ТОМ 12
VOL
ВЫПУСК 1
NUMBER
1989
УСПЕХИ МЕХАНИКИ
Advances in Mechanics

GEOMETRICALLY NONLINEAR THEORIES OF THIN ELASTIC SHELLS

W. Pietraszkiewicz (Poland)

Contents

1. Introduction

Shell theory attempts to describe the mechanical behaviour of a thin three-dimensional solid layer – the shell – by a finite number of fields defined over its reference (usually middle) surface. Since this is not possible, in general, the shell theory is an approximate one virtually by definition. It cannot provide a complete and exact information about all three-dimensional fields describing the mechanical behaviour of the shell. However, the results which follow from such a two-dimensional approximate description of the shell are usually sufficiently accurate for the majority of applications in science and technology. At the same time, the two-dimensional problem resulting from an appropriate shell theory is much easier to handle than the original three-dimensional one.

This report deals with one of the simplest formulations of the shell theory: the geometrically nonlinear first-approximation theory of thin elastic shells. This theory is applicable when:

a) the shell is made of a homogeneous, isotropic and elastic material;

b) the shell is thin, i.e. $h/R \ll 1$, where h is the constant thickness of the undeformed shell and R is the smallest radius of curvature of its reference surface $\mathscr{M}$;

c) the undeformed reference surface is smooth, i.e. $(h/l)^2 \ll 1$, where l is the smallest wave length of geometric patterns of $\mathscr{M}$;

d) the shell deformation is smooth, i.e. $(h/L)^2 \ll 1$, where L is the smallest wave length of deformation patterns on $\mathscr{M}$;

e) the strains are small everywhere, i.e. $\eta \ll 1$, where η is the largest strain in the shell space.

Under an additional restriction of rotations of material fibres to be also small everywhere, the geometrically nonlinear theory reduces to the classical linear first-approximation theory of shells, which was discussed in detail in many papers and books, for example [135, 296, 79, 83, 175, 275, 75, 39, 158, 85, 228, 26].

Within the assumptions given above, the behaviour of an interior domain of the shell can be described with sufficient accuracy by the behaviour of the shell reference surface. Already Aron [13] approximated the shell strain energy density by a sum of two quadratic functions describing the stretching and the bending of the shell reference surface. Love [135] came to the same conclusion by the application of two well known constraints, analogous to those used by Kirchhoff [111, 112] in the plate theory (cf. Novozhilov [175]). The accuracy of such a so-called Kirchhoff–Love shell theory was examined in a number of papers [177, 79, 80, 113, 118, 50, 123, 212, 26]. In particular, Novozhilov and Finkelshtein [177] and Koiter [113] pointed out explicitly that within the basic assumptions the quadratic expression of Love [135] for the shell strain energy is a consistent first approximation. Moreover [113], various versions of the shell theory, which differ from the version given by Love [135] only by terms of the order of η/R in the definition of the two-dimensional measure of change of curvature, should be regarded as equivalent from the point of view of the first approximation to the shell strain energy. Since the consistently approximated strain energy of the shell is expressed entirely in terms of two-dimensional strain measures of the reference surface, the conclusions are valid both for the linear and for the geometrically nonlinear theory of shells.

Although some geometric results about nonlinear deformation of the shell space had been given already by Love [135, Ch. 24], Donnell [54, 55] and Mushtari [146, 147] seem to be the first who proposed the simplest nonlinear theory for stability analysis of cylindrical shells. Marguerre [144], Mushtari [148, 149] and Vlasov [295] developed the nonlinear theory of shallow shells which was applied with great success to a number of problems of flexible shells analysed for strength, deformability and the loss of stability. In particular, by applying Marguerre's theory Kármán and Tsien [107] discovered that the axial compressive forces applied to a cylindrical shell drop considerably in the post-buckling range of deformation. This differed qualitatively from the behaviour of compressed bars and plates, but was in good agreement with the experimental results for cylindrical shells. Many results obtained with the help of the nonlinear theory of shallow shells have been summarized in the books of Vlasov [296], Volmir [297, 298], Mushtari and Galimov [157], Kornishin [121], Brush and Almroth [35] and Kantor [105] where further references may be found.

The foundations for the general geometrically nonlinear theory of elastic shells were laid down by Chien [44]. He expanded all three-dimensional fields into series of the normal coordinate and applied order-of-magnitude estimates valid under the assumption of small strains. As a result, three equilibrium equations and three compatibility conditions were derived in [44] in an invariant tensor notation, which were then expressed in the intrinsic form, in terms of two-dimensional strains and changes of curvatures of the shell reference surface. Under additional assumptions about the thinness of the shell and the smallness of its curvature, 27 types of approximate versions of intrinsic shell equations were given. It was assumed in [44] that when $h \to 0$ the limits of some functions do not change their order upon the surface differentiation. This assumption was criticized in [77, 85] as to be applicable only to a limited class of shell problems. It was also recognized that only special problems can be formulated and solved directly in the intrinsic form. As a result, the very general approach of [44] has gained little attention in the following papers.

Alternative two-dimensional formulations of the nonlinear theory of shells were given in an invariant tensor notation in the series of papers by Mushtari [150–154], Galimov [62–70] and Alumäe [4–8]. It was assumed there from the outset that the behaviour of the shell can be described with sufficient accuracy by the behaviour of its middle surface. While Mushtari and Galimov presented several forms of shell relations in the natural bases of the undeformed and deformed surface, Alumäe derived his nonlinear shell relations in the intermediate non-holonomic basis, which was obtained from the undeformed basis by its rigid-body rotation. Unfortunately, some of these original results were published in the local journals which even today are hardly available outside the Soviet Union. The monograph by Mushtari and Galimov [157] was written in the classical notation, using the initially orthogonal system of coordinates coinciding with lines of principal curvatures of the undeformed surface. It provided well documented sets of shell relations for the simplified nonlinear theory of medium bending and for the one of shallow shells. However, not all of the general results published in the original works of the authors were presented in their monograph with sufficient generality and accuracy. In the classical notation some intermediate formulae became extremely complex and had to be simplified by omitting some terms which were supposed to be small. This raised some doubts about the consistency and the range of applicability of the final relations of the geometrically nonlinear theory of shell, cf. [115].

Various equivalent forms of nonlinear relations for thin shells were independently rederived and developed further by Rüdiger [211], Leonard [128], Sanders [215], Naghdi and Nordgren [162], Koiter [115, 116], Woźniak [299], Budiansky [36], Simmonds and Danielson [247, 248], Reissner [208] and Pietraszkiewicz [182–185]. In particular, concrete error estimates given by John [101, 103] and Berger [30] for the two-dimensional differential equations of the geometrically nonlinear theory of elastic shells strengthened the foundations of the theory and established more precise bounds of its applicability. Danielson [49] and Koiter and Simmonds [120] worked out the refined intrinsic shell equations which were expressed in terms of internal stress resultants and changes of curvatures as independent field variables (cf. also [185, 190]). Simmonds and Danielson [247, 248] proposed the set of nonlinear shell equations in terms of finite rotation and stress function vectors as independent variables and constructed an appropriate variational principle. Pietraszkiewicz and Szwabowicz [201] derived entirely Lagrangian nonlinear shell equations in terms of displacements as independent variables. In case of dead surface and boundary loadings these equations were derivable as stationarity conditions of the Hu-Washizu functional (cf. also [197]). The theory of finite rotations in shells developed by Pietraszkiewicz [184, 185, 190] allowed then to work out a consistent classification of approximate versions of displacement equations for shells undergoing restricted rotations [195, 197].

Various general theoretical aspects of the nonlinear theory of thin shells are discussed also in the books by Kilchevskii [110], Teregulov [273], Naghdi [159], Galimov [71], Pietraszkiewicz [185, 190], Grigolyuk and Kabanov [89], Mason [145], Woźniak [300], Wempner [291], Dikmen [53], Zubov [304], Berdichevskii [28], Başar and Krätzig [26], Galimov and Paimushin [73], Chernykh [42] and Axelrad [15, 17] as well as in the reviews or extensive papers by Goldenveizer [78], Koiter [117], Mushtari [155, 156], Novozhilov [176], Başar [20], Langhaar [127], Pietraszkiewicz [187, 191, 193], Koiter and Simmonds [120], Woźniak [301], Simmonds [243], Naghdi [160, 161], Schmidt and Pietraszkiewicz [224], Atluri [14], Libai and Simmonds [133], Schmidt [222], Stumpf [263] and Szwabowicz [271], where further references may be found. One-dimensional problems of the nonlinear theory of elastic shells are extensively treated by Shilkrut [237], Shilkrut and Vyrlan [238], Valishvili [286], Antman [10] and, in particular, by Libai and Simmonds [134].

The behaviour of the shell near its lateral boundaries, i.e. in an edge zone of depth of the order of the shell thickness, is nearly always essentially three-dimensional. The physical explanation of this statement is quite simple. The external (or reactive) stresses applied to the shell lateral boundary surface are statically equivalent to the external force and moment resultants on the reference boundary contour plus some self-equilibrated part of the stress distribution over the lateral boundary surface. The resultants enter into the boundary conditions of the basic boundary-value problem which describes correctly the shell behaviour in its interior domain, far from its lateral boundary surfaces. The self-equilibrated part generates additional stresses in the shell space, which are localized in the edge zone. Within the linear shell theory, these additional stresses may be calculated approximately as some linear combinations of solutions of the plane and anti-plane problems for a semi-infinite strip [82, 83] and then may be added to the basic stress state associated with the resultants, (cf. also [91]). An extension of this approximate method, based on a superposition of elementary stress states, to the nonlinear range of deformation may not always be correct, in particular near the stress states associated with the bifurcation or limit points of solutions of the basic boundary value problem. Additionally, the exact stress distribution over the shell lateral boundary surface is rarely known in the majority of engineering problems, except in the case of a free edge. As a result, within the geometrically nonlinear theory of shells, little has been achieved in a better two-dimensional description of the shell behaviour in the edge zone. Some approximate results have been given by Koiter and Simmonds [120] and Novotny [172].

In this report basic relations of the nonlinear theory of thin elastic shells are reviewed. Various consistent sets of nonlinear shell equations in terms of displacements, in terms of rotations and some other field variables as well as in terms of two-dimensional strain and/or stress measures as independent variables are discussed. The final nonlinear relations are then consistently simplified under the assumption that strains are small, while displacement equations are simplified further under consistently restricted rotations. For some types of conservative surface and boundary loadings, appropriate variational functionals are constructed for displacement and rotational nonlinear shell equations.

During preparation of this report it became necessary to clarify some theoretical problems which have not been fully treated in the literature. Among those new results is a discussion of integrability of kinematic boundary conditions, the construction of the general form of the work-conjugate static and geometric boundary conditions for displacement shell equations, alternative derivation of rotation shell equations in the rotated and undeformed basis, the construction of the variational functional in terms of rotations, displacements and Lagrange multipliers as well as an alternative derivation of the refined intrinsic shell equations.

The literature on various aspects of the nonlinear theory of shells is very extensive and some kind of selection of references has to be made. The references in this report are given primarily to those original papers and monographs which deal with general aspects of the theory and are written in the invariant tensor notation.

Other original papers and monographs, which are written in classical notation or which deal with special shell geometries, are referred on the basis of their historical or informative value. Although it is believed that the most important papers, which concern the derivation of various invariant forms of nonlinear shell relations, are included into the list of references, no attempt is made to provide the complete list of such references.

It is worthwhile to point out here once again that some of the two-dimensional relations of the nonlinear theory of thin shells are derived by taking a difference between two groups of terms of the same order associated with the deformed and undeformed reference surface. In the derivation process it often happens that the principal terms of those groups cancel out and the seemingly secondary terms are the only ones which appear in the final shell relations. In the geometrically nonlinear theory of shells, in which strains are assumed to be always small, it is quite dangerous to simplify the intermediate relations by dropping terms of the order of strains relative to the unity, since then the final relations may happen to be inconsistent or even incorrect. This has actually been the case in several early papers devoted to the derivation and simplification of the geometrically nonlinear theory of shells. In this report all two-dimensional relations associated whith the reference surface are derived for unrestricted strains. The small strain assumption is then used at the end of the derivation process to simplify the final set of nonlinear shell relations.

Stability analysis of flexible shells is one of the most important possible applications of the geometrically nonlinear theory of shells discussed in this report. The literature on various approximate versions of the stability equations for thin shells is extensive and has to be reviewed separately. The stability equations are usually derived as a result of superposition of two or more nonlinear deformations of the shell. Since different types of approximation may be used to describe the first (basic) deformation and the following (superposed) deformations, a large variety of types of shell stability equations for thin elastic shells may be constructed. We only note here that problems of superposition of deformations and derivation of various types of stability equations have been discussed, among others, by Novozhilov [173], Koiter [114], Mushtari [153], Alumäe [5, 6], Mushtari and Galimov [157], Timoshenko and Gere [278], Darevskii [52], Volmir [298], Bolotin [32, 33], Koiter [116], Budiansky [36], Danielson and Simmonds [51], Seide [229], Abé [1], Başar [21], Brush and Almroth [35], Zubov [303], Grigolyuk and Kabanov [89], Talaslidis [272], van der Heijden [92], Stumpf [257–263], Srubshchik [250], Başar and Krätzig [22, 26], Krätzig et al. [124, 125], Stein et al. [252, 253], Eckstein [59], Arbocz [11], Nolte [164, 165], Schmidt and Stumpf [225] and Pietraszkiewicz [197], where further references are given.

It is not the aim of this report to review recent achievements in the large-strain nonlinear theory of thin shells. Suffice it to point out that many two-dimensional relations collected here are applicable also to this more general case of shell deformation, provided that the behaviour of the shell is still approximated only by the behaviour of its reference surface. Such simple versions of the large-strain K–L type theories of shells have been proposed recently by Chernykh [40, 42], Simmonds [246] as well as by Stumpf and Makowski [264]. However, the change of shell thickness during deformation should then be taken explicitly into account not only in the approximate form of the strain energy density but also in definitions of the external force and couple resultants applied to the shell reference surface and on its boundary contour. Besides this review there are also the more advanced models of shells in which the behaviour of the shell is described not only by the behaviour of its reference surface but also by additional higher-order independent parameters. We share the view expressed by Koiter and Simmonds [120] that the rapid development of numerical techniques in three-dimensional problems, in particular the finite element technique, may obviate the need of (complicated) refined shell theories in the near future.

2. Notation and geometric relations

The notation which will be used in this report follows that of Koiter [115] and Pietraszkiewicz [185, 190, 193, 197]. In order to make the paper self-contained, we review here the notation and some basic geometric relations of the surface and its nonlinear deformation.

Let $\mathscr{P}$ be the region of the three-dimensional Euclidean space $\mathscr{E}$ occupied by the shell in its undeformed configuration. In $\mathscr{P}$ we introduce the normal system of curvilinear coordinates $(\theta^1, \theta^2, \zeta)$ such that $-h/2 \leqslant \zeta \leqslant h/2$ is the distance from the middle surface $\mathscr{M}$ of $\mathscr{P}$ and h is the undeformed shell thickness assumed here to be constant and small as compared to the smallest radius of curvature R of $\mathscr{M}$ and to the linear dimensions of $\mathscr{P}$.

The surface $\mathscr{M}$ is described by the position vector $\mathbf{r} = x^k(\theta^\alpha)\mathbf{i}_k$, $k = 1, 2, 3$, $\alpha = 1, 2$, where $\mathbf{i}_k$ is an orthonormal basis attached to a point $O \in \mathscr{E}$. With each point $M \in \mathscr{M}$ we associate the natural covariant base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r}_{,\alpha}$, the covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the surface metric tensor $\mathbf{a}$ with the determinant $a = |a_{\alpha\beta}|$, the contravariant components $\varepsilon^{\alpha\beta}$ of the permutation tensor such that $\varepsilon^{12} = -\varepsilon^{21} = 1/\sqrt{a}$, $\varepsilon^{11} = \varepsilon^{22} = 0$, the unit normal vector $\mathbf{n} = \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$ and the covariant components $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ of the curvature tensor $\mathbf{b}$. Contravariant components $a^{\alpha\beta}$ of $\mathbf{a}$, satisfying the relations $a^{\alpha\gamma}a_{\beta\gamma} = \delta^\alpha_\beta$, where $\delta^1_1 = \delta^2_2 = 1$, $\delta^1_2 = \delta^2_1 = 0$, are used to raise indices of the surface vectors and tensors, for example $\mathbf{a}^\alpha = a^{\alpha\beta}\mathbf{a}_\beta$, $b^\alpha_\beta = a^{\alpha\gamma}b_{\gamma\beta}$, etc.

The boundary contour $\mathscr{C}$ of $\mathscr{M}$ consists of the finite set of piecewise smooth curves given by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along $\mathscr{C}$. With each regular point $M \in \mathscr{C}$ we associate the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds \equiv \mathbf{r}' = t^\alpha\mathbf{a}_\alpha$ and the outward unit normal vector $\mathbf{v} = \partial\mathbf{r}/\partial s_{v|\mathscr{C}} \equiv \mathbf{r}_{,v} = \mathbf{t} \times \mathbf{n} = v^\alpha\mathbf{a}_\alpha$, $v^\alpha = \varepsilon^{\alpha\beta}t_\beta$, where s_v is the arc length of the coordinate line on $\mathscr{M}$ which is orthogonal to $\mathscr{C}$. The curvature properties of $\mathscr{C}$ are described by the normal curvature $\sigma_t = b_{\alpha\beta}t^\alpha t^\beta$, the geodesic torsion $\tau_t = -b_{\alpha\beta}\nu^\alpha t^\beta$ and the geodesic curvature $\varkappa_t = t_\alpha \nu^\alpha|_\beta t^\beta$, where $(\ )|_\alpha$ denotes the covariant surface derivative on $\mathscr{M}$. For other geometric definitions and relations on $\mathscr{M}$ and $\mathscr{C}$ we refer to Pietraszkiewicz [185, 190, 193].

The deformed configuration $\bar{\mathscr{M}}$ of the surface $\mathscr{M}$ is described by the position vector relative to the same Cartesian frame

$$\bar{\mathbf{r}} = \bar{x}^k(\theta^\alpha)\mathbf{i}_k = \chi(\mathbf{r}) = \mathbf{r}+\mathbf{u}, \tag{2.1}$$

where θ^α are the same surface curvilinear convected (material) coordinates and $\mathbf{u} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}$ is the displacement field. Geometric quantities and relations, which may be analogously defined on $\bar{\mathscr{M}}$ and $\bar{\mathscr{C}}$ on a point $\bar{M}$ with the same values of θ^α or s, will be marked by an additional overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{\varepsilon}^{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\mathbf{b}}$, $\bar{a}^{\alpha\beta}$, $(\ )_{\|\alpha}$, $\bar{s}$, $\bar{\mathbf{t}}$, $\bar{t}^\alpha$, $\bar{\boldsymbol{\nu}}$, $\bar{\nu}^\alpha$, $\bar{\sigma}_t$, $\bar{\tau}_t$, $\bar{\varkappa}_t$ etc.

For the base vectors on $\bar{\mathscr{M}}$ the following relations hold

$$\begin{aligned}\bar{\mathbf{a}}_\alpha &= \mathbf{G}\mathbf{a}_\alpha = l^\lambda_{\cdot\alpha}\mathbf{a}_\lambda + \varphi_\alpha \mathbf{n},\\ \bar{\mathbf{n}} &= \mathbf{G}\mathbf{n} = n^\lambda \mathbf{a}_\lambda + n\mathbf{n},\end{aligned} \tag{2.2}$$

where

$$\begin{aligned} l_{\alpha\beta} &= a_{\alpha\beta}+\vartheta_{\alpha\beta}-\omega_{\alpha\beta}, \qquad \varphi_\alpha = w_{,\alpha}+b^\lambda_\alpha u_\lambda,\\ \vartheta_{\alpha\beta} &= \frac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}w, \qquad \omega_{\alpha\beta} = \frac{1}{2}(u_{\beta|\alpha}-u_{\alpha|\beta}),\end{aligned} \tag{2.3}$$

$$\begin{aligned} n_\alpha &= \frac{1}{d}m_\alpha, \qquad n = \frac{1}{d}m, \qquad d = \sqrt{\frac{\bar{a}}{a}},\\ m_\mu &= \varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\varphi_\alpha l^\lambda_{\cdot\beta}, \qquad m = \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}l^\lambda_{\cdot\alpha}l^\mu_{\cdot\beta},\end{aligned} \tag{2.4}$$

$$\mathbf{G} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}}\otimes\mathbf{n}, \qquad \mathbf{G}^{-1} = \mathbf{a}_\alpha\otimes\bar{\mathbf{a}}^\alpha + \mathbf{n}\otimes\bar{\mathbf{n}}. \tag{2.5}$$

Here $\mathbf{G} \equiv \partial\chi/\partial\mathbf{r}$ is the deformation gradient tensor of the surface $\mathscr{M}$ while $\otimes$ is the tensor product.

