}) + \right.$$

$$\left. + 2\gamma_{3t}(t^\lambda|_\beta + \bar{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha)\right] t^\beta \tag{4.4.38}$$

$$\bar{a}_\nu \bar{a}_t^2 \bar{\kappa}_t = \bar{\mathbf{a}}_t \cdot \frac{d}{ds}\bar{\mathbf{a}}_\nu = -\bar{\mathbf{a}}_\nu \cdot \frac{d}{ds}\bar{\mathbf{a}}_t = -\sqrt{\frac{\bar{a}}{a}}\,\nu_\lambda(t^\lambda|_\beta + \bar{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha)\, t^\beta$$

Thus, the relations (4.4.16), (4.4.37) and (4.4.38) made it possible to express components $k_{\nu t}$, k_{tt} and k_{nt} of the vector of change of curvature of the boundary curve $\mathbf{k}_t$ entirely in terms of components γ_{ab} and $\kappa_{\alpha\beta}$ of the Lagrangean measures of shell strain at the boundary surface $\mathscr{B}_\kappa$:

$$-k_{tt} = \frac{1}{\bar{a}_t \bar{a}_m}\left[\bar{a}_t^2\left(2\frac{d\gamma_{3t}}{ds} + \sigma_t - \kappa_{tt}\right) - 2\gamma_{3t}\frac{d\gamma_{tt}}{ds}\right] - \sigma_t$$

$$k_{\nu t} = \frac{1}{\bar{a}_\nu \bar{a}_m}\sqrt{\frac{\bar{a}}{a}}\left[\bar{a}_t^2(\tau_t + \nu_\lambda \bar{a}^{\lambda c}\gamma_{c3\beta}t^\beta) + 2\gamma_{3t}(\kappa_t - \nu_\lambda \bar{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha t^\beta)\right] - \tau_t \tag{4.4.39}$$

$$-k_{nt} = \frac{1}{\bar{a}_\nu \bar{a}_t}\sqrt{\frac{\bar{a}}{a}}(\kappa_t - \nu_\lambda \bar{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha t^\beta) - \kappa_t$$

It appears then that the surface $\mathscr{B}_\gamma$ may be described implicitly, with accuracy up to a rigid-body motion in space, by specifying along the shell boundary one vector function, or its three components, and three scalar functions

$$\mathbf{k}_t(s) = \mathbf{p}(s),$$
$$\gamma_{tt}(s) = l(s), \quad \gamma_{3t}(s) = m(s), \quad \gamma_{33}(s) = n(s) \tag{4.4.40}$$

where

$$\mathbf{p}(s) = -p(s)\,\mathbf{v} + q(s)\,\mathbf{t} - r(s)\,\mathbf{n} \tag{4.4.41}$$

Such boundary conditions will be called deformational boundary conditions.

In order to allow for a compact description of deformational boundary conditions, relations (4.4.36) ÷ (4.4.39) derived above have only partly been expressed in terms of physical components of strain measures at the shell boundary. For the further discussion of various approximate variants it will be necessary to have these relations expressed entirely in terms of physical components of strain measures at the shell boundary. Such relations are obtained immediately from the developed forms of appropriate tensor relations, but the transformations are extremely involved. Here main steps and more important intermediate relations will only be presented.

For the tensor $\gamma_{c\alpha\beta}$ it follows from (3.4.4) that

$$\gamma_{c\alpha\beta} = \gamma_{c\alpha;\,\beta} + \gamma_{c\beta;\,\alpha} - \gamma_{\alpha\beta;\,c} = \gamma_{c\alpha,\,\beta} + \gamma_{c\beta,\,\alpha} - \gamma_{\alpha\beta,\,c} - 2G^d_{\alpha\beta}\gamma_{cd} \tag{4.4.42}$$

where $G^d_{\alpha\beta}$ are Christoffel symbols of the basis $\mathbf{a}_a$. Hence we obtain after some transformations the following formulae for the physical components of the tensor $\gamma_{c\alpha\beta}$

$$\gamma_{\nu tt}=\gamma_{\lambda\alpha\beta}\,\nu^{\lambda}t^{\alpha}t^{\beta}=2\frac{d\gamma_{\nu t}}{ds}+\kappa_t(2\gamma_{\nu\nu}-2\gamma_{tt})-\sigma_t\cdot 2\gamma_{3\nu}-\frac{d\gamma_{tt}}{ds_\nu}+\kappa_\nu\cdot 2\gamma_{\nu t}$$

$$\gamma_{ttt}=\gamma_{\lambda\alpha\beta}\,t^{\lambda}t^{\alpha}t^{\beta}=\frac{d\gamma_{tt}}{ds}+\kappa_t\cdot 2\gamma_{\nu t}-\sigma_t\cdot 2\gamma_{3t} \tag{4.4.43}$$

$$\gamma_{3tt}=\gamma_{3\alpha\beta}\,t^{\alpha}t^{\beta}=2\frac{d\gamma_{3t}}{ds}+\kappa_t\cdot 2\gamma_{3\nu}-\sigma_t\cdot 2\gamma_{33}-\kappa_{tt}$$

$$\gamma_{\nu 3t}=\gamma_{\alpha 3\beta}\,\nu^{\alpha}t^{\beta}=\tfrac{1}{2}(\kappa_{\nu t}+\kappa_{t\nu})+\frac{d\gamma_{3\nu}}{ds}-\frac{d\gamma_{3t}}{ds_\nu}+\kappa_\nu\cdot\gamma_{3\nu}+$$

$$+2\sigma_t\cdot\gamma_{\nu t}-2\tau_t\cdot\gamma_{\nu\nu}-\kappa_t\cdot\gamma_{3t} \tag{4.4.44}$$

$$\gamma_{t3t}=\gamma_{\alpha 3\beta}\,t^{\alpha}t^{\beta}=\kappa_{tt}+2\sigma_t\cdot\gamma_{tt}-2\tau_t\cdot\gamma_{\nu t}$$

$$\gamma_{33t}=\gamma_{33\beta}\,t^{\beta}=\frac{d\gamma_{33}}{ds}+2\sigma_t\cdot\gamma_{3t}-2\tau_t\cdot\gamma_{3\nu}$$

where

$$\frac{d}{ds_\nu}(\quad)=\nu^{\alpha}\frac{d}{d\vartheta^{\alpha}}(\quad)$$

Covariant components of the spatial metric tensor over $\mathscr{M}_\gamma$ can be conveniently expressed in terms of physical components of strain measures, using the formulae

$$\bar{a}_{3\alpha}=2\gamma_{3t}\,t_\alpha+2\gamma_{3\nu}\,\nu_\alpha, \qquad \bar{a}_{33}=1+2\gamma_{33}$$

$$\bar{a}_{\alpha\beta}=(1+2\gamma_{\nu\nu})\,\nu_\alpha\nu_\beta+2\gamma_{\nu t}\,\nu_\alpha t_\beta+2\gamma_{t\nu}\,t_\alpha\nu_\beta+(1+2\gamma_{tt})\,t_\alpha t_\beta \tag{4.4.45}$$

Contravariant components should be first expressed in terms of covariant ones, according to (4.2.13). Then a number of geometric identities at the boundary $\mathscr{C}_\kappa$ should be used. One uses in particular formula (4.4.16) and

$$t_\alpha t^\alpha=\nu_\alpha\nu^\alpha=1, \qquad t_\alpha\nu^\alpha=t^\alpha\nu_\alpha=0, \qquad \nu^\alpha=\epsilon^{\alpha\beta}t_\beta$$

$$t^\alpha=\epsilon^{\beta\alpha}\nu_\beta, \qquad t_\alpha t^\alpha|_\beta t^\beta=0, \qquad \nu_\alpha\nu^\alpha|_\beta t^\beta=0 \tag{4.4.46}$$

$$\epsilon^{\alpha\beta}=\nu^\alpha t^\beta-t^\alpha\nu^\beta$$

After numerous and tedious transformations we obtain

$$\bar{\mathbf{a}}_\nu\cdot\frac{d}{ds}\bar{\mathbf{a}}_3=\sqrt{\frac{\bar{a}}{a}}\,\tau_t+\sqrt{\frac{a}{\bar{a}}}\left\{\left[\frac{d\gamma_{3\nu}}{ds}-\frac{d\gamma_{3t}}{ds_\nu}+\frac{1}{2}(\kappa_{\nu t}+\kappa_{t\nu})+\right.\right.$$

$$\left.+\kappa_\nu\gamma_{3\nu}+2\sigma_t\gamma_{\nu t}-2\tau_t\gamma_{\nu\nu}-\kappa_t\gamma_{3t}\right]\left[(1+2\gamma_{tt})(1+2\gamma_{33})-4\gamma_{3t}^2\right]-$$

$$-\left(\frac{d\gamma_{33}}{ds}+2\sigma_t\gamma_{3t}-2\tau_t\gamma_{3\nu}\right)\left[2\gamma_{3\nu}(1+2\gamma_{tt})-4\gamma_{\nu t}\cdot\gamma_{3t}\right]-$$

$$\left.-(\kappa_{tt}+2\sigma_t\gamma_{tt}-2\tau_t\gamma_{\nu t})\left[2\gamma_{\nu t}(1+2\gamma_{33})-4\gamma_{3\nu}\gamma_{3t}\right]\right\} \tag{4.4.47}$$

$$\bar{\mathbf{a}}_v \cdot \frac{d}{ds}\bar{\mathbf{a}}_t = -\sqrt{\frac{\bar{a}}{a}}\kappa_t - \sqrt{\frac{a}{\bar{a}}}\left\{\left[\frac{d\gamma_{tt}}{ds_v} - 2\frac{d\gamma_{vt}}{ds} - 2\kappa_v\gamma_{vt} + \right.\right.$$

$$\left. + 2\sigma_t\gamma_{3v} + 2\kappa_t(\gamma_{tt} - \gamma_{vv})\right]\left[(1+2\gamma_{tt})(1+2\gamma_{33}) - 4\gamma_{3t}^2\right] +$$

$$+\left(2\frac{d\gamma_{3t}}{ds} - \kappa_{tt} - 2\sigma_t\gamma_{33} + 2\kappa_t\gamma_{3v}\right)\left[2\gamma_{3v}(1+2\gamma_{tt}) - 4\gamma_{vt}\gamma_{3t}\right] +$$

$$\left. +\left(\frac{d\gamma_{tt}}{ds} - 2\sigma_t\gamma_{3t} + 2\kappa_t\gamma_{vt}\right)\left[2\gamma_{vt}(1+2\gamma_{33}) - 4\gamma_{3v}\gamma_{3t}\right]\right\} \tag{4.4.48}$$

$$\bar{\mathbf{a}}_3 \cdot \frac{d}{ds}\bar{\mathbf{a}}_t = 2\frac{d\gamma_{3t}}{ds} - \bar{\mathbf{a}}_t \cdot \frac{d}{ds}\bar{\mathbf{a}}_3 = 2\frac{d\gamma_{3t}}{ds} + \sigma_t - \kappa_{tt} \tag{4.4.49}$$

$$\bar{\mathbf{a}}_t \cdot \frac{d}{ds}\bar{\mathbf{a}}_t = \frac{d\gamma_{tt}}{ds}, \qquad \bar{\mathbf{a}}_3 \cdot \frac{d}{ds}\bar{\mathbf{a}}_3 = \frac{d\gamma_{33}}{ds} \tag{4.4.50}$$

The invariant $\bar{a}/a$ can be written in terms of γ_{ab}, according to $(4.2.13)_3$. Its developed form is

$$\frac{\bar{a}}{a} = 1 + 2\gamma_a^a + 2(\gamma_a^a)^2 + \tfrac{4}{3}(\gamma_a^a)^3 - 2\gamma_b^a\gamma_a^b - 4(\gamma_a^a)(\gamma_b^a\gamma_a^b) + \tfrac{8}{3}\gamma_b^a\gamma_c^b\gamma_a^c \tag{4.4.51}$$

where the respective invariants are related to physical strain components through the formulae

$$\gamma_a^a = \gamma_{vv} + \gamma_{tt} + \gamma_{33}$$

$$\gamma_b^a\gamma_a^b = \gamma_{vv}^2 + \gamma_{tt}^2 + \gamma_{33}^2 + 2(\gamma_{vt}^2 + \gamma_{3v}^2 + \gamma_{3t}^2) \tag{4.4.52}$$

$$\gamma_b^a\gamma_c^b\gamma_a^c = \gamma_{vv}^3 + \gamma_{tt}^3 + \gamma_{33}^3 + 3\left[\gamma_{vv}(\gamma_{vt}^2 + \gamma_{3v}^2) + \gamma_{tt}(\gamma_{vt}^2 + \gamma_{3t}^2) + \gamma_{33}(\gamma_{3v}^2 + \gamma_{3t}^2)\right] + 6\gamma_{vt}\gamma_{3v}\gamma_{3t}$$

Hence

$$\frac{\bar{a}}{a} = 1 + 2(\gamma_{vv} + \gamma_{tt} + \gamma_{33}) +$$

$$+ 4(\gamma_{vv}\gamma_{tt} + \gamma_{vv}\gamma_{33} + \gamma_{tt}\gamma_{33} - \gamma_{vt}^2 - \gamma_{3v}^2 - \gamma_{3t}^2) + \tag{4.4.53}$$

$$+ 8(\gamma_{vv}\gamma_{tt}\gamma_{33} + 2\gamma_{vt}\gamma_{3v}\gamma_{3t} - \gamma_{vv}\gamma_{3t}^2 - \gamma_{tt}\gamma_{3v}^2 - \gamma_{33}\gamma_{vt}^2)$$

In this way all necessary intermediate relations have been expressed in terms of physical components of strain measures at the boundary $\mathscr{C}_\kappa$. Owing to this, the components of the vector of change of curvature $\mathbf{k}_t$ can easily be written as dependent in the physical components only. To this end the relations (4.4.30), (4.4.33), (4.4.36), (4.4.38), and (4.4.39) should be used.

For known values of displacements $\mathbf{A}(s)$ and $\mathbf{B}(s)$ at the boundary $\mathscr{C}_\kappa$ of a shell, the functions $l(s)$, $m(s)$ and $n(s)$ to be set may be defined by formula (4.4.12). The remaining functions that are to be set at the boundary, namely $p(s)$, $q(s)$ and $r(s)$ as used in (4.4.41), may be found from

$$p(s)=\bar{a}_t(s)\bar{\sigma}_t(s)-\sigma_t(s), \quad q(s)=\bar{a}_t(s)\bar{\tau}_t(s)-\tau_t(s), \quad r(s)=\bar{a}_t(s)\bar{\kappa}_t(s)-\kappa_t(s) \tag{4.4.54}$$

where

$$\begin{aligned}
\bar{a}_t(s)\bar{\sigma}_t(s)&=\frac{(1+2l)(\mathbf{n}+\mathbf{B})\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)-2m\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)}{(1+2l)\sqrt{(1+2l)(1+2n)-(2m)^2}}\\
\bar{a}_t(s)\bar{\tau}_t(s)&=\frac{\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]\left[(1+2l)\left(\frac{d\mathbf{n}}{ds}+\frac{d\mathbf{B}}{ds}\right)-2m\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)\right]}{\sqrt{1+2l}\left[(1+2l)(1+2n)-(2m)^2\right]}\\
\bar{a}_t(s)\bar{\kappa}_t(s)&=\frac{\left[(\mathbf{n}+\mathbf{B})\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\right]\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)}{\sqrt{1+2l}\sqrt{(1+2l)(1+2n)-(2m)^2}}
\end{aligned} \tag{4.4.55}$$

while for $\sigma_t(s)$, $\tau_t(s)$ and $\kappa_t(s)$ relations (4.4.15) hold true.

If parameters $\mathbf{m}(s)$ and $l(s)$, $m(s)$ and $n(s)$ of kinematical boundary conditions as defined by (4.4.9) are known, then the vector $\mathbf{p}(s)$ can be defined by a formula analogous to that expressing an instantaneous angular velocity in terms of the finite rotation vector. Such relations are known in the analytical mechanics [46, 47]. After taking into account changes due to the different length assumed in the definition of the finite rotation vector we obtain [55]

$$\mathbf{p}(s)=\frac{d\mathbf{m}}{ds}-\frac{1}{1+\sqrt{1-\mathbf{m}\cdot\mathbf{m}}}\left[\mathbf{m}\cdot\frac{d}{ds}\sqrt{1-\mathbf{m}\cdot\mathbf{m}}+\mathbf{m}\times\frac{d\mathbf{m}}{ds}\right] \tag{4.4.56}$$

Setting of a group of variables $\mathbf{k}_t$, γ_{tt}, γ_{3t} and γ_{33} at the shell boundary defines the boundary surface $\mathscr{B}_\gamma$ with accuracy up to arbitrary rigid-body motion of the surface in space.

When boundary values of displacements $\mathbf{u}$ and $\boldsymbol{\beta}$ are known then the boundary values of $\boldsymbol{\Omega}_t$, γ_{tt}, γ_{3t}, γ_{33} and $\mathbf{k}_t$ can be computed without difficulty by proper differentiation, basing on relations (4.4.11), (4.4.12), (4.4.54) and (4.4.55). According to formula (4.4.56) boundary values of $\mathbf{k}_t$ are defined by boundary values of $\boldsymbol{\Omega}_t$.

It is much more difficult to solve the problem the other way round, that is to determine the displacement parameters of the boundary $\mathbf{u}$ and $\boldsymbol{\beta}$ when boundary values of $\boldsymbol{\Omega}_t$, γ_{tt}, γ_{3t}, γ_{33} or $\mathbf{k}_t$ are known. It is even difficult to determine $\boldsymbol{\Omega}_t$ from a known $\mathbf{k}_t$. In general, the problem consists in solving some non-linear differential equations and the result in form of a quadrature can not be expected. It can be shown, as in [73], that the determination of $\boldsymbol{\Omega}_t$, from a known $\mathbf{k}_t$, is mathematically equivalent to the determination of the finite rotation vector based on a known instantaneous angular velocity in the problem of motion of a rigid body around a fixed point. In the analytical mechanics it has been shown [46, 47] that simple solutions of the above mentioned problem, in form of quadrature, can be obtained only in the following particular cases: 1) motion of a rigid−body around a fixed axis of rotation, 2) small rotation of a rigid – body. The first case corresponds to deformation of a planar boundary contour in its plane. The second case corresponds to the linear theory of shells. Closed solutions can also be expected for shells of specific geometry or for specific deformation of the boundary contour. However, as far as the general geometry and non-linear deformation of the shell is considered, the problem of determination of displacement parameters of the boundary from known strain components requires further studies.

4.5. Relations for defoımation of Kirchhoff-Love type

If constraints of the Kirchhoff-Lovc type are imposed on a deformation, then the vector $\boldsymbol{\beta}$ will become a function of $\mathbf{u}$ and simplified relations as discussed in p. 3.3 will be satisfied.

Consider the simplifications of various relations using finite rotations under constraints of the Kirchhoff-Love type.

The polar decomposition of the deformation gradient given by (3.3.1) results in defining a right stretch tensor $\mathbf{U}$, for which the vector $\mathbf{n}$ is the unit vector corresponding to the third principal direction, and the eigenvalue corresponding to $\mathbf{n}$ is equal to $+1$. Let $\mathbf{k}_\Gamma$, $\Gamma=1, 2$ be unit vectors corresponding to the first and second principal directions of the tensor $\mathbf{U}$. Then there is

$$\begin{gathered}\mathbf{U}=\sum_\Gamma U_\Gamma \mathbf{k}_\Gamma\otimes\mathbf{k}_\Gamma+\mathbf{n}\otimes\mathbf{n}, \qquad \gamma=\sum_\Gamma \gamma_\Gamma \mathbf{k}_\Gamma\otimes\mathbf{k}_\Gamma=\gamma_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta \\ \Lambda=\sum_\Gamma(\sqrt{1+2\gamma_\Gamma}-1-\gamma_\Gamma)\,\mathbf{k}_\Gamma\otimes\mathbf{k}_\Gamma=\Lambda_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta \\ \breve{\gamma}=\sum_\Gamma(\sqrt{1+2\gamma_\Gamma}-1)\,\mathbf{k}_\Gamma\otimes\mathbf{k}_\Gamma=\breve{\gamma}_{\alpha\beta}\,\mathbf{a}^\alpha\otimes\mathbf{a}^\beta\end{gathered} \tag{4.5.1}$$

and the form of the rotation tensor $\mathbf{R}$ resulting from (4.1.24) is

$$\mathbf{R}=\bar{a}^{\alpha\beta}(\mathbf{a}_\alpha+\mathbf{u}_{,\alpha})\otimes(\mathbf{a}_\beta+\breve{\gamma}_{\beta\lambda}\,\mathbf{a}^\lambda)+\bar{\mathbf{n}}\otimes\mathbf{n} \tag{4.5.2}$$

Reduction of (4.2.21) also leads to a formula describing the finite rotation vector

$$2\boldsymbol{\Omega}=\epsilon_{\lambda\mu}\,[n^\lambda-\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)\,\varphi_\beta]\,\mathbf{a}^\mu+\epsilon_{\lambda\mu}\,\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)\,l^\mu_{.\beta}\,\mathbf{n} \tag{4.5.3}$$

Formulae (4.2.14) assume a simpler form

$$\begin{gathered}\breve{\mathbf{a}}_\alpha=\mathbf{a}_\alpha+\breve{\gamma}_{\alpha\beta}\,\mathbf{a}^\beta=(\delta^\beta_\alpha+\breve{\gamma}^\beta_\alpha)\,\mathbf{a}_\beta \\ \bar{a}_{\alpha\beta}=(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)(\delta^\mu_\beta+\breve{\gamma}^\mu_\beta)\,a_{\lambda\mu}, \qquad 2\gamma_{\alpha\beta}=2\breve{\gamma}_{\alpha\beta}+\breve{\gamma}^\lambda_\alpha\breve{\gamma}_{\lambda\beta}\end{gathered} \tag{4.5.4}$$

A reduction of (4.2.15) yields

$$\begin{gathered}\bar{\epsilon}_{\alpha\beta}=\sqrt{\frac{\bar{a}}{a}}\,\epsilon_{\alpha\beta}=(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)(\delta^\mu_\beta+\breve{\gamma}^\mu_\beta)\,\epsilon_{\lambda\mu} \\ \sqrt{\frac{\bar{a}}{a}}=\tfrac{1}{2}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)(\delta^\mu_\beta+\breve{\gamma}^\mu_\beta) \\ \bar{\epsilon}^{\lambda\mu}=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\lambda\mu}=\frac{a}{\bar{a}}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)(\delta^\mu_\beta+\breve{\gamma}^\mu_\beta)\,\epsilon^{\alpha\beta}\end{gathered} \tag{4.5.5}$$

while a reduction of (4.2.16) results in

$$\mathbf{a}_\alpha=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\lambda\mu}\epsilon_{\alpha\beta}(\delta^\beta_\mu+\check{\gamma}^\beta_\mu)\,\check{\mathbf{a}}_\lambda$$

$$\bar{a}^{\alpha\beta}=\frac{a}{\bar{a}}\,\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}(\delta^\gamma_\lambda+\check{\gamma}^\gamma_\lambda)(\delta^\rho_\mu+\check{\gamma}^\rho_\mu)\,a_{\gamma\rho} \tag{4.5.6}$$

$$\check{\mathbf{a}}^\alpha=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}(\delta^\mu_\beta+\check{\gamma}^\mu_\beta)\,\mathbf{a}^\lambda$$

Using (4.5.4) and (4.5.6) we arrive at the following useful identities

$$\mathbf{a}^\lambda=(\delta^\lambda_\alpha+\check{\gamma}^\lambda_\alpha)\,\check{\mathbf{a}}^\alpha$$

$$\check{\mathbf{a}}^\alpha\epsilon_{\alpha\rho}=\sqrt{\frac{a}{\bar{a}}}\,\epsilon_{\lambda\mu}(\delta^\mu_\rho+\check{\gamma}^\mu_\rho)\,\mathbf{a}^\lambda \tag{4.5.7}$$

$$a^{\lambda\mu}=(\delta^\lambda_\alpha+\check{\gamma}^\lambda_\alpha)(\delta^\mu_\beta+\check{\gamma}^\mu_\beta)\,\bar{a}^{\alpha\beta}$$

$$\bar{a}^{\alpha\kappa}(\delta^\beta_\kappa+\check{\gamma}^\beta_\kappa)=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}(a_{\lambda\mu}+\check{\gamma}_{\lambda\mu})$$

For the vector of change of curvature $\mathbf{k}_\beta$ a reduction of formula (4.2.23) leads to

$$2\mathbf{k}_\beta=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\lambda\mu}[(2\kappa_{\beta\lambda}+2b^\alpha_\beta\check{\gamma}_{\alpha\lambda})\check{\mathbf{a}}_\mu+(2\gamma_{\beta\mu|\lambda}-\check{\gamma}^\rho_\mu\check{\gamma}_{\rho\lambda|\beta})\mathbf{n}] \tag{4.5.8}$$

A covariant differentiation of the intermediate basis $\check{\mathbf{a}}_\alpha$, $\mathbf{n}$ yields the relations resulting from a reduction of (4.2.26)

$$\begin{aligned}\check{\mathbf{a}}_{\alpha|\beta}&=(b_{\alpha\beta}-\kappa_{\alpha\beta})\,\mathbf{n}+\bar{a}^{\lambda\mu}\gamma_{\mu\alpha\beta}\,\check{\mathbf{a}}_\lambda-\mathbf{k}_\beta\times\check{\mathbf{a}}_\alpha\\ \mathbf{n}_{|\beta}&=[-b^\lambda_\beta+\bar{a}^{\lambda\mu}(\kappa_{\beta\mu}+2b^\alpha_\beta\gamma_{\alpha\mu})]\,\check{\mathbf{a}}_\lambda-\mathbf{k}_\beta\times\mathbf{n}\end{aligned} \tag{4.5.9}$$

Formulae (4.5.8) and $(4.5.6)_3$ lead to

$$\begin{aligned}-\mathbf{k}_\beta\times\check{\mathbf{a}}_\alpha&=\epsilon_{\alpha\rho}\,\epsilon^{\mu\lambda}(\gamma_{\beta\mu|\lambda}-\tfrac{1}{2}\check{\gamma}^\kappa_\mu\check{\gamma}_{\kappa\lambda|\beta})\,\check{\mathbf{a}}^\rho+(\kappa_{\beta\alpha}+b^\kappa_\beta\check{\gamma}_{\kappa\alpha})\,\mathbf{n}\\ -\mathbf{k}_\beta\times\mathbf{n}&=-(\kappa_{\beta\lambda}+b^\alpha_\beta\check{\gamma}_{\alpha\lambda})\,\check{\mathbf{a}}^\lambda\end{aligned} \tag{4.5.10}$$

Hence, after combining (4.5.9) with (4.5.10) and performing some transformations, we arrive at the more simple relations

$$\begin{aligned}\check{\mathbf{a}}_{\alpha|\beta}&=\check{\gamma}_{\alpha\lambda|\beta}\,\mathbf{a}^\lambda+(b_{\alpha\beta}+\check{\gamma}_{\alpha\lambda}\,b^\lambda_\beta)\,\mathbf{n}\\ \mathbf{n}_{|\beta}&=-\,b^\lambda_\beta\,\mathbf{a}_\lambda\end{aligned} \tag{4.5.11}$$

The same result may be obtained by direct differentiation of (4.5.4). This is an additional proof of correctness of the relations (4.5.8) - (4.5.10).