The Lagrangian surface strain tensor $\boldsymbol{\gamma}$ and the tensor of change of surface curvature $\boldsymbol{\varkappa}$ are defined by

$$\begin{aligned}\boldsymbol{\gamma} &= \frac{1}{2}(\mathbf{G}^T\mathbf{G}-\mathbf{1}) = \gamma_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta,\\ \boldsymbol{\varkappa} &= -(\mathbf{G}^T\bar{\mathbf{b}}\mathbf{G}-\mathbf{b}) = \varkappa_{\alpha\beta}\mathbf{a}^\alpha\otimes\mathbf{a}^\beta,\end{aligned} \tag{2.6}$$

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta}) = \frac{1}{2}(\bar{\mathbf{r}}_{,\alpha}\cdot\bar{\mathbf{r}}_{,\beta}-a_{\alpha\beta}) = \frac{1}{2}(l^\lambda_{\cdot\alpha}l_{\lambda\beta}+\varphi_\alpha\varphi_\beta-a_{\alpha\beta}), \tag{2.7}$$

$$\varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{n}}_{,\beta} + b_{\alpha\beta} = -\bar{\mathbf{a}}_{\alpha,\beta} \cdot \bar{\mathbf{n}} + b_{\alpha\beta}, \tag{2.8}$$

$$= l_{\lambda\alpha}(n^\lambda|_\beta - b^\lambda_\beta n) + \varphi_\alpha(n_{,\beta} + b^\lambda_\beta n_\lambda) + b_{\alpha\beta}, \tag{2.9}$$

$$= -n(\varphi_{\alpha|\beta} + b^\lambda_\beta l_{\lambda\alpha}) - n_\lambda(l^\lambda_{\cdot\alpha|\beta} - b^\lambda_\beta \varphi_\alpha) + b_{\alpha\beta}, \tag{2.10}$$

where $\mathbf{1} \equiv \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha + \mathbf{n} \otimes \mathbf{n}$ is the metric tensor of the Euclidean space $\mathscr{E}$.

According to [64], the strain-displacement relations (2.7) and (2.10) were given first by Mushtari [150]. They were applied, among others, in the papers [64, 215, 115, 183, 185, 291, 36, 71, 72]. The importance of an equivalent representation (2.9) for $\varkappa_{\alpha\beta}$ was recognized only recently by Pietraszkiewicz and Szwabowicz [201] and was applied, for example, in [197, 198, 267, 271, 218–222, 225, 164, 165, 262, 263].

Note that $\gamma_{\alpha\beta}$ are quadratic polynomials of u_α, w and their first surface derivatives while $\varkappa_{\alpha\beta}$ are non-rational functions of u_α, w and their first as well as the second surface derivatives. The non-rationality is caused by the presence of the invariant d in the definitions of n_α and n appearing in Eqs. (2.9) and (2.10) where

$$d^2 = \frac{\bar{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2(\gamma^\alpha_\alpha \gamma^\beta_\beta - \gamma^\alpha_\beta \gamma^\beta_\alpha). \tag{2.11}$$

The components of the Lagrangian surface strain measures should satisfy the compatibility conditions originally derived by Chien [44] and rederived by Galimov [63, 65] (with the sign error) and Koiter [115]. We present them in the form given in [185]

$$\begin{gathered} \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\varkappa_{\beta\lambda|\mu} + \bar{a}^{\varkappa\nu}(b_{\varkappa\lambda} - \varkappa_{\varkappa\lambda})\gamma_{\nu\beta\mu}] = 0, \\ K\gamma^\varkappa_\varkappa + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\varkappa_{\beta\lambda} + \frac{1}{2}(\varkappa_{\alpha\mu}\varkappa_{\beta\lambda} + \bar{a}^{\varkappa\lambda}\gamma_{\varkappa\alpha\mu}\gamma_{\nu\beta\lambda})] = 0, \end{gathered} \tag{2.12}$$

where $K = |b^\alpha_\beta| = \det \mathbf{b}$ is the Gaussian curvature of $\mathscr{M}$ and

$$\gamma_{\nu\beta\mu} = \gamma_{\nu\beta|\mu} + \gamma_{\nu\mu|\beta} - \gamma_{\beta\mu|\nu}. \tag{2.13}$$

An alternative form of Eq. $(2.12)_2$ is given in [270].

The deformation of the shell lateral boundary element may be described by two vectors:

$$\begin{gathered} \mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} = u_\nu \mathbf{v} + u_t \mathbf{t} + w\mathbf{n}, \\ \boldsymbol{\beta} = \bar{\mathbf{n}} - \mathbf{n} = n_\nu \mathbf{v} + n_t \mathbf{t} + (n-1)\mathbf{n}, \end{gathered} \tag{2.14}$$

which are subjected to two geometric constraints:

$$\bar{\mathbf{n}} \cdot \bar{\mathbf{r}}' = 0, \quad \bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = 1. \tag{2.15}$$

These constraints imply that among six components of $\bar{\mathbf{r}}$ (or $\mathbf{u}$) and $\bar{\mathbf{n}}$ (or $\boldsymbol{\beta}$) on $\mathscr{C}$, only four are independent: three components of $\bar{\mathbf{r}}$ (or $\mathbf{u}$), which determine the translation of the boundary contour $\mathscr{C}$, and a scalar function φ which describes the rotational deformation of the shell lateral boundary element. Since the rotational deformation may be described by various means, also various definitions of φ may be used in the nonlinear theory of shells.

If φ is identified with $n_\nu = \bar{\mathbf{n}}\cdot\mathbf{v}$, then $\bar{\mathbf{n}}$ can be expressed entirely in terms of $\mathbf{u}$, $\mathbf{u}'$ and n_ν, [201]

$$\bar{\mathbf{n}} = \frac{1}{c_t^2+c^2}\left[n_\nu \bar{\mathbf{a}}_t \times (\mathbf{v}\times\bar{\mathbf{a}}_t) + \sqrt{\bar{a}_t^2(1-n_\nu^2)-c_\nu^2}\,\mathbf{v}\times\bar{\mathbf{a}}_t\right], \tag{2.16}$$

where

$$\begin{gathered}\bar{\mathbf{a}}_t \equiv \bar{\mathbf{r}}' = \mathbf{t}+\mathbf{u}' = c_\nu\mathbf{v}+c_t\mathbf{t}+c\mathbf{n}, \qquad c_\nu = u_\nu'+\tau_t w-\varkappa_t u_t,\\ c_t = 1+u_t'+\varkappa_t u_\nu-\sigma_t w, \qquad c = w'+\sigma_t u_t-\tau_t u_\nu,\\ \bar{a}_t = |\bar{\mathbf{a}}_t| = \sqrt{1+2\gamma_{tt}}, \qquad 2\gamma_{tt} = 2\gamma_{\alpha\beta}t^\alpha t^\beta = (\bar{\mathbf{r}}')^2.\end{gathered} \tag{2.17}$$

The relation (2.16) is valid when the rotation of the boundary element does not exceed $\pm\pi/2$. For larger rotations the sign in front of the square root is not unique and may change.

An equivalent description of $\bar{\mathbf{n}}$ in terms of displacement derivatives at $\mathscr{C}$ is given by

$$\bar{\mathbf{n}} = \frac{1}{d}\bar{\mathbf{r}}_{,\nu}\times\bar{\mathbf{r}}', \tag{2.18}$$

$$\begin{gathered}\bar{\mathbf{r}}_{,\nu} = \mathbf{v}+\mathbf{u}_{,\nu} = \bar{\mathbf{a}}_\alpha \nu^\alpha = \frac{1}{\bar{a}_t}(d\bar{\mathbf{v}}+2\gamma_{\nu t}\bar{\mathbf{t}}),\\ d^2 = (\bar{\mathbf{r}}_{,\nu})^2(\bar{\mathbf{r}}')^2-(\bar{\mathbf{r}}_{,\nu}\cdot\bar{\mathbf{r}}')^2,\\ 2\gamma_{\nu t} = 2\gamma_{\alpha\beta}\nu^\alpha t^\beta = \bar{\mathbf{r}}_{,\nu}\cdot\bar{\mathbf{r}}'.\end{gathered} \tag{2.19}$$

Note that $\bar{\mathbf{r}}_{,\nu}$ is not orthogonal to $\bar{\mathscr{C}}$ due to the shear distortion of the surface during deformation.

In what follows we shall use the following transformation

$$\bar{\mathbf{a}}_\alpha = \frac{1}{\bar{a}_t}(d\nu_\alpha\bar{\mathbf{v}}+\bar{a}_{\alpha\beta}t^\beta\bar{\mathbf{t}}) \tag{2.20}$$

which holds at the deformed boundary contour $\bar{\mathscr{C}}$.

3. Basic forms of shell equations

The two-dimensional equilibrium equations and the appropriate natural static boundary conditions for the nonlinear K–L type theory of shells may be derived in several ways. The usual way is to integrate the corresponding three-dimensional relations of a continua over the shell thickness. This leads to six equilibrium equations and six static boundary conditions, expressed in terms of two-dimensional non-symmetric internal force and couple resultants and the shearing forces. Additional transformations allow then to reduce the relations to three equilibrium equations and four static boundary conditions, which are expressed in terms of two-dimensional symmetric internal force and couple resultants. An alternative direct way is to postulate the two-dimensional virtual work principle compatible with the basic assumptions of the shell theory, from which follow at once the same three equilibrium equations and four static boundary conditions. Internal force and couple resultants are symmetric here by definition, since they appear as coefficients of the symmetric virtual surface strain measures in the invariant virtual work expression. In this report we shall apply the second direct approach, since it leads directly to the final shell equations.

A clear distinction should be made between the set of shell equations written in the Eulerian description and the one written in the Lagrangian description. In the Lagrangian description, all quantities and equations are referred to the known, natural basis of the undeformed reference surface. In the Eulerian description, they are referred to the natural basis of the deformed surface, the geometry of which is not known in advance. If transformation formulae between the deformed and undeformed surface are used to express components of the Eulerian quantities in terms of corresponding Lagrangian ones, then the Eulerian shell equations can be presented in the so called mixed form.

3.1. Eulerian shell equations

Let $\bar{\mathcal{M}}$ be the reference surface of a thin shell in an equilibrium state, under the surface force $\bar{\mathbf{p}} = \bar{p}^\alpha \bar{\mathbf{a}}_\alpha + \bar{p}\bar{\mathbf{n}}$ and the surface static moment $\bar{\mathbf{h}} = \bar{h}^\alpha \bar{\mathbf{a}}_\alpha + \bar{h}\bar{\mathbf{n}}$, both per unit area of $\bar{\mathcal{M}}$, as well as under the boundary force $\bar{\mathbf{T}} = \bar{T}_\nu \bar{\mathbf{v}} + \bar{T}_t \bar{\mathbf{t}} + \bar{T}\bar{\mathbf{n}}$ and the boundary static moment $\bar{\mathbf{H}} = \bar{H}_\nu \bar{\mathbf{v}} + \bar{H}_t \bar{\mathbf{t}} + \bar{H}\bar{\mathbf{n}}$, both per unit length of $\bar{\mathscr{C}}$. For an additional virtual displacement field $\delta\bar{\mathbf{u}} = \delta\bar{u}_\alpha \bar{\mathbf{a}}^\alpha + \delta\bar{w}\bar{\mathbf{n}}$, which is subjected to geometric constraints, the internal virtual work, performed by the internal stress and couple resultants on virtual strain measures, is equal to the external virtual work, performed by the external surface and boundary loads on appropriate virtual displacement parameters:

$$\iint_{\bar{\mathcal{M}}} (\bar{\mathbf{N}}\cdot\delta\bar{\boldsymbol{\gamma}} + \bar{\mathbf{M}}\cdot\delta\bar{\boldsymbol{\varkappa}})d\bar{A} = \int_{\bar{\mathscr{C}}} (\bar{\mathbf{p}}\cdot\delta\bar{\mathbf{u}} + \bar{\mathbf{h}}\cdot\delta\bar{\boldsymbol{\beta}})d\bar{A} + \int_{\bar{\mathscr{C}}_f} (\bar{\mathbf{T}}\cdot\delta\bar{\mathbf{u}} + \bar{\mathbf{H}}\cdot\delta\bar{\boldsymbol{\beta}})d\bar{s}, \tag{3.1}$$

where $\bar{\mathbf{N}} = \bar{N}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\beta$, $\bar{\mathbf{M}} = \bar{M}^{\alpha\beta}\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\beta$ are symmetric (Cauchy type) internal stress and couple resultant tensors and

$$\begin{aligned}
\delta\bar{\boldsymbol{\gamma}} &= \left[\frac{1}{2}(\delta\bar{u}_{\alpha\|\beta} + \delta\bar{u}_{\beta\|\alpha}) - \bar{b}_{\alpha\beta}\delta\bar{w}\right]\bar{\mathbf{a}}^\alpha \otimes \bar{\mathbf{a}}^\beta, \\
\delta\bar{\boldsymbol{\varkappa}} &= [-\delta\bar{w}_{\|\alpha\beta} - \bar{b}^\lambda_\alpha \delta\bar{u}_{\lambda\|\beta} - \bar{b}^\lambda_\beta \delta\bar{u}_{\lambda\|\alpha} - \bar{b}^\lambda_{\alpha\|\beta}\delta\bar{u}_\lambda + \bar{b}^\lambda_\alpha \bar{b}_{\lambda\beta}\delta\bar{w}]\bar{\mathbf{a}}^\alpha \otimes \bar{\mathbf{a}}^\beta, \\
\delta\bar{\boldsymbol{\beta}} &= -(\delta\bar{w}_{,\alpha} + \bar{b}^\lambda_\alpha \delta\bar{u}_\lambda)\bar{\mathbf{a}}^\alpha.
\end{aligned} \tag{3.2}$$

After elementary transformations, Eq. (3.1) takes the form

$$(3.3)\quad \begin{aligned} &-\iint_{\bar{\mathcal{M}}}(\bar{\mathbf{N}}^{\beta}\|_{\beta}+\bar{\mathbf{p}})\cdot\delta\bar{\mathbf{u}}\,d\bar{A}+\int_{\bar{\mathscr{C}}_f}[(\bar{\mathbf{P}}-\bar{\mathbf{P}}^*)\cdot\delta\bar{\mathbf{u}}+(\bar{M}-\bar{M}^*)\delta\bar{\beta}_{\nu}]\,d\bar{s}+\sum_j(\bar{\mathbf{F}}_j-\bar{\mathbf{F}}_j^*)\cdot\delta\bar{\mathbf{u}}_j=0,\\ &\bar{\mathbf{N}}^{\beta}=(\bar{N}^{\alpha\beta}-\bar{b}^{\alpha}_{\lambda}\bar{M}^{\lambda\beta})\bar{\mathbf{a}}_{\alpha}+(\bar{M}^{\alpha\beta}\|_{\alpha}+\bar{h}^{\beta})\bar{\mathbf{n}}, \end{aligned}$$

$$(3.4)\quad \begin{aligned} &\bar{\mathbf{P}}=\bar{\mathbf{N}}^{\beta}\bar{\nu}_{\beta}+\frac{d}{d\bar{s}}\bar{\mathbf{F}},\qquad \bar{\mathbf{P}}^*=\bar{\mathbf{T}}+\frac{d}{d\bar{s}}\bar{\mathbf{F}}^*,\\ &\bar{M}=\bar{M}^{\alpha\beta}\bar{\nu}_{\alpha}\bar{\nu}_{\beta},\qquad \bar{M}^*=\bar{H}_{\nu},\\ &\bar{\mathbf{F}}=\bar{M}^{\alpha\beta}\bar{\nu}_{\alpha}\bar{t}_{\beta}\bar{\mathbf{n}},\qquad \bar{\mathbf{F}}^*=\bar{H}_t\bar{\mathbf{n}},\\ &\bar{\mathbf{F}}_j=\bar{\mathbf{F}}(\bar{s}_j+0)-\bar{\mathbf{F}}(\bar{s}_j-0),\qquad \delta\bar{\beta}_{\nu}=\delta\bar{\boldsymbol{\beta}}\cdot\bar{\boldsymbol{\nu}},\qquad \delta\bar{\mathbf{u}}_j=\delta\bar{\mathbf{u}}(\bar{s}_j). \end{aligned}$$

For arbitrary $\delta\bar{\mathbf{u}}$ on $\bar{\mathcal{M}}$ and $\delta\bar{\mathbf{u}}$, $\delta\bar{\beta}_{\nu}$ and $\delta\bar{\mathbf{u}}_j$ on $\bar{\mathscr{C}}_f$ from Eq. (3.3) follow Eulerian equilibrium equations and corresponding static boundary conditions for the free edge [197]

$$(3.5)\quad \begin{aligned} &\bar{\mathbf{N}}^{\beta}\|_{\beta}+\bar{\mathbf{p}}=\mathbf{0}\quad\text{in } \bar{\mathcal{M}},\\ &\bar{\mathbf{P}}=\bar{\mathbf{P}}^*,\qquad \bar{M}=\bar{M}^*\quad\text{on } \bar{\mathscr{C}}_f,\\ &\bar{\mathbf{F}}_j=\bar{\mathbf{F}}_j^*\ \text{at each corner } \bar{M}_j\in\bar{\mathscr{C}}_f. \end{aligned}$$

The virtual rotation $\delta\bar{\beta}_{\nu}$ on $\bar{\mathscr{C}}_f$, appearing in the boundary line integral of Eq. (3.3), may also be given in alternative but equivalent forms

$$(3.6)\quad \begin{aligned} \delta\bar{\beta}_{\nu}&=\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}=-\bar{\mathbf{n}}\cdot\delta\bar{\boldsymbol{\nu}},\\ &=-(\delta\bar{w}_{,\alpha}+\bar{b}^{\lambda}_{\alpha}\delta\bar{u}_{\lambda})\bar{\nu}^{\alpha}=-(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{u}})_{,\bar{\nu}}+\bar{\mathbf{n}}_{,\bar{\nu}}\cdot\delta\bar{\mathbf{u}},\\ &=-\bar{\mathbf{n}}\cdot(\delta\bar{\mathbf{u}})_{,\bar{\nu}}. \end{aligned}$$

Here $(\ )_{,\bar{\nu}}\equiv\partial(\ )/\partial\bar{x}_{\nu|\bar{\mathscr{C}}}$ such that $\bar{\mathbf{r}}_{,\bar{\nu}}=\bar{\boldsymbol{\nu}}$ on $\bar{\mathscr{C}}$, where $\bar{x}_{\nu}$ is the arc length of the coordinate line of $\bar{\mathcal{M}}$ which is orthogonal to $\bar{\mathscr{C}}$.

Using Eq. $(3.6)_2$, the line integral of Eq. (3.3) may also be transformed into the alternative form

$$(3.7)\quad \int_{\bar{\mathscr{C}}_f}[(\bar{\mathbf{P}}_1-\bar{\mathbf{P}}_1^*)\cdot\delta\bar{\mathbf{u}}-(\bar{M}-\bar{M}^*)(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{u}})_{,\bar{\nu}}]\,d\bar{s},$$

$$(3.8)\quad \begin{aligned} &\bar{\mathbf{P}}_1=\bar{\mathbf{N}}^{\beta}\bar{\nu}_{\beta}+\frac{d}{d\bar{s}}\bar{\mathbf{F}}+\bar{M}\bar{\mathbf{n}}_{,\bar{\nu}},\\ &\bar{\mathbf{P}}_1^*=\bar{\mathbf{T}}+\frac{d}{d\bar{s}}\bar{\mathbf{F}}^*+\bar{M}^*\bar{\mathbf{n}}_{,\bar{\nu}}, \end{aligned}$$

which leads to a modified static boundary condition $\bar{\mathbf{P}}_1=\bar{\mathbf{P}}_1^*$ on $\bar{\mathscr{C}}_f$ in Eq. $(3.5)_2$.