Formula (4.5.8) can be used for the determination of $\kappa_{\alpha\beta}$. Taking into account the symmetry of this tensor for deformation of Kirchhoff-Love type, we arrive at a formula

$$\kappa_{\alpha\beta}=\tfrac{1}{2}(\bar{\epsilon}_{\alpha\lambda}\,\mathbf{k}_\beta+\bar{\epsilon}_{\beta\lambda}\,\mathbf{k}_\alpha)\cdot\check{\mathbf{a}}^\lambda-\tfrac{1}{2}(b^\lambda_\alpha\check{\gamma}_{\lambda\beta}+b^\lambda_\beta\check{\gamma}_{\lambda\alpha}) \tag{4.5.12}$$

In (4.2.28) $\mathbf{k}_\beta$ was written in terms of $\mathbf{\Omega}$. Therefore, also $\kappa_{\alpha\beta}$ can be written in terms of $\mathbf{\Omega}$ after substitution of (4.2.28) into (4.5.12)

Let us introduce a tensor

$$\check{\rho}_{\alpha\beta}=\kappa_{\alpha\beta}+\tfrac{1}{2}(b^\lambda_\alpha \check{\gamma}_{\lambda\beta}+b^\lambda_\beta \check{\gamma}_{\lambda\alpha}) \tag{4.5.13}$$

the linear part of which is a measure for the change of curvature appearing in the "best" linear theory of shells [4]. For this tensor there is

$$\check{\rho}_{\alpha\beta}=\tfrac{1}{2}(\bar{\epsilon}_{\alpha\lambda}\mathbf{k}_\beta+\bar{\epsilon}_{\beta\lambda}\mathbf{k}_\alpha)\cdot\check{\mathbf{a}}^\lambda \tag{4.5.14}$$

Also this formula can be rewritten in terms of $\mathbf{\Omega}$ with the aid of (4.2.28).

After a reduction of (4.2.30) we obtain

$$\begin{aligned}\mathbf{u}_{,\beta}&=\check{\gamma}_{\alpha\beta}\,\mathbf{a}^\alpha+(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left[\mathbf{\Omega}\times\mathbf{a}_\lambda+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{a}_\lambda)\right]\\ \boldsymbol{\beta}&=\mathbf{\Omega}\times\mathbf{n}+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{n})\end{aligned} \tag{4.5.15}$$

while a reduction of (4.2.31) yields

$$\begin{aligned}\check{\gamma}_{\alpha\beta}&=\mathbf{a}_\alpha\cdot\mathbf{u}_{,\beta}+(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left[\epsilon_{\alpha\lambda}(\mathbf{\Omega}\cdot\mathbf{n})+\frac{1}{2\cos^2\omega/2}(\mathbf{\Omega}\times\mathbf{a}_\alpha)(\mathbf{\Omega}\times\mathbf{a}_\lambda)\right]\\ \mathbf{a}_\alpha\times\mathbf{u}_{,\beta}&=\epsilon_{\alpha\lambda}\check{\gamma}^\lambda_\beta\mathbf{n}+(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left\{\left[a_{\alpha\lambda}-\frac{1}{2\cos^2\omega/2}\,\epsilon_{\alpha\lambda}(\mathbf{\Omega}\cdot\mathbf{n})\right]\mathbf{\Omega}-\right.\\ &\qquad\left.-(\mathbf{\Omega}\cdot\mathbf{a}_\alpha)\left(\mathbf{a}_\lambda+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times\mathbf{a}_\lambda\right)\right\}\\ \mathbf{n}\times\mathbf{u}_{,\beta}&=\epsilon_{\lambda\alpha}\check{\gamma}^\lambda_\beta\mathbf{a}^\alpha+(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left[\frac{1}{2\cos^2\omega/2}\,\epsilon_{\alpha\lambda}(\mathbf{\Omega}\cdot\mathbf{a}^\alpha)\mathbf{\Omega}-\right.\\ &\qquad\left.-(\mathbf{\Omega}\cdot\mathbf{n})\left(\mathbf{a}_\lambda+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times\mathbf{a}_\lambda\right)\right]\end{aligned} \tag{4.5.16}$$

According to (3.3.2) for deformation of Kirchhoff-Love type there is

$$\begin{aligned}\bar{\mathbf{a}}_\beta&=l_{\alpha\beta}\,\mathbf{a}^\alpha+\varphi_\beta\,\mathbf{n}=(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left[\mathbf{a}_\lambda+\mathbf{\Omega}\times\mathbf{a}_\lambda+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{a}_\lambda)\right]\\ \bar{\mathbf{n}}&=n_\alpha\mathbf{a}^\alpha+n\mathbf{n}=\mathbf{n}+\mathbf{\Omega}\times\mathbf{n}+\frac{1}{2\cos^2\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{n})\end{aligned} \tag{4.5.17}$$

Hence the following relations can be obtained for the respective components

$$l_{\alpha\beta}=(\delta^\lambda_\beta+\check{\gamma}^\lambda_\beta)\left[a_{\lambda\alpha}+\epsilon_{\lambda\alpha}(\mathbf{\Omega}\cdot\mathbf{n})-\frac{1}{2\cos^2\omega/2}(\mathbf{\Omega}\times\mathbf{a}_\lambda)(\mathbf{\Omega}\times\mathbf{a}_\alpha)\right]$$

$$\varphi_\beta=(\delta_\beta^\lambda+\breve{\gamma}_\beta^\lambda)\left[\epsilon_{\alpha\lambda}(\boldsymbol{\Omega}\cdot\mathbf{a}^\alpha)-\frac{1}{2\cos^2\omega/2}(\boldsymbol{\Omega}\times\mathbf{n})(\boldsymbol{\Omega}\times\mathbf{a}_\lambda)\right]$$

$$n_\alpha=\epsilon_{\alpha\lambda}(\boldsymbol{\Omega}\cdot\mathbf{a}^\lambda)-\frac{1}{2\cos^2\omega/2}(\boldsymbol{\Omega}\times\mathbf{a}_\alpha)(\boldsymbol{\Omega}\times\mathbf{n}) \tag{4.5.18}$$

$$n=1-\frac{1}{2\cos^2\omega/2}(\boldsymbol{\Omega}\times\mathbf{n})(\boldsymbol{\Omega}\times\mathbf{n})$$

During deformation of a boundary element subject to Kirchhoff-Love constraints the vectors $\bar{\mathbf{a}}_3=\bar{\mathbf{a}}_m=\bar{\mathbf{n}}$ are normal to the surface $\mathscr{M}_\gamma$, while the vector $\bar{\mathbf{a}}_\nu$ rests on a plane tangent to this surface. Therefore [55]

$$\bar{\mathbf{a}}_t=\sqrt{1+2\gamma_{tt}}\left[\mathbf{t}+\boldsymbol{\Omega}_t\times\mathbf{t}+\frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t\times(\boldsymbol{\Omega}_t\times\mathbf{t})\right]$$

$$\bar{\mathbf{a}}_\nu=\sqrt{1+2\gamma_{tt}}\left[\boldsymbol{\nu}+\boldsymbol{\Omega}_t\times\boldsymbol{\nu}+\frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t\times(\boldsymbol{\Omega}_t\times\boldsymbol{\nu})\right] \tag{4.5.19}$$

$$\bar{\mathbf{n}}=\mathbf{n}+\boldsymbol{\Omega}_t\times\mathbf{n}+\frac{1}{2\cos^2\omega_t/2}\boldsymbol{\Omega}_t\times(\boldsymbol{\Omega}_t\times\mathbf{n})$$

Taking into account that under Kirchhoff-Love constraints the finite rotation vector $\breve{\boldsymbol{\Omega}}_t$ has rotation axis normal to the surface $\mathscr{M}_\kappa$, we obtain

$$\breve{\boldsymbol{\Omega}}_t=\mathbf{n}\sin\breve{\omega}_t,$$

$$\sin\breve{\omega}_t=-\frac{\breve{\gamma}_{t\nu}}{\sqrt{1+2\gamma_{tt}}},\qquad \cos\breve{\omega}_t=\frac{1+\breve{\gamma}_{tt}}{\sqrt{1+2\gamma_{tt}}} \tag{4.5.20}$$

$$\cos^2\breve{\omega}_t/2=\frac{\sqrt{1+2\gamma_{tt}}+1+\breve{\gamma}_{tt}}{2\sqrt{1+2\gamma_{tt}}}$$

Then, in view of (4.3.27) and (4.3.28) we have

$$\boldsymbol{\Omega}_t=\left[1+\frac{(\mathbf{n}\cdot\boldsymbol{\Omega})\sin\breve{\omega}_t}{4\cos^2\breve{\omega}_t/2\cos^2\omega/2}\right][\sin\breve{\omega}_t(\mathbf{n}\cos^2\omega/2+\tfrac{1}{2}\boldsymbol{\Omega}\times\mathbf{n})+\boldsymbol{\Omega}\cos^2\breve{\omega}_t/2]$$

$$2\boldsymbol{\Omega}_t=\frac{1}{\sqrt{1+2\gamma_{tt}}}(\boldsymbol{\nu}\times\bar{\mathbf{a}}_\nu+\mathbf{t}\times\bar{\mathbf{a}}_t)+\mathbf{n}\times\bar{\mathbf{n}} \tag{4.5.21}$$

Consider forms of geometric boundary conditions for deformations of Kirchhoff-Love type. Let

$$\boldsymbol{\beta}(s)=\beta_\nu(s)\,\bar{\mathbf{a}}_\nu+\beta_t(s)\bar{\mathbf{a}}_t+\beta_n(s)\bar{\mathbf{n}} \tag{4.5.22}$$

Transformations of this relation yield

$$\beta_\nu=-\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\beta\,\bar{a}^{\beta\alpha}(\mathbf{u}_{,\alpha}\cdot\mathbf{n}),\qquad \beta_t=-\frac{1}{1+2\gamma_{tt}}\frac{d\mathbf{u}}{ds}\cdot\mathbf{n} \tag{4.5.23}$$

$$\beta_n=1-\sqrt{1-(1+2\gamma_{tt})(\beta_\nu^2+\beta_t^2)}$$

Now it is evident that for known $\mathbf{u}(s)$ and $\beta_\nu(s)$ also $\beta_t(s)$ and $\beta_n(s)$ can easily be calculated. Displacement boundary conditions take for deformation of Kirchhoff-Love type the reduced form

$$\mathbf{u}(s)=\mathbf{A}(s), \quad \beta_\nu(s)=b(s) \tag{4.5.24}$$

Functions $\mathbf{A}(s)$ and $b(s)$ define uniquely also the vector function $\mathbf{B}(s)$ appearing in (4.4.3).

Kinematical boundary conditions are obtained as a result of reduction of formulae (4.4.9) ÷ (4.4.12). They have the following form

$$\boldsymbol{\Omega}_t(s)=\mathbf{m}(s), \quad \gamma_{tt}(s)=l(s) \tag{4.5.25}$$

where

$$2\mathbf{m}(s)=\frac{1}{\sqrt{1+2l}}\left\{\mathbf{v}\times\left[\mathbf{t}\times\mathbf{B}+\frac{d\mathbf{A}}{ds}\times(\mathbf{n}+\mathbf{B})\right]+\mathbf{t}\times\frac{d\mathbf{A}}{ds}\right\}+\mathbf{n}\times\mathbf{B}$$
$$2l(s)=2\mathbf{t}\cdot\frac{d\mathbf{A}}{ds}+\frac{d\mathbf{A}}{ds}\cdot\frac{d\mathbf{A}}{ds} \tag{4.5.26}$$

Deformational boundary conditions result from reduction of relations (4.4.36) ÷ (4.4.39):

$$\mathbf{k}_t(s)=\mathbf{p}(s), \quad \gamma_{tt}(s)=l(s) \tag{4.5.27}$$

where components of the vector $\mathbf{k}_t$ can be computed from the relations

$$k_{tt}=\sigma_t\left(1-\frac{1}{\sqrt{1+2\gamma_{tt}}}\right)+\frac{\kappa_{tt}}{\sqrt{1+2\gamma_{tt}}}$$
$$k_{\nu t}=\tau_t\left(\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}-1\right)+\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\lambda\bar{a}^{\lambda\mu}(\kappa_{\mu\beta}+2b^\kappa_\beta\gamma_{\mu\kappa})t^\beta \tag{4.5.28}$$
$$k_{nt}=\kappa_t\left(1-\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\right)+\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\lambda\bar{a}^{\lambda\mu}(\gamma_{\mu\alpha|\beta}+\gamma_{\mu\beta|\alpha}-\gamma_{\alpha\beta|\mu})t^\alpha t^\beta$$

These formulae for k_{tt} and k_{nt} are identical in form with analogous formulae presented in [55]. The formula expressing $k_{\nu t}$ is formally different than that given in [55], but in fact the both forms are equivalent.

Formulae (4.4.47) and (4.4.48) can be written in a more simple form

$$\bar{\mathbf{a}}_\nu\cdot\frac{d}{ds}\bar{\mathbf{n}}=\sqrt{\frac{\bar{a}}{a}}\,\tau_t+\sqrt{\frac{a}{\bar{a}}}\{(\kappa_{\nu t}+2\sigma_t\gamma_{\nu t}-2\tau_t\gamma_{\nu\nu})(1+2\gamma_{tt})-(\kappa_{tt}+2\sigma_t\gamma_{tt}-2\tau_t\gamma_{\nu t})2\gamma_{\nu t}\}$$
$$\bar{\mathbf{a}}_\nu\cdot\frac{d}{ds}\bar{\mathbf{a}}_t=-\sqrt{\frac{\bar{a}}{a}}\,\kappa_t-\sqrt{\frac{a}{\bar{a}}}\left\{\left[\frac{d\gamma_{tt}}{ds_\nu}-2\frac{d\gamma_{\nu t}}{ds}-2\kappa_\nu\gamma_{\nu t}+2\kappa_t(\gamma_{tt}-\gamma_{\nu\nu})\right](1+2\gamma_{tt})+\right.$$
$$\left.+\left(\frac{d\gamma_{tt}}{ds}+2\kappa_t\gamma_{\nu t}\right)2\gamma_{\nu t}\right\} \tag{4.5.29}$$

This makes it possible to represent also $k_{\nu t}$ and k_{nt} entirely in terms of physical components of strain measures at the shell boundary

$$k_{\nu t}=\tau_t\left(\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}-1\right)+\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{a}{\bar{a}}}\{(1+2\gamma_{tt})(\kappa_{\nu t}+2\sigma_t\gamma_{\nu t}-2\tau_t\gamma_{\nu\nu})-$$
$$-2\gamma_{\nu t}(\kappa_{tt}+2\sigma_t\gamma_{tt}-2\tau_t\gamma_{\nu t})\}$$
$$k_{nt}=\kappa_t\left(1-\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\right)-\frac{2\gamma_{\nu t}}{1+2\gamma_{tt}}\sqrt{\frac{a}{\bar{a}}}\left(\frac{d\gamma_{tt}}{ds}+2\kappa_t\gamma_{\nu t}\right)+ \tag{4.5.30}$$
$$+\sqrt{\frac{a}{\bar{a}}}\left[2\frac{d\gamma_{\nu t}}{ds}-\frac{d\gamma_{tt}}{ds_\nu}+2\kappa_\nu\gamma_{\nu t}+2\kappa_t(\gamma_{\nu\nu}-\gamma_{tt})\right]$$

where in view of (4.4.53)

$$\frac{\bar{a}}{a}=1+2(\gamma_{\nu\nu}+\gamma_{tt})+4(\gamma_{\nu\nu}\gamma_{tt}-\gamma_{\nu t}^2) \tag{4.5.31}$$

Functions (4.4.54), which should be assumed at the deformed boundary, take the following values resulting from a reduction of relations (4.4.55)

$$\bar{a}_t(s)\bar{\sigma}_t(s)=\frac{1}{\sqrt{1+2l}}(\mathbf{n}+\mathbf{B})\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)$$
$$\bar{a}_t(s)\bar{\tau}_t(s)=\frac{1}{\sqrt{1+2l}}\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]\left(\frac{d\mathbf{n}}{ds}+\frac{d\mathbf{B}}{ds}\right) \tag{4.5.32}$$
$$\bar{a}_t(s)\bar{\kappa}_t(s)=\frac{1}{1+2l}\left[(\mathbf{n}+\mathbf{B})\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\right]\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)$$

The value of function $\mathbf{p}(s)$ in (4.5.27) will be found from relation (4.4.56) if value of function $\mathbf{m}(s)$ given by (4.5.26) are used.

5. Basic shell equations

5.1. Equations of motion

There are several ways of arriving at general dwo-dimensional equations of motion together with appropriate natural boundary conditions for shells in the Lagrangean description. In particular, these equations have been derived in papers [19, 24] by a direct integration of the local equations of motion of continuum in the Lagrangean form over the shell thickness. Here another equivalent derivation, based on the principle of stationary action, will be presented.

Consider an elastic shell $\mathscr{S}$ which, according to [74], is a dynamic system with all dynamic properties local in space as well as in time. Hence, for any time period (t_1, t_2)

and any part of shell in the reference configuration κ, a relation

$$\delta \int_{t_1}^{t_2} \mathscr{L} dt = 0 \tag{5.1.1}$$

is satisfied. Here

$$\mathscr{L} = \mathscr{K} - \mathscr{E} + \mathscr{W}_b + \mathscr{W}_t \tag{5.1.2}$$

$$\mathscr{K} = \tfrac{1}{2} \int_{\mathscr{S}_\kappa} \dot{\mathbf{v}} \cdot \dot{\mathbf{v}} \rho_\kappa \, dv, \qquad \mathscr{E} = \int_{\mathscr{S}_\kappa} \varepsilon_\kappa \rho_\kappa \, dv \tag{5.1.3}$$

$$\mathscr{W}_b = \int_{\mathscr{S}_\kappa} \mathbf{b}_\kappa \cdot \mathbf{v} \rho_\kappa \, dv, \quad \mathscr{W}_t = \int_{\partial \mathscr{S}_\kappa} \mathbf{t}_\kappa \cdot \mathbf{v} da \tag{5.1.4}$$

In the above relations $\mathscr{K}$ denotes the kinetic energy, $\mathscr{E}$ is the internal energy of the elastic strain, and $\mathscr{W}_b$ and $\mathscr{W}_t$ denote the work of external conservative mass forces $\mathbf{b}_\kappa$ and surface forces $\mathbf{t}_\kappa$, respectively. Besides, $\mathbf{v}$ denotes the displacement vector of a particle $X = \kappa^{-1}(P)$, $\dot{\mathbf{v}}$ denotes a velocity of the particle with respect to an inertial frame, ρ_κ is the mass density in κ, and $\varepsilon_\kappa = \varepsilon_\kappa(\mathbf{F})$ denotes the internal strain energy density of a homogeneous shell, related to the κ configuration.

A variation of the displacement field $\delta \mathbf{v}$ that satisfies the geometric boundary conditions and the conditions $\delta \mathbf{v}(t_1) = \delta \mathbf{v}(t_2) = 0$ results in (5.1.1) taking, after a Green transformation, the form

$$\int_{t_1}^{t_2} \int_{\mathscr{S}_\kappa} [\operatorname{div} \mathbf{T}_\kappa + \rho_\kappa (\mathbf{b}_\kappa - \mathbf{a})] \cdot \delta \mathbf{v} \, dv \, dt + \int_{t_1}^{t_2} \int_{\partial \mathscr{S}_\kappa} (\mathbf{t}_\kappa - \mathbf{T}_\kappa \mathbf{n}_\kappa) \cdot \delta \mathbf{v} \, da \, dt = 0 \tag{5.1.5}$$

where $\mathbf{a} = \ddot{\mathbf{v}}$ is the acceleration of the particle X with respect to the inertial frame, $\mathbf{T}_\kappa \equiv \equiv \rho_\kappa \, \varepsilon_{\kappa, \mathbf{F}}(\mathbf{F})$ is the first Piola-Kirchhoff stress tensor, related to the Cauchy stress tensor $\mathbf{T}$ and the second Piola-Kirchhoff stress tensor $\mathbf{S}$ by means of formulae [18]

$$\mathbf{S} = \mathbf{F}^{-1} \mathbf{T}_\kappa = \frac{\rho_\kappa}{\rho_\chi} \mathbf{F}^{-1} \mathbf{T} (\mathbf{F}^{-1})^T \tag{5.1.6}$$

The integral relation (5.1.5) leads to the local equations of motion and natural boundary conditions of a continuum in the Lagrangean description. In this work the integral form (5.1.5) will be employed directly.

In view of the relations (5.1.6), (2.3.5) and (3.1.2) the following form for the tensor $\mathbf{T}_\kappa$ in a coordinate system normal in $\mathscr{S}_\kappa$ is obtained

$$\mathbf{T}_\kappa = \mathbf{FS} = [S^{ij} + \mu^i_\alpha (v^\alpha_{;\beta} S^{\beta j} + v^\alpha_{,3} S^{3j})] \, \mathbf{g}_i \otimes \mathbf{g}_j \tag{5.1.7}$$

For a linear approximation of deformation (3.1.4) or displacement (3.1.7) across the shell thickness in $\mathscr{S}_\kappa$ one obtains, taking into account (3.1.10) and (3.1.8), the following relations

$$\begin{aligned} T^{\varphi j}_\kappa &= [\delta^\varphi_\vartheta + \mu^\varphi_\alpha \delta^\lambda_\vartheta (\varphi^\alpha_{.\lambda} + \zeta \psi^\alpha_{.\lambda})] S^{\vartheta j} + \mu^\varphi_\alpha \beta^\alpha S^{3j} \\ T^{3j}_\kappa &= \delta^\lambda_\vartheta (\varphi_\lambda + \zeta \psi_\lambda) S^{\vartheta j} + (1 + \beta) S^{3j} \end{aligned} \tag{5.1.8}$$

The variational problem (5.1.5) can be written out in an expanded form. To this end the relations (2.3.13) can be used for the description of div $\mathbf{T}_\kappa$ together with a variation $\delta\mathbf{v}=\delta\mathbf{u}+\zeta\delta\boldsymbol{\beta}$ compatible with the expansion (3.1.7). After appropriate transformations the first term in (5.1.5) will take the form

$$\int_{t_1}^{t_2}\int_{\mathcal{M}_\kappa}\int_{-h/2}^{h/2}\{[(\mu\mu^\alpha_\varphi T^{\varphi\psi}_\kappa\delta^\beta_\psi)_{|\beta}-b^\alpha_\beta(\mu T^{3\psi}_\kappa\delta^\beta_\psi)+(\mu\mu^\alpha_\varphi T^{\varphi 3}_\kappa)_{,3}+\rho_\kappa\mu\mu^\alpha_\varphi(b^\varphi_\kappa-a^\varphi)]\,\delta u_\alpha+$$
$$+[(\mu T^{3\psi}_\kappa\delta^\beta_\psi)_{|\beta}+b_{\alpha\beta}(\mu\mu^\alpha_\beta T^{\varphi\psi}_\kappa\delta^\beta_\psi)+(\mu T^{33}_\kappa)_{,3}+\rho_\kappa\mu(b^3_\kappa-a^3)]\,\delta w+$$
$$+[(\mu\mu^\alpha_\varphi T^{\varphi\psi}_\kappa\delta^\beta_\psi\zeta)_{|\beta}-b^\alpha_\beta(\mu T^{3\psi}_\kappa\delta^\beta_\psi\zeta)-(\mu\mu^\alpha_\varphi T^{\varphi 3}_\kappa)+(\mu\mu^\alpha_\varphi T^{\varphi 3}_\kappa\zeta)_{,3}+ \tag{5.1.9}$$
$$+\rho_\kappa\mu\mu^\alpha_\varphi(b^\varphi_\kappa-a^\varphi)]\,\delta\beta_\alpha+[(\mu T^{3\psi}_\kappa\delta^\beta_\psi\zeta)_{|\beta}+b_{\alpha\beta}(\mu\mu^\alpha_\varphi T^{\varphi\psi}_\kappa\delta^\beta_\psi\zeta)-$$
$$-(\mu T^{33}_\kappa)+(\mu T^{33}_\kappa\zeta)_{,3}+\rho_\kappa\mu(b^3_\kappa-a^3)]\,\delta\beta\}\,d\zeta\,da\,dt=0.$$

Let us substitute (5.1.8) into (5.1.9) and integrate over the shell thickness. After a number of obvious, however involved transformations, analogous to those shown in earlier works of the author [19, 24], for the respective variations we arrive at the following two-dimensional equations of motion of a shell in the Lagrangean description

$$\delta u_\alpha:\quad [l^\alpha_{.\lambda}N^{\lambda\beta}+\psi^\alpha_{.\lambda}M^{\lambda\beta}-b^\alpha_\lambda M^{\lambda\beta}+\beta^\alpha N^{\beta 3}]_{|\beta}-$$
$$-b^\alpha_\beta[\varphi_\lambda N^{\lambda\beta}+\psi_\lambda M^{\lambda\beta}+(1+\beta)N^{\beta 3}]+p^\alpha=\rho_0\ddot{u}^\alpha+\rho_1\ddot{\beta}^\alpha \tag{5.1.10}$$

$$\delta w:\quad [\varphi_\lambda N^{\lambda\beta}+\psi_\lambda M^{\lambda\beta}+(1+\beta)N^{\beta 3}]_{|\beta}+$$
$$+b_{\alpha\beta}[l^\alpha_{.\lambda}N^{\lambda\beta}+\psi^\alpha_{.\lambda}M^{\lambda\beta}-b^\alpha_\lambda M^{\lambda\beta}+\beta^\alpha N^{\beta 3}]+p^3=\rho_0\ddot{w}+\rho_1\ddot{\beta} \tag{5.1.11}$$

$$\delta\beta_\alpha:\quad [l^\alpha_{.\lambda}M^{\lambda\beta}+\psi^\alpha_{.\lambda}K^{\lambda\beta}-b^\alpha_\lambda K^{\lambda\beta}+\underline{\beta^\alpha M^{\beta 3}}]_{|\beta}-$$
$$-b^\alpha_\beta[\varphi_\lambda M^{\lambda\beta}+\psi_\lambda K^{\lambda\beta}+\underline{(1+\beta)M^{\beta 3}}]-$$
$$-[l^\alpha_{.\lambda}N^{\lambda 3}+\underline{\psi^\alpha_{.\lambda}M^{\lambda 3}-b^\alpha_\lambda M^{\lambda 3}}+\beta^\alpha N^{33}]+l^\alpha=\rho_1\ddot{u}^\alpha+\rho_2\ddot{\beta}^\alpha \tag{5.1.12}$$

$$\delta\beta:\quad [\varphi_\lambda M^{\lambda\beta}+\psi_\lambda K^{\lambda\beta}+\underline{(1+\beta)M^{\beta 3}}]_{|\beta}+$$
$$+b_{\alpha\beta}[l^\alpha_{.\lambda}M^{\lambda\beta}+\psi^\alpha_{.\lambda}K^{\lambda\beta}-b^\alpha_\lambda K^{\lambda\beta}+\underline{\beta^\alpha M^{\beta 3}}]-$$
$$-[\varphi_\lambda N^{\lambda 3}+\underline{\psi_\lambda M^{\lambda 3}}+(1+\beta)N^{33}]+l^3=\rho_1\ddot{w}+\rho_2\ddot{\beta} \tag{5.1.13}$$

The following denotations have been used in equations (5.1.10)÷(5.1.13)

$$\begin{bmatrix}N^{\alpha\beta}\\ M^{\alpha\beta}\\ K^{\alpha\beta}\end{bmatrix}=\delta^\alpha_\varphi\delta^\beta_\psi\int_{-h/2}^{h/2}\mu S^{\varphi\psi}\begin{bmatrix}1\\ \zeta\\ \zeta^2\end{bmatrix}d\zeta \tag{5.1.14}$$

$$\begin{bmatrix}N^{\alpha 3}\\ M^{\alpha 3}\end{bmatrix}=\delta^\alpha_k\int_{-h/2}^{h/2}\mu S^{3k}\begin{bmatrix}1\\ \zeta\end{bmatrix}d\zeta \tag{5.1.15}$$

$$\begin{bmatrix} B^a \\ M^a \end{bmatrix} = \int_{-h/2}^{h/2} \rho_\kappa \mu \mu_k^a b_\kappa^k \begin{bmatrix} 1 \\ \zeta \end{bmatrix} d\zeta \tag{5.1.16}$$

$$\begin{bmatrix} \rho_0 \\ \rho_1 \\ \rho_2 \end{bmatrix} = \int_{-h/2}^{h/2} \rho_\kappa \mu \begin{bmatrix} 1 \\ \zeta \\ \zeta^2 \end{bmatrix} d\zeta = \begin{bmatrix} h\rho_\kappa\left(1+\dfrac{h^2}{12}K\right) \\ h\rho_\kappa\left(-\dfrac{h^2}{6}H\right) \\ \dfrac{h^3\rho_\kappa}{12}\left(1+\dfrac{3}{10}hK\right) \end{bmatrix} \tag{5.1.17}$$

$$\begin{bmatrix} C^\alpha \\ L^\alpha \end{bmatrix} = \{\delta_\varphi^\lambda [l_{.\lambda}^\alpha + (\psi_{.\lambda}^\alpha - b_\lambda^\alpha)\zeta]\mu S^{\varphi 3} + \beta^\alpha \mu S^{33}\} \begin{bmatrix} 1 \\ \zeta \end{bmatrix} \Bigg|_{-h/2}^{h/2} \tag{5.1.18}$$

$$\begin{bmatrix} C^3 \\ L^3 \end{bmatrix} = \{\delta_\varphi^\alpha (\varphi_\alpha + \psi_\alpha \zeta)\mu S^{\varphi 3} + (1+\beta)\mu S^{33}\} \begin{bmatrix} 1 \\ \zeta \end{bmatrix} \Bigg|_{-h/2}^{h/2} \tag{5.1.19}$$

$$p^a = B^a + C^a, \qquad l^a = M^a + L^a \tag{5.1.20}$$

The set of equations (5.1.10)÷(5.1.13), together with definitions (5.1.14)÷(5.1.20), forms the set of equations of shell motion in the Lagrangean description compatible with the linear distribution (3.1.4) of deformation across the shell thickness in the reference configuration κ. These relations are equivalent to the results published earlier by the author [19, 24]. Their slightly changed present form is more clear and shows some symmetry in the equations.