The tensor form of the Eulerian equilibrium equations, but expressed in terms of non-symmetric stress and couple resultant tensors, was first given independently by Lurie [136] and by Synge and Chien [266] while Galimov [63] derived static boundary conditions for the smooth $\bar{\mathscr{C}}_f$. In terms of symmetric stress and couple resultant tensors and for smooth $\bar{\mathscr{C}}_f$, the relations Eqs. $(3.5)_{1,2}$ were first given by Galimov [64] (cf. also [71]). The equilibrium equations $(3.5)_1$ were rederived also by Sanders [215] and Koiter [115]. The final form of Eqs. (3.5) without $\bar{\mathbf{h}}$ was given by the author [185]. The modified static boundary conditions on $\bar{\mathscr{C}}_f$ resulting from the relation (3.7) were given first by Koiter [115] and rederived by Zubov [304]. It was noted already by Lurie [136] that the structure of Eulerian shell equations (3.5) is exactly the same as the one of the classical linear theory of shells, only all the quantities are referred now to the geometry of the deformed reference surface $\bar{\mathscr{M}}$ and of its boundary contour $\bar{\mathscr{C}}$.

As it has been mentioned above, the geometry of $\bar{\mathscr{M}}$ and $\bar{\mathscr{C}}$ is usually not known in advance and should be determined as an outcome of the solution of the nonlinear shell problem. As a result, the simple Eulerian shell equations (3.5) can not be used directly to analyse the shell problems, but they can serve as the basis for deriving of other mixed forms of shell equations. The virtual displacement parameters $\delta\bar{u}_\alpha = \bar{\mathbf{a}}_\alpha \cdot \delta\bar{\mathbf{u}}$, $\delta\bar{w} = \bar{\mathbf{n}} \cdot \delta\bar{\mathbf{u}}$ and $\delta\bar{\beta}_\nu = \bar{\mathbf{v}} \cdot \delta\boldsymbol{\beta}$ should not be identified here with variations of displacement and rotation components, since the respective bases $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ and $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$ of $\bar{\mathscr{M}}$ and $\bar{\mathscr{C}}$ are themselves subjected to the variation. In particular, $\delta\bar{\beta}_\nu$ should not be identified with the variation of $\bar{\mathbf{v}} \cdot \boldsymbol{\beta}$. This is the reason why no work-conjugate geometric boundary conditions expressed in terms of displacement parameters can be associated with the Eulerian shell equations (3.5).

3.2. Lagrangian shell equations

Usually only the undeformed configuration of the shell is the one which is known in advance, while the deformed configuration is the one which should be determined in the process of solution. Therefore it is desirable to construct the equilibrium equations and corresponding boundary and corner conditions which are expressed entirely in the geometry of $\mathscr{M}$ and $\mathscr{C}$. Such Lagrangian shell equations can be derived with the help of transformation rules between deformed and undeformed surface geometries [185]

$$d\bar{A} = \sqrt{\frac{\bar{a}}{a}}\, dA, \qquad d\bar{s} = \bar{a}_t\, ds,$$

$$\bar{\nu}_\beta\, d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, \nu_\beta\, ds, \qquad \bar{t}_\beta\, d\bar{s} = (\delta^\alpha_\beta + 2\gamma^\alpha_\beta)\, t_\alpha\, ds, \tag{3.9}$$

$$\bar{\nu}^\beta\, d\bar{s} = \sqrt{\frac{a}{\bar{a}}}\,(\delta^\beta_\alpha + 2\varepsilon_{\alpha\lambda}\varepsilon^{\beta\mu}\gamma^\lambda_\mu)\, \nu^\alpha\, ds, \qquad \bar{t}^\beta\, d\bar{s} = t^\beta\, ds.$$

Let us introduce the symmetric (2nd Piola–Kirchhoff type) internal stress and couple resultant tensors $\mathbf{N} = N^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$, $\mathbf{M} = M^{\alpha\beta}\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$, the Lagrangian surface force $\mathbf{p} = p^\alpha \mathbf{a}_\alpha + p\mathbf{n}$ and the surface static moment $\mathbf{h} = h^\alpha \mathbf{a}_\alpha + h\mathbf{n}$, both per unit area of $\mathcal{M}$, as well as the Lagrangian boundary force $\mathbf{T} = T_\nu \mathbf{v} + T_t \mathbf{t} + T\mathbf{n}$ and the boundary static moment $\mathbf{H} = H_\nu \mathbf{v} + H_t \mathbf{t} + H\mathbf{n}$, both per unit length of $\mathcal{C}$, by the following relations

$$
\bar{\mathbf{N}} = \frac{1}{d}\mathbf{G}\mathbf{N}\mathbf{G}^T, \quad \bar{\mathbf{M}} = \frac{1}{d}\mathbf{G}\mathbf{M}\mathbf{G}^T,
$$

(3.10)

$$
\bar{\mathbf{p}} = \frac{1}{d}\mathbf{p}, \quad \bar{\mathbf{h}} = \frac{1}{d}\mathbf{h}, \quad \bar{\mathbf{T}} = \frac{1}{\bar{a}_t}\mathbf{T}, \quad \bar{\mathbf{H}} = \frac{1}{\bar{a}_t}\mathbf{H}.
$$

Let us also note that the virtual strain measures in Eq. (3.1) are transformed according to

$$
\delta\bar{\boldsymbol{\gamma}} = \mathbf{G}^{-T}\delta\boldsymbol{\gamma}\mathbf{G}^{-1}, \quad \delta\bar{\boldsymbol{\varkappa}} = \mathbf{G}^{-T}\delta\boldsymbol{\varkappa}\mathbf{G}^{-1}. \tag{3.11}
$$

With the help of Eqs. (3.9), (3.10) and (3.11), the principle of virtual work (3.1) is transformed into the Lagrangian principle of virtual displacements:

$$
\iint_{\mathcal{M}} (\mathbf{N}\cdot\delta\boldsymbol{\gamma} + \mathbf{M}\cdot\delta\boldsymbol{\varkappa})dA = \iint_{\mathcal{M}} (\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{h}\cdot\delta\boldsymbol{\beta})dA + \int_{\mathcal{C}_f} (\mathbf{T}\cdot\delta\mathbf{u} + \mathbf{H}\cdot\delta\boldsymbol{\beta})ds, \tag{3.12}
$$

where now

(3.13)

$$
\delta\mathbf{u} = \delta u_\alpha \mathbf{a}^\alpha + \delta w\mathbf{n}, \quad \delta\boldsymbol{\beta} = \delta n_\nu \mathbf{v} + \delta n_t \mathbf{t} + \delta n\mathbf{n},
$$

$$
\delta\boldsymbol{\gamma} = \delta\gamma_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta, \quad \delta\boldsymbol{\varkappa} = \delta\varkappa_{\alpha\beta}\mathbf{a}^\alpha \otimes \mathbf{a}^\beta,
$$

(3.14)

$$
\delta\gamma_{\alpha\beta} = \frac{1}{2}(\bar{\mathbf{a}}_\alpha\cdot\delta\mathbf{u}_{,\beta} + \bar{\mathbf{a}}_\beta\cdot\delta\mathbf{u}_{,\alpha}),
$$

$$
\delta\varkappa_{\alpha\beta} = \frac{1}{2}(\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \bar{\mathbf{n}}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha} + \bar{\mathbf{a}}_\alpha\cdot\delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_\beta\cdot\delta\bar{\mathbf{n}}_{,\alpha})
$$

are variations of the displacement and strain measures, since the bases $\mathbf{a}_\alpha$, $\mathbf{n}$ and $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$ are fixed and not subjected to the variation during shell deformation.

After involved transformations given in [197] and taking into account that $\delta\boldsymbol{\beta} = \delta\bar{\mathbf{n}} = -(\bar{\mathbf{a}}^\beta \otimes \delta\mathbf{u}_{,\beta})\bar{\mathbf{n}}$ in $\mathcal{M}$, the principle (3.12) can be transformed into

$$
-\iint_{\mathcal{M}} (\mathbf{T}^\beta|_\beta + \mathbf{p})\cdot\delta\mathbf{u}dA + \int_{\mathcal{C}_f} [(\mathbf{P}-\mathbf{P}^*)\cdot\delta\mathbf{u} + (M-M^*)\delta n_\nu]ds + \sum_j (\mathbf{F}_j - \mathbf{F}_j^*)\cdot\delta\mathbf{u}_j = 0, \tag{3.15}
$$

where now

$$
\mathbf{T}^\beta = T^{\lambda\beta}\mathbf{a}_\lambda + T^\beta\mathbf{n} + (\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}} = N^{\alpha\beta}\bar{\mathbf{a}}_\alpha + M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha} + [(M^{\varkappa\varrho}\bar{\mathbf{a}}_\varkappa)|_\varrho\cdot\bar{\mathbf{a}}^\beta]\bar{\mathbf{n}} + (\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}, \tag{3.16}
$$

$$
\mathbf{P} = \mathbf{T}^\beta \nu_\beta + \mathbf{F}', \quad \mathbf{P}^* = \mathbf{T} + \mathbf{F}^{*\prime},
$$

$$
\mathbf{F} = -\frac{1}{a_\nu}[(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\mathbf{v}]M^{\alpha\beta}\nu_\beta\bar{\mathbf{n}}, \quad M = \frac{1}{a_\nu}(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\bar{\mathbf{a}}_t M^{\alpha\beta}\nu_\beta, \tag{3.17}
$$

$$\mathbf{F}^* = -\frac{1}{a_\nu}[(\bar{\mathbf{n}}\times\mathbf{H})\cdot\mathbf{v}]\bar{\mathbf{n}}, \qquad M^* = \frac{1}{a_\nu}(\bar{\mathbf{n}}\times\mathbf{H})\cdot\bar{\mathbf{a}}_t, \qquad a_\nu = (\bar{\mathbf{a}}_t\times\bar{\mathbf{n}})\cdot\mathbf{v},$$

$$\mathbf{F}_j = \mathbf{F}(s_j+0)-\mathbf{F}(s_j-0), \qquad \delta\mathbf{u}_j = \delta\mathbf{u}(s_j).$$

Here $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ are understood to be expressed in terms of $\mathbf{a}_\alpha$, $\mathbf{n}$ and $\mathbf{u}$, n_ν, what gives

$$(3.18)\qquad \begin{aligned} T^{\lambda\beta} &= l^\lambda_{\cdot\alpha}(N^{\alpha\beta}-\bar{b}^\alpha_\lambda M^{\lambda\beta})+n^\lambda[M^{\alpha\beta}|_\alpha+\bar{a}^{\beta\varkappa}(2\gamma_{\varkappa\lambda|\mu}-\gamma_{\lambda\mu|\varkappa})M^{\lambda\mu}],\\ T^\beta &= \varphi_\alpha(N^{\alpha\beta}-\bar{b}^\alpha_\lambda M^{\lambda\beta})+n[M^{\alpha\beta}|_\alpha+\bar{a}^{\beta\varkappa}(2\gamma_{\varkappa\lambda|\mu}-\gamma_{\lambda\mu|\varkappa})M^{\lambda\mu}], \end{aligned}$$

$$(3.19)\qquad \begin{aligned} \mathbf{F} &= (g_\nu R_{t\nu}+r_\nu R_\nu)\mathbf{v}+(g_t R_{t\nu}+r_t R_\nu)\mathbf{t}+(gR_{t\nu}+rR_\nu)\mathbf{n},\\ \mathbf{F}^* &= (g_\nu H_t+r_\nu H)\mathbf{v}+(g_t H_t+r_t H)\mathbf{t}+(gH_t+rH)\mathbf{n}, \end{aligned}$$

$$(3.20)\qquad \begin{aligned} &M = R_{\nu\nu}+fR_{t\nu}+kR_\nu, \qquad M^* = H_\nu+fH_t+kH,\\ &R_{\nu\nu} = \nu^\lambda l_{\lambda\alpha}M^{\alpha\beta}\nu_\beta, \qquad R_{t\nu} = t^\lambda l_{\lambda\alpha}M^{\alpha\beta}\nu_\beta, \qquad R_\nu = \varphi_\alpha M^{\alpha\beta}\nu_\beta, \end{aligned}$$

where g_ν, g_t, g, r_ν, r_t, r are complex functions of $\mathbf{u}$, n_ν given in [197].

An alternative representation for $T^{\lambda\beta}$, T^β in Eq. $(3.16)_1$ can also be derived [197]:

$$(3.21)\qquad \begin{aligned} T^{\lambda\beta} &= [N^{\alpha\beta}+\bar{a}^{\alpha\beta}(A_\mu n^\mu+An)]l^\lambda_{\cdot\alpha}+M^{\alpha\beta}(n^\lambda|_\alpha-b^\lambda_\alpha n)+\frac{1}{d}\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(Al_{\mu\alpha}-A_\mu\varphi_\alpha),\\ T^\beta &= [N^{\alpha\beta}+\bar{a}^{\alpha\beta}(A_\mu n^\mu+An)]\varphi_\alpha+M^{\alpha\beta}(n_{,\alpha}+b^\lambda_\alpha n_\lambda)+\frac{1}{d}\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}A_\mu l_{\lambda\alpha}, \end{aligned}$$

$$(3.22)\qquad \begin{aligned} &\bar{a}^{\alpha\beta} = \frac{1}{d^2}[(1+2\gamma^\varkappa_\varkappa)a^{\alpha\beta}-2\gamma^{\alpha\beta}],\\ &A_\mu = (M^{\varkappa\varrho}l_{\mu\varkappa})|_\varrho-M^{\varkappa\varrho}\varphi_\varkappa b_{\mu\varrho}, \qquad A = (M^{\varkappa\varrho}\varphi_\varkappa)|_\varrho+M^{\varkappa\varrho}l_{\gamma\varkappa}b^\gamma_\varrho. \end{aligned}$$

For arbitrary $\delta\mathbf{u}$ in $\mathscr{M}$ and $\delta\mathbf{u}$, δn_ν and $\delta\mathbf{u}_j$ on $\mathscr{C}_f$ from Eq. (3.15) follow now entirely Lagrangian equilibrium equations and corresponding static boundary and corner conditions:

$$(3.23)\qquad \begin{aligned} &\mathbf{T}^\beta|_\beta+\mathbf{p} = \mathbf{0} \quad \text{in } \mathscr{M},\\ &\mathbf{P} = \mathbf{P}^*, \qquad M = M^* \quad \text{on } \mathscr{C}_f,\\ &\mathbf{F}_j = \mathbf{F}^*_j \quad \text{at each corner } M_j\in\mathscr{C}_f. \end{aligned}$$

Corresponding work-conjugate geometric boundary conditions are

$$(3.24)\qquad \begin{aligned} &\mathbf{u} = \mathbf{u}^*, \qquad n_\nu = n^*_\nu \quad \text{on } \mathscr{C}_u,\\ &\mathbf{u}_i = \mathbf{u}^*_i \quad \text{at each corner } M_i\in\mathscr{C}_u. \end{aligned}$$

The equivalent entirely Lagrangian shell equations (3.23) and (3.24) (without $\mathbf{h}$) were first derived by Pietraszkiewicz and Szwabowicz [201] using a modified tensor of change of curvature $\chi_{\alpha\beta}$ which, by definition, is a third-degree polynomial in displacements and their surface derivatives (see also [267, 271]). Alternative equivalent formulations, in terms of the modified tensor of change of curvature proposed by Budiansky [36], were given in [218, 97]. In terms of $\varkappa_{\alpha\beta}$, the Lagrangian shell relations were derived by the author [197, 198] and in [221, 223, 96].

Let us note that already Galimov [63] proposed a version of Lagrangian shell equations by transforming the final Eulerian vector relations into the undeformed configuration and resolving them in components with respect to the undeformed basis. Under such a transformation, the fourth static boundary condition for the couple still remained to be defined with respect to the tangent of the deformed boundary contour $\bar{\mathscr{C}}$. In [67] it was shown that such a condition appears as a multiplier of the kinematic parameter $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$ in the transformed principle of virtual work (cf. Eqs. (3.3) and $(3.6)_1$). In order to construct the corresponding geometric boundary conditions, the parameter called „rotation" was defined formally as $\Omega = \int \bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, such that $\delta\Omega = \bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, and was extensively used in [71, 72] for the construction of variational principles. But it is obvious that so-defined Ω can not describe the total rotation of the boundary for an arbitrary deformation of the shell and Galimov himself was apparently aware that this representation is not consistent (see discussion on p. 14 of [67]). Various forms of Lagrangian equilibrium equations, but without boundary conditions, were also proposed by Shrivastava and Glockner [242], Sanders [215] and Budiansky [36]. Pietraszkiewicz [183] derived the complete set of Lagrangian shell equations with the fourth static boundary condition compatible with the kinematic parameter $(\bar{\mathbf{n}}\cdot\delta\mathbf{u})_{,\nu}$ and in [193] with the kinematic parameter $\bar{\mathbf{a}}_t\cdot\delta\mathbf{\Omega}_t$, where $\delta\mathbf{\Omega}_t$ was the virtual total rotation vector, but the corresponding work-conjugate geometric boundary conditions were not constructed. In the section 4.4 below we shall prove that the kinematic parameters $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, $(\bar{\mathbf{n}}\cdot\delta\mathbf{u})_{,\nu}$ and $\bar{\mathbf{a}}_t\cdot\delta\mathbf{\Omega}_t$ are not integrable, in general, i.e. there exists no scalar function such that its variation would give us the kinematic parameters $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, $(\bar{\mathbf{n}}\cdot\delta\mathbf{u})_{,\nu}$ or $\bar{\mathbf{a}}_t\cdot\delta\mathbf{\Omega}_t$, even multiplied by another scalar function.

The Eulerian and Lagrangian shell equations are equivalent within the basic assumptions of the K–L type theory of shells. However, the procedures allowing for a reduction to four the number of independent boundary conditions are different in both descriptions. As a result, numerical values of the Eulerian static boundary parameters $\bar{\mathbf{F}}$, $\bar{\mathbf{F}}_j$ and $\bar{M}$ may differ, in general, from the numerical values of the corresponding Lagrangian static boundary parameters $\mathbf{F}$, $\mathbf{F}_j$ and M, see [197].

3.3. Mixed shell equations

For some problems it is convenient to express the component form of the Eulerian shell equations (3.5), written in the basis $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ and $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$ of the deformed reference surface, in terms of components of vectors and tensors measured with respect to the undeformed surface geometry.

Let us introduce the symmetric (Kirchhoff type) internal stress and couple resultant tensors $\mathbf{N}_K = N^{\alpha\beta}\bar{\mathbf{a}}_\alpha\otimes\bar{\mathbf{a}}_\beta$, $\mathbf{M}_K = M^{\alpha\beta}\bar{\mathbf{a}}_\alpha\otimes\bar{\mathbf{a}}_\beta$ related to the Eulerian and

Lagrangian resultant tensors by

$$\text{(3.25)}\qquad \begin{aligned} \mathbf{N}_K &= d\bar{\mathbf{N}} = \mathbf{G}\mathbf{N}\mathbf{G}^T, \qquad \mathbf{M}_K = d\bar{\mathbf{M}} = \mathbf{G}\mathbf{M}\mathbf{G}^T,\\ N^{\alpha\beta} &= d\bar{N}^{\alpha\beta}, \qquad M^{\alpha\beta} = d\bar{M}^{\alpha\beta}. \end{aligned}$$

Note that in the convected system of coordinates used here components of the Kirchhoff type resultant tensors $\mathbf{N}_K$, $\mathbf{M}_K$ in the deformed basis $\bar{\mathbf{a}}_\alpha \otimes \bar{\mathbf{a}}_\beta$ are exactly the same as components of the 2nd Piola–Kirchhoff type resultant tensors $\mathbf{N}$, $\mathbf{M}$ in the undeformed basis $\mathbf{a}_\alpha \otimes \mathbf{a}_\beta$.

Let $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$ defined by the relations (3.10) are supposed to be given through their components in the deformed basis

$$\text{(3.26)}\qquad \begin{aligned} \mathbf{p} &= q^\alpha \bar{\mathbf{a}}_\alpha + q\bar{\mathbf{n}}, \qquad \mathbf{h} = k^\alpha \bar{\mathbf{a}}_\alpha + k\bar{\mathbf{n}}.\\ \mathbf{T} &= Q_\nu \bar{\mathbf{v}} + Q_t \bar{\mathbf{t}} + Q\bar{\mathbf{n}}, \qquad \mathbf{H} = K_\nu \bar{\mathbf{v}} + K_t \bar{\mathbf{t}} + K\bar{\mathbf{n}}. \end{aligned}$$

Then it follows from the relations (3.25) and $(3.10)_2$ that the virtual work principle (3.1) can be transformed into

$$\text{(3.27)}\qquad \iint_{\mathcal{M}} (\mathbf{N}_K \cdot \delta\bar{\boldsymbol{\gamma}} + \mathbf{M}_K \cdot \delta\bar{\boldsymbol{\varkappa}})\, dA = \iint_{\mathcal{M}} (\mathbf{p}\cdot\delta\bar{\mathbf{u}} + \mathbf{h}\cdot\delta\bar{\boldsymbol{\beta}})\, dA + \int_{\mathscr{C}_f} (\mathbf{T}\cdot\delta\bar{\mathbf{u}} + \mathbf{H}\cdot\delta\bar{\boldsymbol{\beta}})\, ds.$$

After additional transformations we obtain

$$\text{(3.28)}\qquad \begin{aligned} &-\iint_{\mathcal{M}} (\mathbf{N}^\beta|_\beta + \mathbf{p})\cdot\delta\bar{\mathbf{u}}\, dA - \sum_j (H_j - H_j^*)\bar{\mathbf{n}}_j \cdot \delta\bar{\mathbf{u}}_j +\\ &\qquad + \int_{\mathscr{C}_f} \{[\mathbf{N}^\beta \nu_\beta - (H\bar{\mathbf{n}})' - \mathbf{T} + (H^*\bar{\mathbf{n}})']\cdot\delta\bar{\mathbf{u}} + (G - G^*)\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}\}\, ds = 0, \end{aligned}$$

where now

$$\text{(3.29)}\qquad \begin{aligned} \mathbf{N}^\beta &= (N^{\alpha\beta} - \bar{b}^\alpha_\lambda M^{\lambda\beta})\bar{\mathbf{a}}_\alpha + (M^{\alpha\beta}|_\alpha + \bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu} M^{\lambda\mu})\bar{\mathbf{n}} + k^\beta \bar{\mathbf{n}},\\ G &= \frac{1}{\bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\, M^{\alpha\beta}\nu_\alpha \nu_\beta, \qquad G^* = \mathbf{H}\cdot\bar{\mathbf{v}},\\ H &= -\frac{1}{\bar{a}_t^2} M^{\alpha\beta}\bar{a}_{\alpha\lambda} t^\lambda \nu_\beta, \qquad H^* = -\frac{1}{\bar{a}_t}\mathbf{H}\cdot\bar{\mathbf{t}}. \end{aligned}$$

For arbitrary $\delta\bar{\mathbf{u}}$, $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$ and $\delta\bar{\mathbf{u}}_j$, from Eq. (3.28) follow the mixed equilibrium equations and corresponding static boundary conditions for the free edge:

$$\text{(3.30)}\qquad \begin{aligned} &\mathbf{N}^\beta|_\beta + \mathbf{p} = \mathbf{0} \quad \text{in } \mathcal{M},\\ &\mathbf{N}^\beta \nu_\beta - (H\bar{\mathbf{n}})' = \mathbf{T} - (H^*\bar{\mathbf{n}})', \quad G = G^* \quad \text{on } \mathscr{C}_f,\\ &H_j \bar{\mathbf{n}}_j = H_j^* \bar{\mathbf{n}}_j \quad \text{at each corner } M_j \in \mathscr{C}_f. \end{aligned}$$

Since the mixed shell equations (3.30) are referred to the deformed basis $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$, their component form is

$$(N^{\alpha\beta}-\bar{b}^{\alpha}_{\varkappa}M^{\varkappa\beta})_{|\beta}+\bar{a}^{\alpha\varkappa}\gamma_{\varkappa\lambda\beta}(N^{\lambda\beta}-\bar{b}^{\lambda}_{\mu}M^{\mu\beta})- \\ -\bar{b}^{\alpha}_{\beta}(M^{\lambda\beta}|_{\lambda}+\bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu}M^{\lambda\mu})+q^{\alpha}-\bar{b}^{\alpha}_{\beta}k^{\beta}=0, \tag{3.31}$$
$$M^{\alpha\beta}|_{\alpha\beta}+(\bar{a}^{\beta\varkappa}\gamma_{\varkappa\lambda\mu}M^{\lambda\mu})_{|\beta}+\bar{b}_{\alpha\beta}(N^{\alpha\beta}-\bar{b}^{\alpha}_{\varkappa}M^{\varkappa\beta})+q+k^{\beta}|_{\beta}=0.$$

The mixed shell equations (3.30) and their component form (3.31) were given first by Galimov [64, 67, 70] and rederived by Danielson [49]. Since only two-dimensional stress and strain measures appear explicitly in the form (3.31), these equilibrium equations are particularly useful if the shell problems are solved in the intrinsic way, cf. [120, 185, 187].

3.4. Constitutive equations

Within the first-approximation theory of thin isotropic elastic shells the strain energy density, per unit area of $\mathscr{M}$, is given by the sum of two quadratic functions describing the stretching and the bending energies of the shell reference surface. This conclusion was already given by Aron [13] and Love [135] within the classical linear theory of shells. The accuracy of such an approximation was discussed, among others, by Basset [27], Lamb [126], Novozhilov and Finkelshtein [177], Goldenveizer [79–81], Koiter [113, 118], Danielson [50], Krätzig [122] and Rychter [212]. Within the geometrically nonlinear theory, according to John [101] and Koiter [115], the strain energy density of the shell is given by

$$\Sigma=\frac{h}{2}H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu}+\frac{h^2}{12}\varkappa_{\alpha\beta}\varkappa_{\lambda\mu}\right)+O(\mathrm{E}h\eta^2\theta^2), \\ H^{\alpha\beta\lambda\mu}=\frac{\mathrm{E}}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu}+a^{\alpha\mu}a^{\beta\lambda}+\frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right), \tag{3.32}$$

where E is Young's modulus and ν is Poisson's ratio of the linearly-elastic material. The error of Σ at any point of $\mathscr{M}$ is expressed in terms of the small parameter θ defined in [101, 115, 119] to be

$$\theta=\max\left(\frac{h}{b},\frac{h}{L},\frac{h}{l},\sqrt{\frac{h}{R}},\sqrt{\eta}\right), \tag{3.33}$$

where b is the distance of the point from the lateral shell boundary and other quantities are defined in the Introduction.

The modified elasticity tensor $H^{\alpha\beta\lambda\mu}$ defined by Eq. $(3.32)_2$ takes implicitly into account the change of the shell thickness during deformation according to the plane stress state in the shell, cf. [189].

Differentiating Eq. $(3.32)_1$ with respect to the strain measures, we obtain the constitutive equations

(3.34)
$$N^{\alpha\beta} = \frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2}[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\varkappa}_{\varkappa}] + O(Eh\eta\theta^2),$$
$$M^{\alpha\beta} = \frac{\partial \Sigma}{\partial \varkappa_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}[(1-\nu)\varkappa^{\alpha\beta} + \nu a^{\alpha\beta}\varkappa^{\varkappa}_{\varkappa}] + O(Eh^2\eta\theta^2).$$

Inversion of Eqs. (3.34) leads to

(3.35)
$$\gamma_{\alpha\beta} = \frac{1}{Eh}[(1+\nu)N_{\alpha\beta} - \nu a_{\alpha\beta}N^{\lambda}_{\lambda}] + O(\eta\theta^2),$$
$$\varkappa_{\alpha\beta} = \frac{12}{Eh^3}[(1+\nu)M_{\alpha\beta} - \nu a_{\alpha\beta}M^{\lambda}_{\lambda}] + O\left(\frac{\eta\theta^2}{h}\right).$$

In some variational principles it is convenient to apply the Legendre transformation

(3.36)
$$\Sigma^C(N^{\alpha\beta}, M^{\alpha\beta}) = N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\varkappa_{\alpha\beta} - \Sigma(\gamma_{\alpha\beta}, \varkappa_{\alpha\beta}),$$

from which follows the complementary energy density of the shell

(3.37)
$$\Sigma^C = \frac{1}{h}E_{\alpha\beta\lambda\mu}\left(N^{\alpha\beta}N^{\lambda\mu} + \frac{12}{h^2}M^{\alpha\beta}M^{\lambda\mu}\right) + O(Eh\eta^2\theta^2),$$
$$E_{\alpha\beta\lambda\mu} = \frac{1+\nu}{2E}\left(a_{\alpha\lambda}a_{\beta\mu} + a_{\alpha\mu}a_{\beta\lambda} - \frac{2\nu}{1+\nu}a_{\alpha\beta}a_{\lambda\mu}\right).$$

Now the inverse constitutive equations (3.35) may also be defined in terms of Σ^C by

(3.38)
$$\gamma_{\alpha\beta} = \frac{\partial \Sigma^C}{\partial N^{\alpha\beta}}, \qquad \varkappa_{\alpha\beta} = \frac{\partial \Sigma^C}{\partial M^{\alpha\beta}}.$$

It is worthwhile to note that while the equilibrium equations and compatibility conditions are exact on the reference surface (although incomplete from the three-dimensional point of view), the constitutive equations are always approximate. In general, the energy densities Σ and Σ^C are infinite series of the two-dimensional strain and stress measures, respectively, and have to be consistently approximated for any type of the two-dimensional theory of shells.

Within the error already introduced into Σ in $(3.32)_1$ by the simplifying assumptions of the first-approximation theory of shells, some alternative definitions for the two-dimensional measure of change of curvature may be used, for example

(3.39)
$$\hat{\varrho}_{\alpha\beta} = \varkappa_{\alpha\beta} + \frac{1}{2}(b^{\lambda}_{\alpha}\gamma_{\lambda\beta} + b^{\lambda}_{\beta}\gamma_{\lambda\alpha}),$$
$$K_{\alpha\beta} = -(d\bar{b}_{\alpha\beta} - b_{\alpha\beta}) + b_{\alpha\beta}\gamma^{\varkappa}_{\varkappa} + \frac{1}{2}(b^{\lambda}_{\alpha}\gamma_{\lambda\beta} + b^{\lambda}_{\beta}\gamma_{\lambda\alpha}),$$
$$\chi_{\alpha\beta} = -(d\bar{b}_{\alpha\beta} - b_{\alpha\beta}) + b_{\alpha\beta}\gamma^{\varkappa}_{\varkappa}.$$

Each of the measures (3.39) can be expressed in terms of displacements either using the formula (2.9) or (2.10). The measure $-\hat{\varrho}_{\alpha\beta}$ with Eq. (2.10) was introduced by Koiter [115] and used in [183, 187, 190, 97]. Without displacemental representation, the measure $-\hat{\varrho}_{\alpha\beta}$ was applied by Koiter and Simmonds [120] to derive the canonical intrinsic shell equations (cf. [187, 190]). The measure $K_{\alpha\beta}$ with Eq. (2.10) was introduced by Budiansky [36], while with Eq. (2.9) it was applied in [218, 225]. The measure $\chi_{\alpha\beta}$ with Eq. (2.9) was introduced by Pietraszkiewicz and Szwabowicz [201] and then applied in [202, 267, 268, 271, 164, 165, 262].

The main advantage of using $K_{\alpha\beta}$ and $\chi_{\alpha\beta}$ is that they are third-degree polynomials in displacements and their first and second derivatives while $\hat{\varrho}_{\alpha\beta}$ and $K_{\alpha\beta}$, when linearized, reduce to the measure of change of curvature supposed to be the best one for the linear theory of shells according to Budiansky and Sanders [37]. The disadvantage of using the modified measures (3.39) in the general theory of shells is that their definitions involve additional geometric parameters of the reference configuration. With γ and $\varkappa$ we can always associate the equivalent Eulerian strain measures $\bar{\gamma}$ and $\bar{\varkappa}$ defined by [185]

$$\bar{\gamma} = \frac{1}{2}(\mathbf{1}-\mathbf{G}^{-T}\mathbf{G}^{-1}), \quad \bar{\varkappa} = -(\bar{\mathbf{b}}-\mathbf{G}^{-T}\mathbf{b}\mathbf{G}^{-1}) \tag{3.40}$$

which satisfy the following transformation rules:

$$\gamma = \mathbf{G}^T\bar{\gamma}\mathbf{G}, \quad \varkappa = \mathbf{G}^T\bar{\varkappa}\mathbf{G}. \tag{3.41}$$

No equivalent exact definitions of the modified measures (3.39) in the Eulerian description can be given which would satisfy the transformation rule (3.41). This becomes an important disadvantage of the modified measures (3.39) when exact superposition of two arbitrary deformations is discussed, what is necessary in correct incremental analysis of the highly nonlinear shell problems [141, 197]. An alternative symmetric measure $\varrho_{\alpha\beta}$ for the change of curvature, which is free from such disadvantages and when linearized reduces to the best measure of the linear shell theory, was introduced by Alumäe [8] and will be used in the Chapters 5 and 6 of this report.

4. Shell equations in terms of displacements

The majority of nonlinear shell problems discussed in the literature has been formulated and solved in terms of displacements as basic independent field variables. The primary advantage of such displacement nonlinear shell equations is that their solution gives us the complete solution of the problem in terms of well-defined and easily interpretable fields. When displacements in $\mathscr{M}$ and on $\mathscr{C}$ are determined from the shell equations, other field variables such as strain measures, rotations, stress measures etc. are calculated by the prescribed algebraic and differential procedures.

4.1. Lagrangian displacement shell equations

Since displacements and their surface derivatives appear explicitly in the definitions (3.21), the set (3.23) and (3.24) of the Lagrangian shell equations can only be solved in terms of displacements as basic independent variables. The component form of Eq. $(3.23)_1$ in the undeformed basis $\mathbf{a}_\alpha$, $\mathbf{n}$ is given by

$$(4.1)\qquad \begin{aligned} T^{\lambda\beta}|_\beta - b^\lambda_\beta T^\beta + p^\lambda + g^\lambda &= 0,\\ T^\beta|_\beta + b_{\lambda\beta} T^{\lambda\beta} + p + g &= 0, \end{aligned}$$

where

$$(4.2)\qquad \begin{aligned} g^\lambda = (n^\lambda B^\beta)|_\beta - b^\lambda_\beta n B^\beta, \qquad g &= (nB^\beta)|_\beta + b^\lambda_\beta n_\lambda B^\beta,\\ B^\beta &= (h_\alpha l^\alpha_{\cdot\varkappa} + h\varphi_\varkappa)\bar{a}^{\varkappa\beta}, \end{aligned}$$

and the relations (3.18) or (3.21) should be introduced.

The Lagrangian equilibrium equations (4.1) and the corresponding static boundary conditions $(3.23)_{2,3}$ are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but are nonlinear non-rational expressions in terms of displacements and their surface derivatives. When the constitutive equations (3.34) together with the strain-displacement relations (2.7), (2.9) are introduced into Eqs. (4.1) we obtain three extremely complex nonlinear equations which are non-rational in terms of displacements and their surface derivatives. These complex displacement shell equations are two-dimensionally exact for the shell reference surface.

Within the geometrically nonlinear theory of shells, when strains are omitted with respect to the unity, we have

$$(4.2)\qquad \begin{aligned} d \approx 1 + \gamma^\alpha_\alpha \approx 1, \qquad n \approx m(1-\gamma^\alpha_\alpha) \approx m, \qquad n_\mu \approx m_\mu,\\ \varkappa_{\alpha\beta} \approx l_{\lambda\alpha}(m^\lambda|_\beta - b^\lambda_\beta m) + \varphi_\alpha(m_{,\beta} + b^\lambda_\beta m_\lambda) + b_{\alpha\beta}(1+\gamma^\varkappa_\varkappa),\\ \bar{\mathbf{n}} \approx \frac{1}{1-c_\nu^2}\left[n_\nu \bar{\mathbf{a}}_t \times (\mathbf{v}\times\bar{\mathbf{a}}_t) + \sqrt{1-n_\nu^2-c_\nu^2}\,\mathbf{v}\times\bar{\mathbf{a}}_t\right]. \end{aligned}$$

If the relation $(4.2)_2$ is used in the left-hand side of Eq. (3.12), it generates the following reduced definitions of Eqs. (3.21), [201, 197], and of $\mathbf{g}$

$$(4.3)\qquad \begin{aligned} T^{\lambda\beta} &= l^\lambda_{\cdot\alpha}(N^{\alpha\beta} + a^{\alpha\beta} b_{\varkappa\varrho} M^{\varkappa\varrho}) + (m^\lambda|_\alpha - b^\lambda_\alpha m)M^{\alpha\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(Al_{\mu\alpha} - A_\mu\varphi_\alpha),\\ T^\beta &= \varphi_\alpha(N^{\alpha\beta} + a^{\alpha\beta} b_{\varkappa\varrho} M^{\varkappa\varrho}) + (m_{,\alpha} + b^\lambda_\alpha m_\lambda) M^{\alpha\beta} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu} A_\mu l_{\lambda\alpha},\\ \mathbf{g} &= \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[(h_\mu\varphi_\alpha - hl_{\mu\alpha})\mathbf{a}_\lambda - h_\mu l_{\lambda\alpha}\mathbf{n}]|_\beta. \end{aligned}$$

Therefore, in the geometrically nonlinear theory of shells the Lagrangian equilibrium equations (4.1) with (4.3) are linear in $M^{\alpha\beta}$, $N^{\alpha\beta}$ and quadratic in u_α, w and their surface derivatives, while the Lagrangian static boundary conditions $(3.23)_{2,3}$ with the relations (3.17), (4.3) and $(4.2)_3$ are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but still non-rational in $\mathbf{u}$, n_ν, since in the reduced expression for $\bar{\mathbf{n}}$ in the relation $(4.2)_3$ there still remains the square-root function of the displacement parameters.

It is interesting to note that when the reduced expression $(4.2)_3$ for $\bar{\mathbf{n}}$ is used in the right-hand side of Eq. (3.12), then analytically derived expressions for the generalized static boundary resultants $\mathbf{P}$, M, $\mathbf{F}_j$ will not exactly coincide with the ones which could be constructed by omitting in Eqs. (3.17) some terms which are small with respect to the unity. However, this discrepancy lies within the error margin of the first-approximation theory described by the error of the strain energy density $(3.32)_1$. As a result, both ways of deriving the reduced Lagrangian shell equations in terms of displacements should be regarded as equivalent within the first-approximation geometrically nonlinear theory of shells.

4.2. Variational principles

In many cases of practical importance it is more convenient to formulate the Lagrangian nonlinear theory of shells in the variational form, as the problem of stationarity of some functional which may be free or subjected to additional subsidiary conditions. Stationarity conditions of such a functional are then equivalent to some set of basic shell equations.

The possibility of the construction of such a functional depends upon the type of external surface and boundary loads. In general, the vector fields $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$ may be assumed to depend arbitrarily upon the shell deformation. Such loads may be non-conservative, in general, i.e. they may not be derivable as gradients of some potentials. However, in several special cases of practical importance the external loads can be given in terms of the scalar fields $\Phi[\mathbf{u}, \boldsymbol{\beta}(\nabla\mathbf{u})]$ and $\Psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{u}, \mathbf{u}', n_\nu)]$ by

$$(4.4)\qquad \mathbf{p} = -\frac{\partial\Phi}{\partial\mathbf{u}}, \quad \mathbf{h} = -\frac{\partial\Phi}{\partial\boldsymbol{\beta}}, \quad \mathbf{T} = -\frac{\partial\Psi}{\partial\mathbf{u}}, \quad \mathbf{H} = -\frac{\partial\Psi}{\partial\boldsymbol{\beta}}.$$

When all the external loads do not depend upon the shell deformation, i.e. they are dead, they can be derived using the relations (4.4) from the following simple potentials [201, 197]

$$(4.5)\qquad \Phi = -\mathbf{p}\cdot\mathbf{u} - \mathbf{h}\cdot\boldsymbol{\beta}, \qquad \Psi = -\mathbf{T}\cdot\mathbf{u} - \mathbf{H}\cdot\boldsymbol{\beta}.$$

In case of a uniformly distributed surface load of the pressure-type, we may set $\bar{\mathbf{p}}(\mathbf{u}) = \bar{p}\bar{\mathbf{n}}$, where $\bar{p} = \text{const}$, but measured per unit area of $\bar{\mathscr{M}}$. Then the existence of a potential depends upon the type of geometric boundary conditions. When the shell is closed [116] or when two of three displacement components are prescribed on $\mathscr{C}_u$ [168, 302], then the pressure load is derivable according to the relations $(4.4)_1$ from the potential

$$(4.6)\qquad \Phi = -\bar{p}\left(\mathbf{n} + \frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{u}_{,\beta} + \frac{1}{6}\varepsilon^{\alpha\beta}\mathbf{u}_{,\alpha} \times \mathbf{u}_{,\beta}\right)\cdot\mathbf{u}.$$

Potentiality of different displacement-dependent surface loads is discussed in [38, 226, 227, 231, 210]. Potentiality conditions for the boundary couple $\mathbf{K} = \bar{\mathbf{n}} \times \mathbf{H}$ are discussed in [9, 245, 269, 271]. General problems associated with potential loads, treated as nonlinear operators acting from the spaces of geometric variables to the conjugate force spaces, are discussed in [285, 231, 210].