A surface $\partial\mathscr{S}_\kappa$ enclosing a region $\mathscr{S}_\kappa$ of the shell consists of two subregions $\partial\mathscr{S}_\kappa^+$ and $\partial\mathscr{S}_\kappa^-$ defined by $\zeta = +h/2$ and $\zeta = -h/2$, and the boundary surface $\mathscr{B}_\kappa$ of the shell. The loading exerted on $\partial\mathscr{S}_\kappa^+$ and $\partial\mathscr{S}_\kappa^-$ has been accounted for in (5.1.20), whereas in integration over $\mathscr{B}_\kappa$ in the second term of (5.1.5) with (5.1.7) and (5.1.8) taken into account, yields, after some transformations, the following natural boundary conditions in the Lagrangean description

$$\begin{aligned} \delta u_\alpha: &\quad [l_{.\lambda}^\alpha N^{\lambda\beta} + (\psi_{.\lambda}^\alpha - b_\lambda^\alpha) M^{\lambda\beta} + \beta^\alpha N^{\beta 3}] \nu_\beta = f^\alpha \\ \delta w: &\quad [\varphi_\lambda N^{\lambda\beta} + \psi_\lambda M^{\lambda\beta} + (1+\beta) N^{\beta 3}] \nu_\beta = f^3 \end{aligned} \tag{5.1.21}$$

$$\begin{aligned} \delta \beta_\alpha: &\quad [l_{.\lambda}^\alpha M^{\lambda\beta} + (\psi_{.\lambda}^\alpha - b_\lambda^\alpha) K^{\lambda\beta} + \underline{\beta^\alpha M^{\beta 3}}] \nu_\beta = k^\alpha \\ \delta \beta: &\quad [\varphi_\lambda M^{\lambda\beta} + \psi_\lambda K^{\lambda\beta} + \underline{(1+\beta) M^{\beta 3}}] \nu_\beta = k^3 \end{aligned} \tag{5.1.22}$$

$$\begin{bmatrix} f^a \\ k^a \end{bmatrix} = \int_{-h/2}^{h/2} \mu \mu_i^a t_\kappa^i \, d\zeta \tag{5.1.23}$$

This form of the natural boundary conditions, (5.1.21) and (5.1.22), is in agreement with the assumption of a linear distribution of deformation (3.1.4) over the shell thickness in the reference configuration κ.

Lagrangean relations based on similar assumptions have also been discussed in works [34, 35]. A comparison of results presented in [34, 35] with our results shows that in the relations obtained in [34, 35], which are analogous to (5.1.12), (5.1.13) and (5.1.22), the terms (underlined in the present work) containing $M^{\beta 3}$ have been omitted, while the terms containing $M^{\beta 3}|_{\beta}$ have been left. In what follows we shall show that for small strains all terms of this type are of the same order of magnitude as other terms appearing in the equations. Therefore, an omission of any of them at this stage of generality is unjustified.

Now let us study the consequences of symmetry of the second Piola-Kirchhoff stress tensor $\mathbf{S}$ used in the previous definitions. For a three-dimensional problem we have an identity

$$\epsilon_{ijk} S^{ij} = 0 \tag{5.1.24}$$

which can be expressed in terms of two-dimensional components as follows

$$\begin{gathered} \epsilon_{\alpha\beta}\, \mu^{\alpha}_{\varphi}\, \mu^{\beta}_{\psi}\, S^{\varphi\psi} = 0 \\ \epsilon_{\gamma\alpha}\, \delta^{\gamma}_{\vartheta}\, \delta^{\alpha}_{\varphi}\, \mu S^{\varphi 3} + \epsilon_{\beta\gamma}\, \delta^{\beta}_{\psi}\, \delta^{\gamma}_{\vartheta}\, \mu S^{3\psi} = 0 \end{gathered} \tag{5.1.25}$$

Integration of (5.1.25) with respect to ζ, and with (5.1.4) and (5.1.5) taken into account, yields

$$\begin{gathered} \epsilon_{\alpha\beta}(N^{\alpha\beta} - b^{\alpha}_{\lambda} M^{\lambda\beta} - b^{\beta}_{\mu} M^{\alpha\mu} + b^{\alpha}_{\lambda}\, b^{\beta}_{\mu} K^{\lambda\mu}) = 0 \\ (\epsilon_{\alpha\beta} + \epsilon_{\beta\alpha})\, N^{\alpha 3} = 0 \end{gathered} \tag{5.1.26}$$

For the definition of internal forces as adopted here these relations are identities.

5.2. Modifications of equations of motion

The symmetric structure of the Lagrangean equations of motion and natural boundary conditions obtained above implies a possibility of writing them in a form utilizing a vector--tensor notation.

Let us introduce the following resultant vectors of internal forces and moments

$$\int_{\partial\mathcal{M}_\kappa} \mathbf{GN}^{\beta} \nu_{\beta}\, ds = \int_{\mathcal{B}_\kappa} \mathbf{T}_\kappa\, \mathbf{n}_\kappa\, da, \qquad \int_{\partial\mathcal{M}_\kappa} \mathbf{GM}^{\beta} \nu_{\beta}\, ds = \int_{\mathcal{B}_\kappa} \mathbf{T}_\kappa\, \mathbf{n}_\kappa\, \zeta\, da \tag{5.2.1}$$

By subsequently taking advantage of relations (5.1.6), (3.2.3), (3.2.2) and (2.2.10) for (5.1.14) and (5.1.15) we obtain

$$\begin{aligned} \mathbf{GN}^{\beta} &= N^{\alpha\beta} \bar{\mathbf{a}}_{\alpha} + M^{\alpha\beta} \bar{\mathbf{a}}_{3,\alpha} \\ \mathbf{GM}^{\beta} &= M^{\alpha\beta} \bar{\mathbf{a}}_{\alpha} + K^{\alpha\beta} \bar{\mathbf{a}}_{3,\alpha} \end{aligned} \tag{5.2.2}$$

In view of (3.1.12) relation (5.2.2) can be written out

$$\begin{aligned} \mathbf{GN}^{\beta} = {} & [l^{\alpha}_{\cdot\lambda} N^{\lambda\beta} + (\psi^{\alpha}_{\cdot\lambda} - b^{\alpha}_{\lambda}) M^{\lambda\beta} + \beta^{\alpha} N^{\beta 3}]\, \mathbf{a}_{\alpha} + \\ & + [\varphi_{\lambda} N^{\lambda\beta} + \Psi_{\lambda} M^{\lambda\beta} + (1+\beta) N^{\beta 3}]\, \mathbf{n} \\ \mathbf{GM}^{\beta} = {} & [l^{\alpha}_{\cdot\lambda} M^{\lambda\beta} + (\psi^{\alpha}_{\cdot\lambda} - b^{\alpha}_{\lambda}) K^{\lambda\beta} + \beta^{\alpha} M^{\beta 3}]\, \mathbf{a}_{\alpha} + \\ & + [\varphi_{\lambda} M^{\lambda\beta} + \Psi_{\lambda} K^{\lambda\beta} + (1+\beta) M^{\beta 3}]\, \mathbf{n} \end{aligned} \tag{5.2.3}$$

$$\mathbf{Gm}^3 = N^{\alpha 3}\bar{\mathbf{a}}_\alpha + M^{\alpha 3}\bar{\mathbf{a}}_{3,\alpha} = [l^{\alpha}_{.\lambda} N^{\lambda 3} + (\psi^{\alpha}_{.\lambda} - b^{\alpha}_{\lambda}) M^{\lambda 3} + \beta^{\alpha} N^{33}]\,\mathbf{a}_\alpha + \\ + [\varphi_\lambda N^{\lambda 3} + \beta_\lambda M^{\lambda 3} + (1+\beta) N^{33}]\,\mathbf{n} \tag{5.2.4}$$

Clearly, the components of vectors describing forces and moments (5.2.3) and (5.2.4) are arranged in a way similar to that appearing in Lagrangean equations of motion (5.1.10) ÷(5.1.13) and natural boundary conditions (5.1.21) and (5.1.22). Hence we can write the equations of motion in the following form

$$\begin{aligned} &(\mathbf{GN}^\beta)_{|\beta} + \mathbf{p} = \rho_0 \ddot{\mathbf{u}} + \rho_1 \ddot{\boldsymbol{\beta}} \\ &(\mathbf{GM}^\beta)_{|\beta} - \mathbf{Gm}^3 + \mathbf{l} = \rho_1 \ddot{\mathbf{u}} + \rho_2 \ddot{\boldsymbol{\beta}} \end{aligned} \tag{5.2.5}$$

with the natural boundary conditions taking the form

$$\mathbf{GN}^\beta \nu_\beta = \mathbf{f}, \qquad \mathbf{GM}^\beta \nu_\beta = \mathbf{k} \tag{5.2.6}$$

Relations (5.2.5) and (5.2.6) written in terms of vectors and tensors can easily be memorized. They may be used as a starting point in various modifications.

In the convected coordinates material particles of a shell have the same surface coordinates in the both configurations κ and γ. A surface element, belonging to the shell middle surface, and components of a unit vector, directed along an external normal to the boundary line and lying on a plane tangent to the shell middle surface in κ and γ, satisfy the following transformation rules [21, 51]

$$d\bar{a} = \sqrt{\frac{\tilde{a}}{a}}\, da, \qquad \bar{\nu}_\alpha d\bar{s} = \sqrt{\frac{\tilde{a}}{a}}\, \nu_\alpha ds \tag{5.2.7}$$

where $\tilde{a} = |\bar{a}_{\alpha\beta}|$ is a two-dimensional invariant over $\mathscr{M}_\gamma$.

Now let us introduce the Eulerian vectors of internal forces and moments, measured in the geometry of the shell middle surface $\mathscr{M}_\gamma$ in the actual configuration. These take the form

$$\begin{aligned} &\int_{\mathscr{C}_\kappa} \mathbf{GN}^\alpha \nu_\alpha ds = \int_{\mathscr{C}_\gamma} \bar{\mathbf{N}}^\alpha \bar{\nu}_\alpha d\bar{s}, \qquad \int_{\mathscr{C}_\kappa} \mathbf{GM}^\alpha \nu_\alpha ds = \int_{\mathscr{C}_\gamma} \bar{\mathbf{M}}^\alpha \bar{\nu}_\alpha d\bar{s} \\ &\int_{\mathscr{M}_\kappa} \mathbf{Gm}^3 da = \int_{\mathscr{M}_\gamma} \bar{\mathbf{m}}^3 d\bar{a} \end{aligned} \tag{5.2.8}$$

In view of (5.2.7) we have

$$\bar{\mathbf{N}}^\alpha = \sqrt{\frac{a}{\tilde{a}}}\, \mathbf{GN}^\alpha, \qquad \bar{\mathbf{M}}^\alpha = \sqrt{\frac{a}{\tilde{a}}}\, \mathbf{GM}^\alpha, \qquad \bar{\mathbf{m}}^3 = \sqrt{\frac{a}{\tilde{a}}}\, \mathbf{Gm}^3 \tag{5.2.9}$$

Next we introduce tensors

$$\begin{aligned} &\mathbf{N} = \mathbf{N}^\alpha \otimes \mathbf{a}_\alpha, \qquad \mathbf{M} = \mathbf{M}^\alpha \otimes \mathbf{a}_\alpha, \qquad \mathbf{m} = \mathbf{m}^3 \otimes \mathbf{n} \\ &\bar{\mathbf{N}} = \bar{\mathbf{N}}^\alpha \otimes \bar{\mathbf{a}}_\alpha, \qquad \bar{\mathbf{M}} = \bar{\mathbf{M}}^\alpha \otimes \bar{\mathbf{a}}_\alpha, \qquad \bar{\mathbf{m}} = \bar{\mathbf{m}}^3 \otimes \bar{\mathbf{a}}_3 \end{aligned} \tag{5.2.10}$$

According to (4.1.4) and (5.2.9) the tensors satisfy the following transformation rules.

$$\bar{\mathbf{N}}=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{G}\mathbf{N}\mathbf{G}^T, \qquad \mathbf{N}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{G}^{-1}\bar{\mathbf{N}}(\mathbf{G}^{-1})^T$$

$$\bar{\mathbf{M}}=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{G}\mathbf{M}\mathbf{G}^T, \qquad \mathbf{M}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{G}^{-1}\bar{\mathbf{M}}(\mathbf{G}^{-1})^T \tag{5.2.11}$$

$$\bar{\mathbf{m}}=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{G}\mathbf{m}\mathbf{G}^T, \qquad \mathbf{m}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{G}^{-1}\bar{\mathbf{m}}(\mathbf{G}^{-1})^T$$

A comparison of these relations with (5.1.6), valid for three-dimensional continuum, shows that both sets of transformations have identical structures. Thus the Lagrangean tensors **N**, **M** and **m** are in the shell theory some analogs of the second Piola-Kirchhoff stress tensor **S**. Therefore we will call them the second Piola-Kirchhoff tensors of internal forces and moments. Similarly, there is an analogy between the Eulerian tensors $\bar{\mathbf{N}}$, $\bar{\mathbf{M}}$ and $\bar{\mathbf{m}}$ in the theory of shells, and the Cauchy stress tensor **T**, so the former can be called the Cauchy tensors of internal forces and moments. In this work components of the Lagrangean tensors are the basic quantities, definitions of which include the assumption of linearity of the deformation distribution across the shell thickness in the reference configuration. The components of Eulerian tensors are secondary quantities. One might formulate in a similar way a dual problem, assuming a system of coordinates normal in the actual configuration and making for this configuration an assumption of linearity of the distribution of deformation across the shell thickness [19]. Obviously, this would be different approximation of a three-dimensional problem and Eulerian strain tensors defined in γ would not coincide with tensors obtained under transformations (5.2.9) from the Lagrangean ones. Thus in this sense the two approaches to the approximation of the three-dimensional problem – in the Eulerian description and in the Lagrangean one – are not equivalent from the point of view of the theory of shells.

An expanded form of the Lagrangean tensors is obtained from (5.2.2) together with $(3.2.2)_2$ and (4.1.4)

$$\mathbf{N}=(N^{\alpha\beta}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}M^{\lambda\beta})\,\mathbf{a}_\alpha\otimes\mathbf{a}_\beta$$

$$\mathbf{M}=(M^{\alpha\beta}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}K^{\lambda\beta})\,\mathbf{a}_\alpha\otimes\mathbf{a}_\beta \tag{5.2.12}$$

$$\mathbf{m}^3=(N^{\alpha 3}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}M^{\lambda 3})\,\mathbf{a}_\alpha\otimes\mathbf{n}$$

Transformation formulae (5.2.11) applied for (5.2.12) yield the following expanded form of the Eulerian tensors

$$\bar{\mathbf{N}}=(\bar{N}^{\alpha\beta}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}\bar{M}^{\lambda\beta})\,\bar{\mathbf{a}}_\alpha\otimes\bar{\mathbf{a}}_\beta$$

$$\bar{\mathbf{M}}=(\bar{M}^{\alpha\beta}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}\bar{K}^{\lambda\beta})\,\bar{\mathbf{a}}_\alpha\otimes\bar{\mathbf{a}}_\beta \tag{5.2.13}$$

$$\bar{\mathbf{m}}^3=(\bar{N}^{\alpha 3}-\bar{a}^{\alpha c}\bar{\lambda}_{c\lambda}\bar{M}^{\lambda 3})\,\bar{\mathbf{a}}_\alpha\otimes\bar{\mathbf{a}}_3$$

Representations of these groups of tensors related to different bases satisfy simple relations resulting from the following transformation formulae

$$\bar{N}^{ab}=\sqrt{\frac{a}{\tilde{a}}}\,N^{ab}, \qquad \bar{M}^{\alpha\beta}=\sqrt{\frac{a}{\tilde{a}}}\,M^{\alpha\beta}, \qquad \bar{K}^{\alpha\beta}=\sqrt{\frac{a}{\tilde{a}}}\,K^{\alpha\beta} \tag{5.2.14}$$

The difference between Lagrangean and Eulerian tensors is difficult to specify basing exclusively on representations related to each other through formulae (5.2.14), if the component notation in the convected coordinate system is used exclusively. Only the complete description (5.2.12) and (5.2.13) shows the essential geometrical difference between these tensors.

The invariant form (5.2.5) of equations of motion of shells in the Lagrangean description, (5.1.10) and (5.1.13), is obtained in the following manner. First vectors $\mathbf{GN}^\beta$ and $\mathbf{GM}^\beta$ are decomposed in the basis $\mathbf{a}_a$ with the aid of deformation gradient tensor components (4.2.32). Next the operation of convariant differentiation over $\mathscr{M}_\kappa$ is performed and the resultant vector relations are decomposed in the basis $\mathbf{a}_a$. Changes in the sequence of operations or in terms grouping provide a simple way of obtaining the other equivalent forms of equilibrium equations, expressed also in terms of the Lagrangean quantities.

The Lagrangean vectors (5.2.2) can be presented in the form

$$\mathbf{GN}^\beta = Q^{\alpha\beta}\bar{\mathbf{a}}_\alpha + Q^{3\beta}\bar{\mathbf{a}}_3, \qquad \mathbf{GM}^\beta = R^{\alpha\beta}\bar{\mathbf{a}}_\alpha + R^{3\beta}\bar{\mathbf{a}}_3, \qquad \mathbf{Gm}^3 = Q^{\alpha 3}\bar{\mathbf{a}}_\alpha + Q^{33}\bar{\mathbf{a}}_3 \tag{5.2.15}$$

where denotations following from (5.2.2) are used:

$$\begin{aligned} Q^{ab} &= N^{ab} - \bar{a}^{ac}\bar{\lambda}_{c\lambda} M^{\lambda b} = N^{ab} + (G^a_{3\lambda} + \bar{a}^{ac}\gamma_{c3\lambda}) M^{\lambda b} \\ R^{a\beta} &= M^{a\beta} - \bar{a}^{ac}\bar{\lambda}_{c\lambda} K^{\lambda\beta} = M^{a\beta} + (G^a_{3\lambda} + \bar{a}^{ac}\gamma_{c3\lambda}) K^{\lambda\beta} \end{aligned} \tag{5.2.16}$$

Let us perform the operation of covariant differentiation of the quantities (5.2.15) using the following geometrical formulae

$$\begin{aligned} \bar{\mathbf{a}}_{\alpha|\beta} &= \bar{a}^{\lambda d}\gamma_{d\alpha\beta}\bar{\mathbf{a}}_\lambda + (b_{\alpha\beta} + \bar{a}^{3d}\gamma_{d\alpha\beta})\bar{\mathbf{a}}_3 \\ \bar{\mathbf{a}}_{3|\beta} &= (-b^\lambda_\beta + \bar{a}^{\lambda d}\gamma_{d3\beta})\bar{\mathbf{a}}_\lambda + \bar{a}^{3d}\gamma_{d3\beta}\bar{\mathbf{a}}_3 \end{aligned} \tag{5.2.17}$$

The result of differentiation can be written either in terms of components in the basis $\bar{\mathbf{a}}_\alpha, \bar{\mathbf{a}}_3$, or in terms of components in the basis $\mathbf{a}_\alpha$, $\mathbf{n}$. If the first decomposition is used then equilibrium equations in a following form are obtained

$$\begin{aligned} &\bar{\mathbf{a}}_\alpha: \quad Q^{\alpha\beta}|_\beta + \bar{a}^{\alpha d}\gamma_{d\lambda\beta}Q^{\lambda\beta} + (-b^\alpha_\lambda + \bar{a}^{\alpha d}\gamma_{d3\lambda})Q^{3\lambda} + \sqrt{\frac{\tilde{a}}{a}}\,\bar{p}^\alpha = 0 \\ &\bar{\mathbf{a}}_3: \quad Q^{3\alpha}|_\alpha + (b_{\alpha\beta} + \bar{a}^{3d}\gamma_{d\alpha\beta})Q^{\alpha\beta} + \bar{a}^{3d}\gamma_{d3\alpha}Q^{3\alpha} + \sqrt{\frac{\tilde{a}}{a}}\,\bar{p}^3 = 0 \\ &\bar{\mathbf{a}}_\alpha: \quad R^{\alpha\beta}|_\beta + \bar{a}^{\alpha d}\gamma_{d\lambda\beta}R^{\lambda\beta} + (-b^\alpha_\lambda + \bar{a}^{\alpha d}\gamma_{d3\lambda})R^{3\lambda} - Q^{\alpha 3} + \sqrt{\frac{\tilde{a}}{a}}\,\bar{l}^\alpha = 0 \\ &\bar{\mathbf{a}}_3: \quad R^{3\alpha}|_\alpha + (b_{\alpha\beta} + \bar{a}^{3d}\gamma_{d\alpha\beta})R^{\alpha\beta} + \bar{a}^{3d}\gamma_{d3\alpha}R^{3\alpha} - Q^{33} + \sqrt{\frac{\tilde{a}}{a}}\,\bar{l}^3 = 0 \end{aligned} \tag{5.2.18}$$

where

$$\bar{\mathbf{p}} = \sqrt{\frac{a}{\tilde{a}}}\,\mathbf{G}^{-1}\mathbf{p} = \bar{p}^\alpha\bar{\mathbf{a}}_\alpha + \bar{p}^3\bar{\mathbf{a}}_3, \qquad \bar{\mathbf{l}} = \sqrt{\frac{a}{\tilde{a}}}\,\mathbf{G}^{-1}\mathbf{l} = \bar{l}^\alpha\bar{\mathbf{a}}_\alpha + \bar{l}^3\bar{\mathbf{a}}_3 \tag{5.2.19}$$

This is the so called mixed form of equilibrium equations. Here analogous shell equations of Kirchhoff-Love type theory obtained in [27] are generalized to the theory based on the linear distribution of deformation over the shell thickness.

We shall designate the left-hand part of equations (5.2.18) without load terms subsequently by P^α, P^3, B^α, B^3. Next we will decompose in the basis $\mathbf{a}_\alpha$, $\mathbf{n}$ the vector relations resulting from differentiation of (5.2.15) with (5.2.17) taken into account. As a result we will obtain from (5.2.5) the equilibrium equations in the following form

$$\begin{aligned} &\mathbf{a}_\alpha: \quad l^\alpha_{.\lambda}P^\lambda+\beta^\alpha P^3+p^\alpha=0, \qquad \mathbf{n}: \quad \varphi_\lambda P^\lambda+(1+\beta)P^3+p^3=0 \\ &\mathbf{a}_\alpha: \quad l^\alpha_{.\lambda}B^\lambda+\beta^\alpha B^3+l^\alpha=0, \qquad \mathbf{n}: \quad \varphi_\lambda B^\lambda+(1+\beta)B^3+l^3=0 \end{aligned} \tag{5.2.20}$$

This set of equilibrium equations is only another form of equations (5.1.10)÷(5.1.13). However in this form derivatives of displacement gradients do not appear. This may be useful when formulating various simplified equations for approximate variants of the non-linear shell theory.

Relations equivalent to (5.2.20) can be obtained in still another form. To this end the relations (5.2.15) and (5.2.16) should be taken into account, in which $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{a}}_3$ should be expressed in terms of $\mathbf{a}_\alpha$, $\mathbf{n}$ according to (3.1.12), and only then a covariant differentiation over $\mathscr{M}_\kappa$ should be performed. Decomposition of this result in the basis $\mathbf{a}_\alpha$, $\mathbf{n}$ leads in view of (5.2.5) to the following equilibrium equations

$$\begin{aligned} &\mathbf{a}_\alpha: \quad (l^\alpha_{.\lambda}Q^{\lambda\beta}+\beta^\alpha Q^{3\beta})|_\beta-b^\alpha_\beta[\varphi_\lambda Q^{\lambda\beta}+(1+\beta)Q^{3\beta}]+p^\alpha=0 \\ &\mathbf{n}: \quad [\varphi_\lambda Q^{\lambda\beta}+(1+\beta)Q^{3\beta}]|_\beta+b_{\alpha\beta}(l^\alpha_{.\lambda}Q^{\lambda\beta}+\beta^\alpha Q^{3\beta})+p^3=0 \\ &\mathbf{a}_\alpha: \quad (l^\alpha_{.\lambda}R^{\lambda\beta}+\beta^\alpha R^{3\beta})|_\beta-b^\alpha_\beta[\varphi_\lambda R^{\lambda\beta}+(1+\beta)R^{3\beta}]-(l^\alpha_{.\lambda}Q^{\lambda 3}+\beta^\alpha Q^{33})+l^\alpha=0 \\ &\mathbf{n}: \quad [\varphi_\lambda R^{\lambda\beta}+(1+\beta)R^{3\beta}]|_\beta+b_{\alpha\beta}(l^\alpha_{.\lambda}R^{\lambda\beta}+\beta^\alpha R^{3\beta})-[\varphi_\lambda Q^{\lambda 3}+(1+\beta)Q^{33}]+l^3=0 \end{aligned} \tag{5.2.21}$$

This form of equilibrium equations is a generalization of analogous equations for the Kirchhoff-Love type theory of shells obtained by the author in [21, 32].

A polar decomposition of the deformation gradient tensor $\mathbf{G}$ into a pure strain and a finite rotation was presented in chapter 4. We will use those results to modify equations (5.2.5).

Let $\mathbf{G}=\mathbf{RU}$. Then (5.2.11) take the form

$$\bar{\mathbf{N}}=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{R}\check{\mathbf{N}}\mathbf{R}^T, \quad \bar{\mathbf{M}}=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{R}\check{\mathbf{M}}\mathbf{R}^T, \quad \bar{\mathbf{m}}^3=\sqrt{\frac{a}{\tilde{a}}}\,\mathbf{R}\check{\mathbf{m}}^3\mathbf{R}^T \tag{5.2.22}$$

where

$$\begin{aligned} &\check{\mathbf{N}}=\mathbf{UNU}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{N}}\mathbf{R}, \quad \check{\mathbf{M}}=\mathbf{UMU}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{M}}\mathbf{R} \\ &\check{\mathbf{m}}^3=\mathbf{Um}^3\mathbf{U}=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{m}}^3\mathbf{R} \end{aligned} \tag{5.2.23}$$

Let us introduce the vectors

$$\begin{aligned} &\check{\mathbf{N}}^\alpha=\mathbf{UN}^\alpha=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{N}}^\alpha, \quad \check{\mathbf{M}}^\alpha=\mathbf{UM}^\alpha=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{M}}^\alpha \\ &\check{\mathbf{m}}^3=\mathbf{Um}^3=\sqrt{\frac{\tilde{a}}{a}}\,\mathbf{R}^T\bar{\mathbf{m}}^3 \end{aligned} \tag{5.2.24}$$

Clearly for the tensors and vectors just introduced the following relations hold true

$$\check{\mathbf{N}}^{\beta}=Q^{\alpha\beta}\check{\mathbf{a}}_{\alpha}+Q^{3\beta}\check{\mathbf{a}}_{3}, \qquad \check{\mathbf{M}}^{\beta}=R^{\alpha\beta}\check{\mathbf{a}}_{\alpha}+R^{3\beta}\check{\mathbf{a}}_{3}$$
$$\check{\mathbf{m}}^{3}=Q^{\alpha 3}\check{\mathbf{a}}_{\alpha}+Q^{33}\check{\mathbf{a}}_{3} \tag{5.2.25}$$

Moreover

$$\check{\mathbf{N}}=\check{\mathbf{N}}^{\alpha}\otimes\check{\mathbf{a}}_{\alpha}, \qquad \check{\mathbf{M}}=\check{\mathbf{M}}^{\alpha}\otimes\check{\mathbf{a}}_{\alpha}, \qquad \check{\mathbf{m}}=\check{\mathbf{m}}^{3}\otimes\check{\mathbf{a}}_{3} \tag{5.2.26}$$

In view of the above relations the equilibrium equations (5.2. 5) can be rewritten in the form

$$(\mathbf{R}\check{\mathbf{N}}^{\beta})|_{\beta}+\mathbf{p}=\mathbf{0}$$
$$(\mathbf{R}\check{\mathbf{M}}^{\beta})|_{\beta}-\mathbf{R}\check{\mathbf{m}}^{3}+\mathbf{l}=\mathbf{0} \tag{5.2.27}$$

Thus it happened to be possible also in equilibrium equations to separate the deformation in a general way into a rotational part associated with the tensor **R** and a part describing a pure strain effected by the tensor **U**.

Each operation performed with the aid of tensor **R** may also be done with the aid of vector $\mathbf{\Omega}$, e.g.

$$\mathbf{R}\check{\mathbf{N}}^{\beta}=\check{\mathbf{N}}^{\beta}+\mathbf{\Omega}\times\check{\mathbf{N}}^{\beta}+\frac{1}{2\cos^{2}\omega/2}\,\mathbf{\Omega}\times(\mathbf{\Omega}\times\check{\mathbf{N}}^{\beta}) \tag{5.2.28}$$

which shows that the equations (5.2.27) can also be written in terms of the vector $\mathbf{\Omega}$.

Let us present (5.2.27) in component form with respect to the basis $\check{\mathbf{a}}_{\alpha}$, $\check{\mathbf{a}}_{3}$. First note that according to (4.2.2) there is

$$\bar{\mathbf{a}}_{d}\cdot\check{\mathbf{a}}^{a}=\cos\omega\,\delta_{d}^{a}+\bar{a}^{ac}\bar{\epsilon}_{cde}\check{\Omega}^{e}+\frac{1}{2\cos^{2}\omega/2}\,\check{\Omega}_{d}\check{\Omega}^{a}$$
$$\bar{\mathbf{a}}_{d}\cdot\mathbf{a}^{a}=(\delta_{d}^{e}+\check{\gamma}_{d}^{e})\left(\cos\omega\,\delta_{e}^{a}+a^{ac}\,\epsilon_{cef}\,\Omega^{f}+\frac{1}{2\cos^{2}\omega/2}\,\Omega_{e}\Omega^{d}\right) \tag{5.2.29}$$
$$\mathbf{a}_{d}\cdot\check{\mathbf{a}}^{a}=\frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{abc}\epsilon_{def}(\delta_{b}^{e}+\check{\gamma}_{b}^{e})(\delta_{c}^{f}+\check{\gamma}_{c}^{f})$$

From (5.2.18) the following equilibrium equations in the basis $\check{\mathbf{a}}_{a}$ can be obtained

$$\check{\mathbf{a}}_{a}:\quad (\bar{\mathbf{a}}_{d}\cdot\check{\mathbf{a}}^{a})\,P^{d}+(\mathbf{a}_{d}\cdot\check{\mathbf{a}}^{a})\,p^{d}=0$$
$$\check{\mathbf{a}}_{a}:\quad (\bar{\mathbf{a}}_{d}\cdot\check{\mathbf{a}}^{a})\,B^{d}+(\mathbf{a}_{d}\cdot\check{\mathbf{a}}^{a})\,l^{d}=0 \tag{5.2.30}$$

Another form will be obtained directly from (5.2.5) if all the relations are represented with respect to the basis $\check{\mathbf{a}}_{a}$. Let

$$\mathbf{G}\mathbf{N}^{\beta}=T^{a\beta}\check{\mathbf{a}}_{a}, \qquad \mathbf{G}\mathbf{M}^{\beta}=C^{a\beta}\check{\mathbf{a}}_{a}$$
$$\mathbf{G}\check{\mathbf{m}}^{3}=T^{a3}\check{\mathbf{a}}_{a} \tag{5.2.31}$$

where

$$T^{ab}=(\bar{\mathbf{a}}_{d}\cdot\check{\mathbf{a}}^{a})\,Q^{db}, \qquad C^{a\beta}=(\bar{\mathbf{a}}_{d}\cdot\check{\mathbf{a}}^{a})\,R^{d\beta} \tag{5.2.32}$$

are defined in terms of strain and rotation parameters by the relations (5.2.16) and (5.2.29). It follows then from (5.2.5) that

$$\begin{aligned} T^{a\beta}{}_{|\beta} + (\check{\mathbf{a}}_{e|\beta} \cdot \check{\mathbf{a}}^{a})\, T^{e\beta} + p^{a} &= 0 \\ C^{a\beta}\big|_{\beta} + (\check{\mathbf{a}}_{e|\beta} \cdot \check{\mathbf{a}}^{a})\, C^{e\beta} - T^{a3} + l^{a} &= 0 \end{aligned} \tag{5.2.33}$$

where, according to (4.2.25) and (4.2.23)

$$\check{\mathbf{a}}_{e|\beta} \cdot \check{\mathbf{a}}^{a} = G^{a}_{e\beta} - \Gamma^{a}_{e\beta} + \bar{a}^{ac}\gamma_{ce\beta} - \bar{a}^{ac}\bar{\epsilon}_{ecd}\,\bar{\epsilon}^{ghd}(\gamma_{h\beta;\,g} - \tfrac{1}{2}a^{pq}\check{\gamma}_{hp}\check{\gamma}_{gq;\,\beta}) \tag{5.2.34}$$

Equilibrium equations in the form (5.2.33) contain derivatives of the vector $\mathbf{\Omega}$, while in the form (5.2.30) such derivatives do not appear. This may be of some importance for a formulation of simplified variants of shell equations.

The sets of equilibrium equations (5.2.20), (5.2.30) and (5.2.33) are novel forms of such equations and did not appear before in the shell literature, even within the Kirchhoff--Love type theories.

5.3. Constitutive equations of linear-elastic shells

Constitutive equations of an elastic solid have been discussed thoroughly in monograph [18] and in the papers of the author [65, 68, 69]. It is known that the necessity to satisfy the material-frame indifference principle causes the dependence of the energy function ε_{κ} on $\mathbf{F}$ through a tensor $\mathbf{F}^T\mathbf{F}$. Thus this function can be expressed entirely in terms of the Green strain tensor $\mathbf{E}$, namely

$$\varepsilon_{\kappa} = \varepsilon_{\kappa}(\mathbf{F}^T\mathbf{F}) = \sigma_{\kappa}(\mathbf{E}) \tag{5.3.1}$$

In such a case the constitutive equation for $\mathbf{S}$ takes a particularly simple form

$$\mathbf{S} = \rho_{\kappa}\,\sigma_{\kappa,\,\mathbf{E}}(\mathbf{E}) \tag{5.3.2}$$

Let us expand (5.3.2) into a Taylor series in the vicinity of the configuration κ [68, 69]. This will result in

$$\mathbf{S} = \mathbf{K} + \mathbf{L}\cdot\mathbf{E} + \tfrac{1}{2}\mathbf{M}\cdot(\mathbf{E}\otimes\mathbf{E}) + \dots \tag{5.3.3}$$

where

$$\mathbf{K} = \rho_{\kappa}\,\sigma_{\kappa,\,\mathbf{E}}(\mathbf{0}) \in \mathscr{T}_2, \quad \mathbf{L} = \rho_{\kappa}\,\sigma_{\kappa,\,\mathbf{EE}}(\mathbf{0}) \in \mathscr{T}_4, \quad \mathbf{M} = \rho_{\kappa}\,\sigma_{\kappa,\,\mathbf{EEE}}(\mathbf{0}) \in \mathscr{T}_6 \tag{5.3.4}$$

are elasticity tensors of the zeroth, first and second order, respectively.

For the operation of differentiation to be carried out in (5.3.4), the form of the energy function $\sigma_{\kappa}(\mathbf{E})$ with respect to the reference configuration κ should be known explicity. However, for a given material symmetry group of the solid, the representation theorems are usually formulated for the energy function defined with respect to the natural state κ_0 of the solid. Thus the calculation of elasticity tensors (5.3.4) in the configuration κ, that does not represent the natural state, requires many additional operations. These have been specified in [68] for an isotropic material. Here we will confine ourselves to a case where

$$\kappa \equiv \kappa_0, \quad \mathbf{K} \equiv \mathbf{0}, \quad \mathbf{M} \equiv \mathbf{0} \tag{5.3.5}$$

assuming that the constitutive equation of an elastic material is linear with respect to $\mathbf{E}$

$$\mathbf{S} = \mathbf{L} \cdot \mathbf{E} \tag{5.3.6}$$

The components of the tensor $\mathbf{L}$ satisfy the relations

$$L^{ijkl} = L^{jikl} = L^{ijlk} = L^{klij} \tag{5.3.7}$$

For an anisotropic material having elastic symmetry with respect to $\mathscr{M}_\kappa$, in a coordinate system normal in κ (5.3.6) takes the form

$$\begin{gathered} S^{\varphi\psi} = L^{\varphi\psi\vartheta\sigma} E_{\vartheta\sigma} + L^{\varphi\psi 33} E_{33}, \qquad S^{\varphi 3} = L^{\varphi 3\vartheta 3} E_{\vartheta 3} + L^{\varphi 33\sigma} E_{3\sigma} \\ S^{33} = L^{33\vartheta\sigma} E_{\vartheta\sigma} + L^{3333} E_{33} \end{gathered} \tag{5.3.8}$$

The number of independent components of the tensor $\mathbf{L}$ for arbitrarily anisotropic material is equal to 21, for a material symmetric with respect to the middle surface it is equal to 13, for an orthotropic material – equal to 9, for a transversally isotropic material – equal to 5, and for an isotropic material this number is equal to 2.

Substitution of (5.3.8) to (5.1.14) and (5.1.15), with (3.113) taken into account, integration and some transformations lead eventually to a general form of two-dimensional constitutive equations for linear-elastic shells in the Lagrangean description (cf. [35])

$$\begin{aligned} N^{\alpha\beta} &= B_0^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + B_1^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} + B_2^{\alpha\beta\lambda\mu}\mu_{\lambda\mu} + B_0^{\alpha\beta 33}\gamma_{33} \\ M^{\alpha\beta} &= B_1^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + B_2^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} + B_3^{\alpha\beta\lambda\mu}\mu_{\lambda\mu} + B_1^{\alpha\beta\lambda\mu}\gamma_{33} \\ K^{\alpha\beta} &= B_2^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + B_3^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} + B_4^{\alpha\beta\lambda\mu}\mu_{\lambda\mu} + B_2^{\alpha\beta\lambda\mu}\gamma_{33} \\ N^{\alpha 3} &= 2B_0^{\alpha 3\lambda 3}\gamma_{\lambda 3} + B_1^{\alpha 3\lambda 3}\kappa_{\lambda 3} \\ M^{\alpha 3} &= 2B_1^{\alpha 3\lambda 3}\gamma_{\lambda 3} + B_2^{\alpha 3\lambda 3}\kappa_{\lambda 3} \\ N^{33} &= B_0^{\alpha\beta 33}\gamma_{\alpha\beta} + B_1^{\alpha\beta 33}\kappa_{\alpha\beta} + B_2^{\alpha\beta 33}\mu_{\alpha\beta} + B_0^{3333}\gamma_{33} \end{aligned} \tag{5.3.9}$$

where

$$B_n^{abcd} = \delta_i^a \delta_j^b \delta_k^c \delta_l^d \int_{-h/2}^{h/2} \mu L^{ijkl} \zeta^n d\zeta \tag{5.3.10}$$

and γ_{ab}, $\kappa_{\alpha\beta}$, $\mu_{\alpha\beta}$ are components of the strain tensor (3.2.5), (3.2.6) defined at $\mathscr{M}_\kappa$ according to the assumed linear distribution of the deformation in the vicinity of $M \in \mathscr{M}_\kappa$.

The constitutive equations (5.3.9) can also be obtained by differentiation of certain function which depends on two-dimensional shell strain measures only. Note that for the linear–elastic material having symmetry with respect to $\mathscr{M}_\kappa$

$$\begin{aligned} \sigma_\kappa = \tfrac{1}{2}\mathbf{L}\cdot(\mathbf{E}\otimes\mathbf{E}) = \tfrac{1}{2}(&L^{\varphi\psi\vartheta\sigma}E_{\varphi\psi}E_{\vartheta\sigma} + 2L^{\varphi\psi 33}E_{\varphi\psi}E_{33} + \\ &+ 4L^{\varphi 3\vartheta 3}E_{\varphi 3}E_{\vartheta 3} + L^{3333}E_{33}E_{33}) \end{aligned} \tag{5.3.11}$$

where in a coordinate system normal in κ the formula (3.1.13) holds, together with

$$L^{ijkl}=\delta_a^i\delta_b^j\delta_c^k\delta_d^l(L_0^{abcd}+\zeta L_1^{abcd}+\zeta^2L_2^{abcd}+\ldots) \tag{5.3.12}$$

For this reason σ_κ can also be presented in a form of a series

$$\sigma_\kappa=\sigma_0+\zeta\sigma_1+\zeta^2\sigma_2+\ldots \tag{5.3.13}$$

where

$$\sigma_0=\tfrac{1}{2}L_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+L_0^{\alpha\beta33}\gamma_{\alpha\beta}\gamma_{33}+2L_0^{\alpha3\lambda3}\gamma_{\alpha3}\gamma_{\lambda3}+\tfrac{1}{2}L_0^{3333}\gamma_{33}\gamma_{33} \tag{5.3.14}$$

$$\sigma_1=L_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu}+\tfrac{1}{2}L_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+L_0^{\alpha\beta33}\kappa_{\alpha\beta}\gamma_{33}+$$
$$+L_1^{\alpha\beta33}\gamma_{\alpha\beta}\gamma_{33}+2L_0^{\alpha3\lambda3}\gamma_{3\alpha}\kappa_{3\lambda}+2L_1^{\alpha3\lambda3}\gamma_{3\alpha}\gamma_{3\lambda} \tag{5.3.15}$$

$$\sigma_2=L_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\mu_{\lambda\mu}+\tfrac{1}{2}L_0^{\alpha\beta\lambda\mu}\kappa_{\alpha\beta}\kappa_{\lambda\mu}+L_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu}+\tfrac{1}{2}L_2^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+L_0^{\alpha\beta33}\mu_{\alpha\beta}\gamma_{33}+$$
$$+L_1^{\alpha\beta33}\kappa_{\alpha\beta}\gamma_{33}+L_2^{\alpha\beta33}\gamma_{\alpha\beta}\gamma_{33}+\tfrac{1}{2}L_0^{\alpha3\lambda3}\kappa_{3\alpha}\kappa_{3\lambda}+2L_1^{\alpha3\lambda3}\gamma_{3\alpha}\kappa_{3\lambda}+2L_2^{\alpha3\lambda3}\gamma_{3\alpha}\gamma_{3\lambda} \tag{5.3.16}$$

For the future purposes it is convenient to modify the form of σ_κ. Let us solve $(5.3.8)_3$ with respect to E_{33}, which gives

$$E_{33}=\frac{S^{33}}{L^{3333}}-\frac{L^{33\vartheta\sigma}}{L^{3333}}E_{\vartheta\sigma} \tag{5.3.17}$$

For a material having elastic symmetry with respect to the shell middle surface we can eliminate E_{33} from (5.3.11) by using (5.3.17) to obtain

$$\sigma_\kappa=\frac{(S^{33})^2}{2L^{3333}}+\frac{1}{2}H^{\varphi\psi\vartheta\sigma}E_{\varphi\psi}E_{\vartheta\sigma}+2L^{3\psi3\sigma}E_{3\psi}E_{3\sigma} \tag{5.3.18}$$

where the components of a modified elasticity tensor are defined by

$$H^{\varphi\psi\vartheta\sigma}=L^{\varphi\psi\theta\sigma}-\frac{L^{\varphi\psi33}L^{33\theta\sigma}}{L^{3333}}$$
$$H^{\varphi\psi\vartheta\sigma}=\delta_\alpha^\varphi\delta_\beta^\psi\delta_\lambda^\vartheta\vartheta_\mu^\sigma(H_0^{\alpha\beta\lambda\mu}+\zeta H_1^{\alpha\beta\lambda\mu}+\zeta^2H_2^{\alpha\beta\lambda\mu}+\ldots) \tag{5.3.19}$$

In terms of this tensor the expansion (5.3.13) can be transformed to the form

$$\sigma_\kappa=\frac{(S^{33})^2}{2L^{3333}}+\sigma_0'+\zeta\sigma_1'+\zeta^2\sigma_2'+\ldots\,,\qquad \sigma_0'=\tfrac{1}{2}H_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+2L_0^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu}$$
$$\sigma_1'=H_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu}+\tfrac{1}{2}H_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+2L_0^{3\beta3\mu}\gamma_{3\beta}\kappa_{3\mu}+2L_1^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu}$$
$$\sigma_2'=H_0^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\mu_{\lambda\mu}+\tfrac{1}{2}\kappa_{\alpha\beta}\kappa_{\lambda\mu})+H_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu}+\tfrac{1}{2}H_2^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu}+$$
$$+\tfrac{1}{2}L_0^{3\beta3\mu}\kappa_{3\beta}\kappa_{3\mu}+2L_1^{3\beta3\mu}\gamma_{3\beta}\kappa_{3\mu}+2L_2^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu} \tag{5.3.20}$$

Let us introduce a two-dimensional function of elastic strain energy Σ_κ of a shell, measured per unit area of the surface $\mathcal{M}_\kappa$

$$\Sigma_\kappa=\int_{-h/2}^{h/2}\mu\sigma_\kappa\,d\zeta \tag{5.3.21}$$

After substitution of (2.2.11) and (2.2.12) together with (5.3.18)÷(5.3.20) into (5.3.21) it becomes evident that Σ_κ becomes a function of $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$ and $\mu_{\alpha\beta}$ only, and constitutive equations (5.3.9) can be obtained from the relations

$$N^{\alpha\beta}=\frac{\partial\Sigma_\kappa}{\partial\gamma_{\alpha\beta}},\quad 2N^{\alpha3}=\frac{\partial\Sigma_\kappa}{\partial\gamma_{\alpha3}},\quad M^{\alpha\beta}=\frac{\partial\Sigma_\kappa}{\partial\kappa_{\alpha\beta}},\quad M^{\alpha3}=\frac{\partial\Sigma_\kappa}{\partial\kappa_{3\alpha}},\quad K^{\alpha\beta}=\frac{\partial\Sigma_\kappa}{\partial\mu_{\alpha\beta}} \tag{5.3.22}$$

It is obvious that even for a linear-elastic material the constitutive equations (5.3.9) have an extremely complex form when strains are large. The use of the function Σ_κ makes it possible to simplify significantly the relations for small strains. A more thorough discussion of this case will be presented in chapter 6.

5.4. Basic equations for shells under Kirchhoff-Love constraints

Basic relations of the non-linear theory of shells of Kirchhoff-Love type in the Lagrangean description can be obtained from general relations discussed earlier under the assumption of linear distribution (3.1.4). It is sufficient to assume that the deformation is subject to appropriate additional constraints which results in a number of terms in the basic equations having been omitted. The simplifications of description of deformation and of geometrical and kinematical relations can be found in p. 3.3 and p. 4.5. Here we will deal with simplified Lagrangean equations of motion, natural boundary conditions and constitutive equations.

It is known that simplifications of the basic equations for a theory of Kirchhoff-Love type can not be made automatically and consequently basing on the geometrical assumption as presented in p. 3.3 only. Note that the omission in the equations of motion of all terms associated with transverse shear forces would result in too crude simplifications, for which equilibrium would already be impossible to maintain (for instance, for a particular case of a plate under transverse load). Already in the linear theory of shells many ways of application of simplifications of Kirchhoff-Love type have been proposed. Much more such methods may be formulated within the frames of the non-linear theory.

In works by KOITER [3, 11], SANDERS [20] and the present author [21, 32] the Lagrangean set of basic equations of the Kirchhoff-Love type non-linear theory of shells have been derived directly from a two-dimensional principle of virtual work. Taking the two-dimensional principle of virtual work as a starting point allows also for a proper, compatible with the constraints imposed on the deformation, formulation of equilibrium equations and natural boundary conditions. It is also easy to take into account the modified tensor of change of curvature, linear part of which becomes identical with that in the "best" variant of the linear theory of shells [4]. A more detailed discussion of these problems can be found in a review [12] made by the author.

In a theory of Kirchhoff-Love type a deformation of the shell space is defined entirely by the deformation of the shell middle surface. Hence the virtual work principle in the Lagrangean description takes the form

$$\iint_{\mathcal{M}_\kappa}(n^{\alpha\beta}\delta\gamma_{\alpha\beta}+m^{\alpha\beta}\delta\rho_{\alpha\beta})\,da=\iint_{\mathcal{M}_\kappa}\mathbf{p}\cdot\delta\mathbf{u}\,da+\int_{\mathscr{C}_\kappa}(\mathbf{f}\cdot\delta\mathbf{u}+\mathbf{k}\cdot\delta\mathbf{\Omega}_t)\,ds \tag{5.4.1}$$

where

$$\rho_{\alpha\beta}=\kappa_{\alpha\beta}+\tfrac{1}{2}(b^{\lambda}_{\alpha}\gamma_{\lambda\beta}+b^{\lambda}_{\beta}\gamma_{\lambda\alpha}) \tag{5.4.2}$$

are components of the modified tensor of change of curvature.

Taking into account a variation of the shell middle surface strain measures

$$\delta\gamma_{\alpha\beta}=\tfrac{1}{2}(\delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}_{\beta}+\bar{\mathbf{a}}_{\alpha}\cdot\delta\mathbf{u}_{,\beta}),\qquad \delta\kappa_{\alpha\beta}=[(\delta\mathbf{u})_{|\alpha\beta}-\bar{a}^{\lambda\mu}\gamma_{\mu\alpha\beta}\,\delta\mathbf{u}_{,\lambda}]\cdot\bar{\mathbf{n}} \tag{5.4.3}$$

and applying the Stockes' theorem, the principle (5.4.1) can be transformed into the form [82, 84]

$$\begin{gathered}
-\iint_{\mathscr{M}_{\kappa}}(\mathbf{G}\mathbf{N}^{\beta})_{|\beta}\cdot\delta\mathbf{u}\,da+I=\iint_{\mathscr{M}_{\kappa}}\mathbf{p}\cdot\delta\mathbf{u}\,da+I^{*}\\
I=\int_{\mathscr{C}_{\kappa}}(\mathbf{P}_{\nu}\cdot\delta\mathbf{u}+\bar{m}_{\nu\nu}\bar{\mathbf{a}}_{t}\cdot\delta\mathbf{\Omega}_{t})\,ds+\sum_{M_n}\Delta\bar{m}_{t\nu}\bar{\mathbf{n}}\cdot\delta\mathbf{u}\Big|_{\mathscr{C}_{\kappa}}\\
I^{*}=\int_{\mathscr{C}_{\kappa}}(\mathbf{R}\cdot\delta\mathbf{u}+\bar{k}_{\nu}\bar{\mathbf{a}}_{t}\cdot\delta\mathbf{\Omega}_{t})\,ds+\sum_{M_n}\Delta\bar{k}_{t}\bar{\mathbf{n}}\cdot\delta\mathbf{u}\Big|_{\mathscr{C}_{\kappa}}\\
\Delta\bar{m}_{t\nu}(s_n)=\bar{m}_{t\nu}(s_n+0)-\bar{m}_{t\nu}(s_n-0),\qquad \Delta\bar{k}_{t}(s_n)=\bar{k}_{t}(s_n+0)-\bar{k}_{t}(s_n-0)
\end{gathered} \tag{5.4.4}$$

where M_n, $n=1,2,\ldots,N$ are corners of $\mathscr{C}_{\kappa}$ labelled by $s=s_n$ and

$$\begin{gathered}
\mathbf{N}^{\beta}=Q^{\alpha\beta}\mathbf{a}_{\alpha}+Q^{\beta}\mathbf{n}\\
Q^{\alpha\beta}=n^{\alpha\beta}+\tfrac{1}{2}\bar{b}^{\alpha}_{\lambda}m^{\lambda\beta}-\tfrac{1}{2}\bar{b}^{\beta}_{\lambda}m^{\alpha\lambda},\qquad Q^{\beta}=m^{\alpha\beta}|_{\alpha}+\bar{a}^{\beta\nu}\gamma_{\nu\lambda\mu}m^{\lambda\mu}\\
\mathbf{P}_{\nu}=\mathbf{G}\mathbf{N}^{\beta}\nu_{\beta}+\frac{d}{ds}(\bar{m}_{t\nu}\bar{\mathbf{n}}),\qquad \mathbf{R}=\mathbf{f}+\frac{d}{ds}(\bar{k}_{t}\bar{\mathbf{n}})
\end{gathered} \tag{5.4.5}$$

$$\bar{m}_{t\nu}=\frac{1}{1+2\gamma_{tt}}m^{\alpha\beta}(\delta^{\lambda}_{\alpha}+2\gamma^{\lambda}_{\alpha})\,t_{\lambda}\nu_{\beta},\quad \bar{m}_{\nu\nu}=\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,m^{\alpha\beta}\nu_{\alpha}\nu_{\beta} \tag{5.4.6}$$

$$k^{\alpha}=\bar{\epsilon}^{\alpha\beta}\mathbf{k}\cdot\mathbf{a}_{\beta},\qquad \bar{k}_{t}=\frac{1}{1+2\gamma_{tt}}k^{\alpha}(\delta^{\lambda}_{\alpha}+2\gamma^{\lambda}_{\alpha})\,t_{\lambda},\qquad \bar{k}_{\nu}=\frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\,k^{\alpha}\nu_{\alpha}$$

The local form of (5.4.4) gives the Lagrangean equilibrium equations to be satisfied within $\mathscr{M}_{\kappa}$, the natural boundary conditions to be satisfied on $\mathscr{C}_{\kappa}$ and the discontinuity conditions for $\bar{m}_{t\nu}$ to be satisfied at each corner point M_n of $\mathscr{C}_{\kappa}$:

$$\begin{gathered}
(\mathbf{G}\mathbf{N}^{\beta})_{|\beta}+\mathbf{p}=\mathbf{0}\ \text{in}\ \mathscr{M}_{\kappa},\qquad \mathbf{P}_{\nu}=\mathbf{R},\ m^{\alpha\beta}\nu_{\alpha}\nu_{\beta}=k^{\alpha}\nu_{\alpha}\ \text{on}\ \mathscr{C}_{\kappa}\\
\Delta\bar{m}_{t\nu}\,\mathbf{G}\mathbf{n}=\Delta\bar{k}_{t}\,\mathbf{G}\mathbf{n}\quad \text{at each}\quad M_n
\end{gathered} \tag{5.4.7}$$

After an expansion in the basis $\mathbf{a}_{\alpha}$, $\mathbf{n}$ the equations $(5.4.7)_1$ take the form [21]

$$\begin{gathered}
(l^{\alpha}_{.\lambda}Q^{\lambda\beta}+n^{\alpha}Q^{\beta})_{|\beta}-b^{\alpha}_{\beta}(\varphi_{\lambda}Q^{\lambda\beta}+nQ^{\beta})+p^{\alpha}=0\\
(\varphi_{\lambda}Q^{\lambda\beta}+nQ^{\beta})_{|\beta}+b_{\alpha\beta}(l^{\alpha}_{.\lambda}Q^{\lambda\beta}+n^{\alpha}Q^{\beta})+p=0
\end{gathered} \tag{5.4.8}$$

Constitutive equations for tensors $n^{\alpha\beta}$ and $m^{\alpha\beta}$ defined by the variational principle (5.4.1) have the form

$$n^{\alpha\beta}=\frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}},\qquad m^{\alpha\beta}=\frac{\partial\Sigma}{\partial\rho_{\alpha\beta}} \tag{5.4.9}$$

where $\Sigma = \Sigma(\gamma_{\alpha\beta}, \rho_{\alpha\beta})$ is a known function of the shell strain energy, measured per unit area of the surface $\mathscr{M}_\kappa$ and expressed in terms of $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ only.