If the external loads are derivable from potentials, then the principle of virtual displacements (3.12) can be transformed into the variational principle $\delta I = 0$ for the functional

$$I = \iint_{\mathscr{M}} \{\Sigma(\gamma, \varkappa) + \Phi[\mathbf{u}, \boldsymbol{\beta}(\nabla\mathbf{u})]\} dA + \int_{\mathscr{C}_f} \Psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{u}, \mathbf{u}', n_\nu)] ds, \tag{4.7}$$

where the strain-displacement relations (2.7) and (2.9) as well as the geometric boundary conditions (3.24) have to be imposed as subsidiary conditions. The variational principle $\delta I = 0$ states that among all possible values of displacement and strain fields, which are subjected to the subsidiary conditions, the actual solution renders the functional (4.7) stationary.

Let us introduce the subsidiary conditions (2.7), (2.9) and (3.24) into the functional (4.7) by using the method of Lagrange multipliers. Then we obtain the free functional

$$\begin{aligned} I_1 = &\iint_{\mathscr{M}} \{\Sigma(\gamma, \varkappa) + \Phi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{u})] - \mathbf{N}\cdot[\gamma - \gamma(\mathbf{u})] - \mathbf{M}\cdot[\varkappa - \varkappa(\mathbf{u})]\} dA + \\ &+ \int_{\mathscr{C}_f} \Psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{u}, n_\nu)] ds - \int_{\mathscr{C}_u} [\mathbf{P}\cdot(\mathbf{u} - \mathbf{u}^*) + M(n_\nu - n_\nu^*)] ds - \sum_i \mathbf{F}_i\cdot(\mathbf{u}_i - \mathbf{u}_i^*). \end{aligned} \tag{4.8}$$

The functional I_1 is defined on three types of independent fields: displacement measures u, strain measures ε and Lagrange multipliers σ (stress measures) defined by

$$\begin{aligned} u &\equiv \{\mathbf{u} \text{ in } \mathscr{M};\ \mathbf{u}, n_\nu \text{ on } \mathscr{C};\ \mathbf{u}_i \text{ at each } M_i\}, \\ \varepsilon &\equiv \{\gamma, \varkappa \text{ in } \mathscr{M}\}, \\ \sigma &\equiv \{\mathbf{N}, \mathbf{M} \text{ in } \mathscr{M};\quad \mathbf{P}, M \text{ on } \mathscr{C}_u;\quad \mathbf{F}_i \text{ at each } M_i\}. \end{aligned} \tag{4.9}$$

The associated Hu-Washizu (within the nonlinear elasticity, for dead body and surface forces, the principle was given by Teregulov [273], extending the principles of Hu [93] and Washizu [288] of the linear elasticity) variational principle $\delta I_1 = 0$ states that among all possible values of displacement, strain and stress fields (u, ε, σ), which are not restricted by any subsidiary conditions, the actual solution renders the functional (4.8) stationary. The stationarity conditions of I_1 are: equilibrium equations (4.1), strain-displacement relations (2.7) and (2.9), static boundary and corner conditions $(3.23)_{2,3}$, geometric boundary and corner conditions (3.24) and additional relations which identify the Lagrange multipliers with the fields already described by their symbols in the functional (4.8). These additional relations are constitutive equations (3.34), definitions of the effective generalized boundary force and couple resultants $(3.17)_{1,2}$ and definitions of the effective corner forces $(3.17)_4$.

The free three-field functional I_1 was originally constructed by Pietraszkiewicz and Szwabowicz [201, 202] using the modified tensor of change of curvature $\chi_{\alpha\beta}$ given by Eq. $(3.39)_3$ and for dead-load type external surface and boundary loads. It was also given in [218] using $K_{\alpha\beta}$ defined by Eq. $(3.39)_2$, in [197, 198, 221, 223] using $\varkappa_{\alpha\beta}$ and in [97] using $\hat{\varrho}_{\alpha\beta}$ defined by Eq. $(3.39)_1$. Each of those formulations of I_1, which are equivalent within the first-approximation theory, can be used as a starting point for derivation of various free or constrained variational functionals, according to the general procedure discussed in [48, 179, 289, 2, 276, 277]. Various functionals defined on different three and two fields as well as functionals defined on the displacement field alone were constructed by Szwabowicz [267, 268] and Schmidt [217–219] for dead-load type external surface and boundary loads and by Szwabowicz [271] for conservative $\mathbf{p}$, $\mathbf{T}$ and $\mathbf{K} = \bar{\mathbf{n}} \times \mathbf{H}$.

Several variational functionals were also constructed by Galimov [67, 71, 72] in terms of the formally defined geometric boundary parameter Ω such that $\delta\Omega = \bar{\mathbf{v}} \cdot \delta\bar{\mathbf{n}}$. We shall prove in Section 4.4 that such a parameter does not exist since the kinematic constraint $\bar{\mathbf{v}} \cdot \delta\bar{\mathbf{n}} = 0$ is not integrable, in general. As a result, the functionals given in [67, 71, 72] in terms of Ω are meaningless within the general geometrically nonlinear theory of thin elastic shells expressed in terms of displacements as basic independent variables.

4.3. Consistent classification of displacement equations for shells undergoing restricted rotations

The set of Lagrangian nonlinear shell equations expressed in displacements given in Section 4.1 is extremely complex even in tensor notation. This is caused by the generality of those relations since no restrictions have been imposed on displacements and/or rotations of the shell material elements. In many engineering problems of flexible shells displacements and/or rotations cannot be arbitrary due to implicit constraints imposed by the shell geometry, limits of an elastic behaviour of the material, types of external loadings, boundary conditions etc.

Several approximation schemes leading to simplified sets of displacement shell equations were proposed in the literature. In [157, 71, 128, 215, 115, 186, 187] restrictions of components of the linearized rotation vector and of the displacement gradients were used to derive several simplified versions of nonlinear shell equations. Among the best known simplified versions obtained in this way are displacement shell equations of medium bending given by Mushtari and Galimov [157], for moderately small rotations proposed by Sanders [215] and with small finite deflections derived by Koiter [115], the special case of which are the nonlinear equations of shallow shells developed earlier in [144, 148, 295]. A variety of simplified versions proposed by Duszek [56, 57] followed from restrictions of displacements and their surface derivatives, while those given by Novotny [171] were obtained from three-dimensional equations by a formal asymptotic procedure.

The deformation about a point of the shell middle surface can be exactly decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation [5, 247, 184, 185]. Within the first-approximation theory discussed in this report, strains are already assumed to be small, what leads to reduced shell relations $(3.32)_1$, (3.34), (4.1) – (4.3). Therefore, several consistently approximated versions of the nonlinear displacement shell equations were constructed in [185, 190] by imposing additional restrictions upon the finite rotations of the shell material elements.

A finite rotation in the shell may be described by the angle of rotation ω about an axis of rotation described by the unit vector **e**. The rotations in [185, 190] were classified in terms of the small parameter θ defined in the expression (3.33) as follows: a) $\omega \leqslant O(\theta^2)$ – small rotations, $\omega = O(\theta)$ – moderate rotations (cf. [207]), $\omega = O(\sqrt{\theta})$ – large rotations, $\omega \geqslant O(1)$ – finite rotations. This classification restricts the magnitude of the rotation angle ω. However, shell structures are usually quite rigid for in-surface deformation being flexible for out-of-surface deformation. In order to take this into account, the finite rotation vector $\boldsymbol{\Omega} = \mathbf{e}\sin\omega$ may be defined. Since for $|\omega| < \pi/2$, $O(|\boldsymbol{\Omega}|) = O(\sin\omega) = O(\omega)$ the name „small, moderate, large or finite rotation" may be associated with the particular component $\Omega = \boldsymbol{\Omega}\cdot\mathbf{n}$ or $\Omega_\beta = \boldsymbol{\Omega}\cdot\mathbf{a}_\beta$ of $\boldsymbol{\Omega}$.

Within small strains (but not small rotations) the vector $\boldsymbol{\Omega}$ is expressed in terms of displacements by [185, 193]

$$\boldsymbol{\Omega} \approx \varepsilon^{\beta\alpha}\left[\varphi_\alpha\left(1+\frac{1}{2}\vartheta^\varkappa_\varkappa\right)-\frac{1}{2}\varphi^\lambda(\vartheta_{\lambda\alpha}-\omega_{\lambda\alpha})\right]\mathbf{a}_\beta+\frac{1}{2}\varepsilon^{\alpha\beta}\omega_{\alpha\beta}\mathbf{n}. \tag{4.10}$$

For any restriction imposed on $\boldsymbol{\Omega}$ estimates for φ_α and $\omega_{\alpha\beta}$ are given by the (4.10) and estimates for $\vartheta_{\alpha\beta}$ follow from the expression (2.7) with $\gamma_{\alpha\beta} = O(\eta)$. Then simplified expressions for the strain measures $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ can be obtained taking into account the accuracy of the strain energy density $(3.32)_1$. In the estimation procedure, covariant surface derivatives are estimated by dividing their maximal value by a large parameter λ defined by

$$\lambda = \frac{h}{\theta} = \min\left(b,\, L,\, l,\, \sqrt{hR},\, \frac{1}{\sqrt{\eta}}\right). \tag{4.11}$$

Introducing such energetically consistent simplified expressions of the strain measures into the Lagrangian principle of virtual displacements (3.12), one gets the corresponding reduced expressions for the internal force vector $\mathbf{T}^\beta$ and the generalized static boundary parameters $\mathbf{P}$, M and $\mathbf{F}_j$, together with the consistently simplified expression for the geometric boundary parameter n_ν.

Simplified versions of the Lagrangian shell equations proposed in [183] were discussed in [185, 193]. Simplifications of the entirely Lagrangian shell equations derived in [201] were given in detail in [195, 197]. Let us remind here some of those consistently approximated nonlinear shell equations.

Within **small rotations** $\varphi_\alpha = O(\theta^2)$, $\omega_{\alpha\beta} = O(\theta^2)$, $\vartheta_{\alpha\beta} = O(\theta^2)$ and the strain measures are approximated by $\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + O(\eta\theta^2)$, $\varkappa_{\alpha\beta} = -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha}) + O(\eta\theta/\lambda)$, which describe the linear bending theory of shells treated extensively in many monographs.

Within **moderate rotations** $\varphi_\alpha = O(\theta)$, $\omega_{\alpha\beta} = O(\theta)$, $\vartheta_{\alpha\beta} = O(\theta^2)$ and the consistently simplified shell relations take the form [185, 193, 224]

$$\begin{aligned} \gamma_{\alpha\beta} &= \vartheta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \underline{\frac{1}{2}\omega^\lambda_{\cdot\alpha}\omega_{\lambda\beta} - \frac{1}{2}(\vartheta^\lambda_\alpha\omega_{\lambda\beta} + \vartheta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta^2), \\ \varkappa_{\alpha\beta} &= -\frac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} - \underline{b^\lambda_\alpha\omega_{\lambda\beta} - b^\lambda_\beta\omega_{\lambda\alpha}}) + O\left(\frac{\eta\theta}{\lambda}\right), \end{aligned} \tag{4.12}$$

$$\begin{aligned} \mathbf{T}^\beta &= \Bigg[N^{\lambda\beta} - \underline{\frac{1}{2}(b^\lambda_\alpha M^{\alpha\beta} + b^\beta_\alpha M^{\alpha\lambda}) - \frac{1}{2}\omega^{\lambda\beta}N^\alpha_\alpha - \frac{1}{2}(\omega^{\lambda\alpha}N^\beta_\alpha + \omega^{\beta\alpha}N^\lambda_\alpha) +} \\ &\qquad \underline{+\frac{1}{2}(\vartheta^{\lambda\alpha}N^\beta_\alpha - \vartheta^{\beta\alpha}N^\lambda_\alpha)} \Bigg] \mathbf{a}_\lambda + (\varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha)\mathbf{n} + h^\beta\mathbf{n}, \\ \bar{\mathbf{n}} &= -\varphi_\alpha\mathbf{a}^\alpha + \mathbf{n} + O(\theta^2), \qquad n_\nu = -\varphi_\nu, \\ \mathbf{F} &= M_{t\nu}\mathbf{n}, \qquad M = M_{\nu\nu}. \end{aligned} \tag{4.13}$$

If, additionally, rotations about the normal are assumed to be also small then also $\omega_{\alpha\beta} = O(\theta^2)$. For such a **moderate/small rotation** theory of shells the relations (4.12) and (4.13) may be considerably simplified by omitting there the underlined terms.

The set of nonlinear relations (3.23), (3.24) with Eqs. (4.13) and (4.12) describes the consistently reduced Lagrangian nonlinear theory of shells undergoing moderate rotations. The theory contains as special cases various simpler versions of shell equations proposed in the literature. Among them are the theory of medium bending [157], for moderately small rotations [215], with small finite deflections [115] and the classical nonlinear theory of shallow shells. A detailed review of those simpler versions was presented by Schmidt and Pietraszkiewicz [224], where also a set of sixteen basic free functionals and several functionals with subsidiary conditions was constructed for conservative dead-type surface and boundary loadings, (cf. [216]). These functionals and the variational principles associated with them extend to the moderate rotation range of deformation earlier results on particular variational principles formulated for shallow shells [296, 6, 157, 287, 94, 74, 255, 3, 256, 67, 34, 249, 71] and for simplified versions of the theory of shells undergoing moderate rotations [258, 259, 251]. Stability equations for the moderate rotation theory of shells are given in [139, 260], which extend various simpler versions of stability equations given in the literature. More complex moderate-rotation shell equations were proposed in [23, 26, 163], where the expression for $\varkappa_{\alpha\beta}$ contains also some nonlinear terms, whose contribution to the strain energy density $(3.32)_1$ lies within the indicated error of the first-approximation theory.

Within **large rotations** $\varphi_\alpha = O(\sqrt{\theta})$, $\omega_{\alpha\beta} = O(\sqrt{\theta})$, $\vartheta_{\alpha\beta} = O(\theta)$. Appropriately simplified relations of the Lagrangian theory of [183] were discussed already in [185, 193]. It was found in [193] that the consistently simplified (but still nonlinear) expression for $\varkappa_{\alpha\beta}$ generated the boundary integral which contained six (instead of four) independent variations: $\delta\mathbf{u}$ and $\delta\left(\dfrac{\partial \mathbf{u}}{\partial s_\nu}\right)$. This did not allow for a variational formulation of the shell problem even if the external boundary forces were conservative. An explanation for this paradox was found in the definition of the fourth geometric boundary parameter used in [183, 185], which was not entirely Lagrangian. As a result, an entirely Lagrangian nonlinear theory of shells was proposed in [201] where the new parameter n_ν was used on the shell boundary. Appropriately simplified relations of [201] within the large rotation range of deformation were discussed in detail in [195], various alternative results, within the prescribed accuracy of the strain energy density, were presented also in [219–222, 197, 198, 165, 169, 170, 140, 141].

The most interesting special case of the large rotation shell theory appears when rotations about the normal are assumed to be always small, i.e. $\omega_{\alpha\beta} = O(\theta^2)$. If, additionally, we allow for a greater error in the strain energy function $(3.32)_1$ to be $O(\mathrm{E}h\eta^2\theta\sqrt{\theta})$ instead of $O(\mathrm{E}h\eta^2\theta^2)$, then the set of shell equations for such a **simplified large/small rotation** theory (without **h**) is described by the following relations [197, 198]

$$\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\vartheta^\lambda_\alpha\vartheta_{\lambda\beta} - \underline{\frac{1}{2}(\vartheta^\lambda_\alpha\omega_{\lambda\beta} + \vartheta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta\sqrt{\theta}),$$

(4.14)
$$\varkappa_{\alpha\beta} = -\frac{1}{2}\{[(\delta^\lambda_\alpha + \underline{\vartheta^\lambda_\alpha})\varphi_{\lambda|\beta} + (\delta^\lambda_\beta + \underline{\vartheta^\lambda_\beta})\varphi_{\lambda|\alpha}] + \varphi^\lambda(\varphi_\alpha\varphi_{\lambda|\beta} + \varphi_\beta\varphi_{\lambda|\alpha}) + \underline{(b^\lambda_\alpha\vartheta_{\lambda\beta} + b^\lambda_\beta\vartheta_{\lambda\alpha}) + (b^\lambda_\alpha\varphi_\beta + b^\lambda_\beta\varphi_\alpha)\varphi_\lambda - b_{\alpha\beta}\varphi^\lambda\varphi_\lambda}\} + O\left(\frac{\eta\sqrt{\theta}}{\lambda}\right),$$

$$n_\nu = -\varphi_\nu + O(\theta^2\sqrt{\theta}), \qquad \bar{\mathbf{n}} \approx -\varphi_\nu\mathbf{v} - \varphi_t\mathbf{t} + \left(1 - \frac{1}{2}\varphi_\nu^2 - \frac{1}{2}\varphi_t^2\right)\mathbf{n},$$

$$T^{\lambda\beta} \doteq (\delta^\lambda_\alpha + \vartheta^\lambda_\alpha)N^{\alpha\beta} - \underline{\frac{1}{2}(\omega^{\lambda\alpha}N^\beta_\alpha + \omega^{\beta\alpha}N^\lambda_\alpha)} - \frac{1}{2}[(\underline{b^\lambda_\alpha} + \varphi^\lambda|_\alpha)M^{\alpha\beta} + (\underline{b^\beta_\alpha} + \varphi^\beta|_\alpha)M^{\alpha\lambda}],$$

(4.15)
$$T^\beta = \varphi_\alpha N^{\alpha\beta} + [(\delta^\beta_\lambda + \vartheta^\beta_\lambda)M^{\lambda\alpha}]_{|\alpha} + (\varphi_\lambda M^{\lambda\alpha})_{|\alpha}\varphi^\beta - \varphi^\lambda|_\alpha\varphi_\lambda M^{\alpha\beta} - \underline{(b^\lambda_\alpha M^{\alpha\beta} + b^\beta_\alpha M^{\alpha\lambda})\varphi_\lambda + b_{\alpha\lambda}\varphi^\beta M^{\alpha\lambda}},$$

$$R_{\nu\nu} = (1 + \vartheta_{\nu\nu})M_{\nu\nu} + \vartheta_{\nu t}M_{t\nu}, \qquad R_{t\nu} = \vartheta_{t\nu}M_{\nu\nu} + (1 + \vartheta_{tt})M_{t\nu},$$

$$R_\nu = \varphi_\nu M_{\nu\nu} + \varphi_t M_{t\nu}, \qquad M = (1 + \vartheta_{\nu\nu} + \varphi_\nu^2)M_{\nu\nu} + (\vartheta_{\nu t} + \varphi_\nu\varphi_t)M_{t\nu},$$

$$\mathbf{F} = F\mathbf{n}, \qquad F = (\vartheta_{\nu t} + \varphi_\nu\varphi_t)M_{\nu\nu} + (1 + \vartheta_{tt} + \varphi_t^2)M_{t\nu}.$$

The relations (4.14) and (4.15) have an important property: for conservative surface and boundary loadings they allow to construct the functional (4.8), whose stationarity conditions lead **exactly** to all shell equations described by Eqs. (4.14) and (4.15). Another such formulation was proposed in [197, 221]. Alternative versions of shell relations of the simplified large/small rotation theory discussed in [195, 220, 165, 170] are also energetically consistent, although some additional transformations should be applied in order to derive the shell equations from the variational functional (4.8). In particular, the version proposed by Nolte and Stumpf [170], in which $\varkappa_{\alpha\beta}$ are quadratic polynomials in displacements and their surface derivatives, was shown [165, 169, 140, 141, 166, 167] to be numerically efficient and leading to good results also far beyond the large rotation range of shell deformation.