Equations analogous to $(5.4.7)_1$ and (5.4.8) will also result from a reduction of general relations (5.2.5) if it is assumed that for a theory of Kirchhoff-Love type the following relations are satisfied

$$N^{33} = M^{\alpha 3} = K^{\alpha\beta} = 0, \quad \mathbf{l} = \mathbf{m}^3 = \mathbf{0}, \quad \rho_1 = \rho_2 = 0 \tag{5.4.10}$$

In that case equilibrium equations (5.2.5) will reduce to the form $(5.4.7)_1$ and (5.4.8), with

$$Q^{\alpha\beta} = N^{\alpha\beta} - \bar{b}^{\alpha}_{\lambda} M^{\lambda\beta}, \quad Q^{\alpha} = M^{\alpha\beta}|_{\beta} + \bar{a}^{\alpha\nu}\gamma_{\nu\lambda\mu} M^{\lambda\mu} \tag{5.4.11}$$

Simultaneously, constitutive equations for $N^{\alpha\beta}$ and $M^{\alpha\beta}$ will be obtained as a result of reduction (5.3.9)

$$N^{\alpha\beta} = B_0^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + B_1^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu}, \quad M^{\alpha\beta} = B_1^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + B_2^{\alpha\beta\lambda\mu}\kappa_{\lambda\mu} \tag{5.4.12}$$

It is worth noting that principal components of internal forces $n^{\alpha\beta}$ and $m^{\alpha\beta}$ appearing in (5.4.5) differ from components $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in (5.4.11). This comes from their definition as coefficients of strain variations in (5.4.1). In the definition components of the modified tensor of change of curvature $\rho_{\alpha\beta}$ as described by (5.4.2) have been taken into account. This difference is however insignificant in the quantitative meaning, because for small strains both sets of quantities differ by terms giving negligible contribution to the strain energy.

A relation equivalent to (5.1.26), written in a vector-tensor form, can be obtained as a result of the principle of conservation of moment of momentum in a shell. It is known [18] that for a three dimensional medium this principle leads to a confirmation of the symmetry of tensor S. Consequences of the principle of angular momentum conservation for shells have been studied within the frames of the Eulerian description [7] by investigations of invariancy of the principle of energy conservation. Within the Kirchhoff-Love type shell theory the results obtained in [7] in the Eulerian description are physically equivalent to those obtainable directly in the Lagrangean description. According to [7] the principle of angular momentum conservation leads to a relation, which in the notation used in this work takes the form

$$\bar{\mathbf{a}}_{\beta} \times \mathbf{G}\mathbf{N}^{\beta} + \bar{\mathbf{n}}_{,\beta} \times \mathbf{G}\mathbf{M}^{\beta} = \mathbf{0} \tag{5.4.13}$$

Hence, taking into account the relation resulting from the reduction of (5.2.5) we arrive at the formula

$$\bar{\mathbf{a}}_{\beta} \times \mathbf{G}\mathbf{N}^{\beta} + (\bar{\mathbf{n}} \times \mathbf{G}\mathbf{M}^{\beta})_{|\beta} = \mathbf{0} \tag{5.4.14}$$

where

$$\mathbf{M}^{\beta} = M^{\alpha\beta}\mathbf{a}_{\alpha} \tag{5.4.15}$$

Obviously, in view of $(5.3.7)_1$ and (5.4.11), the relation (5.4.14) is an identity which does not impose any additional restrictions upon components of tensors of internal forces and moments. This relations has been used in works [32, 53].

In the shell theory of Kirchhoff-Love type the axial vector is often used for the determination of a vector of internal moments. Then

$$\bar{\mathbf{n}} \times \mathbf{G}\mathbf{M}^{\beta} = \sqrt{\frac{\bar{a}}{a}}\, \epsilon_{\alpha\lambda} M^{\alpha\beta} \bar{a}^{\lambda\rho} \mathbf{a}_{\rho} \tag{5.4.16}$$

which is in agreement with a formula given in [12] (formula $(20)_1$). In this work definition (5.4.15) will be used for this purpose.

Equations (5.4.8), expressed in terms of quantities known at $\mathscr{M}_\kappa$ and decomposed in the basis $\mathbf{a}_\alpha$, $\mathbf{n}$, are an entirely Lagrangean form of equilibrium equations of the Kirchhoff--Love type theory of shells. It is however possible to formulate several different mixed forms of equilibrium equations in terms of Lagrangean quantities. These may happen to be more convenient in some applications.

According to $(3.3.1)_2$ vector equations (5.4.7) can be written in the form

$$(Q^{\alpha\beta}\bar{\mathbf{a}}_\alpha + Q^{\beta}\bar{\mathbf{n}})_{|\beta} + \sqrt{\frac{\bar{a}}{a}}\, \bar{\mathbf{p}} = \mathbf{0} \tag{5.4.17}$$

where

$$\bar{\mathbf{p}} = \bar{p}^{\alpha}\bar{\mathbf{a}}_\alpha + \bar{p}\bar{\mathbf{n}} = \sqrt{\frac{a}{\bar{a}}}\, \mathbf{G}^{-1}\mathbf{p} \tag{5.4.18}$$

Since in this case

$$\bar{\mathbf{a}}_{\alpha|\beta} = \bar{a}^{\lambda\mu}\gamma_{\mu\alpha\beta}\bar{\mathbf{a}}_\lambda + \bar{b}_{\alpha\beta}\bar{\mathbf{n}}, \qquad \bar{\mathbf{n}}_{|\beta} = -\bar{b}^{\lambda}_{\beta}\bar{\mathbf{a}}_\lambda \tag{5.4.19}$$

equations (5.4.17) in terms of components in the basis $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ take the form

$$Q^{\alpha\beta}\big|_\beta + \bar{a}^{\alpha\mu}\gamma_{\mu\lambda\beta}Q^{\lambda\beta} - \bar{b}^{\alpha}_{\beta}Q^{\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p}^{\alpha} = 0, \qquad Q^{\beta}\big|_\beta + \bar{b}_{\alpha\beta}Q^{\alpha\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p} = 0 \tag{5.4.20}$$

Equilibrium equations in the form of (5.4.20) and possible simplifications of these equations have been discussed in [12,82].

After performing the differentiation in (5.4.17) according to (5.4.19) one can, taking advantage of (3.3.2), decompose the obtained vector result in the basis $\mathbf{a}_\alpha$, $\mathbf{n}$ to arrive at the following formulae

$$\begin{aligned} &l^{\alpha}_{.\kappa}(Q^{\kappa\beta}\big|_\beta + \bar{a}^{\kappa\mu}\gamma_{\mu\lambda\beta}Q^{\lambda\beta} - \bar{b}^{\kappa}_{\beta}Q^{\beta}) + n^{\alpha}(Q^{\beta}\big|_\beta + \bar{b}_{\alpha\beta}Q^{\alpha\beta}) + p^{\alpha} = 0 \\ &\varphi_\kappa(Q^{\kappa\beta}\big|_\beta + \bar{a}^{\kappa\mu}\gamma_{\mu\lambda\beta}Q^{\lambda\beta} - \bar{b}^{\kappa}_{\beta}Q^{\beta}) + n\,(Q^{\beta}\big|_\beta + \bar{b}_{\alpha\beta}Q^{\alpha\beta}) + p = 0 \end{aligned} \tag{5.4.21}$$

Equations (5.4.21) are another form of Lagrangean equations (5.4.8). However in this form derivatives of displacement gradients do not appear. Therefore if $l^{\alpha}_{.\kappa}$, φ_κ, n^α and n are expressed in terms of $\check{\gamma}^{\lambda}_{\alpha}$ and components of the vector $\mathbf{\Omega}$ according to geometrical formulae (4.5.18), a form of equilibrium equations in which derivatives of the finite rotation vector $\mathbf{\Omega}$ do not appear will be obtained.

Consider a decomposition of equations (5.4.17) in the intermediate base $\check{\mathbf{a}}_\alpha$, $\mathbf{n}$. Note that according to (4.2.3), (4.5.5), (4.5.6) and (4.5.7) the quantities shown below can be expressed in terms of the components of $\mathbf{\Omega}$ and strain parameters only

$$\bar{\mathbf{a}}_\kappa \cdot \check{\mathbf{a}}^\alpha = \cos\omega\delta^\alpha_\kappa + \bar{\epsilon}_{\kappa\lambda}\bar{a}^{\lambda\alpha}\Omega + \frac{1}{2\cos^2\omega/2}\check{\Omega}_\kappa\check{\Omega}^\alpha =$$

$$= \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\nu\mu}(\delta^\lambda_\kappa + \check{\gamma}^\lambda_\kappa)(\delta^\mu_\beta + \check{\gamma}^\mu_\beta)\left[\cos\omega\delta^\nu_\lambda + \epsilon_{\lambda\rho}a^{\rho\nu}\Omega + \frac{1}{2\cos^2\omega/2}\Omega_\lambda\Omega^\nu\right]$$

$$\bar{\mathbf{n}} \cdot \check{\mathbf{a}}^\alpha = \bar{\epsilon}^{\alpha\beta}\check{\Omega}_\beta + \frac{1}{2\cos^2\omega/2}\Omega\check{\Omega}^\alpha = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\nu\mu}(\delta^\mu_\beta + \check{\gamma}^\mu_\beta)\left[\epsilon^{\nu\lambda}\Omega_\lambda + \frac{1}{2\cos^2\omega/2}\Omega\Omega^\nu\right] \qquad (5.4.22)$$

$$\mathbf{a}_\kappa \cdot \check{\mathbf{a}}^\alpha = \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\beta}\epsilon_{\kappa\mu}(\delta^\mu_\beta + \check{\gamma}^\mu_\beta)$$

$$\bar{\mathbf{a}}_\kappa \cdot \mathbf{n} = \bar{\epsilon}_{\alpha\kappa}\check{\Omega}^\alpha + \frac{1}{2\cos^2\omega/2}\Omega\check{\Omega}_\kappa = (\delta^\lambda_\kappa + \check{\gamma}^\lambda_\kappa)\left[\epsilon_{\alpha\lambda}\Omega^\alpha + \frac{1}{2\cos^2\omega/2}\Omega\Omega_\alpha\right]$$

$$\bar{\mathbf{n}} \cdot \mathbf{n} = \cos\omega + \Omega^2$$

where

$$\mathbf{\Omega} = \check{\Omega}^\alpha\check{\mathbf{a}}_\alpha + \Omega\mathbf{n} = \check{\Omega}_\alpha\check{\mathbf{a}}^\alpha + \Omega\mathbf{n} = \Omega^\alpha\mathbf{a}_\alpha + \Omega\mathbf{n} = \Omega_\alpha\mathbf{a}^\alpha + \Omega\mathbf{n} \qquad (5.4.23)$$

Hence the component of (5.4.17) in the direction of $\check{\mathbf{a}}_\alpha$ takes the form

$$(\bar{\mathbf{a}}_\kappa \cdot \check{\mathbf{a}}^\alpha)(Q^{\kappa\beta}|_\beta + \bar{a}^{\kappa\mu}\gamma_{\mu\lambda\beta}Q^{\lambda\beta} - \bar{b}^\kappa_\beta Q^\beta) + (\bar{\mathbf{n}} \cdot \check{\mathbf{a}}^\alpha)(Q^\beta|_\beta + \bar{b}_{\alpha\beta}Q^{\alpha\beta}) + (\mathbf{a}_\kappa \cdot \check{\mathbf{a}}^\alpha)\,p^\kappa = 0 \qquad (5.4.24)$$

while the component along $\mathbf{n}$ is identical as that in $(5.4.21)_2$.

Another equivalent form results from writing (5.4.7) out in the basis $\check{\mathbf{a}}_\alpha$, $\mathbf{n}$. Let

$$T^{\alpha\beta} = \check{\mathbf{a}}^\alpha \cdot \mathbf{G}\mathbf{N}^\beta = (\bar{\mathbf{a}}_\kappa \cdot \check{\mathbf{a}}^\alpha)\,Q^{\kappa\beta} + (\bar{\mathbf{n}} \cdot \check{\mathbf{a}}^\alpha)\,Q^\beta$$
$$T^\beta = \mathbf{n} \cdot \mathbf{G}\mathbf{N}^\beta = (\bar{\mathbf{a}}_\kappa \cdot \mathbf{n})\,Q^{\kappa\beta} + (\bar{\mathbf{n}} \cdot \mathbf{n})\,Q^\beta \qquad (5.4.25)$$

Then equation (5.4.7) can be decomposed in the basis $\check{\mathbf{a}}_\alpha$, $\mathbf{n}$ with the aid of (4.5.9) and (4.5.10), which leads to

$$T^{\alpha\beta}|_\beta + \bar{a}^{\alpha\mu}(\delta^\lambda_\mu + \check{\gamma}^\lambda_\mu)\check{\gamma}_{\lambda\kappa|\beta}T^{\kappa\beta} - \sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\alpha\mu}\epsilon_{\lambda\beta}(\delta^\beta_\mu + \check{\gamma}^\beta_\mu)\,b^\lambda_\kappa T^\kappa + (\mathbf{a}_\kappa \cdot \check{\mathbf{a}}^\alpha)\,p^\kappa = 0$$
$$T^\beta|_\beta + (\delta^\lambda_\alpha + \check{\gamma}^\lambda_\alpha)\,b_{\lambda\beta}T^{\alpha\beta} + p = 0 \qquad (5.4.26)$$

Equilibrium equations in the form (5.4.26) contain, due to (5.4.25) and (5.4.22), derivatives of components of the vector $\mathbf{\Omega}$. Equations (5.4.24) and $(5.4.21)_2$, which are their equivalents, do not contain such derivatives.

It is worth noting that sets of equations (5.4.8) and (5.4.20) have already been published by the author [21, 12], while sets (5.4.21), (5.4.24) and (5.4.26) are an entirely new contribution to the theory of shells. They also contain deformation parameters separated into a part related to a pure strain, and that of purely rotational nature. These equations can serve for consequent derivation of various simplified variants of equilibrium equations with restrictions imposed on strains and/or finite rotations independently.

6. Small strain theory

6.1. Elastic energy of small strains

Consider possible simplifications of the elastic shell strain energy function Σ_κ, as defined in (5.3.21), under an assumption of small strains in the shell. In this chapter we will assume that $\max|E_M| = \eta \ll 1$, where E_M, $M = 1, 2, 3$ are eigenvalues of the strain tensor $\mathbf{E}$.

Integration in (5.3.21) with (2.2.12) and (5.3.20) taken into account yields

$$\Sigma_\kappa = S + h\left[\sigma_0' + (\sigma_2' - 2H\sigma_1' + K\sigma_0')\frac{h^2}{12} + (\sigma_4' - 2H\sigma_3' + K\sigma_2')\frac{h^4}{80} + \dots\right]$$

$$S = \int_{-h/2}^{h/2} \frac{(S^{33})^2}{2L^{3333}}\,\mu d\zeta \tag{6.1.1}$$

As it is evident from the above formula, Σ_κ has a form of an infinite series with respect to small parameter h. For thin shells terms of higher orders can be omitted due to their low numerical value. We will prove this for an isotropic material, for which

$$L^{ijkl} = \frac{E}{2(1+\nu)}\left[g^{ik}g^{jl} + g^{il}g^{jk} + \frac{2\nu}{1-2\nu}g^{ij}g^{kl}\right] \tag{6.1.2}$$

or, in a coordinate system normal in $\mathscr{S}_\kappa$

$$L^{\varphi\psi\vartheta\sigma} = \frac{E}{2(1+\nu)}\left[g^{\varphi\vartheta}g^{\psi\sigma} + g^{\varphi\sigma}g^{\psi\vartheta} + \frac{2\nu}{1-2\nu}g^{\varphi\psi}g^{\vartheta\sigma}\right]$$

$$L^{\varphi\psi 33} = \frac{E\nu}{(1+\nu)(1-2\nu)}g^{\varphi\psi}, \qquad L^{\varphi 3\vartheta 3} = \frac{E}{2(1+\nu)}g^{\varphi\vartheta} \tag{6.1.3}$$

$$L^{3333} = \frac{E(1-\nu)}{(1+\nu)(1-2\nu)} \tag{6.1.4}$$

Substitution of (2.2.11) into (6.1.3) and (5.3.19), will yield expansions with respect to ζ compatible with $(5.3.19)_2$. For the respective components of these expansions appearing in (5.3.20) we obtain

$$H_0^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right]$$

$$H_1^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left[2(a^{\alpha\lambda}b^{\beta\mu} + b^{\alpha\lambda}a^{\beta\mu}) + 2(a^{\alpha\mu}b^{\beta\lambda} + b^{\alpha\mu}a^{\beta\lambda}) + \frac{4\nu}{1-\nu}(a^{\alpha\beta}b^{\lambda\mu} + b^{\alpha\beta}a^{\lambda\mu})\right]$$

$$H_2^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\Big[(3a^{\alpha\lambda}b^\beta_\kappa b^{\kappa\mu} + 4b^{\alpha\lambda}b^{\beta\mu} + 3b^\alpha_\kappa b^{\kappa\lambda}a^{\beta\mu}) + (3a^{\alpha\mu}b^\beta_\kappa b^{\kappa\lambda} + 4b^{\alpha\mu}b^{\beta\lambda} +$$

$$+ 3b^\alpha_\kappa b^{\kappa\mu}a^{\beta\lambda}) + \frac{2\nu}{1-\nu}(3a^{\alpha\beta}b^\lambda_\kappa b^{\kappa\mu} + 4b^{\alpha\beta}b^{\lambda\mu} + 3b^\alpha_\kappa b^{\kappa\beta}a^{\lambda\mu})\Big] \tag{6.1.5}$$

$$L_0^{\alpha 3\lambda 3} = \frac{E}{2(1+\nu)}a^{\alpha\lambda}, \qquad L_1^{\alpha 3\lambda 3} = \frac{E}{2(1+\nu)}2b^{\alpha\lambda}, \qquad L_2^{\alpha 3\lambda 3} = \frac{E}{2(1+\nu)}3b^\alpha_\kappa b^{\kappa\lambda} \tag{6.1.6}$$

Let us evaluate the orders of magnitude of the respective terms in (6.1.1) in order to simplify this relation.

Assume at $\mathcal{M}_\kappa$ such coordinate system ϑ^α that the following estimates

$$a_{\alpha\beta}\sim a^{\alpha\beta}\sim 1,\ b_{\alpha\beta}\sim b^{\alpha\beta}\sim b^\alpha_\beta\sim\frac{1}{R} \tag{6.1.7}$$

hold true. Here R is the smallest radius of curvature of the surface $\mathcal{M}_\kappa$. Let L be the characteristic wavelength of deformation patterns at $\mathcal{M}_\kappa$ as well as of the surface and boundary loading, and d be the distance of the shell point under consideration from the shell boundary. It is known that two-dimensional equations of the theory of shells are valid provided that the following estimates are satisfied

$$\left(\frac{h}{L}\right)^2\ll 1,\ \left(\frac{h}{d}\right)^2\ll 1,\ \frac{h}{R}\ll 1 \tag{6.1.8}$$

Let us introduce a small parameter [9, 12]

$$\vartheta=\max\left(\frac{h}{L},\ \frac{h}{d},\ \sqrt{\frac{h}{R}},\ \sqrt{\eta}\right) \tag{6.1.9}$$

This parameter will be used for the estimation of orders of magnitude of various terms.

Exact estimates of the stress state components and their derivatives in an isotropic thin elastic shell subject to loading at its lateral boundaries only have been presented by John [62]. They have the form

$$S^{\varphi\psi}\sim E\eta,\ S^{\varphi 3}\sim E\eta\vartheta,\ S^{33}\sim E\eta\vartheta^2 \tag{6.1.10}$$

Consider the case of the bending theory of shells, for which we have the following estimates of the strain measures

$$\gamma_{\alpha\beta}\sim h\kappa_{\alpha\beta}\sim\eta,\ \gamma_{3\alpha}\sim h\kappa_{3\alpha} \tag{6.1.11}$$

Hence, in view of (3.3.8), (5.3.8), (3.1.13), (5.3.19) and (6.1.10) the following estimates for the remaining parameters of the shell strain state can be obtained

$$\begin{gathered}
h^2\mu_{\alpha\beta}\sim\eta\vartheta^2,\qquad \gamma_{3\alpha}\sim h\kappa_{3\alpha}\sim\eta\vartheta\\
\gamma_{33}=-\frac{\nu}{1-\nu}\gamma^\kappa_\kappa+O(\eta\vartheta^2)=O(\nu\eta)\\
E_{33}=\frac{(1+\nu)(1-2\nu)}{E(1-\nu)}S^{33}-\frac{\nu}{1-\nu}\{\gamma^\kappa_\kappa+\zeta(2b^\alpha_\beta\gamma^\beta_\alpha+\kappa^\kappa_{.\kappa})+\\
+\zeta^2[\mu^\kappa_\kappa+b^\alpha_\beta(\kappa^{\cdot\beta}_\alpha+\kappa^\beta_{.\alpha})+3b^\alpha_\kappa b^\kappa_\beta\gamma^\beta_\alpha]+O(\eta\vartheta^4)\}
\end{gathered} \tag{6.1.12}$$

In the elastic strain energy function Σ_κ as given by (6.1.1) terms proportional to h and $h^3/12$ should be estimated. The first term denoted by S, terms proportional to $h^5/80$ and those of higher order are obviously small and may be omitted at once.

A thorough estimation of the above mentioned terms shows, that in the case under consideration the strain energy function Σ_κ contains two leading terms of the order of

$Eh\eta^2$ and five terms of secondary importance of the order of $Eh\eta^2\vartheta^2$. The remaining terms are of the order of $Eh\eta^2\vartheta^4$ or of the higher order and can be omitted. Therefore, the two-dimensional function Σ_κ of the elastic shell strain energy can be presented in the following consistently approximated form [88]

$$\Sigma_\kappa=\frac{h}{2}H_0^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu}+\frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right)+2hL_0^{3\beta3\mu}\left(k^2\gamma_{3\beta}\gamma_{3\mu}+l^2\frac{h^2}{48}\kappa_{3\beta}\kappa_{3\mu}\right)+$$

$$+\frac{h^3}{12}H_0^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(\mu_{\lambda\mu}-2H\kappa_{\lambda\mu})+\frac{h^3}{12}H_1^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\kappa_{\lambda\mu}+O(Eh\eta^2\vartheta^4) \qquad (6.1.13)$$

The third and the fourth term in (6.1.13) has been multiplied here by some correction factors $k^2=5/6$ and $l^2=7/10$, respectively, which were obtained in [88,7]. By these factors we take into account cubic distribution of $\mu S^{3\beta}$ in the shell space, such that $\mu S^{3\beta}=0$ at $\zeta=\pm h/2$.

The first two terms of (6.1.13) take into account the membrane and bending strain energies, as well as the one due to the transverse strains, which is included in the modified shell elasticity tensor (5.3.19). These two terms appear in the classical first-approximation theory of isotropic thin elastic shells.

It was already shown by KOITER [3], who adopted the assumption about the plane stress state in the shell, that these first two terms determine the shell strain energy within the error $O(Eh\eta\vartheta^2)$. Hence different variants of the first-approximation theory, which differ by terms of the order of η/R in the definition of the tensor of change of curvature $\kappa_{\alpha\beta}$, are equivalent in the sense of the first approximation to the strain energy. This statement has formed a basis on which the "best" variant of equations of the first-approximation linear theory of shells has been proposed [4]. The remaining five terms in (6.1.13) form the consistent, compatible with (6.1.10)÷(6.1.12), correction to the first-approximation shell strain energy. In this correction the secondary terms, which take into account the strain energy due to transverse shear and transverse bending strains, as well as secondary terms from coupling between membrane, bending, transverse and shearing strains have been taken into account in a consistent manner. Thus (6.1.13) defines properly the strain energy of the second-approximation theory of thin isotropic elastic shells.

Differentiating (6.1.13) according to (5.3.22), with symmetry conditions of tensors taken into account, and using (6.1.4)÷(6.1.6), after transformations we arrive at the following constitutive equations for the second-approximation theory of isotropic thin elastic shells:

$$N^{\alpha\beta}=B[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma_\lambda^\lambda]+D\{(1-\nu)[\mu^{\alpha\beta}+2(b_\lambda^\alpha\kappa^{(\lambda\beta)}+b_\lambda^\beta\kappa^{(\alpha\lambda)})-2H\kappa^{(\alpha\beta)}]+$$

$$+\nu[a^{\alpha\beta}(\mu_\lambda^\lambda+2b_\mu^\lambda\kappa_\lambda^\mu-2H\kappa_{.\lambda}^\lambda)+2b^{\alpha\beta}\kappa_{.\lambda}^\lambda]\}+O(Eh\eta\vartheta^4)$$

$$M^{\alpha\beta}=D\{(1-\nu)[\kappa^{(\alpha\beta)}+2(b_\lambda^\alpha\gamma^{\lambda\beta}+b_\lambda^\beta\gamma^{\alpha\lambda})-2H\gamma^{\alpha\beta}]+$$

$$+\nu[a^{\alpha\beta}(\kappa_{.\lambda}^\lambda+2b_\mu^\lambda\gamma_\lambda^\mu-2H\gamma_\lambda^\lambda)+2b^{\alpha\beta}\gamma_\lambda^\lambda]\}+O(Eh^2\eta\vartheta^4) \qquad (6.1.14)$$

$$K^{\alpha\beta}=D[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma_\lambda^\lambda]+O(Eh^3\eta\vartheta^2)$$

$$N^{3\beta}=k^2B(1-\nu)\gamma^{3\beta}+O(Eh\eta\vartheta^3),\qquad M^{3\beta}=\tfrac{1}{2}l^2D(1-\nu)\kappa^{3\beta}+O(Eh^2\eta\vartheta^3)$$

where

$$B=\frac{Eh}{1-\nu^2}, \qquad D=\frac{Eh^3}{12(1-\nu^2)} \tag{6.1.15}$$

The results given here are accurate for an isotropic linearly elastic shell in the bending state of strain and in the absence of surface forces at the upper and at the lower shell boundary surfaces. They may also be applicable in the case of sufficiently weak anisotropy and in the presence of small and slowly varying surface forces with components $p^\alpha = O(E\eta\vartheta^3)$ and $p^3 = O(Eh\eta^4)$ at $\zeta = \pm h/2$. The surface forces with larger magnitude or stronger variability would appear explicitly in (6.1.13) and (6.1.14).