In some engineering applications the shell relations (4.14) and (4.15) may still be simplified at the expense of a larger loss in accuracy of the strain energy function $(3.32)_1$ to be $O(Eh\eta^2\theta)$. Within this larger error the shell relations of such **simplest large/small rotation** theory of shells [197] are described again by Eqs. (4.14) and (4.15), where the underlined terms should be omitted and the term $\vartheta^\lambda_\alpha N^{\alpha\beta}$ in Eq. $(4.15)_1$ should be replaced by its symmetric part $\frac{1}{2}(\vartheta^\lambda_\alpha N^{\alpha\beta} + \vartheta^\beta_\alpha N^{\alpha\lambda})$. Alternative energetically consistent versions were proposed in [193, 195, 196, 165, 170, 218, 221]. On the other hand, the comparative discussion given in [198] suggests that some known versions [115, 23, 71, 235] of the nonlinear theory of shells, which are based on various quadratic expressions of $\varkappa_{\alpha\beta}$, cannot be regarded as energetically consistent within the large-rotation range of deformation since some energetically important terms $O(\theta\sqrt{\theta}/\lambda)$ do not appear in the expressions for $\varkappa_{\alpha\beta}$ used there. Various simplified versions of the nonlinear shell relations were also proposed in [99, 100, 58, 87, 88, 98, 45, 46, 230, 31].

When only rotations about the normal are assumed to be small, while other ones are unrestricted, then $\varphi_\alpha = O(1)$, $\omega_{\alpha\beta} = O(\theta^2)$, $\vartheta_{\alpha\beta} = O(1)$. For such a **finite/small rotation** theory of shells only a few terms may be omitted in $\varkappa_{\alpha\beta}$ within the error $O(Eh\eta^2\theta^2)$ of the function $(3.32)_1$ or even within the greater error $O(Eh\eta^2\theta\sqrt{\theta})$. It seems, therefore, that considerably simplified shell relations derived in [104, 236] for such a theory cannot be justified within the assumed error of the first-approximation theory.

An extensive comparative numerical analysis, based on energetically consistent simplified versions of nonlinear shell equations discussed above and on several other simplified versions proposed in the literature, was carried out in the series of papers [139–141, 252, 253, 165–169, 35, 59, 89, 90, 86, 47] for a large number of one- and two-dimensional problems of flexible shells. In order to provide a reliable reference solution, the full version of entirely Lagrangian shell equations [201] and in [47] also the refined three-dimensional NONSAP numerical code were used. The results of the numerical analysis showed that all energetically consistent versions of nonlinear shell equations led to results which, within the range of their applicability, were always in good agreement with the reference solution. In some examples the agreement was adequate also far beyond the range of applicability of those versions. On the other hand, some of the simplified versions suggested in the literature, which were even more complex but still energetically inconsistent, led to load-displacement paths which occasionally diverged from the reference path already on an early stage of the shell deformation.

4.4. Integrability of kinematic boundary constraints

In the entirely Lagrangian nonlinear theory of shells discussed in Section 3.2, the component $n_\nu = \bar{\mathbf{n}} \cdot \boldsymbol{\nu}$ has been used as the fourth independent boundary parameter, in terms of which $\delta\bar{\mathbf{n}}$ can be given [201, 197] by

$$\delta\bar{\mathbf{n}} = \frac{1}{a_\nu}[\bar{\mathbf{a}}_\nu \delta n_\nu + \boldsymbol{\nu} \times \bar{\mathbf{n}}(\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}}')]. \tag{4.16}$$

This has allowed to reduce the principle (3.12) into the form (3.15) and to construct four work-conjugate static $(3.23)_{2,3}$ and geometric (3.24) boundary conditions.

In the derivation of the mixed shell equations in Section 3.3, an alternative expression for $\delta\bar{\mathbf{n}}$ has been used:

$$\delta\bar{\mathbf{n}} = \bar{\boldsymbol{\nu}}(\bar{\boldsymbol{\nu}} \cdot \delta\bar{\mathbf{n}}) - \frac{1}{\bar{a}_t^2}\bar{\mathbf{r}}'(\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}}'). \tag{4.17}$$

This has allowed to reduce the relation (3.27) into the form (3.28).

Still another expression for $\delta\bar{\mathbf{n}}$ results from a direct variation of Eq. (2.18) to be

$$\begin{aligned} \delta\bar{\mathbf{n}} &= -\nu_\beta \bar{\mathbf{a}}^\beta(\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}}_{,\nu}) - t_\beta \bar{\mathbf{a}}^\beta(\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}}'), \\ \nu_\beta \bar{\mathbf{a}}^\beta &= \bar{a}_t \frac{1}{d}\bar{\boldsymbol{\nu}}, \qquad t_\beta \bar{\mathbf{a}}^\beta = \frac{1}{\bar{a}_t}\left(\bar{\mathbf{t}} - \frac{1}{d}2\gamma_{\nu t}\bar{\boldsymbol{\nu}}\right). \end{aligned} \tag{4.18}$$

When the expression $(4.18)_1$ is introduced into the relation (3.12), the internal boundary integral transforms into

$$\begin{aligned} &\int_{\mathscr{C}} \{[\mathbf{T}^\beta \nu_\beta + (M_{\nu t}\bar{\mathbf{n}})'] \cdot \delta\bar{\mathbf{r}} - M_{\nu\nu}\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}}_{,\nu}\}\, ds + \sum_j (M_{\nu t}\bar{\mathbf{n}})_j \cdot \delta\bar{\mathbf{r}}_j = \\ &\quad = \int_{\mathscr{S}} \{[\mathbf{T}^\beta \nu_\beta + M_{\nu\nu}\bar{\mathbf{n}}_{,\nu} + (M_{\nu t}\bar{\mathbf{n}})'] \cdot \delta\bar{\mathbf{r}} - M_{\nu\nu}(\bar{\mathbf{n}} \cdot \delta\bar{\mathbf{r}})_{,\nu}\}\, ds + \sum_j (M_{\nu t}\bar{\mathbf{n}})_j \cdot \delta\bar{\mathbf{r}}_j. \end{aligned} \tag{4.19}$$

The transformed line integral $(4.19)_2$ was used in [183] while the simpler integral $(4.19)_1$ was not used in the literature.

Static boundary and corner conditions in (3.23) and (3.30) have been constructed on $\mathscr{C}_f$ by demanding that all the multipliers of $\delta\bar{\mathbf{r}}$, $\delta\bar{\mathbf{r}}_j$ and of δn_ν or $\bar{\boldsymbol{\nu}} \cdot \delta\bar{\mathbf{n}}$ in the corresponding line integral identically vanish. Using the transformation (4.18) and

$(4.19)_1$, we may construct alternative static boundary and corner conditions on $\mathscr{C}_f$ again by demanding that all the multipliers of $\delta\bar{\mathbf{r}}$, $\delta\bar{\mathbf{r}}_j$ and $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,v}$ identically vanish. It is implicitly assumed that the work-conjugate geometric boundary conditions on $\mathscr{C}_u$ should satisfy the kinematic constraints $\delta\bar{\mathbf{r}} = \mathbf{0}$, $\delta\bar{\mathbf{r}}_i = \mathbf{0}$ and $\delta n_v = 0$, $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}} = 0$ or $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,v} = 0$, respectively. It is easy to note that from the kinematic constraints $\delta\bar{\mathbf{r}} = \mathbf{0}$, $\delta n_v = 0$ and $\delta\bar{\mathbf{r}}_i = \mathbf{0}$ follow the geometric boundary conditions $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$, $n_v = n_v^*$ on $\mathscr{C}_u$ and $\bar{\mathbf{r}}_i = \bar{\mathbf{r}}_i^*$ at each $M_i \in \mathscr{C}_u$. It is not apparent, however, what kind of a scalar parameter should be assumed to be given on $\mathscr{C}_u$ in order to satisfy the fourth kinematic constraints $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}} = 0$ or $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,v} = 0$. Therefore, the question arises whether there exists a scalar parameter φ such that its variation on $\mathscr{C}_f$ would coincide with the variational expressions $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$ or $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,v}$, possibly multiplied by some scalar function μ. If such functions φ and μ exist, the question arises how to construct them. This general problem has been solved only recently by Makowski and Pietraszkiewicz [142]. Here we summarize some of the results given there.

The variational expressions $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,v}$ or δn_v discussed above are particular cases of the following general variational expression:

$$\omega = \mathbf{A}\cdot\delta\bar{\mathbf{r}}_{,v} + \mathbf{B}\cdot\delta\bar{\mathbf{r}}', \tag{4.20}$$

where $\mathbf{A} = \mathbf{A}(\bar{\mathbf{r}}_{,v}, \bar{\mathbf{r}}')$ and $\mathbf{B} = \mathbf{B}(\bar{\mathbf{r}}_{,v}, \bar{\mathbf{r}}')$ are vector-valued functions of the vector arguments.

Extending the method suggested in [304], it was shown in [142] that at each point $M \in \mathscr{C}$ the variational expression (4.20) may be regarded as a differential one-form on the six-dimensional manifold X with the local coordinates $\xi_i \in X$, $i = 1, 2, \ldots, 6$ defined by

$$\xi_i = (\mathbf{v}\cdot\bar{\mathbf{r}}_{,v},\ \mathbf{t}\cdot\bar{\mathbf{r}}_{,v},\ \mathbf{n}\cdot\bar{\mathbf{r}}_{,v},\ \mathbf{v}\cdot\bar{\mathbf{r}}',\ \mathbf{t}\cdot\bar{\mathbf{r}}',\ \mathbf{n}\cdot\bar{\mathbf{r}}'). \tag{4.21}$$

Let also the components of $(\mathbf{A}, \mathbf{B})$ in the basis $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$ be defined by

$$A_i = (\mathbf{v}\cdot\mathbf{A},\ \mathbf{t}\cdot\mathbf{A},\ \mathbf{n}\cdot\mathbf{A},\ \mathbf{v}\cdot\mathbf{B},\ \mathbf{t}\cdot\mathbf{B},\ \mathbf{n}\cdot\mathbf{B}), \tag{4.22}$$

so that $\omega = \sum_{i=1}^{6} A_i \delta\xi_i$.

The one-form (4.20) is said to be exact if there exists a primitive scalar-valued function $\varphi(\bar{\mathbf{r}}_{,v}, \bar{\mathbf{r}}')$ such that $\omega = \delta\varphi$. The necessary conditions for ω to be exact are

$$A_{j,i} - A_{i,j} = 0 \tag{4.23}$$

for any $i, j \in (1, 2, \ldots, 6)$. The one-form (4.20) is said to be integrable if there exist scalar-valued functions $\mu(\bar{\mathbf{r}}_{,v}, \bar{\mathbf{r}}')$, called the integrating factor, and $\varphi(\bar{\mathbf{r}}_{,v}, \bar{\mathbf{r}}')$ such that $\mu\omega = \delta\varphi$. The necessary conditions for ω to be integrable are

$$A_i(A_{k,j} - A_{j,k}) + A_j(A_{i,k} - A_{k,i}) + A_k(A_{j,i} - A_{i,j}) = 0 \tag{4.24}$$

for any $i, j, k \in (1, 2, \ldots, 6)$.

Let us check the exactness and integrability of the one-form $\omega = \bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}}$, for which

A and B are given by

$$\mathbf{A} = -\bar{a}_t \frac{1}{d} \bar{\mathbf{n}}, \qquad \mathbf{B} = \frac{1}{\bar{a}_t} \frac{1}{d} 2\gamma_{vt} \bar{\mathbf{n}}. \tag{4.25}$$

Differentiation of the relations (4.25) with respect to $\bar{\mathbf{r}}_{,v}$ and $\bar{\mathbf{r}}'$ gives

$$\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{r}}_{,v}} = \bar{a}_t^2 \frac{1}{d^2} (\bar{\mathbf{v}} \otimes \bar{\mathbf{n}} + \bar{\mathbf{n}} \otimes \bar{\mathbf{v}}),$$

$$\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{r}}'} = \frac{1}{d} \bar{\mathbf{t}} \otimes \bar{\mathbf{n}} - \frac{1}{d^2} 2\gamma_{vt} (\bar{\mathbf{v}} \otimes \bar{\mathbf{n}} + \bar{\mathbf{n}} \otimes \bar{\mathbf{v}}) = \left[\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{r}}_{,v}} \right]^T, \tag{4.26}$$

$$\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{r}}'} = \frac{1}{\bar{a}_t^2} \bar{\mathbf{n}} \otimes \bar{\mathbf{v}} + \frac{1}{\bar{a}_t^2} \frac{1}{d^2} (2\gamma_{vt})^2 (\bar{\mathbf{v}} \otimes \bar{\mathbf{n}} + \bar{\mathbf{n}} \otimes \bar{\mathbf{v}}) - \frac{1}{\bar{a}_t^2} \frac{1}{d} 2\gamma_{vt} (\bar{\mathbf{t}} \otimes \bar{\mathbf{n}} + \bar{\mathbf{n}} \otimes \bar{\mathbf{t}}).$$

Since Eq. $(4.26)_3$ is not symmetric, the conditions (4.23) are not satisfied for $(i, j) = (4, 5)$, for example. Moreover, with the relations (4.25) and (4.26) the integrability conditions (4.24) are not satisfied as well for $(i, j, k) = (1, 4, 5)$, for example. As a result, the differential one-form $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}}$ is neither exact nor integrable, in general. The discussion given in more detail in [142] provides the proof for the same statement given by Zubov [305].

The variational expression $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}}$, which appeared originally in the paper by Galimov [64], may be presented in several different but equivalent forms. Note that in terms of the difference vector $\boldsymbol{\beta}$ given in Eq. $(2.14)_2$ $\delta \bar{\mathbf{n}} = \delta \boldsymbol{\beta}$ and $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}} = \bar{\mathbf{v}} \cdot \delta \boldsymbol{\beta} \equiv \delta \bar{\beta}_v$, which was used in [185]. Here δ should not be understood as the variation of $\bar{\beta}_v$ since so defined $\delta \bar{\beta}_v \neq \delta(\bar{\mathbf{v}} \cdot \boldsymbol{\beta})$. The rotation of the boundary may also be described [185, 188, 192] by the total rotation tensor $\mathbf{R}_t = \bar{\mathbf{v}} \otimes \mathbf{v} + \bar{\mathbf{t}} \otimes \mathbf{t} + \bar{\mathbf{n}} \otimes \mathbf{n}$ such that $\bar{\mathbf{n}} = \mathbf{R}_t \mathbf{n}$. Then we can introduce axial vectors $\delta \boldsymbol{\omega}_t$ and $\delta \mathbf{w}_t$ of the skew-symmetric tensors $\delta \mathbf{R}_t \mathbf{R}_t^T$ and $\mathbf{R}_t^T \delta \mathbf{R}_t$, respectively, according to [138, 199, 200],

$$\delta \mathbf{R}_t \mathbf{R}_t^T = \delta \boldsymbol{\omega}_t \times \mathbf{1}, \qquad \mathbf{R}_t^T \delta \mathbf{R}_t = \delta \mathbf{w}_t \times \mathbf{1}, \qquad \delta \boldsymbol{\omega}_t = \mathbf{R}_t \delta \mathbf{w}_t. \tag{4.27}$$

Since $\delta \bar{\mathbf{n}} = \delta \boldsymbol{\omega}_t \times \bar{\mathbf{n}} = \mathbf{R}_t (\delta \mathbf{w}_t \times \mathbf{n})$, it follows that we have $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}} = \delta \boldsymbol{\omega}_t \cdot \bar{\mathbf{t}} = \delta \mathbf{w}_t \cdot \mathbf{t}$. Here again δ should not be understood as the symbol of variation of $\boldsymbol{\omega}_t$ or $\mathbf{w}_t$ since the symbols $\boldsymbol{\omega}_t$ or $\mathbf{w}_t$ alone have no geometric meaning here. The expression $\delta \boldsymbol{\omega}_t \cdot \bar{\mathbf{t}}$ was applied, among others, in [291, 214, 17] while $\delta \mathbf{w}_t \cdot \mathbf{t}$ was used in [271].

According to the discussion given above and in the forms (3.6), the variational expressions $\delta \bar{\beta}_v$, $\delta \boldsymbol{\omega}_t \cdot \bar{\mathbf{t}}$, $\delta \mathbf{w}_t \cdot \mathbf{t}$, $-\bar{\mathbf{n}} \cdot \delta \bar{\mathbf{v}}$, $-\delta \bar{\varphi}_v$, $-\bar{\mathbf{n}} \cdot \delta \bar{\mathbf{u}}_{,\bar{v}}$ which appeared in the literature are all equivalent to the differential one-form $\bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}}$ since they have all the same representation (4.20) with (4.25) in terms of variations of $\bar{\mathbf{r}}_{,v}$ and $\bar{\mathbf{r}}'$. As a result, neither of the one-forms is exact or integrable as well. It is apparent from this discussion that the variational principles given by Galimov [68, 71] in terms of Ω such that $\delta \Omega = \bar{\mathbf{v}} \cdot \delta \bar{\mathbf{n}}$ are not correct, in general, since such a function Ω does not exist.

In [142] it was confirmed that the differential one-form $\omega = \mathbf{v} \cdot \delta \bar{\mathbf{n}} \equiv \delta n_v$ is exact indeed and its primitive function is $\varphi \equiv n_\nu$. It was also proved that the one-form $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ is neither exact nor integrable since the conditions (4.24) are not satisfied. Using the same method, many other variational expressions of the type (4.20) may be checked. On the other hand, a similar direct discussion of integrability of the variational expressions $(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{u}})_{,\bar{\nu}}$ or $(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}})_{,\nu}$ has to be performed with the help of a nine-dimensional manifold with local coordinates identified with the components of $\bar{\mathbf{r}}$, $\bar{\mathbf{r}}_{,\nu}$, $\bar{\mathbf{r}}'$ in the basis $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$. However, such a discussion is not necessary since those variational expressions can always be transformed further by taking the partial derivative with respect to s_ν on $\mathscr{C}$. Since $\delta\bar{\mathbf{r}}$ is exact on a three-dimensional manifold of positions $\bar{\mathbf{r}}$, the problem can always be reduced to the integrability of the one-form of the type (4.20) on X.

4.5. Work-conjugate boundary conditions

Each of the variational expressions of the type (4.20), which appears in the boundary line integral and is connected with the boundary couple, may be transformed further by multiplying (and dividing) it by a non-vanishing scalar function $\eta(\bar{\mathbf{r}}_{,\nu}, \bar{\mathbf{r}}')$ and by adding (and subtracting) terms of the type $\mathbf{c}(\bar{\mathbf{r}}_{,\nu}, \bar{\mathbf{r}}')\cdot\delta\bar{\mathbf{r}}'$ since terms with $\delta\bar{\mathbf{r}}'$ can always be eliminated by integration by parts. By such a transformation, a non-integrable one-form may be transformed to the exact one-form for which a primitive may be constructed.