6.2. Simplified relations of the general theory

Consider possible simplifications of various general geometric relations derived up to now, which result from the assumption of small strains, $\eta \ll 1$.

For various strain measures from (4.1.10), (4.1.18) and (4.1.21) the following estimates are obtained

$$\mathbf{U}=\mathbf{1}+O(\eta), \quad \check{\boldsymbol{\gamma}}=\boldsymbol{\gamma}+O(\eta^2), \quad \boldsymbol{\Lambda}=\boldsymbol{\gamma}^2+O(\eta^3) \tag{6.2.1}$$

Also from (4.1.9), (4.2.4), (4.2.13) and (4.2.14) it follows

$$\begin{aligned} &\check{\mathbf{a}}_a=\mathbf{a}_a+O(\eta), \quad \frac{\bar{a}}{a}=1+2\gamma^a_a+O(\eta^2)=1+O(\eta) \\ &\bar{a}^{ab}=a^{ab}-2\gamma^{ab}+O(\eta^2)=a^{ab}+O(\eta) \end{aligned} \tag{6.2.2}$$

Hence, the general formula for the rotation tensor $\mathbf{R}$, (4.1.24), reduces to

$$\begin{aligned} \mathbf{R}=&[l_{\lambda\alpha}(\delta^\alpha_c-\gamma^\alpha_c)+\beta_\lambda(a_{3c}-\gamma_{3c})]\,\mathbf{a}^\lambda\otimes\mathbf{a}^c+ \\ &+[\varphi_\alpha(\delta^\alpha_c-\gamma^\alpha_c)+(1+\beta)(a_{3c}-\gamma_{3c})]\,\mathbf{n}\otimes\mathbf{a}^c+O(\eta^2) \end{aligned} \tag{6.2.3}$$

In this and some other simplified relations given below it is suggested to keep also some secondary terms, since the order of the rotations have not been specified yet.

The general formula for the finite rotation vector $\mathbf{\Omega}$ as given by (4.2.21) reduces to

$$\begin{aligned} 2\mathbf{\Omega}=&\epsilon^{\lambda\mu}[-l_{\lambda\alpha}\gamma^\alpha_3+\beta_\lambda(1-\gamma_{33})-\varphi_\alpha(\delta^\alpha_\lambda-\gamma^\alpha_\lambda)+(1+\beta)\gamma_{3\lambda}]\,\mathbf{a}_\mu+ \\ &+\epsilon^{\lambda\mu}[\varphi_{\mu\alpha}(\delta^\alpha_\lambda-\gamma^\alpha_\lambda)-\beta_\mu\gamma_{3\lambda}]\,\mathbf{n}+O(\eta^2) \end{aligned} \tag{6.2.4}$$

while the vector of change of curvature $\mathbf{k}_\beta$, (4.2.23), takes the form

$$\begin{aligned} \mathbf{k}_\beta=&[\epsilon^{aef}\gamma_{e\beta;a}\mathbf{a}_f][1+O(\eta)]=\epsilon^{\lambda\alpha}\{[\gamma_{3\beta|\alpha}-b_{\beta\alpha}\gamma_{33}-\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})-b^\kappa_\beta\gamma_{\alpha\kappa}]\,\mathbf{a}_\lambda+ \\ &+(\gamma_{\beta\alpha|\lambda}-b_{\beta\lambda}\gamma_{\alpha3})\,\mathbf{n}\}\,[1+O(\eta)] \end{aligned} \tag{6.2.5}$$

Differential relations (4.2.27) and (4.2.28) remain unchanged within the small strain theory under consideration which permits unrestricted rotations. Simplifications of other geometric relations (4.2.30)$\div$(4.2.33) are obvious and therefore will not be shown here.

Simplified relations appearing in p. 4.3, where the deformation of a boundary element was described, are as follows:

$$\bar{a}_t=1+\gamma_{tt}+O(\eta^2), \qquad \bar{a}_3=1+\gamma_{33}+O(\eta^2)$$
$$\bar{a}_v=1+\gamma_{tt}+\gamma_{33}+O(\eta^2), \quad \bar{a}_m=1+2\gamma_{tt}+\gamma_{33}+O(\eta^2) \tag{6.2.6}$$

$$\breve{\mathbf{R}}_t=[(\mathbf{v}-\gamma_{vt}\mathbf{t}+\gamma_{3v}\mathbf{n})\otimes\mathbf{v}+(\gamma_{vt}\mathbf{v}+\mathbf{t}-\gamma_{3t}\mathbf{n})\otimes\mathbf{t}+(\gamma_{3v}\mathbf{v}+\gamma_{3t}\mathbf{t}+\mathbf{n})\otimes\mathbf{n}]\,[1+O(\eta)] \tag{6.2.7}$$

$$\breve{\mathbf{\Omega}}_t=(\gamma_{3t}\mathbf{v}+\gamma_{3v}\mathbf{t}-\gamma_{vt}\mathbf{n})\,[1+O(\eta)] \tag{6.2.8}$$

$$\sin\breve{\omega}_t=\breve{\omega}_t+O(\eta^2)=\sqrt{(\gamma_{3t})^2+(\gamma_{3v})^2+(\gamma_{vt})^2} \tag{6.2.9}$$

$$\cos\breve{\omega}_t=1+O(\eta)$$

$$\mathbf{\Omega}_t=\cos^2\omega^+/2\,\breve{\mathbf{\Omega}}_t+\mathbf{\Omega}+\tfrac{1}{2}\mathbf{\Omega}^+\times\breve{\mathbf{\Omega}}_t-\frac{\breve{\mathbf{\Omega}}_t\cdot\mathbf{\Omega}^+}{4\cos^2\omega^+/2}\,\mathbf{\Omega}^++O(\eta^2)$$
$$\mathbf{\Omega}^+=\tfrac{1}{2}\epsilon^{\lambda\mu}[(\beta_\lambda-\varphi_\lambda)\mathbf{a}_\mu+\varphi_{\lambda\mu}\mathbf{n}], \qquad \cos^2\omega^+/2=\tfrac{1}{4}(\mathrm{tr}\,\mathbf{G}+1) \tag{6.2.10}$$

Consider now simplifications of geometric boundary conditions as formulated in p. 4.4. Using the kinematical boundary conditions (4.4.9) we employ the total finite rotation vector $\mathbf{\Omega}_t$ in a form simplified according to (6.2.10). Simultaneously values of the boundary function $\mathbf{m}(s)$ can be determined from consistently simplified relation

$$2\mathbf{m}\approx\mathbf{v}\times\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]+\mathbf{t}\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)+\mathbf{n}\times\left\{\left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]\times\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\right\} \tag{6.2.11}$$

Deformational quantities (4.4.39) appearing in deformational boundary conditions (4.4.40) have been formulated in p. 4.4 in a tensor notation and in an expanded form in terms of physical components at the shell boundary. After a consistent omission of all terms small as compared to the unity, and after a number of involved transformations which are omitted here, we obtain the following simplified relations for physical components of the vector $\mathbf{k}_t$ at the boundary

$$-k_{tt}\approx-\kappa_{tt}+2\frac{d\gamma_{3t}}{ds}-\sigma_t(\gamma_{tt}+\gamma_{33})$$

$$k_{vt}\approx\tfrac{1}{2}(\kappa_{vt}+\kappa_{tv})-\frac{d\gamma_{3t}}{ds_v}+\frac{d\gamma_{3v}}{ds}+\kappa_v\gamma_{3v}+2(\sigma_t-\underline{\kappa_{tt}})\gamma_{vt}-\tau_t(\gamma_{vv}+\gamma_{33})+\kappa_t\gamma_{3t} \tag{6.2.12}$$

$$-k_{nt}\approx\frac{d\gamma_{tt}}{ds_v}-2\frac{d\gamma_{vt}}{ds}-2\kappa_v\gamma_{vt}+2(\sigma_t-\underline{\kappa_{tt}})\gamma_{3v}+\kappa_t(\gamma_{tt}-\gamma_{vv})$$

where (6.2.6) has been used together with

$$\sqrt{\frac{a}{\bar{a}}}=1-(\gamma_{vv}+\gamma_{tt}+\gamma_{33})+O(\eta^2) \tag{6.2.13}$$

Boundary values of functions (4.4.54), (4.4.55) reduce to

$$
\begin{aligned}
p(s) &\approx (\mathbf{n}+\mathbf{B})\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)-\sigma_t(s)\\
q(s) &\approx \left[\left(\mathbf{t}+\frac{d\mathbf{A}}{ds}\right)\times(\mathbf{n}+\mathbf{B})\right]\left(\frac{d\mathbf{n}}{ds}+\frac{d\mathbf{B}}{ds}\right)-\tau_t(s)\\
r(s) &\approx \left[(\mathbf{n}+\mathbf{B})\times\left(\mathbf{t}+\frac{d\mathbf{A}}{dt}\right)\right]\left(\frac{d\mathbf{t}}{ds}+\frac{d^2\mathbf{A}}{ds^2}\right)-\kappa_t(s)
\end{aligned}
\qquad (6.2.14)
$$

Deformational values (6.2.12), together with boundary strains γ_{tt}, γ_{3t}, γ_{33}, are assumed as deformational boundary conditions for a geometrically non-linear theory of shells of the Reissner type.

It is worth noting that the simplifications leading to the linear theory of shells can be accompanied by additional omission of the underlined terms in the relations (6.2.12) which define the change of normal curvature of the boundary line during shell deformation. Deformational boundary conditions of a linear theory of shells of the Reissner type have been formulated in a different manner by SHAMINA [72]. When we expand vector relations presented in [72], after taking into account differences in nomenclature and sign convention adopted ,we obtain the relations identical to those resulting from the simplification of our deformational quantities (6.2.12). This provides an additional proof of correctness of the relations (6.2.12) as well as of the general relations (4.4.36)÷(4.4.39) derived in this work.

When simplifying the geometrical relations (6.2.1)÷(6.2.14) only the assumption of small strains has been taken into account.

Now consider possible simplification of homogeneous equilibrium equations (5.2.18) for a bending theory of shells composed of a homogeneous isotropic elastic material. In this case not only the small strain assumption is valid, but also sharper estimates (6.1.11) and $(6.1.12)_1$, leading to constitutive equations of the form (6.1.14), hold true. Writing out the relations (5.2.18) according to (5.2.16) and taking into account the estimates (6.2.2), for quantities appearing in (5.2.18) we obtain the following approximate relations estimated to within an error compatible with the error of constitutive equations

$$
\begin{aligned}
Q^{\alpha\beta} &= N^{\alpha\beta}-D(1-\nu)\left[b^{\alpha}_{\lambda}-\tfrac{1}{2}(\kappa^{\alpha}_{.\lambda}+\kappa^{.\alpha}_{\lambda})\right]\left[\tfrac{1}{2}(\kappa^{\lambda\beta}+\kappa^{\beta\lambda})+\frac{\nu}{1-\nu}\,a^{\lambda\beta}\kappa^{\mu}_{.\mu}\right]+O(Eh\eta\vartheta^4)\\
Q^{3\alpha} &= k^2B(1-\nu)\gamma^{3\alpha}+O(Eh\eta\vartheta^3)\\
R^{\alpha\beta} &= M^{\alpha\beta}-D(1-\nu)\left[b^{\alpha}_{\lambda}-\tfrac{1}{2}(\kappa^{\alpha}_{.\lambda}+\kappa^{.\alpha}_{\lambda})\right]\left[\gamma^{\lambda\beta}+\frac{\nu}{1-\nu}\,a^{\lambda\beta}\gamma^{\mu}_{\mu}\right]+O(Eh^2\eta\vartheta^4)\\
R^{3\alpha} &= \tfrac{1}{2}l^2D(1-\nu)\kappa^{3\alpha}+O(Eh^2\eta\vartheta^3)
\end{aligned}
\qquad (6.2.15)
$$

Derivatives of various components in (5.2.18) are estimated in the following manner

$$
(\ )_{|\beta}\sim\frac{1}{\lambda}(\ ) \qquad (6.2.16)
$$

where [9, 12]

$$\lambda=\frac{h}{\vartheta}=\min\left(L,\, d,\, \sqrt{hR},\, \frac{h}{\sqrt{\eta}}\right) \tag{6.2.17}$$

The error of the equation $(5.2.18)_1$ resulting from the approximate form of constitutive equations is equal to $O\left(\frac{Eh\eta\vartheta^4}{\lambda}\right)$, while for the equation $(5.2.18)_2$ it is equal to $O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right)$. Thus groups of terms appearing in $(5.2.18)_{1,2}$, which are other than (6.2.15), can be taken into account with identical accuracy

$$\bar{a}^{\alpha d}\gamma_{d\lambda\beta}\, Q^{\lambda\beta}=Ba^{\alpha\kappa}(2\gamma_{\kappa\lambda|\beta}-\gamma_{\lambda\beta|\kappa})\left[(1-\nu)\gamma^{\lambda\beta}+\nu a^{\lambda\beta}\gamma^{\mu}_{\mu}\right]+O\left(\frac{Eh\eta\vartheta^4}{\lambda}\right)$$

$$-b^{\alpha}_{\lambda}Q^{3\lambda}=-k^2B(1-\nu)\, b^{\alpha}_{\lambda}\gamma^{\lambda 3}+O\left(\frac{Eh\eta\vartheta^4}{\lambda}\right), \qquad \bar{a}^{\alpha d}\gamma_{d3\lambda}\, Q^{3\lambda}=O\left(\frac{Eh\eta\vartheta^4}{\lambda}\right) \tag{6.2.18}$$

$$b_{\alpha\beta}\, Q^{\alpha\beta}=Bb_{\alpha\beta}\left[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\mu}_{\mu}\right]=O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right)$$

$$\bar{a}^{3d}\gamma_{d\alpha\beta}\, Q^{\alpha\beta}=-B\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})\left[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\mu}_{\mu}\right]+O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right) \tag{6.2.19}$$

$$\bar{a}^{3d}\gamma_{d3\alpha}\, Q^{3\alpha}=O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right)$$

Substitution of (6.2.15), (6.2.18) and (6.2.19) into the equations $(5.2.18)_{1,2}$ yields

$$\begin{aligned} N^{\alpha\beta}|_{\beta}&-D(1-\nu)\left[b^{\alpha}_{\lambda|\beta}-\tfrac{1}{2}(\kappa^{\alpha}_{\cdot\lambda}+\kappa^{\cdot\alpha}_{\lambda})|_{\beta}\right]\left[\tfrac{1}{2}(\kappa^{\lambda\beta}+\kappa^{\beta\lambda})+\frac{\nu}{1-\nu}\, a^{\lambda\beta}\kappa^{\mu}_{\cdot\mu}\right]-\\ &-D(1-\nu)\left[b^{\alpha}_{\lambda}-\tfrac{1}{2}(\kappa^{\alpha}_{\cdot\lambda}+\kappa^{\cdot\alpha}_{\lambda})\right]\left[\tfrac{1}{2}(\kappa^{\lambda\beta}+\kappa^{\beta\lambda})|_{\beta}+\frac{\nu}{1-\nu}\, a^{\lambda\beta}\kappa^{\mu}_{\cdot\mu|\beta}\right]+\\ &+B(2\gamma^{\alpha}_{\lambda|\beta}-\gamma_{\lambda\beta}|^{\alpha})\left[(1-\nu)\gamma^{\lambda\beta}+\nu a^{\lambda\beta}\gamma^{\mu}_{\mu}\right]-k^2B(1-\nu)\, b^{\alpha}_{\lambda}\gamma^{\lambda 3}=O\left(\frac{Eh\eta\vartheta^4}{\lambda}\right) \end{aligned} \tag{6.2.20}$$

$$k^2B(1-\nu)\gamma^{3\alpha}|_{\alpha}+B\left[b_{\alpha\beta}-\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})\right]\left[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\mu}_{\mu}\right]=O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right) \tag{6.2.21}$$

Analogous estimations performed for the equations $(5.8.12)_{3,4}$ show that, owing to the estimate of $Q^{3\alpha}$ obtained in (6.2.18), the equation $(5.2.18)_3$ can be determined with the accuracy $O\left(\frac{Eh^2\eta\vartheta^2}{\lambda}\right)$ what gives us

$$D(1-\nu)\left[\tfrac{1}{2}(\kappa^{\alpha\beta}|_{\beta}+\kappa^{\beta\alpha}|_{\beta})+\frac{\nu}{1-\nu}\, a^{\alpha\beta}\kappa^{\mu}_{\cdot\mu}|_{\beta}\right]-k^2B(1-\nu)\gamma^{3\alpha}=O\left(\frac{Eh^2\eta\vartheta^2}{\lambda}\right) \tag{6.2.22}$$

In the equation $(5.2.18)_4$ we have no constitutive equation for the quantity Q^{33}, since by (5.3.17) strain E_{33} (and therefore γ_{33}) has been eliminated from the strain energy function.

However, it follows from $(5.2.16)_1$, $(5.1.15)_1$ and (6.1.10) that $Q^{33} \sim Eh\eta\vartheta^2$, which is the order of other most important terms appearing in $(5.2.18)_4$. Thus the equation becomes here indefinite. For this reason it is necessary to obtain, within the frames of the second approximation theory, a better estimate of Q^{33}, which would take into account the actual distribution of surface loading over the surfaces $\zeta = \pm h/2$. Such a more accurate estimate suggested by results presented by WESTBROOK [75] will be proposed here.

Homogeneous equilibrium equations of three-dimensional continua, written out in component form along the actual deformed basis $\bar{\mathbf{g}}_i$ but expressed in terms of the Lagrangean quantities, have the form

$$S^{ij}_{,j} + (G^i_{mj} + C^i_{mj}) S^{mj} + (G^j_{mj} + C^j_{mj}) S^{im} = 0 \tag{6.2.23}$$

where

$$C^i_{mj} = \bar{g}^{ik}(E_{km,\,j} + E_{kj,\,m} - E_{mj,\,k} - 2G^n_{mj} E_{kn}) \tag{6.2.24}$$

Let us use (6.2.23) to estimate partial derivatives $S^{\varphi 3}_{,3}$ and $S^{33}_{,3}$. Expansion of (6.2.24) in the region $\mathscr{S}_\kappa$ leads to the following estimates

$$\begin{gathered} C^{\varphi}_{\psi\vartheta} \sim \frac{\eta}{\lambda}, \quad C^{\varphi}_{33} \sim \frac{\eta}{\lambda}, \quad C^{3}_{\psi 3} \sim \frac{\eta\vartheta^2}{\lambda}, \quad C^{\varphi}_{\psi 3} = \tfrac{1}{2}\delta^{\varphi}_{\alpha}\,\delta^{\beta}_{\psi}\,(\kappa^{\cdot\alpha}_{\cdot\beta} + \kappa^{\cdot\alpha}_{\beta}) + O\left(\frac{\eta\vartheta}{\lambda}\right) \\ C^{3}_{\psi\vartheta} = -\tfrac{1}{2}\delta^{\beta}_{\psi}\,\delta^{\lambda}_{\vartheta}(\kappa_{\beta\lambda} + \kappa_{\lambda\beta}) + O\left(\frac{\eta\vartheta}{\lambda}\right), \quad C^{3}_{33} = -\frac{\nu}{1-\nu}\,\kappa^{\lambda}_{\cdot\lambda} + O\left(\frac{\eta\vartheta}{\lambda}\right) \end{gathered} \tag{6.2.25}$$

In view of (6.2.25) and taking into account that for thin shells $\mu^{\alpha}_{\varphi} \approx \delta^{\alpha}_{\varphi}$, $\mu \approx 1$, equations (6.2.23) can be written out as

$$\begin{gathered} S^{\varphi 3}_{,3} + S^{\varphi\psi}|_{\psi} = O\left(\frac{E\eta\vartheta^2}{\lambda}\right) \\ S^{33}_{,3} + S^{3\psi}|_{\psi} + \delta^{\alpha}_{\varphi}\,\delta^{\beta}_{\psi}\,[b_{\alpha\beta} - \tfrac{1}{2}(\kappa_{\alpha\beta} + \kappa_{\beta\alpha})]\,S^{\varphi\psi} = O\left(\frac{E\eta\vartheta^3}{\lambda}\right) \end{gathered} \tag{6.2.26}$$

Assume the following general loading conditions at the upper and lower shell surfaces

$$S^{i3} = \delta^{i}_{a}\left(\frac{\bar{p}^{a}}{2} \pm \frac{\bar{l}^{a}}{h}\right) \quad \text{for} \quad \zeta = \pm\frac{h}{2} \tag{6.2.27}$$

Bearing in mind that

$$\delta^{\alpha}_{\varphi}\,\delta^{\beta}_{\psi}\,S^{\varphi\psi} = \frac{1}{h}\,N^{\alpha\beta} + \zeta\,\frac{12}{h^3}\,M^{\alpha\beta} + O(E\eta\vartheta^2) \tag{6.2.28}$$

we integrate $(6.2.26)_1$ with respect to ζ under conditions (6.2.27) to obtain

$$\delta^{\alpha}_{\varphi}\,S^{\varphi 3} = \frac{1}{2}\,\bar{p}^{\alpha} + \frac{2\zeta}{h^2}\,\bar{l}^{\alpha} + \frac{6}{h}\left(\frac{1}{4} - \frac{\zeta^2}{h^2}\right) M^{\alpha\beta}|_{\beta} + O(E\eta\vartheta^3) \tag{6.2.29}$$

where

$$\bar{p}^{\alpha} \sim \frac{\bar{l}^{\alpha}}{h} \sim E\eta\vartheta \tag{6.2.30}$$

Now substitute (6.2.29) and (6.2.28) into $(6.2.26)_2$. After integration of $(6.2.26)_2$ with respect to ζ under the conditions (6.2.27), after some transformations we have

$$S^{33}=\frac{1}{2}\,\bar{p}^3+\left(\frac{1}{4}-\frac{1}{3}\frac{\zeta^2}{h^2}\right)\frac{12\zeta}{h^2}\,\bar{l}^3+\left(\frac{1}{4}-\frac{\zeta^2}{h^2}\right)\Big\{\bar{l}^\alpha|_\alpha+\zeta\bar{p}^\alpha|_\alpha+$$

$$+\left[b_{\alpha\beta}+\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})\right]\left(\frac{6}{h}\,M^{\alpha\beta}+\zeta\,\frac{2}{h}\,N^{\alpha\beta}\right)\Big\}+O\left(\frac{Eh\eta\vartheta^3}{\lambda}\right) \qquad (6.2.31)$$

where

$$\bar{p}^3\sim\frac{\bar{l}^3}{h}\sim E\eta\vartheta^2 \qquad (6.2.32)$$

Integration of (6.2.31) in agreement with (5.1.15) yields

$$Q^{33}=N^{33}+O\left(\frac{Eh^2\eta\vartheta^3}{\lambda}\right)=$$

$$=\frac{h}{2}\,\bar{p}^3+\frac{h}{6}\,\bar{l}^\alpha|_\alpha+6\left[b_{\alpha\beta}-\tfrac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})\right]M^{\alpha\beta}+O\left(\frac{Eh^2\eta\vartheta^3}{\lambda}\right) \qquad (6.2.33)$$

If we admit only $\bar{p}_\alpha\sim\bar{l}^\alpha/h\sim E\eta\vartheta^3$ and $\bar{p}^3\sim\bar{l}^3/h\sim E\eta\vartheta^4$, as it was assumed for derivation of (6.1.13), the surface loading terms become negligible in (6.2.33) and then it becomes possible to present also $(5.2.18)_4$ in a simplified form with the accuracy $O\left(\frac{Eh^2\eta\vartheta^3}{\lambda}\right)$, namely

$$\frac{1}{2}\,l^2D(1-\nu)\,\kappa^{3\alpha}|_\alpha+D(1-\nu)\left[b_{\alpha\beta}-\frac{1}{2}(\kappa_{\alpha\beta}+\kappa_{\beta\alpha})\right]\left[\frac{1}{2}(\kappa^{\alpha\beta}+\kappa^{\beta\alpha})+\frac{\nu}{1-\nu}\,a^{\alpha\beta}\kappa^{\lambda}_{\cdot\lambda}\right]-$$

$$-Q^{33}=O\left(\frac{Eh^2\eta\vartheta^3}{\lambda}\right) \qquad (6.2.34)$$

Thus simplifications carried out consistently have led to a set of six non-linear equations (6.2.10), (6.2.21), (6.2.22) and (6.2.34) in terms of nine independent components of the strain state $\gamma_{\alpha\beta}$ and $\kappa_{(\alpha\beta)}$. Expressing the strain parameters with the aid of six displacement components u^α, w, β^α, β we obtain a set of equilibrium equations which defines completely the solution of the problem in terms of displacements. However, if the problem is supposed to be solved in terms of strain measures then the above set of equilibrium equations should be complemented with three compatibility conditions. Also these conditions ought to be simplified according to the principles presented above. The conditions can be formulated basing for instance on relations given in (4.2.28).

6.3. Simplified relations for Kirchhoff-Love theory

Simplified relations for small strain theory of Kirchhoff-Love type can be found either from simplified relations of the general theory, p. 6.2, subject to additional Kirchhoff-Love constraints, or by a direct reduction of relations for the non-linear theory of Kirchhoff-Love type, as derived above.

Under the assumption of small strains, $\eta \ll 1$, we obtain, as in p. 6.2, the following estimates

$$\check{\gamma}_{\alpha\beta}=\gamma_{\alpha\beta}+O(\eta^2), \qquad \Lambda_{\alpha\beta}=\gamma_\alpha^\lambda \gamma_{\lambda\beta}+O(\eta^3)$$

$$\frac{\bar{a}}{a}=1+2\gamma_\lambda^\lambda+O(\eta^2)=1+O(\eta), \qquad \check{\mathbf{a}}_\alpha=\mathbf{a}_\alpha+O(\eta) \tag{6.3.1}$$

$$\bar{a}^{\alpha\beta}=a^{\alpha\beta}-2\gamma^{\alpha\beta}+O(\eta^2)=a^{\alpha\beta}+O(\eta)$$

Expanding (3.3.3) and denoting displacement gradients according to (3.4.9) we arrive at the following reduced relations

$$\begin{aligned} n_\alpha &= [\varphi^\lambda(\vartheta_{\lambda\alpha}-\epsilon_{\lambda\alpha}\,\varphi)-\varphi_\alpha(1+\vartheta_\lambda^\lambda)][1+O(\eta)] \\ n &= [1+\vartheta_\lambda^\lambda+\varphi^2-\tfrac{1}{2}(\vartheta_\lambda^\lambda \vartheta_\mu^\mu-\vartheta_\mu^\lambda \vartheta_\lambda^\mu)][1+\gamma_\lambda^\lambda+O(\eta^2)] \end{aligned} \tag{6.3.2}$$

The finite rotation vector (4.5.3) is defined here only by the displacement components u_α, w as follows

$$2\mathbf{\Omega}=\epsilon^{\lambda\mu}\{[(1-\gamma_\kappa^\kappa)(\varphi_\alpha l^\alpha_{.\lambda}-\varphi_\lambda l^\alpha_{.\alpha})-\varphi_\alpha(\delta_\lambda^\alpha-\gamma_\lambda^\alpha)]\,\mathbf{a}_\mu+\varphi_{\mu\alpha}(\delta_\lambda^\alpha-\gamma_\lambda^\alpha)\,\mathbf{n}\}+O(\eta\vartheta) \tag{6.3.3}$$

Relations (6.3.2) and (6.3.3) contain only the displacements **u** and their surface gradients with no restrictions put on them. Thus each of parameters φ_α, φ, $\vartheta_{\alpha\beta}$ can adopt values up to the order of 1, provided that the largest strain in the shell is small.