In [142] the following simple differential one-form on the six-dimensional manifold X has been discussed:

$$\begin{gathered}\theta = \mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu}, \qquad \mathbf{d} = d\bar{\mathbf{n}} = \bar{\mathbf{r}}_{,\nu}\times\bar{\mathbf{r}}', \\ \mathbf{A} = (\xi_2\xi_6-\xi_3\xi_5)\mathbf{v}+(\xi_3\xi_4-\xi_1\xi_6)\mathbf{t}+(\xi_1\xi_5-\xi_2\xi_4)\mathbf{n}, \\ \mathbf{B} = A_4\mathbf{v}+A_5\mathbf{t}+A_6\mathbf{n} \equiv \mathbf{0}.\end{gathered} \tag{4.28}$$

It is easy to check that the one-form θ is not integrable. In [142] it has been proved that an arbitrary function $\varphi(\bar{\mathbf{r}}', \alpha)$, where $\alpha = A_1/A_3 = n_\nu/n$, is the primitive of some transformed one-form ψ such that

$$\begin{gathered}\delta\varphi = \psi = \eta\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu}+\mathbf{c}\cdot\delta\bar{\mathbf{r}}', \\ \eta = -\frac{1}{A_3^2}\xi_5\chi, \qquad \mathbf{c} = \boldsymbol{\lambda}+\frac{1}{A_3^2}\xi_2\chi\mathbf{d}, \\ \boldsymbol{\lambda} = \frac{\partial\varphi}{\partial\bar{\mathbf{r}}'}, \qquad \chi = \frac{\partial\varphi}{\partial\alpha}.\end{gathered} \tag{4.29}$$

If we solve the problem $(4.29)_1$ for $\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ and introduce it into Eq. $(4.18)_1$, then we obtain still another general expression for $\delta\bar{\mathbf{n}}$ to be

$$\delta\bar{\mathbf{n}} = \nu_\alpha \bar{\mathbf{a}}^\alpha f\,\delta\varphi - \bar{\mathbf{a}}^\beta\{[\nu_\beta f \boldsymbol{\lambda} + (\nu_\beta g + t_\beta)\bar{\mathbf{n}}]\cdot\delta\bar{\mathbf{r}}'\},$$

$$f = \frac{dn^2}{c_t\chi}, \qquad g = \frac{\xi_2}{\xi_5} = \frac{1}{\bar{a}_t^2}\left(\frac{1}{d}\frac{c_\nu n - cn_\nu}{c_t} + 2\gamma_{\nu t}\right). \tag{4.30}$$

The expression $(4.30)_1$ is remarkable by the fact that it is given directly in terms of variation of an arbitrary function $\varphi(\bar{\mathbf{r}}', \alpha)$. If now the expression $(4.30)_1$ for $\delta\bar{\mathbf{n}}$ is used to transform the Lagrangian principle of virtual displacements (3.12), then it can take the form

$$-\iint_{\mathcal{M}} (\mathbf{T}^\beta|_\beta + \mathbf{p})\cdot\delta\bar{\mathbf{r}}\,dA + \sum_j (\mathbf{F}_j - \mathbf{F}_j^*)\cdot\delta\bar{\mathbf{r}}_j +$$

$$+ \int_{\mathscr{C}_f} [(\mathbf{T}^\beta\nu_\beta + \mathbf{F}' - \mathbf{T} - \mathbf{F}^{*\prime})\cdot\delta\bar{\mathbf{r}}' + (M - M^*)\delta\varphi]\,ds = 0, \tag{4.31}$$

where

$$\mathbf{F} = f M_{\nu\nu}\boldsymbol{\lambda} + (gM_{\nu\nu} + M_{\nu t})\bar{\mathbf{n}}, \qquad M = f M_{\nu\nu},$$

$$\mathbf{F}^* = (\mathbf{H}\cdot\bar{\mathbf{a}}^\beta)[\nu_\beta f\boldsymbol{\lambda} + (\nu_\beta g + t_\beta)\bar{\mathbf{n}}], \qquad M^* = f(\mathbf{H}\cdot\bar{\mathbf{a}}^\beta)\nu_\beta. \tag{4.32}$$

For arbitrary $\delta\bar{\mathbf{r}}$, $\delta\bar{\mathbf{r}}_j$ and $\delta\varphi$, from the form (4.31) follow the equilibrium equations $(3.23)_1$ and static boundary conditions

$$\mathbf{T}^\beta\nu_\beta + \mathbf{F}' = \mathbf{T} + \mathbf{F}^{*\prime}, \qquad M = M^* \quad \text{on } \mathscr{C}_f,$$

$$\mathbf{F}_j = \mathbf{F}_j^* \quad \text{at each corner } M_j \in \mathscr{C}_f. \tag{4.33}$$

Corresponding work-conjugate geometric boundary conditions are

$$\bar{\mathbf{r}} = \bar{\mathbf{r}}^*, \qquad \varphi = \varphi^* \text{ on } \mathscr{C}_u,$$

$$\bar{\mathbf{r}}_i = \bar{\mathbf{r}}_i^* \quad \text{at each corner } M_i \in \mathscr{C}_u. \tag{4.34}$$

The arbitrariness of φ allows for wide freedom in choosing the form of boundary conditions to be used in the shell theory. This enables one to choose such a definition of φ which would suit best to a particular shell problem. In particular, it was shown in [142] that the parameters n_ν, θ_ν used in [178] and the total rotation angle ω_t of $\mathbf{R}_t$ are all special cases of φ.

5. Shell relations in terms of rotations

Some shell problems are solved in a more convenient way if one uses finite rotations together with other fields as basic independent variables of the nonlinear shell equations. Already Reissner [206, 207] proposed the set of nonlinear equations for an axisymmetric deformation of shells of revolution written in terms of a rotation and a stress resultant (or a stress function) as independent variables. This formulation led to a number of papers on axisymmetric problems of shells of revolution, the results of which have been summarized, among others, in the books by Shilkrut and Vyrlan [238] and Libai and Simmonds [134].

Within the general nonlinear theory of thin shells, Alumäe [5] derived the nonlinear equilibrium equations and compatibility conditions in the intermediate rotated basis while Simmonds and Danielson [247, 248] proposed a set of nonlinear shell equations in terms of a finite rotation vector and a stress function vector as independent variables and constructed an appropriate variational principle. The theory of finite rotations in shells developed by Pietraszkiewicz [184, 185] led to several alternative forms of nonlinear shell equations, boundary conditions, consistently approximated shell relations and some new kinematic relations which have been summarized in [186, 188, 190–194]. Contributions to the nonlinear theory of shells in terms of rotations were also made by Wempner [290–293], Shamina [233, 234], Valid [281–284], Shkutin [240], Reissner [208, 209], Libai and Simmonds [133], Atluri [14], Makowski and Stumpf [143] and Badur and Pietraszkiewicz [19] where further references are given.

The primary advantage of the nonlinear shell equations in terms of finite rotations is that they contain, at the most, first derivatives of the independent field variables. In the computerized analysis of shells, this makes it possible to use the simplest shape functions or the simplest difference schemes which assure high efficiency of the numerical analysis.

The nonlinear theory of shells in terms of rotations is now in the process of development and several questions are still open. Only few two-dimensional problems have been analysed [60, 61] using this approach. Therefore, we found it worthwhile to review here in more detail, in the unified notation, the most important results of this field given in the literature and to supplement them with some new results which are not available elsewhere. It is hoped that it will stimulate further research in the field.

5.1. Additional geometric relations

Applying the polar decomposition theorem [279, 280, 138], the deformation gradient tensor $\mathbf{G}$ defined in the relations (2.5) can be represented [184, 185, 190] in the form

$$\mathbf{G} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \qquad \mathbf{G}^{-1} = \mathbf{U}^{-1}\mathbf{R}^{T} = \mathbf{R}^{T}\mathbf{V}^{-1}. \tag{5.1}$$

Here $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively, while $\mathbf{R}$ is the finite rotation tensor. The tensors $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite while $\mathbf{R}$ is the proper orthogonal, i.e. $\det\mathbf{R} = +1$.

By the relations (2.1) and (5.1), the deformation of a neighbourhood about a particle of the shell middle surface has been decomposed into a rigid-body translation, a pure stretch along principal directions of $\mathbf{U}$ (or $\mathbf{V}$) and a rigid-body rotation. From the relations (2.2) and (5.1), it follows that there exist two intermediate non-holonomic bases, the stretched basis $\mathbf{s}_\alpha$, $\mathbf{n}$ and the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$, which are defined by

$$\mathbf{s}_\alpha = \mathbf{U}\mathbf{a}_\alpha = \mathbf{R}^T\bar{\mathbf{a}}_\alpha, \qquad \mathbf{s}_\alpha\cdot\mathbf{s}_\beta = \bar{a}_{\alpha\beta}, \tag{5.2}$$

$$\mathbf{r}_\alpha = \mathbf{R}\mathbf{a}_\alpha = \mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha, \qquad \mathbf{r}_\alpha\cdot\mathbf{r}_\beta = a_{\alpha\beta}. \tag{5.3}$$

Within the shell theory the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$ was introduced first by Alumäe [5] and was used in [8, 247, 248, 240, 133, 19]. The stretched basis $\mathbf{s}_\alpha$, $\mathbf{n}$ was introduced first by Novozhilov and Shamina [178] and used in [184, 185, 188, 190–194, 14]. In terms of the bases, the following expressions for $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ can be given [185, 190]:

$$\begin{gathered}\mathbf{U} = \mathbf{s}_\alpha\otimes\mathbf{a}^\alpha+\mathbf{n}\otimes\mathbf{n}, \qquad \mathbf{U}^{-1} = \mathbf{a}_\alpha\otimes\mathbf{s}^\alpha+\mathbf{n}\otimes\mathbf{n},\\ \mathbf{V} = \bar{\mathbf{a}}_\alpha\otimes\mathbf{r}^\alpha+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}, \qquad \mathbf{V}^{-1} = \mathbf{r}_\alpha\otimes\bar{\mathbf{a}}^\alpha+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}},\\ \mathbf{R} = \bar{\mathbf{a}}_\alpha\otimes\mathbf{s}^\alpha+\bar{\mathbf{n}}\otimes\mathbf{n} = \mathbf{r}_\alpha\otimes\mathbf{a}^\alpha+\bar{\mathbf{n}}\otimes\mathbf{n}.\end{gathered} \tag{5.4}$$

Any rotation tensor $\mathbf{R}$ may be represented by

$$\mathbf{R} = \cos\omega\mathbf{1}+\sin\omega\mathbf{e}\times\mathbf{1}+(1-\cos\omega)\mathbf{e}\otimes\mathbf{e}, \tag{5.5}$$

where the unit vector $\mathbf{e}$ describes the axis of rotation of $\mathbf{R}$ and ω is the angle of rotation of $\mathbf{R}$ about the axis of rotation.

Sometimes it is more convenient to describe rotations by means of an equivalent finite rotation vector, the direction of which is $\mathbf{e}$ and the length is a function of ω. For example, the finite rotation vector $\mathbf{\Omega} \equiv \sin\omega\mathbf{e}$ was used in [247, 248, 178, 43, 184–186, 190–194], the vector $\boldsymbol{\theta} \equiv 2\,\mathrm{tg}\frac{\omega}{2}\mathbf{e}$ was used in [241, 133, 19] while $\boldsymbol{\omega} \equiv \omega\mathbf{e}$ was applied in [240]. As it was pointed out in [199], each of the definitions has some advantages: $\mathbf{\Omega}$ is particularly convenient to be expressed in terms of displacements (cf. [185, 192]), $\boldsymbol{\theta}$ leads to geometric relations which do not contain trigonometric expressions while $\boldsymbol{\omega}$ is the single-valued function of ω and can be defined in terms of the natural logarithm of $\mathbf{R}$, cf. [199]. In [108, 109] the rotations were described in terms of four Rodrigues parameters. In the following part of this report we shall use primarily the finite rotation vector $\boldsymbol{\theta}$, in terms of which transformation rules for the basic vectors are

$$\begin{gathered}\bar{\mathbf{a}}_\beta = \mathbf{s}_\beta+\frac{1}{t}\boldsymbol{\theta}\times\left(\mathbf{s}_\beta+\frac{1}{2}\boldsymbol{\theta}\times\mathbf{s}_\beta\right), \qquad t = 1+\frac{1}{4}\boldsymbol{\theta}\cdot\boldsymbol{\theta},\\ \mathbf{r}_\beta = \mathbf{a}_\beta+\frac{1}{t}\boldsymbol{\theta}\times\left(\mathbf{a}_\beta+\frac{1}{2}\boldsymbol{\theta}\times\mathbf{a}_\beta\right),\\ \bar{\mathbf{n}} = \mathbf{R}\mathbf{n} = \mathbf{n}+\frac{1}{t}\boldsymbol{\theta}\times\left(\mathbf{n}+\frac{1}{2}\boldsymbol{\theta}\times\mathbf{n}\right).\end{gathered} \tag{5.6}$$

Let us introduce the relative (symmetric) surface stretch tensors

$$(5.7)\qquad \begin{aligned} \boldsymbol{\eta} &= \mathbf{U}-\mathbf{1} = \boldsymbol{\eta}_\beta \otimes \mathbf{a}^\beta, \qquad \boldsymbol{\eta}_\beta = \mathbf{s}_\beta - \mathbf{a}_\beta = \eta_{\alpha\beta}\mathbf{a}^\alpha, \\ \boldsymbol{\varepsilon} &= \mathbf{V}-\mathbf{1} = \boldsymbol{\varepsilon}_\beta \otimes \mathbf{r}^\beta, \qquad \boldsymbol{\varepsilon}_\beta = \bar{\mathbf{a}}_\beta - \mathbf{r}_\beta = \eta_{\alpha\beta}\mathbf{r}^\alpha, \end{aligned}$$

$$(5.8)\qquad \boldsymbol{\varepsilon} = \mathbf{R}\boldsymbol{\eta}\mathbf{R}^T, \qquad \boldsymbol{\varepsilon}_\beta = \mathbf{R}\boldsymbol{\eta}_\beta.$$

In terms of so defined $\eta_{\alpha\beta}$ many useful geometric relations may be derived, [185, 190, 193, 199, 19].

The corresponding relative (unsymmetric) surface bending tensors are defined by

$$(5.9)\qquad \begin{aligned} \boldsymbol{\mu} &= (\mathbf{R}^T\bar{\mathbf{n}}_{,\beta} - \mathbf{n}_{,\beta})\otimes \mathbf{a}^\beta = \boldsymbol{\mu}_\beta \otimes \mathbf{a}^\beta, \qquad \boldsymbol{\mu}_\beta = \mu_{\alpha\beta}\mathbf{a}^\alpha, \\ \boldsymbol{\lambda} &= (\bar{\mathbf{n}}_{,\beta} - \mathbf{R}\mathbf{n}_{,\beta})\otimes \mathbf{r}^\beta = \boldsymbol{\lambda}_\beta \otimes \mathbf{r}^\beta, \qquad \boldsymbol{\lambda}_\beta = \mu_{\alpha\beta}\mathbf{r}^\alpha, \end{aligned}$$

$$(5.10)\qquad \boldsymbol{\lambda} = \mathbf{R}\boldsymbol{\mu}\mathbf{R}^T, \qquad \boldsymbol{\lambda}_\beta = \mathbf{R}\boldsymbol{\mu}_\beta.$$

The relative surface strain measures $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$ were introduced first by Alumäe [5]. They are related to the Lagrangian surface strain measures (2.7) and (2.8) by

$$(5.11)\qquad \begin{aligned} \gamma_{\alpha\beta} &= \eta_{\alpha\beta} + \frac{1}{2}\eta^\lambda_\alpha \eta_{\lambda\beta}, \\ \varkappa_{\alpha\beta} &= \frac{1}{2}[(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\mu_{\lambda\beta} + (\delta^\lambda_\beta + \eta^\lambda_\beta)\mu_{\lambda\alpha}] - \frac{1}{2}(b^\lambda_\alpha \eta_{\lambda\beta} + b^\lambda_\beta \eta_{\lambda\alpha}). \end{aligned}$$

Since $\mathbf{R}^T\mathbf{R}_{,\beta}$ and $\mathbf{R}_{,\beta}\mathbf{R}^T$ are skew-symmetric, they are expressible, according to [199], by their respective axial vectors $\mathbf{k}_\beta$ and $\mathbf{l}_\beta$, called also the vectors of change of curvature of the coordinate lines [232, 190], by the relations

$$(5.12)\qquad \mathbf{R}^T\mathbf{R}_{,\beta} = \mathbf{k}_\beta \times \mathbf{1}, \qquad \mathbf{R}_{,\beta}\mathbf{R}^T = \mathbf{l}_\beta \times \mathbf{1}, \qquad \mathbf{l}_\beta = \mathbf{R}\mathbf{k}_\beta.$$

Then, from Eqs. (5.6), (5.11) and (5.12) we obtain

$$(5.13)\qquad \boldsymbol{\mu}_\beta = \mathbf{k}_\beta \times \mathbf{n}, \qquad \boldsymbol{\lambda}_\beta = \mathbf{l}_\beta \times \bar{\mathbf{n}},$$

$$(5.14)\qquad \begin{aligned} \mathbf{k}_\beta &= \varepsilon^{\alpha\lambda}\mu_{\alpha\beta}\mathbf{a}_\lambda + k_\beta \mathbf{n} = \frac{1}{t}\left(\boldsymbol{\theta}_{,\beta} + \frac{1}{2}\boldsymbol{\theta}_{,\beta}\times\boldsymbol{\theta}\right), \\ \mathbf{l}_\beta &= \varepsilon^{\alpha\lambda}\mu_{\alpha\beta}\mathbf{r}_\lambda + k_\beta \bar{\mathbf{n}} = \frac{1}{t}\left(\boldsymbol{\theta}_{,\beta} - \frac{1}{2}\boldsymbol{\theta}_{,\beta}\times\boldsymbol{\theta}\right). \end{aligned}$$

rotation. From the relations (2.2) and (5.1), it follows that there exist two intermediate non-holonomic bases, the stretched basis $\mathbf{s}_\alpha$, $\mathbf{n}$ and the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$, which are defined by

$$\mathbf{s}_\alpha = \mathbf{U}\mathbf{a}_\alpha = \mathbf{R}^T\bar{\mathbf{a}}_\alpha, \qquad \mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \bar{a}_{\alpha\beta}, \tag{5.2}$$

$$\mathbf{r}_\alpha = \mathbf{R}\mathbf{a}_\alpha = \mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha, \qquad \mathbf{r}_\alpha \cdot \mathbf{r}_\beta = a_{\alpha\beta}. \tag{5.3}$$

Within the shell theory the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$ was introduced first by Alumäe [5] and was used in [8, 247, 248, 240, 133, 19]. The stretched basis $\mathbf{s}_\alpha$, $\mathbf{n}$ was introduced first by Novozhilov and Shamina [178] and used in [184, 185, 188, 190–194, 14]. In terms of the bases, the following expressions for $\mathbf{U}$, $\mathbf{V}$ and $\mathbf{R}$ can be given [185, 190]:

$$\begin{gathered} \mathbf{U} = \mathbf{s}_\alpha \otimes \mathbf{a}^\alpha + \mathbf{n} \otimes \mathbf{n}, \qquad \mathbf{U}^{-1} = \mathbf{a}_\alpha \otimes \mathbf{s}^\alpha + \mathbf{n} \otimes \mathbf{n}, \\ \mathbf{V} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{r}^\alpha + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}, \qquad \mathbf{V}^{-1} = \mathbf{r}_\alpha \otimes \bar{\mathbf{a}}^\alpha + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}, \\ \mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{s}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} = \mathbf{r}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}. \end{gathered} \tag{5.4}$$

Any rotation tensor $\mathbf{R}$ may be represented by

$$\mathbf{R} = \cos\omega \mathbf{1} + \sin\omega \mathbf{e} \times \mathbf{1} + (1 - \cos\omega)\mathbf{e} \otimes \mathbf{e}, \tag{5.5}$$

where the unit vector $\mathbf{e}$ describes the axis of rotation of $\mathbf{R}$ and ω is the angle of rotation of $\mathbf{R}$ about the axis of rotation.