In the similar manner the following relations, among others, can be obtained

$$\begin{aligned} \mathbf{k}_\beta &= \epsilon^{\alpha\lambda}[(\kappa_{\beta\alpha}+b_\beta^\kappa \gamma_{\alpha\kappa})\,\mathbf{a}_\lambda+\gamma_{\beta\alpha|\lambda}\,\mathbf{n}][1+O(\eta)] \\ \kappa_{\alpha\beta} &\approx \tfrac{1}{2}(\epsilon_{\alpha\lambda}\mathbf{k}_\beta+\epsilon_{\beta\lambda}\mathbf{k}_\alpha)\cdot\mathbf{a}^\lambda-\tfrac{1}{2}(b_\alpha^\lambda \gamma_{\lambda\beta}+b_\beta^\lambda \gamma_{\lambda\alpha}) \\ 4\sin^2\omega &\approx a^{\alpha\beta}[(2+\vartheta_\lambda^\lambda)\,\varphi_\alpha-\varphi^\lambda(\vartheta_{\lambda\alpha}-\epsilon_{\lambda\alpha}\,\varphi)][(2+\vartheta_\kappa^\kappa)\,\varphi_\beta-\varphi^\kappa(\vartheta_{\kappa\beta}-\epsilon_{\kappa\beta}\,\varphi)]+4\varphi^2 \end{aligned} \tag{6.3.4}$$

Reduction of other geometric relations presented in p. 4.5 is obvious.

Deformational quantities at the shell boundary, $(4.5.28)_1$ and (4.5.30), reduce to

$$\begin{aligned} k_{tt} &\approx \kappa_{tt}+\sigma_t\,\gamma_{tt}, \qquad k_{vt}\approx\kappa_{vt}+2(\sigma_t-\underline{\kappa_{tt}})\,\gamma_{vtt}-\tau_t\,\gamma_{vv} \\ k_{nt} &\approx -\frac{d\gamma_{tt}}{ds_v}+2\frac{d\gamma_{vt}}{ds}+2\kappa_v\,\gamma_{vt}-\kappa_t(\gamma_{tt}-\gamma_{vv}) \end{aligned} \tag{6.3.5}$$

During the transition to the linear theory it is also permissible to omit in (6.3.5) the underlined term which determines the change of normal curvature of the boundary line during the shell deformation. Deformational boundary conditions in the linear theory of shells have been derived in a different way by CHERNYKH [64]. He used physical components of tensors $\mu_{\alpha\beta}$ and ζ_α, related to the tensors $\kappa_{\alpha\beta}$ and $\gamma_{\alpha\beta}$ of this work through the formulae

$$\kappa_{\alpha\beta}=\mu_{\alpha\beta}-b_\alpha^\lambda \gamma_{\beta\lambda}, \qquad -\zeta_\alpha=\epsilon^{\lambda\beta}\gamma_{\lambda\alpha|\beta} \tag{6.3.6}$$

The same relations in terms of physical components at the boundary have the form

$$\begin{aligned} \kappa_{tt} &= \mu_{tt}+\tau_t\,\gamma_{vt}-\sigma_t\,\gamma_{tt}, \qquad \kappa_{tv}=\mu_{tv}-\sigma_t\,\gamma_{vt}+\tau_t\,\gamma_{vv} \\ -\zeta_v &= -\frac{d\gamma_{tt}}{ds_v}+\frac{d\gamma_{vt}}{ds}+2\kappa_v\,\gamma_{vt}-\kappa_t(\gamma_{tt}-\gamma_{vv}) \end{aligned} \tag{6.3.7}$$

In terms of the new variables (6.3.7) deformational quantities (6.3.5) take the form

$$k_{tt} \approx \mu_{tt} + \tau_t \gamma_{vt}, \qquad k_{vt} \approx \mu_{tv} + (\sigma_t - \underline{2\mu_{tt}}) \gamma_{vt}, \qquad k_{nt} \approx -\zeta_t + \frac{d\gamma_{vt}}{ds} \tag{6.3.8}$$

Relations (6.3.8) reduced to the form appropriate for the linear theory, with the underlined term omitted, are identical as those in [64]. This can be regarded as an additional proof of correctness of conditions (6.3.5) of the geometrically non-linear theory and conditions (4.5.28) and (4.5.30) of the non-linear theory of Kirchhoff-Love type.

Now consider simplifications of equilibrium equations (5.4.20) for the bending theory of shells made of a homogeneous isotropic elastic material. In this case estimates (6.1.11) and (6.1.12) hold true besides the assumption of small strains. In the first-approximation theory the effect of terms $O(Eh\eta^2\vartheta^2)$ in the elastic shell strain energy function Σ_κ as derived in (6.1.13) should be neglected. However, an omission of any of five terms of secondary importance appearing in (6.1.13) must be followed by the omission of the rest of them.

Taking into account (6.1.13) with the accuracy to $O(Eh\eta^2\vartheta^2)$ we can obtain the following consistent first approximation to the strain energy of a thin isotropic and elastic shell [3]

$$\begin{aligned}\Sigma_\kappa &= \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right) + O(Eh\eta^2\vartheta^2) = \\ &= \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\rho_{\alpha\beta}\rho_{\lambda\mu}\right) + O(Eh\eta^2\vartheta^2)\end{aligned} \tag{6.3.9}$$

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right) \tag{6.3.10}$$

Using (5.4.9) we obtain from the strain energy (6.3.9) the following constitutive equations of the first-approximation theory of thin elastic shells

$$\begin{aligned} n^{\alpha\beta} &= B[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^\lambda_\lambda] + O(Eh\eta\vartheta^2) \\ m^{\alpha\beta} &= D[(1-\nu)\rho^{\alpha\beta} + \nu a^{\alpha\beta}\rho^\lambda_\lambda] + O(Eh^2\eta\vartheta^2)\end{aligned} \tag{6.3.11}$$

Inversion of (6.3.11) yields

$$\begin{aligned}\gamma_{\alpha\beta} &= A[(1+\nu)n_{\alpha\beta} - \nu a_{\alpha\beta} n^\lambda_\lambda] + O(\eta\vartheta^2) \\ \rho_{\alpha\beta} &= \frac{1}{D(1-\nu^2)}\left[(1+\nu)m_{\alpha\beta} - \nu a_{\alpha\beta} m^\lambda_\lambda\right] + O\left(\frac{\eta\vartheta^2}{h}\right)\end{aligned} \tag{6.3.12}$$

where

$$A = \frac{1}{Eh} = \frac{1}{B(1-\nu^2)} \tag{6.3.13}$$

Let us perform a reduction of equations (5.4.20). Using $(5.4.5)_2$ we obtain easily the following estimate

$$Q^{\alpha\beta}=n^{\alpha\beta}+\tfrac{1}{2}D(1-\nu)(b^\alpha_\lambda\rho^{\lambda\beta}-b^\beta_\lambda\rho^{\alpha\lambda})+O(Eh\eta\vartheta^4)$$
$$=B[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^\lambda_\lambda]+O(Eh\eta\vartheta^2) \tag{6.3.14}$$
$$Q^\alpha=D\rho^\lambda_\lambda|^\alpha+O\left(Eh^2\frac{\eta\vartheta^2}{\lambda}\right)$$

If the problem of bending theory under consideration is to be solved in terms of strain components $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ then under the assumption of small strain theory we make the error $O\left(Eh\frac{\eta\vartheta^2}{\lambda}\right)$ in $(5.4.20)_1$ and the error $O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right)$ in $(5.4.20)_2$. Thus the equations can be reduced to the form

$$n^\beta_\alpha|_\beta+\bar{p}_\alpha=O\left(Eh\frac{\eta\vartheta^2}{\lambda}\right)$$
$$D\rho^\alpha_\alpha|^\beta_\beta+(b^\alpha_\beta-\rho^\alpha_\beta)\,n^\beta_\alpha+\bar{p}^3=O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right) \tag{6.3.15}$$

where $n^{\alpha\beta}$ is expressed in terms of $\gamma_{\alpha\beta}$ according to $(6.3.11)_1$.

The three equilibrium equations ought to be complemented with three strain compatibility conditions resulting from adequate simplification of the relations (3.4.1). These give after some transformations

$$\rho^\beta_{\alpha\,|\beta}-\rho^\beta_{\beta|\alpha}=O\left(\frac{\eta\vartheta^2}{h\lambda}\right)$$
$$An^\alpha_\alpha|^\beta_\beta+b^\alpha_\beta\rho^\beta_\alpha-b^\alpha_\alpha\rho^\beta_\beta-\tfrac{1}{2}\rho^\alpha_\beta\rho^\beta_\alpha+\tfrac{1}{2}\rho^\alpha_\alpha\rho^\beta_\beta+A(1+\nu)\,\bar{p}^\alpha|_\alpha=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right) \tag{6.3.16}$$

Also here $n^{\alpha\beta}$ is expressed in terms of $\gamma_{\alpha\beta}$ according to $(6.3.11)_1$.

The system of six equations (6.3.15) and (6.3.16), for six variables $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$, together with deformational boundary conditions expressed by means of (6.3.5) and γ_{tt}, can serve as a basis for the analysis of geometrically non-linear problems of the bending theory of isotropic elastic shells. The static boundary conditions, which are energetically compatible with the deformational ones, have recently been given in [87], see also [84, 86].

It is possible to obtain a solution of equations (6.3.15) and (6.3.16) of the bending theory in terms of two functions. Let us assume that

$$|K|\,L^2\ll 1\,. \tag{6.3.17}$$

According to (6.1.8) the theory of shells is capable of describing a strain and stress state provided that L is at least one order of magnitude larger than the shell thickness h. Relation (6.3.17) assumes that within the frames of the bending theory of shells L is also bounded from above. This means that within the accuracy of the bending theory the covariant differentiation in equations (6.3.15) and (6.3.16) becomes interchangeable. Thus the general solution of these equations can be presented in the form

$$\rho^\beta_\alpha=W|^\beta_\alpha+O\left(\frac{\eta\vartheta^2}{h}\right),\quad n^\beta_\alpha=\delta^{\beta\mu}_{\alpha\lambda}F|^\lambda_\mu+P^\beta_\alpha+O(Eh\eta\vartheta^2) \tag{6.3.18}$$

where W and F are strain and stress functions, respectively, and P_α^β is a particular solution of $(6.3.15)_1$. From the remaining two equations $(6.3.15)_2$ and $(6.3.16)_2$ the following set of equations for W and F is obtained [11]

$$\begin{aligned} &DW\big|_{\alpha\beta}^{\alpha\beta}+\delta_{\beta\mu}^{\alpha\lambda}(b_\alpha^\beta-W\big|_\alpha^\beta)F\big|_\lambda^\mu+(b_\beta^\alpha-W\big|_\beta^\alpha)P_\alpha^\beta+\bar{p}^3=O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right)\\ &AF\big|_{\alpha\beta}^{\alpha\beta}-\delta_{\beta\mu}^{\alpha\lambda}(b_\alpha^\beta-\tfrac{1}{2}W\big|_\alpha^\beta)W\big|_\lambda^\mu+A\left[P_\alpha^\alpha\big|_\beta^\beta-(1+v)P^{\alpha\beta}\big|_{\alpha\beta}\right]=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right)\end{aligned} \tag{6.3.19}$$

The boundary conditions can also be expressed in terms of W and F. Then the set (6.3.19) can be used as a basis for the analysis of geometrically non-linear problems of the bending theory of shells.

The above derived sets of equations (6.3.15), (6.3.16) or (6.3.19) are sufficiently accurate to describe the bending state in a shell. However, they can prove to be not accurate enough if strains due to the bending and due to membrane forces are of different orders of magnitude, even if the total strains are small within the shell region. It is therefore worth noting that the reduction of equations (5.4.20) and (3.4.1) can be performed with higher accuracy if a solution of the problem in terms of $n^{\alpha\beta}$ and $\rho_{\alpha\beta}$ regarded as independent variables is employed. The choice of these quantities as independent variables is justified by the fact that for a particular case of membrane linear theory $n^{\alpha\beta}$ can be determined directly from the equilibrium equations, whereas for a particular case of linear bending of a shell without extension of its middle surface $\rho_{\alpha\beta}$ is determined from the strain compatibility conditions only. Tensors $\gamma_{\alpha\beta}$ and $m^{\alpha\beta}$, which are dependent variables, are eliminated with the aid of constitutive equations $(6.3.12)_1$ and $(6.3.11)_2$. After a number of transformations, with smooth and slowly varying surface forces $\bar{p}^\alpha=O(E\eta\vartheta)$ and $\bar{p}^3=O(E\eta\vartheta^2)$ taken into account, the equilibrium equations (5.4.20) and the compatibility conditions (3.4.1) can be reduced to the following cononical form [9, 12, 27]

$$\begin{aligned} &n_{\alpha|\beta}^\beta-\tfrac{1}{2}D(1-v)(b_\alpha^\lambda\rho_\lambda^\beta-b_\lambda^\beta\rho_\alpha^\lambda)_{|\beta}-Db_\alpha^\beta\rho_{\lambda|\beta}^\lambda+\underline{D(\rho_\alpha^\beta\rho_\lambda^\lambda-\tfrac{1}{2}\delta_\alpha^\beta\rho_\lambda^\lambda\rho_\kappa^\kappa)_{|\beta}}+\underline{2A(n_\alpha^\lambda n_\lambda^\beta)_{|\beta}}-\\ &\underline{-\tfrac{1}{2}A\left[(1-v)n_\lambda^\kappa n_\kappa^\lambda+vn_\kappa^\kappa n_\lambda^\lambda\right]_{|\alpha}}+2A\left[(1+v)n_\alpha^\lambda\bar{p}_\lambda-vn_\lambda^\lambda\bar{p}_\alpha\right]+(1+\gamma_\lambda^\lambda)\bar{p}_\alpha=O\left(Eh\frac{\eta\vartheta^4}{\lambda}\right)\\ &D\rho_\alpha^\alpha\big|_\beta^\beta+(b_\beta^\alpha-\rho_\beta^\alpha)n_\alpha^\beta+\bar{p}^3=O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right)\\ &\rho_{\alpha|\beta}^\beta-\rho_{\beta|\alpha}^\beta+\tfrac{1}{2}A(1+v)(b_\alpha^\lambda n_\lambda^\beta-b_\lambda^\beta n_\alpha^\lambda)_{|\beta}-Ab_\alpha^\beta n_{\lambda|\beta}^\lambda+A(1+v)(\rho_\alpha^\beta n_{\lambda|\beta}^\lambda+\rho_\lambda^\beta n_{\beta|\alpha}^\lambda)-\\ &\qquad-2A(1+v)b_\alpha^\beta\bar{p}_\beta+2A(1+v)\rho_\alpha^\beta\bar{p}_\beta-vA\rho_\lambda^\lambda n_{\beta|\alpha}^\beta=O\left(\frac{\eta\vartheta^4}{h\lambda}\right)\\ &An_\alpha^\alpha\big|_\beta^\beta+b_\beta^\alpha\rho_\alpha^\beta-b_\alpha^\alpha\rho_\beta^\beta-\underline{\tfrac{1}{2}\rho_\beta^\alpha\rho_\alpha^\beta+\tfrac{1}{2}\rho_\alpha^\alpha\rho_\beta^\beta}+A(1+v)\bar{p}^\alpha\big|_\alpha=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right)\end{aligned} \tag{6.3.20}$$

Set of equations (6.3.20) together with deformational quantities (see DANIELSON [27]) at the boundary, expressed in terms of variables $n^{\alpha\beta}$ and $\rho_{\alpha\beta}$, can serve as a basis for the analysis of geometrically non-linear problems of thin isotropic elastic shells with arbitrary membrane-to-bending strain ratio.

Let

$$\gamma = \max|\gamma_A|, \qquad \rho = \max|\rho_A|, \qquad A = 1, 2 \tag{6.3.21}$$

be the highest eigenvalues of tensors $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ at $M \in \mathcal{M}_\kappa$. Then the parameter $\rho h/\gamma$ defines at M the relation between strains caused by bending and membrane forces. Depending on the assumed estimates of value of this parameter five types of equations of the geometrically non-linear theory of thin elastic shells can be distinguished:

a) $\rho h/\gamma \lesssim \vartheta^2$ – membrane theory,
b) $\rho h/\gamma \sim \vartheta$ – small bending theory,
c) $\rho h \sim \gamma$ – bending theory,
d) $\gamma/\rho h \sim \vartheta$ – large bending theory,
e) $\gamma/\rho h \lesssim \vartheta^2$ – inextensional bending theory.

For each of the above estimates the equations (6.3.20) can be subjected to a substantial reduction within the same error shown in the equations.

For the membrane theory the estimation of the order of terms containing $\rho_{\alpha\beta}$ makes it possible to omit in (6.3.20) a number of further terms. This leads to the following set of equations

$$n^\beta_{\alpha|\beta} + 2A(n^\lambda_\alpha n^\beta_\lambda)_{|\beta} - \tfrac{1}{2}A[(1-\nu)n^\kappa_\lambda n^\lambda_\kappa + \nu n^\kappa_\kappa n^\lambda_\lambda]_{|\alpha} +$$

$$+ 2A[(1+\nu)n^\lambda_\alpha \bar{p}_\lambda - \nu n^\lambda_\lambda \bar{p}_\alpha] + (1+\gamma^\lambda_\lambda)\bar{p}_\alpha = O\left(Eh\frac{\eta\vartheta^4}{\lambda}\right)$$

$$b^\alpha_\beta n^\beta_\alpha + \bar{p}^3 = O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right) \tag{6.3.22}$$

$$\rho^\beta_{\alpha|\beta} - \rho^\beta_{\beta|\alpha} + \tfrac{1}{2}A(1+\nu)(b^\lambda_\alpha n^\beta_\lambda - b^\beta_\lambda n^\lambda_\alpha)_{|\beta} - Ab^\beta_\alpha n^\lambda_{\lambda|\beta} - 2A(1+\nu)b^\beta_\alpha \bar{p}_\beta = O\left(\frac{\eta\vartheta^4}{h\lambda}\right)$$

$$An^\alpha_\alpha|^\beta_\beta + A(1+\nu)\bar{p}^\alpha|_\alpha = O\left(\frac{\eta\vartheta^2}{\lambda^2}\right)$$

It is evident that the non-linear membrane equilibrium equations $(6.3.22)_{1,2}$ can be solved with respect to $n^{\alpha\beta}$ independently on the state of strain in the shell. In this sense the membrane problems are statically determined. The additional condition $(6.3.22)_4$ for $n^{\alpha\beta}$ shows that within the frames of geometrically non-linear theory of elastic shells the membrane state can occur in certain particular cases only. After determination of $n^{\alpha\beta}$ strains $\gamma_{\alpha\beta}$ are determined from constitutive equations $(6.3.12)_1$, whereas $\rho_{\alpha\beta}$ may happen to be indefinite within the error of the membrane theory. This is so because in general two compatibility conditions $(6.3.22)_3$ may happen to be insufficient for a unique determination of three components of $\rho_{\alpha\beta}$. However, such a solution should be possible in some particular cases.

For the small bending theory it is possible to omit the underlined terms in $(6.3.20)_{1,4}$ which are quadratic with respect to $\rho_{\alpha\beta}$.

Simplifications of the bending theory have already been discussed. They resulted in the set of equation (6.3.15) and (6.3.16) expressed in terms of strain components, and subsequently in equations (6.3.19) expressed in terms of strain and stress functions.

For the large bending theory it is possible to omit in $(6.3.20)_1$ the terms which are underlined with a sinuous line, and which are quadratic with respect to $n^{\alpha\beta}$.

For the inextensional bending theory the estimation of the order of terms containing $n^{\alpha\beta}$ makes it possible to omit in (6.3.20) many terms. This results in the following set of equations

$$n^{\beta}_{\alpha|\beta}-\tfrac{1}{2}D(1-\nu)(b^{\lambda}_{\alpha}\rho^{\beta}_{\lambda}-b^{\beta}_{\lambda}\rho^{\lambda}_{\alpha})_{|\beta}-Db^{\beta}_{\alpha}\rho^{\lambda}_{\lambda|\beta}+$$
$$+D(\rho^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}-\tfrac{1}{2}\delta^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}\rho^{\kappa}_{\kappa})_{|\beta}+(1+\gamma^{\lambda}_{\lambda})\bar{p}_{\alpha}=O\left(Eh\frac{\eta\vartheta^4}{\lambda}\right)$$

$$D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta}+\bar{p}^3=O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right),\quad \rho^{\beta}_{\alpha|\beta}-\rho^{\beta}_{\beta|\alpha}=O\left(\frac{\eta\vartheta^4}{h\lambda}\right) \qquad (6.3.23)$$

$$b^{\alpha}_{\beta}\rho^{\beta}_{\alpha}-b^{\alpha}_{\alpha}\rho^{\beta}_{\beta}-\tfrac{1}{2}\rho^{\alpha}_{\beta}\rho^{\beta}_{\alpha}+\tfrac{1}{2}\rho^{\alpha}_{\alpha}\rho^{\beta}_{\beta}=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right)$$

As for the membrane theory, the non-linear continuity conditions $(6.3.23)_{3,4}$ can be solved with respect to $\rho_{\alpha\beta}$ independently on the state of stress in the shell. In this sense the inextensional bending shell problems are geometrically determined. The additional condition $(6.3.23)_2$ for $\rho_{\alpha\beta}$ indicates that within the frames of geometrically nonlinear theory of shells the inextensional bending strain and stress state can occur only in exceptional cases. After determination of $\rho_{\alpha\beta}$ moments $m^{\alpha\beta}$ can be determined from constitutive equations $(6.3.11)_2$, while $n^{\alpha\beta}$ are usually indefinite quantities within the inextensional bending theory error limit. The reason is the insufficiency of two equilibrium equations $(6.3.23)_1$ for a unique determination of three components of $n^{\alpha\beta}$. Such determination is possible in some particular cases only.

The simplified sets of equations (6.3.22) and (6.3.23) derived in this work are not supposed to be used to shells with singular middle surface points, because in such a case they may occasionally lead to inadequate results. For example, if we apply the membrane equations (6.3.22) to a flat membrane (for which $b^{\alpha}_{\beta}=0$) the equation $(6.3.22)_2$ becomes indefinite. The similar problem appears also within the linear theory of shells, where as typical the following singular surfaces are noted [64]: plate, infinite cylinder, cone, toroid. However, the simplified equations are not supposed to be applied even to shells with points near to singular, such as very long cylinders or very shallow shells.

7. Theory of moderate rotations

7.1. Restrictions of rotations

Application of the polar decomposition theorem (4.1.2) to the deformation gradient tensor **G** made it possible to decompose the deformation of shell material fibres into pure strain and pure rotation. Reduced relations of the geometrically non-linear theory resulting from the assumption of small strains have been discussed in chapter 6. In this chapter we will consider additional reduction of various geometrically non-linear relations resulting from restrictions imposed also upon the parameters of the finite rotation.

The angle of rotation ω is the basic parameter defining the finite rotation magnitude. In general relations of chapter 4 and 5 the angle of rotation ω appears as an argument of trigonometric functions. Expansion of these functions into Taylor series in the vicinity of $\omega=0$ yields

$$\sin\omega=\omega-\frac{1}{3!}\omega^3+\frac{1}{5!}\omega^5-\dots$$

$$\cos\omega=1-\frac{1}{2!}\omega^2+\frac{1}{4!}\omega^4-\dots \qquad (7.1.1)$$

$$2\cos^2\omega/2=2-\frac{1}{2!}\omega^2+\frac{1}{4!}\omega^4-\dots$$

Substantial simplification of all geometric relations can only be obtained when restrictions imposed on the angle ω permit approximation of trigonometric functions by the first terms of respective expansions (7.1.1). Approximation of (7.1.1) by two first terms leads to only insignificant simplification of the general theory. This may however prove to be useful in some particular problems.

Within the geometrically non-linear theory of shells the following four particular cases can be distinguished, depending on the estimate of permissible values of the rotation angle:

a) ω unrestricted – finite rotation theory,

b) $\omega\lesssim\sqrt{\vartheta}$ – large rotation theory,

c) $\omega\lesssim\vartheta$ – moderate rotation theory,

d) $\omega\lesssim\vartheta^2$ – small rotation theory.

Within the large rotation theory $\omega^4\ll 1$ and relations (7.1.1) can be approximated with the first two terms

$$\sin\omega=\omega-\frac{1}{3!}\omega^3+O(\vartheta^{5/2}),\quad \cos\omega=1-\frac{1}{2!}\omega^2+O(\vartheta^2)$$

$$2\cos^2\omega/2=2-\frac{1}{2!}\omega^2+O(\vartheta^2) \qquad (7.1.2)$$

In this work we will not discuss the possible simplifications of geometrical relations and basic equations for the large rotation theory, which are given in [82, 84].

Within the moderate rotation theory $\omega^2\ll 1$ and relations (7.1.1) have the form

$$\sin\omega=\omega+O(\vartheta^3),\qquad \cos\omega=1+O(\vartheta^2),\qquad 2\cos^2\omega/2=2+O(\vartheta^2) \qquad (7.1.3)$$

Corresponding simplifications of geometrical relations and basic equations will be presented in p. 7.2 below.

Within the small rotation theory $\omega\ll 1$ and relations (7.1.1) reduce to

$$\sin\omega=\omega+O(\vartheta^5),\qquad \cos\omega=1+O(\vartheta^4),\qquad 2\cos^2\omega/2=2+O(\vartheta^4) \qquad (7.1.4)$$

whereas all geometrical and physical relations reduce to the linear forms known from the classical linear theory of shells. Detailed discussions of the basic relations of the linear theory of shells can be found in many monographs (see [3, 58, 64, 81]) and therefore these relations are not considered here.

7.2. Simplified equations of the general theory

Consider simplifications of geometrical relations and basic equations of the geometrically non-linear bending theory of shells that are possible under an assumption of moderate rotations. In this case

$$|\Omega| \sim \vartheta \tag{7.2.1}$$

and the following estimates result from (6.2.4)

$$\varphi_\alpha \sim \vartheta, \qquad \varphi \sim \vartheta, \qquad \beta_\alpha \sim \vartheta, \qquad \beta \sim \vartheta^2 \tag{7.2.2}$$

while from estimates (6.1.11) and (6.1.12)$_4$ using (3.1.15) we obtain

$$\vartheta_{\alpha\beta} \sim \eta, \qquad \psi_{\alpha\beta} \sim \frac{\eta}{h}, \qquad \psi_\alpha \sim \frac{\eta\vartheta}{h} \tag{7.2.3}$$

Besides, according to (3.4.9)

$$l^\alpha_{.\lambda} = \delta^\alpha_\lambda + \vartheta^\alpha_\lambda - \omega^\alpha_{.\lambda} = \delta^\alpha_\lambda - \omega^\alpha_{.\lambda} + O(\vartheta^2) \tag{7.2.4}$$

Quantities Q^{ab} and $R^{a\beta}$ appearing in the Lagrangean shell equations (5.2.21) have been reduced to (6.2.15) for the bending theory of small strains. Let us substitute (6.2.15) into (5.2.21), simultaneously taking into account (7.2.2), (7.2.3) and (7.2.4). After an omission of many terms having the order of the error indicated, equations (5.2.21) reduce to

$$\begin{gathered}
[(\delta^\alpha_\lambda - \omega^\alpha_{.\lambda}) Q^{\lambda\beta} + \vartheta^\alpha_\lambda N^{\lambda\beta} + \beta^\alpha Q^{3\beta}]_{|\beta} - b^\alpha_\beta(\varphi_\lambda N^{\lambda\beta} + Q^{3\beta}) + p^\alpha = O\left(Eh\frac{\eta\vartheta^4}{\lambda}\right) \\
(\varphi_\lambda N^{\lambda\beta} + Q^{3\beta})_{|\beta} + b_{\alpha\beta}(\delta^\alpha_\lambda - \omega^\alpha_{.\lambda}) N^{\lambda\beta} + p^3 = O\left(Eh\frac{\eta\vartheta^3}{\lambda}\right) \\
[(\delta^\alpha_\lambda - \omega^\alpha_{.\lambda}) M^{\lambda\beta}]_{|\beta} - (\delta^\alpha_\lambda - \omega^\alpha_{.\lambda}) Q^{\lambda 3} + l^\alpha = O\left(Eh^2\frac{\eta\vartheta^2}{\lambda}\right) \\
(\varphi_\lambda M^{\lambda\beta} + R^{3\beta})_{|\beta} + b_{\alpha\beta}(\delta^\alpha_\lambda - \omega^\alpha_{.\lambda}) M^{\lambda\beta} - \varphi_\lambda Q^{\lambda 3} - Q^{33} + l^3 = O\left(Eh^2\frac{\eta\vartheta^3}{\lambda}\right)
\end{gathered} \tag{7.2.5}$$

Here $Q^{\lambda\beta}$ is expressed in terms of strains by means of (6.2.14)$_1$ and (6.1.12)$_1$, $Q^{3\beta}$ and $R^{3\beta}$ are given by (6.2.15)$_{2,4}$, Q^{33} – by (6.2.33), and for tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$ appearing in (7.2.5) and (6.2.33) we put the relations

$$\begin{gathered}
N^{\alpha\beta} = B\left[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^\lambda_\lambda\right] + O(Eh\eta\vartheta^2) \\
M^{\alpha\beta} = D(1-\nu)\left[\frac{1}{2}(\kappa^{\alpha\beta} + \kappa^{\beta\alpha}) + \frac{\nu}{1-\nu} a^{\alpha\beta}\kappa^\lambda_{.\lambda}\right] + O(Eh^2\eta\vartheta^2)
\end{gathered} \tag{7.2.6}$$

If the strain components appearing above are expressed in terms of six parameters of displacement state as shown in (3.1.15), the final set of six displacemental equations will be obtained. This set describes geometrically non-linear problems of the bending theory of isotropic elastic shells under moderate rotations with the accuracy of the second approximation to the shell strain energy. Equations (7.2.5), which are novel in the literature concerning the theory of shells, may serve as a basis for calculations of a very wide class of problems for elastic shells, since the only assumptions made in their derivation are those of small strains and moderate rotations. They can also be used for a starting point when seeking for further reduced equations under additional simplifying assumptions.

7.3. Simplifications of relations of Kirchhoff-Love type theory

Taking into account the estimates (7.1.3) and (7.2.1), within the frames of deformation of Kirchhoff-Love type we obtain the following estimates for surface displacement gradients

$$\varphi_\alpha \sim \vartheta, \qquad \varphi \sim \vartheta, \qquad \vartheta_{\alpha\beta} \sim \vartheta^2 \tag{7.3.1}$$

The finite rotation vector reduces to

$$\mathbf{\Omega} = (\epsilon^{\lambda\alpha}\varphi_\alpha + \tfrac{1}{2}\varphi^\lambda\varphi)\,\mathbf{a}_\lambda + \varphi\mathbf{n} + O(\eta\vartheta) \tag{7.3.2}$$

Besides, from (6.3.4) it follows

$$\frac{d\mathbf{\Omega}}{d\vartheta^\beta} = [\epsilon^{\lambda\alpha}\varphi_{\alpha|\beta} + \tfrac{1}{2}(\varphi^\lambda\varphi)_{|\beta} - b^\lambda_\beta\,\varphi]\,\mathbf{a}_\lambda + (\epsilon^{\lambda\alpha}b_{\lambda\beta}\,\varphi_\alpha + \varphi_{|\beta})\mathbf{n} + O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.3}$$

$$\mathbf{k}_\beta = \frac{d\mathbf{\Omega}}{d\vartheta^\beta} + \frac{1}{2}\,\frac{d\mathbf{\Omega}}{d\vartheta^\beta}\times\mathbf{\Omega} + O\left(\frac{\eta\vartheta}{\lambda}\right) = (\epsilon^{\lambda\alpha}\varphi_{\alpha|\beta} + \varphi^\lambda\varphi_{|\beta} - b^\lambda_\beta\,\varphi)\,\mathbf{a}_\lambda + $$
$$+ [\epsilon^{\lambda\alpha}(b_{\lambda\beta}\,\varphi_\alpha - \tfrac{1}{2}\varphi_\lambda\,\varphi_{\alpha|\beta}) + \varphi_{|\beta}]\,\mathbf{n} + O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.4}$$

but, according to [11]

$$\varphi^\lambda\varphi_{|\beta} = \tfrac{1}{2}\varphi^\lambda\epsilon^{\kappa\alpha}\omega_{\kappa\alpha|\beta} = \tfrac{1}{2}\varphi^\lambda\epsilon^{\kappa\alpha}(\vartheta_{\alpha\beta|\kappa} - \vartheta_{\kappa\beta|\alpha} + b_{\alpha\beta}\,\varphi_\kappa - b_{\kappa\beta}\,\varphi_\alpha) = O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.5}$$

Hence

$$\mathbf{k}_\beta = (\epsilon^{\lambda\alpha}\varphi_{\alpha|\beta} - b^\lambda_\beta\,\varphi)\,\mathbf{a}_\lambda + [\epsilon^{\lambda\alpha}(b_{\lambda\beta}\,\varphi_\alpha - \tfrac{1}{2}\varphi_\lambda\,\varphi_{\alpha|\beta}) + \varphi_{|\beta}]\,\mathbf{n} + O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.6}$$

In view of $(6.3.9)_2$ and (7.3.6) the components of the tensor of change of curvature take, under the assumptions made above, the following form

$$\rho_{\alpha\beta} = -\tfrac{1}{2}[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\kappa_\alpha\,\omega_{\beta\kappa} + b^\kappa_\beta\,\omega_{\alpha\kappa}] + O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.7}$$

The simplified form of components of the strain tensor results directly from (3.3.4)

$$\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + \tfrac{1}{2}a_{\alpha\beta}\,\varphi^2 - \tfrac{1}{2}a^{\lambda\mu}(\omega_{\lambda\alpha}\,\vartheta_{\mu\beta} + \vartheta_{\lambda\alpha}\,\omega_{\mu\beta}) + \tfrac{1}{2}\varphi_\alpha\,\varphi_\beta + O(\eta\vartheta^2) \tag{7.3.8}$$

or with a slightly worse accuracy

$$\gamma_{\alpha\beta}=\vartheta_{\alpha\beta}+\tfrac{1}{2}a_{\alpha\beta}\,\varphi^2+\tfrac{1}{2}\varphi_\alpha\,\varphi_\beta+O(\eta\vartheta)\,,\qquad \rho_{\alpha\beta}=-\tfrac{1}{2}(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha})+O\left(\frac{\eta}{\lambda}\right). \tag{7.3.9}$$

For the components n_α and n there is

$$\begin{aligned} n_\alpha&=-\varphi_\alpha-\epsilon_{\lambda\alpha}\,\varphi^\lambda\varphi+O(\eta\vartheta)\\ n&=1-\tfrac{1}{2}\varphi^\lambda\varphi_\lambda+O(\eta\vartheta^2)=1+O(\eta)\end{aligned} \tag{7.3.10}$$

By introducing (7.3.7) and (7.3.8) into the left-hand side of (5.4.1), describing the internal virtual work IVW, after applying the Stockes' theorem we obtain

$$\begin{aligned} IVW=&-\iint\limits_{\mathscr{M}_\kappa}(\mathbf{GN}^\beta)|_\beta\cdot\delta\mathbf{u}\,da+\int\limits_{\mathscr{C}_\kappa}\left\{\left[\mathbf{GN}^\beta\nu_\beta+\frac{d}{ds}(m_{t\nu}\mathbf{n})\right]\cdot\delta\mathbf{u}-\right.\\ &\left.-m_{\nu\nu}\,\delta\varphi_\nu\right\}ds+\sum_{M_n}\Delta m_{t\nu}\,\mathbf{n}\cdot\delta\mathbf{u} \end{aligned} \tag{7.3.11}$$

$$\begin{aligned}\mathbf{GN}^\beta=&[n^{\alpha\beta}-\underline{\tfrac{1}{2}(b^\alpha_\kappa\,m^{\kappa\beta}-b^\beta_\kappa\,m^{\alpha\kappa})}-\tfrac{1}{2}\,\omega^{\alpha\beta}n^\lambda_\lambda-\underline{\tfrac{1}{2}(\omega^{\alpha\lambda}m^\beta_\lambda+\omega^{\beta\lambda}n^\alpha_\lambda)}+\\ &+\underline{\tfrac{1}{2}(\vartheta^{\alpha\lambda}n^\beta_\lambda-\vartheta^{\beta\lambda}n^\alpha_\lambda)}]\,\mathbf{a}_\alpha+(\varphi_\alpha\,n^{\alpha\beta}+m^{\alpha\beta}|_\alpha)\,\mathbf{n}\end{aligned}$$

Note that the additional internal force appearing in $(7.3.11)_1$ from elimination of $m_{t\nu}$, is directed here along $\mathbf{n}$ and not along $\bar{\mathbf{n}}$ as in the general case $(5.4.4)_2$. This results from approximating $\rho_{\alpha\beta}$ only by linear terms in (7.3.7), which is compatible with the error of the strain energy function $(6.3.9)_2$. Although the strain energy arguments cannot be applied directly to the external couple $\mathbf{k}$, it seems to us unnatural to use higher accuracy for the bending part in EVW than in IVW. Keeping this in mind, within the moderate rotations we obtain from (5.4.1) the following equilibrium equations and natural boundary conditions:

$$\begin{gathered}(\mathbf{GN}^\beta)|_\beta+\mathbf{p}=\mathbf{0}\ \text{ in }\ \mathscr{M}_\kappa\\ \mathbf{GN}^\beta\nu_\beta+\frac{d}{ds}(m_{t\nu}\mathbf{n})=\mathbf{f}+\frac{d}{ds}(k_t\mathbf{n})\,,\ \ m_{\nu\nu}=k_\nu\ \text{ on }\ \mathscr{C}_\kappa\\ \Delta m_{t\nu}\,\mathbf{n}=\Delta k_t\,\mathbf{n}\ \text{ at each }\ M_n\end{gathered} \tag{7.3.12}$$

When expressed in components with respect to $\mathbf{a}_\alpha$, $\mathbf{n}$ the equilibrium equations $(7.3.12)_1$ take the form

$$\begin{gathered}[n^{\alpha\beta}-\underline{\tfrac{1}{2}(b^\alpha_\kappa\,m^{\kappa\beta}-b^\beta_\kappa\,m^{\alpha\kappa})}-\tfrac{1}{2}\omega^{\alpha\beta}n^\lambda_\lambda-\underline{\tfrac{1}{2}(\omega^{\alpha\lambda}n^\beta_\lambda+\omega^{\beta\lambda}n^\alpha_\lambda)}+\\ +\underline{\tfrac{1}{2}(\vartheta^{\alpha\lambda}n^\beta_\lambda-\vartheta^{\beta\lambda}n^\alpha_\lambda)}]|_\beta-b^\alpha_\beta(\varphi_\lambda\,n^{\lambda\beta}+m^{\lambda\beta}|_\lambda)+p^\alpha=0\\ (\varphi_\alpha\,n^{\alpha\beta}+m^{\alpha\beta}|_\alpha)|_\beta+b_{\alpha\beta}[n^{\alpha\beta}-\underline{\tfrac{1}{2}(\omega^{\alpha\lambda}n^\beta_\lambda+\omega^{\beta\lambda}n^\alpha_\lambda)}]+p=0\end{gathered} \tag{7.3.13}$$

These shell equations can be considerably simplified, if less accurate formulae (7.3.9) for $\gamma_{\alpha\beta}$ and $\rho_{\alpha\beta}$ are used [11]. As a result terms underlined in (7.3.11) and (7.3.13) will not appear in the shell relations.

In the shell literature often an additional, more severe restriction of rotations in the shell middle surface plane is used. This consists in an additional restriction of the finite rotation vector

$$\boldsymbol{\Omega}\cdot\mathbf{n}=\varphi\lesssim\vartheta^2 \tag{7.3.14}$$

Then the finite rotation vector reduces to

$$\boldsymbol{\Omega}=\epsilon^{\lambda\alpha}\varphi_\alpha\mathbf{a}_\lambda+\varphi\mathbf{n}+O(\eta\vartheta) \tag{7.3.15}$$

which is, within the error assumed, in agreement with the relation for the linearized rotation vector as used in many works [11, 12, 13, 20].

Under the additional restriction (7.3.15) the geo metrical relations take the form

$$\begin{aligned}\gamma_{\alpha\beta}&=\vartheta_{\alpha\beta}+\tfrac{1}{2}\varphi_\alpha\varphi_\beta+O(\eta\vartheta^2)\\ \rho_{\alpha\beta}&=-\tfrac{1}{2}(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha})+O\left(\frac{\eta\vartheta}{\lambda}\right)\end{aligned} \tag{7.3.16}$$

and for the force vector $\mathbf{GN}^\beta$ and the local form of equilibrium equations we obtain

$$\mathbf{GN}^\beta=n^{\alpha\beta}\mathbf{a}_\alpha+(\varphi_\alpha n^{\alpha\beta}+m^{\alpha\beta}|_\alpha)\mathbf{n} \tag{7.3.17}$$

$$\begin{aligned}&n^{\alpha\beta}|_\beta-b^\alpha_\beta(\varphi_\lambda n^{\lambda\beta}+m^{\lambda\beta}|_\lambda)+p^\alpha=0\\ &(\varphi_\alpha n^{\alpha\beta}+m^{\alpha\beta}|_\alpha)|_\beta+b_{\alpha\beta}n^{\alpha\beta}+p=0\end{aligned} \tag{7.3.18}$$

After substitution of (6.3.11) and (7.3.16) these relations give the appropriate shell equations in terms of displacements.

The surface load p^β is in many practical problems relatively small. Taking this into account we can assume an additional restriction

$$p^\beta\leqslant Eh\,\frac{\eta\vartheta^2}{\lambda} \tag{7.3.19}$$

for which relations (7.3.18) reduce further to yield

$$n^\beta_{\alpha|\beta}=O\left(Eh\,\frac{\eta\vartheta^2}{\lambda}\right) \tag{7.3.20}$$

$$D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta}+(b^\beta_\alpha-\rho^\beta_\alpha)n^\alpha_\beta+p=O\left(Eh^2\,\frac{\eta\vartheta^2}{\lambda^2}\right)$$

Finally let us point out that the classical geometrically non-linear theory of shallow shells, various variants of which have been discussed in [11, 13, 17, 31, 79], is based on the following restrictions

$$\begin{aligned}&\varphi_\alpha\leqslant\vartheta,\quad \varphi\leqslant\vartheta^2;\quad |K|L^2\leqslant\vartheta^2,\\ &u_\alpha\leqslant w\vartheta,\quad p^\beta\leqslant Eh\,\frac{\eta\vartheta^2}{\lambda},\end{aligned} \tag{7.3.21}$$

for which

$$\gamma_{\alpha\beta}=\vartheta_{\alpha\beta}+\tfrac{1}{2}w_{,\alpha}w_{,\beta}+O(\eta\vartheta^2)$$
$$\rho_{\alpha\beta}=-w_{|\alpha\beta}+O\left(\frac{\eta\vartheta}{\lambda}\right) \tag{7.3.22}$$

and the first two compatibility conditions $(6.3.16)_1$ become identities, whereas the first two equilibrium equations $(7.3.20)_1$ can be satisfied without difficulty by the introduction of the stress function F as described by (6.3.18) with $P^\beta_\alpha=0$. The strain compatibility condition $(6.3.16)_2$ together with equilibrium equation $(7.3.20)_2$ yield

$$Dw\big|^{\alpha\beta}_{\alpha\beta}-\delta^{\alpha\lambda}_{\beta\mu}(b^\beta_\alpha+w\big|^\beta_\alpha)F\big|^\mu_\lambda=p+O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right)$$
$$AF\big|^{\alpha\beta}_{\alpha\beta}-\delta^{\alpha\lambda}_{\beta\mu}(b^\beta_\alpha+\tfrac{1}{2}w\big|^\beta_\alpha)w\big|^\mu_\lambda=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right) \tag{7.3.23}$$

The most of numerical results obtained as yet for geometrically non-linear shell problems have been based mainly on the solution of the equations (7.3.23).

8. Concluding remarks

In this work some fundamental theoretical problems of the non-linear theory of shells have been discussed. A number of new results has been obtained which are the author's original contribution to the non-linear theory of shells.

Among the most important basic results obtained in this study we recall: the development of the exact theory of finite rotations in shells, the formulation of the complete set of basic relations of the Lagrangean non-linear theory of shells based on the assumption of linear deformation distribution across the shell thickness, the formulation of deformational boundary conditions in a general form, and the derivation of consistently reduced set of basic relations for the small strain theory within the second approximation to the shell strain energy.

Important from the practical point of view seem to be the new simplified relations derived within the frames of the Kirchhoff-Love type theory of shells, and also geometrically non-linear relations for the moderate rotation theory. These reduced relations are the ones that may already serve as the basis for the development of effective computer programs for numerical calculations of the non-linear shell problems.

As a result of the problems solved in this work many further problems of the fundamental nature have appeared. A thorough study of these problems can lead to results which can be both interesting from the cognitive point of view and important from the viewpoint of practical applications. Let us point out some of them with hope that they will be solved in the future.

If the displacement values are known, the determination of strain measures, parameters of the finite rotation, the vector of change of curvature and boundary deformation parameters may easily be obtained by differential operations as presented in chapter 4. The inverse problem consisting in determination of displacements from known components of strain measures or finite rotation parameters is much more difficult. For the non-linear problems this question usually reduces to a solution of certain non-linear differential equations. There exist some analogies to the analytical mechanics of rigid-body motion [73] and the results obtained there will probably be useful when attempting to solve the problem for shells.

In chapter 6 reduced relations of the small strain theory have been presented for an isotropic material taken as an example. Supposedly, more convincing results could be obtained, within the second-approximation theory, for a transversally isotropic or even orthotropic material having substantially different elastic properties in the transverse direction.

It would be interesting to present a detailed analysis of strain compatibility conditions for the Lagrangean theory of shells as developed in this work based on the linear distribution of deformation. If the compatibility conditions were obtained in an exact expanded form and then consistently simplified within the second-approximation small strain theory, then the formulation of solutions of shell problems entirely in terms of strain measures would become possible also within the second-approximation theory.

Up to now restrictions as to the rotation parameters have been used within the small strain theory only. In this study the strain and rotational parts of shell deformation have been separated. Thus it becomes possible to develop a theory with small or moderate rotations for the bending theory of shells with moderate or large strains.

Entirely open in the non-linear theory is the problem of obtaining various singular solutions due to concentrated external forces or due the singularities of shell geometry. Also related problems of non-linear solutions in shells having multi-connected regions have not been analyzed as yet. As it is known [76, 77] such solutions within the linear theory are sought for in the form of curvilinear integrals containing a linearized rotation vector or its derivative at the boundary line. It seems then that the theory of finite rotations in shells as developed in this work can prove to be useful when seeking for singular and multi-valued solutions also for non-linear problems.

Problems of stability analysis form an important class of problems of the non-linear theory of shells. Variants of shell stability equations most often used up to now [14, 17, 30, 78] contain various simplifications as compared to our equations of the geometrically non-linear theory of shells of Kirchhoff-Love type. Thus the results of this work may be a stimulant of attempts towards a formulation of a set of equations for stability problems that would not contain too drastic simplifications.

In the author's belief approaching and solving these and other related problems would have a substantial cognitive value, simultaneously resulting in a wider use of shell structures and elements in various branches of technology and everyday life.

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Obroty skończone i opis Lagrange'a w nieliniowej teorii powłok

Streszczenie

We wstępie podano przegląd niektórych zagadnień nieliniowej teorii powłok. W szczególności, rozważono różnicę między opisem Lagrange'a i Eulera, omówiono rolę obrotów skończonych w powłokach oraz podkreślono nowe rezultaty uzyskane w tej rozprawie.

W rozdziale drugim zestawiono podstawowe definicje, zależności geometryczne i niezbędne wzory z algebry i analizy tensorowej w zapisie absolutnym. Wprowadzono niektóre zależności dla układu współrzędnych normalnych.

W rozdziale trzecim opisano deformację powłoki, zakładając liniowy rozkład przemieszczeń na grubości powłoki. Przeanalizowano wyrażenia miar odkształcenia powłoki w opisie Lagrange'a i Eulera. Uzyskano niektóre zależności kinematyczne słuszne przy przyjęciu więzów Kirchhoffa-Love'a.

Ogólna teoria obrotów skończonych w powłokach przedstawiona została w rozdziale czwartym. Stosując twierdzenie o rozkładzie polarnym deformację otoczenia powierzchni środkowej powłoki rozłożono na sztywne przesunięcie, czyste rozciągnięcie wzdłuż głównych kierunków odkształcenia oraz obrót skończony kierunków głównych. Wyprowadzono ścisłe wzory dla tensora obrotu oraz wektora obrotu skończonego głównych kierunków odkształcenia i podano wiele zależności wyrażonych przez te obroty. Skrupulatnie przeanalizowano deformację powierzchni brzegowej powłoki i uzyskano ścisłe wzory dla całkowitego obrotu skończonego elementu brzegowego powłoki. Na podstawie czysto geometrycznych rozważań skonstruowano ogólną postać przemieszczeniowych, kinematycznych i deformacyjnych warunków brzegowych. Wiele zależności wyrażonych przez obroty skończone podano również w szczególnym przypadku deformacji typu Kirchhoffa-Love'a.

Korzystając z zasady najmniejszego działania w rozdziale piątym wyprowadzono podstawowy układ sześciu równań ruchu nieliniowej teorii powłok w opisie Lagrange'a oraz odpowiednie naturalne warunki brzegowe. Skonstruowano również pięć innych równoważnych układów równań podstawowych wyrażonych poprzez wielkości Lagrange'owskie. Odpowiednie zależności teorii typu Kirchhoffa-Love'a otrzymano jako przypadki szczególne.

W rozdziale szóstym rozważono możliwe uproszczenia równań podstawowych teorii powłok sprężystych przy założeniu, że odkształcenia w powłoce są małe. Zbudowano udokładnioną zależność dla drugiego przybliżenia do energii odkształcenia sprężystego powłoki. Składa się ona z dwóch członów głównych, ujmujących energię sprężystą od rozciągania, zginania i zmiany grubości powłoki, oraz pięciu odpowiednio wybranych członów drugorzędnych. Umożliwiło to uzyskanie udokładnionych równań konstytutywnych dla teorii drugiego przybliżenia oraz konsekwentne uproszczenie różnych zależności geometrycznie nieliniowej teorii powłok typu Reissnera. Odpowiednie zależności teorii typu Kirchhoffa-Love'a otrzymano jako przypadki szczególne.

W rozdziale siódmym rozważono uproszczenia zależności geometrycznie nieliniowej teorii powłok przy dodatkowych ograniczeniach nakładanych na wektor obrotu skończonego. Wyróżniono cztery warianty równań teorii powłok przy małych, umiarkowanych, dużych i skończonych obrotach. Dla przypadku umiarkowanych obrotów podano uproszczone zależności teorii ogólnej oraz przedyskutowano szereg przypadków szczególnych, takich jak teoria typu Kirchhoffa-Love'a oraz klasyczna nieliniowa teoria powłok o małej wyniosłości.

Uzyskane w rozprawie wyniki wyłoniły kilka nowych problemów do rozwiązania w przyszłości. Zagadnienia te omówiono w zakończeniu pracy.

Конечные повороты и лагранжево описание в нелинейной теории оболочек

Резюме

В введении дан обзор некоторых проблем нелинейной теории оболочек. В частности, рассмотрено различие между Лагранжевым и Эйлеровым описании, обсуждена роль конечных поворотов в теории оболочек и подчеркнуты новые результаты полученные в этой работе.

Во второй главе приводятся основные определения, геометрические соотношения и необходимые формулы тензорной алгебры и анализа в абсолютной системе обозначений. Выведены некоторые зависимости для нормальной системы координат.

В третей главе описана деформация оболочки на основе линейного распределения поля перемещений по толщине оболочки. Анализируется представление мер деформации оболочки в Лагранжевом и Эйлеровом описании и приводятся некоторые кинематические соотношения справедливые при применении связей Кирхгоффа-Лява.

Общая теория конечных поворотов в оболочках излагается в четвертой главе. Применяя теорему о полярном разложении деформация окрестности срединной поверхности оболочки разложена на жесткое перемещение, чистое растяжение вдоль главных осей деформации и последующий конечный поворот этих главных осей. Получены точные формулы для тензора поворота и для эквивалентного вектора конечного поворота главных осей деформации, а также выведены многие соотношения выраженные через эти повороты. На основе тщательного анализа деформации боковой поверхности оболочки получены точные формулы для вектора полного конечного поворота граничного злемента оболочки. На основе чисто геометрических рассуждений построены в общем виде геометрические, кинематические и деформационные граничные условия. При дополнительных связях Кирхгоффа-Лява получены многие упрощенные зависимости теории оболочек выраженные через конечные повороты.

Основные уравнения движения в Лагранжевом описании и соответствующие натуральные граничные условия нелинейной теории оболочек построены в пятой главе на основе принципа наименьшего действия. Выведены пять разных но эквивалентных систем уравнений для оболочек, выраженных через Лагранжевые величины. Как частные случаи получены разные виды уравнений равновесия для теории оболочек типа Кирхгоффа-Лява.

В шестой главе рассмотрены возможные упрощения основных уравнений теории упругих оболочек при предположении, что растяжения всюду малы. Получена уточненная формула второго приближения к упругой энергии деформации оболочки. Она состоит из двух главных членов, учитывающих энергию растяжения, изгиба и изменения толщины оболочки, и пяти соответственно выбранных второстепенных членов. На этой основе выведены уточненные определяющие уравнения упругих оболочек по теории второго приближения и соответственно упрощенные соотношения геометрически нелинейной моментной теории оболочек типа Рейсснера. Соответственные соотношения теории оболочек типа Кирхгоффа-Ляба получены как частный случай.

В седьмой главе обсуждены упрощения соотношений геометрически нелинейной теории оболочек при дополнительном ограничении величины вектора конечного поворота. Различаются четыре варианты уравнений теории оболочек при малых, умеренных, больших и конечных поворотах. Для случая умеренных поворотов получены упрошенные соотношения нелинейной теории оболочек, рассмотрены также упрощенные варианты теории типа Кирхгоффа-Лява, а в частности классическая нелинейная теория пологих оболочек.

Полученные в работе результаты поставили некоторые новые проблемы, которые должны быть решены в будущем. Эти проблемы обсуждены в заключительной восьмой главе.