Sometimes it is more convenient to describe rotations by means of an equivalent finite rotation vector, the direction of which is $\mathbf{e}$ and the length is a function of ω. For example, the finite rotation vector $\mathbf{\Omega} \equiv \sin\omega\mathbf{e}$ was used in [247, 248, 178, 43, 184–186, 190–194], the vector $\boldsymbol{\theta} \equiv 2\,\mathrm{tg}\frac{\omega}{2}\mathbf{e}$ was used in [241, 133, 19] while $\boldsymbol{\omega} \equiv \omega\mathbf{e}$ was applied in [240]. As it was pointed out in [199], each of the definitions has some advantages: $\mathbf{\Omega}$ is particularly convenient to be expressed in terms of displacements (cf. [185, 192]), $\boldsymbol{\theta}$ leads to geometric relations which do not contain trigonometric expressions while $\boldsymbol{\omega}$ is the single-valued function of ω and can be defined in terms of the natural logarithm of $\mathbf{R}$, cf. [199]. In [108, 109] the rotations were described in terms of four Rodrigues parameters. In the following part of this report we shall use primarily the finite rotation vector $\boldsymbol{\theta}$, in terms of which transformation rules for the basic vectors are

$$\begin{gathered} \bar{\mathbf{a}}_\beta = \mathbf{s}_\beta + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{s}_\beta + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{s}_\beta\right), \qquad t = 1 + \frac{1}{4}\boldsymbol{\theta} \cdot \boldsymbol{\theta}, \\ \mathbf{r}_\beta = \mathbf{a}_\beta + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{a}_\beta + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{a}_\beta\right), \\ \bar{\mathbf{n}} = \mathbf{R}\mathbf{n} = \mathbf{n} + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{n} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{n}\right). \end{gathered} \tag{5.6}$$

In the components in the reference basis we have

$$\boldsymbol{\theta} = \varepsilon^{\alpha\beta}\theta_\beta \mathbf{a}_\alpha + \theta_3 \mathbf{n} = \varepsilon^{\alpha\beta}\theta_\beta \mathbf{r}_\alpha + \theta_3 \bar{\mathbf{n}},$$

$$\mathbf{r}_\beta = r_{\lambda\beta}\mathbf{a}^\lambda + r_\beta \mathbf{n}, \qquad \boldsymbol{\theta}_{,\beta} = \psi^{\lambda}_{\cdot\beta}\mathbf{a}_\lambda + \psi_\beta \mathbf{n}, \tag{5.15}$$

$$r_{\lambda\beta} = a_{\lambda\beta} - \frac{1}{t}\left(\varepsilon_{\lambda\beta}\theta_3 + \frac{1}{2}\theta_\lambda\theta_\beta + \frac{1}{2}a_{\lambda\beta}\theta_3^2\right),$$

$$r_\beta = \frac{1}{t}\left(\theta_\beta + \frac{1}{2}\varepsilon_{\beta\alpha}\theta^\alpha\theta_3\right), \tag{5.16}$$

$$\psi^{\lambda}_{\cdot\beta} = \varepsilon^{\lambda\varkappa}\theta_{\varkappa|\beta} - b^\lambda_\beta\theta_3, \qquad \psi_\beta = \theta_{3,\beta} + b^\lambda_\beta\varepsilon_{\lambda\varkappa}\theta^\varkappa, \tag{5.17}$$

$$n^\lambda = -\frac{1}{t}\left(\theta^\lambda + \frac{1}{2}\varepsilon^{\alpha\lambda}\theta_\alpha\theta_3\right), \qquad n = 1 - \frac{1}{2t}\theta^\lambda\theta_\lambda. \tag{5.18}$$

Using Eqs. (5.15)–(5.18) together with the relations (2.2), $(5.7)_1$, $(5.9)_1$ and $(5.14)_1$, the relative symmetric surface strain measures may be expressed explicitly in terms of components of $\mathbf{u}$ and $\boldsymbol{\theta}$ according to

$$\eta_{\alpha\beta} = \mathbf{r}_\alpha\cdot\bar{\mathbf{a}}_\beta - a_{\alpha\beta} = r_{\alpha\lambda}l^{\lambda}_{\cdot\beta} + r_\alpha\varphi_\beta - a_{\alpha\beta} =$$

$$= u_{\alpha|\beta} - b_{\alpha\beta}w - \frac{1}{1+\frac{1}{4}(\theta^\lambda\theta_\lambda+\theta_3^2)}\left[\varepsilon_{\beta\alpha}\theta_3 + \frac{1}{2}\theta_\alpha\theta_\beta + \frac{1}{2}a_{\alpha\beta}\theta_3^2 + \right. \tag{5.19}$$

$$\left. + \left(\varepsilon_{\lambda\alpha}\theta_3 + \frac{1}{2}\theta_\lambda\theta_\alpha + \frac{1}{2}a_{\lambda\alpha}\theta_3^2\right)(u^\lambda|_\beta - b^\lambda_\beta w) - \left(\theta_\alpha + \frac{1}{2}\varepsilon_{\alpha\varkappa}\theta^\varkappa\theta_3\right)(w_{,\beta} + b^\lambda_\beta u_\lambda)\right],$$

The expression (5.19) should still be symmetrized in $\alpha\beta$ indices.

$$\varrho_{\alpha\beta} = \frac{1}{2}(\varepsilon_{\alpha\lambda}\mathbf{k}_\beta + \varepsilon_{\beta\lambda}\mathbf{k}_\alpha)\cdot\mathbf{a}^\lambda = \frac{1}{2}(\mu_{\alpha\beta} + \mu_{\beta\alpha}) =$$

$$= \frac{1}{2t}\left[\varepsilon_{\alpha\lambda}\left(\psi^{\lambda}_{\cdot\beta} + \frac{1}{2}\theta^\lambda\psi_\beta\right) + \varepsilon_{\beta\lambda}\left(\psi^{\lambda}_{\cdot\alpha} + \frac{1}{2}\theta^\lambda\psi_\alpha\right) - \frac{1}{2}(\psi_{\alpha\beta} + \psi_{\beta\alpha})\theta_3\right] =$$

$$= -\frac{1}{2+\frac{1}{2}(\theta^\lambda\theta_\lambda + \theta_3^2)}\left[\theta_{\alpha|\beta} + \theta_{\beta|\alpha} + \varepsilon_{\alpha\lambda}\left(b^\lambda_\beta\theta_3 - \frac{1}{2}\theta^\lambda\theta_{3,\beta} + \frac{1}{2}\theta^\lambda|_\beta\theta_3\right) + \right. \tag{5.20}$$

$$\left. + \varepsilon_{\beta\lambda}\left(b^\lambda_\alpha\theta_3 - \frac{1}{2}\theta^\lambda\theta_{3,\alpha} + \frac{1}{2}\theta^\lambda|_\alpha\theta_3\right) - b_{\alpha\beta}(\theta^\varkappa\theta_\varkappa + \theta_3^2) + \frac{1}{2}(b_{\alpha\lambda}\theta_\beta + b_{\beta\lambda}\theta_\alpha)\theta^\lambda\right].$$

Corresponding expressions for $\mu_{\alpha\beta}$ and k_β in terms of components of $\boldsymbol{\theta}$ follow directly from the relations $(5.14)_1$ and $(5.15)_1$. In components of $\boldsymbol{\Omega}$ the relations (5.20) were given first by Simmonds and Danielson [247], while the relations (5.19) by the author [184, 185]. Equivalent relations for $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$ and k_β in terms of components of $\boldsymbol{\omega}$ were given by Shkutin [240]. Linearization of $\varrho_{\alpha\beta}$ given by the relations (5.20) with respect to displacements leads to the tensor of change of curvature which, according to Budiansky and Sanders [37], is the best choice for the linear theory of shells.

Rules of differentiation of the intermediate bases may be given with the help of the relations (5.12) in the form [190, 199]

$$\text{(5.21)} \qquad \begin{aligned} \mathbf{s}_{\alpha\|\beta} &= -\mathbf{k}_\beta \times \mathbf{s}_\alpha + \bar{b}_{\alpha\beta}\mathbf{n}, \qquad & \mathbf{n}_{,\beta} &= -\mathbf{k}_\beta \times \mathbf{n} - \bar{b}^\alpha_\beta \mathbf{s}_\alpha, \\ \mathbf{r}_{\alpha|\beta} &= \mathbf{l}_\beta \times \mathbf{r}_\alpha + b_{\alpha\beta}\bar{\mathbf{n}}, \qquad & \bar{\mathbf{n}}_{,\beta} &= \mathbf{l}_\beta \times \bar{\mathbf{n}} - b^\alpha_\beta \mathbf{r}_\alpha. \end{aligned}$$

Since $\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}$, we can solve Eqs. (5.6) and (5.14) for $\mathbf{u}_{,\alpha}$ and $\boldsymbol{\theta}_{,\alpha}$, what leads to

$$\text{(5.22)} \qquad \begin{aligned} \mathbf{u}_{,\alpha} &= \boldsymbol{\eta}_\alpha + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{s}_\alpha + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{s}_\alpha\right) = \boldsymbol{\varepsilon}_\alpha + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{r}_\alpha - \frac{1}{2}\boldsymbol{\theta} \times \mathbf{r}_\alpha\right), \\ \boldsymbol{\theta}_{,\alpha} &= \mathbf{k}_\alpha + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{k}_\alpha + \frac{1}{4}(\boldsymbol{\theta}\cdot\mathbf{k}_\alpha)\boldsymbol{\theta} = \mathbf{l}_\alpha - \frac{1}{2}\boldsymbol{\theta} \times \mathbf{l}_\alpha + \frac{1}{4}(\boldsymbol{\theta}\cdot\mathbf{l}_\alpha)\boldsymbol{\theta}. \end{aligned}$$

The integrability conditions $\varepsilon^{\alpha\beta}\mathbf{u}_{,\alpha\beta} = \mathbf{0}$ and $\varepsilon^{\alpha\beta}\boldsymbol{\theta}_{,\alpha\beta} = \mathbf{0}$ of the relations (5.22) give us the following two sets of vector equations:

$$\text{(5.23)} \qquad \varepsilon^{\alpha\beta}(\boldsymbol{\eta}_{\alpha|\beta} + \mathbf{k}_\beta \times \mathbf{s}_\alpha) = \mathbf{0}, \qquad \varepsilon^{\alpha\beta}\left(\mathbf{k}_{\alpha|\beta} + \frac{1}{2}\mathbf{k}_\beta \times \mathbf{k}_\alpha\right) = \mathbf{0},$$

$$\text{(5.24)} \qquad \varepsilon^{\alpha\beta}(\boldsymbol{\varepsilon}_{\alpha|\beta} + \mathbf{l}_\beta \times \mathbf{r}_\alpha) = \mathbf{0}, \qquad \varepsilon^{\alpha\beta}\left(\mathbf{l}_{\alpha|\beta} + \frac{1}{2}\mathbf{l}_\alpha \times \mathbf{l}_\beta\right) = \mathbf{0}.$$

These two sets of equations constitute two alternative vector forms of compatibility conditions in the nonlinear theory of thin shells. The second equation of the set (5.23) was derived independently by Chernykh and Shamina [43] and Pietraszkiewicz [184]. The vector equations (5.24) were derived first by Shkutin [239, 240] and independently by Axelrad [16] and Libai and Simmonds [133]. In component form the relations (5.24) were given already by Alumäe [5, 7] and in orthogonal coordinates by Reissner [208]. Since $\boldsymbol{\varepsilon}_\beta = \mathbf{R}\boldsymbol{\eta}_\beta$ and $\mathbf{l}_\beta = \mathbf{R}\mathbf{k}_\beta$, both sets of compatibility conditions are transformable to each other. Several other equivalent vector or tensor forms of compatibility conditions may also be constructed from the ones given by Pietraszkiewicz and Badur [199, 200] for the three-dimensional deformation of a continuum. The three-dimensional compatibility conditions of [199, 200] should be written on the reference surfaces $\mathscr{M}$ or $\bar{\mathscr{M}}$ and Kirchhoff–Love constraints should be taken into account.

Within the K–L type shell theory, finite rotations are expressible in terms of displacements by non-rational relations [185, 190, 192] expressed in the stretched basis

$$\mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{s}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} = \bar{a}^{\alpha\beta}(\delta^\lambda_\beta + \eta^\lambda_\beta)(\mathbf{a}_\alpha + \mathbf{u}_{,\alpha}) \otimes \mathbf{a}_\lambda + (n^\alpha \mathbf{a}_\alpha + n\mathbf{n}) \otimes \mathbf{n},$$

(5.25)
$$\boldsymbol{\Omega} = \frac{1}{2}(\mathbf{s}_\alpha \times \bar{\mathbf{a}}^\alpha + \mathbf{n} \times \bar{\mathbf{n}}) = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}[(\bar{\mathbf{n}} \cdot \mathbf{s}_\alpha - \varphi_\alpha)\mathbf{s}_\beta + (\mathbf{u}_{,\alpha} \cdot \mathbf{s}_\beta)\mathbf{n}] =$$

$$= \frac{1}{2}\varepsilon_{\alpha\beta}\{[n^\alpha - \bar{a}^{\lambda\mu}(\delta^\alpha_\lambda + \eta^\alpha_\lambda)\varphi_\mu]\mathbf{a}^\beta + \bar{a}^{\lambda\mu}(\delta^\alpha_\lambda + \eta^\alpha_\lambda) l^\beta_{\boldsymbol{\cdot}\mu}\mathbf{n}\},$$

or in the rotated basis

$$\mathbf{R} = \mathbf{r}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n} = \mathbf{r}^\alpha \otimes [(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\mathbf{r}_\lambda - \mathbf{u}_{,\alpha}] + \bar{\mathbf{n}} \otimes [\varphi_\alpha \bar{a}^{\alpha\beta}(\delta^\lambda_\beta + \eta^\lambda_\beta)\mathbf{r}_\lambda + n\bar{\mathbf{n}}],$$

(5.26)
$$\boldsymbol{\Omega} = \frac{1}{2}(\mathbf{a}_\alpha \times \mathbf{r}^\alpha + \mathbf{n} \times \bar{\mathbf{n}}) = \frac{1}{2}\bar{\varepsilon}^{\alpha\beta}[(\mathbf{u}_{,\alpha} \cdot \mathbf{r}_\beta)\bar{\mathbf{n}} + (n_\alpha - \mathbf{n} \cdot \mathbf{r}_\alpha)\mathbf{r}_\beta] =$$

$$= \frac{1}{2}\varepsilon_{\alpha\beta}\{[n^\alpha - \bar{a}^{\lambda\mu}(\delta^\alpha_\lambda + \eta^\alpha_\lambda)\varphi_\mu]\mathbf{r}^\beta + \bar{a}^{\lambda\mu}(\delta^\alpha_\lambda + \eta^\alpha_\lambda) l^\beta_{\boldsymbol{\cdot}\mu}\bar{\mathbf{n}}\}.$$

The dependence of rotations upon displacements can also be expressed implicitly, in the form of three constraint conditions [19]:

(5.27)
$$\mathbf{n} \cdot \boldsymbol{\eta}_\beta = \bar{\mathbf{n}} \cdot \boldsymbol{\varepsilon}_\beta = (\mathbf{R}\mathbf{n}) \cdot (\mathbf{a}_\beta + \mathbf{u}_{,\beta}) = n_\lambda l^\lambda_{\boldsymbol{\cdot}\beta} + n\varphi_\beta = 0,$$

$$\varepsilon^{\alpha\beta}\eta_{\alpha\beta} = \varepsilon^{\alpha\beta}\mathbf{a}_\alpha \cdot \boldsymbol{\eta}_\beta = \varepsilon^{\alpha\beta}\mathbf{r}_\alpha \cdot \boldsymbol{\varepsilon}_\beta = \varepsilon^{\alpha\beta}(\mathbf{R}\mathbf{a}_\alpha) \cdot (\mathbf{a}_\beta + \mathbf{u}_{,\beta} - \mathbf{R}\mathbf{a}_\beta) = \varepsilon^{\alpha\beta}(r_{\lambda\alpha} l^\lambda_{\boldsymbol{\cdot}\beta} + r_\alpha \varphi_\beta) = 0,$$

where n_λ, n, $r_{\lambda\alpha}$, r_α are given in terms of rotation components by Eqs. (5.16) and (5.17) while $l^\lambda_{\boldsymbol{\cdot}\beta}$, φ_β are expressed in terms of displacement components by the relations $(2.3)_1$.

5.2. Decomposition of deformation at the boundary

During the shell deformation the orthogonal triad $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$ of $\mathscr{C}$ transforms into the orthogonal triad $\bar{\mathbf{a}}_v$, $\bar{\mathbf{a}}_t$, $\bar{\mathbf{n}}$ of $\bar{\mathscr{C}}$, where $\bar{\mathbf{a}}_v = \bar{\mathbf{a}}_t \times \bar{\mathbf{n}}$. According to the polar decomposition (5.1), we obtain

(5.28)
$$\bar{\mathbf{a}}_t = \bar{\mathbf{a}}_\alpha t^\alpha = \mathbf{R}\mathbf{s}_t = \mathbf{V}\mathbf{r}_t, \qquad \mathbf{s}_t = \mathbf{U}\mathbf{t}, \qquad \mathbf{r}_t = \mathbf{r}_\alpha t^\alpha = \mathbf{R}\mathbf{t},$$

$$\bar{\mathbf{a}}_v = \mathbf{R}\mathbf{s}_v = \mathbf{V}\mathbf{r}_v, \qquad \mathbf{s}_v = \mathbf{U}\mathbf{v}, \qquad \mathbf{r}_v = \mathbf{r}_\alpha v^\alpha = \mathbf{R}\mathbf{v}.$$

Since $\mathbf{v}$ and $\mathbf{t}$ do not coincide, in general, with the principal directions of $\mathbf{U}$, the action of $\mathbf{U}$ on $\mathbf{v}$ and $\mathbf{t}$ consists of an extension by a factor $\bar{a}_t$ and a finite rotation about $\mathbf{n}$. This rotation may be described by the proper orthogonal tensor $\mathbf{Q}_U$. Similarly, the action of $\mathbf{V}$ on $\mathbf{r}_v$ and $\mathbf{r}_t$ consists of an extension by a factor $\bar{a}_t$ and the finite rotation performed with the help of the proper orthogonal tensor $\mathbf{Q}_V$. Both rotations are defined by

$$
\mathbf{Q}_U = \frac{1}{\bar{a}_t}(\mathbf{s}_\nu \otimes \mathbf{v} + \mathbf{s}_t \otimes \mathbf{t}) + \mathbf{n} \otimes \mathbf{n},
$$

(5.29)

$$
\mathbf{Q}_V = \frac{1}{\bar{a}_t}(\bar{\mathbf{a}}_\nu \otimes \mathbf{r}_\nu + \bar{\mathbf{a}}_t \otimes \mathbf{r}_t) + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}.
$$

It is convenient to replace two subsequent rotations performed by $\mathbf{Q}_U$ and $\mathbf{R}$, or $\mathbf{R}$ and $\mathbf{Q}_V$, by one total rotation performed by the proper orthogonal tensor

$$
\mathbf{R}_t = \mathbf{R}\mathbf{Q}_U = \mathbf{Q}_V \mathbf{R} = \frac{1}{\bar{a}_t}(\bar{\mathbf{a}}_\nu \otimes \mathbf{v} + \bar{\mathbf{a}}_t \otimes \mathbf{t}) + \bar{\mathbf{n}} \otimes \mathbf{n},
$$

(5.30)

$$
\bar{\mathbf{a}}_\nu = \bar{a}_t \mathbf{R}_t \mathbf{v}, \qquad \bar{\mathbf{a}}_t = \bar{a}_t \mathbf{R}_t \mathbf{t}, \qquad \bar{\mathbf{n}} = \mathbf{R}_t \mathbf{n}.
$$

Since $\mathbf{R}_t^T \mathbf{R}_t'$ and $\mathbf{R}_t' \mathbf{R}_t^T$ are skew-symmetric along $\mathscr{C}$, they are expressible in terms of their respective axial vectors $\mathbf{k}_t$ and $\mathbf{l}_t$, called the vectors of change of curvature of the boundary contour [178, 185], by the relations (cf. [199])

(5.31)
$$
\mathbf{R}_t^T \mathbf{R}_t' = \mathbf{k}_t \times \mathbf{1}, \qquad \mathbf{R}_t' \mathbf{R}_t^T = \mathbf{l}_t \times \mathbf{1}, \qquad \mathbf{l}_t = \mathbf{R}_t \mathbf{k}_t .
$$

Now derivatives of $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{n}}$ along $\mathscr{C}$ can be given by

$$
\bar{\mathbf{a}}_t' = \bar{a}_t \mathbf{R}_t \left[\frac{1}{\bar{a}_t^2} \gamma_{tt}' \mathbf{t} + (\boldsymbol{\varrho}_t + \mathbf{k}_t) \times \mathbf{t} \right],
$$

(5.32)

$$
\bar{\mathbf{n}}' = \mathbf{R}_t [(\boldsymbol{\varrho}_t + \mathbf{k}_t) \times \mathbf{n}],
$$

where in components in the reference basis

$$
\boldsymbol{\varrho}_t = \sigma_t \mathbf{v} + \tau_t \mathbf{t} + \varkappa_t \mathbf{n},
$$

(5.33)

$$
\mathbf{k}_t = -k_{tt} \mathbf{v} + k_{\nu t} \mathbf{t} - k_{nt} \mathbf{n}.
$$

Exact expressions for components of $\mathbf{k}_t$ in Eq. $(5.33)_2$ were given by Novozhilov and Shamina [178] and the author [184, 185]. In terms of physical components of the Lagrangian strain measures on $\mathscr{C}$ these expressions are [193]

$$
k_{tt} = \frac{1}{\bar{a}_t} [\sigma_t (\bar{a}_t - 1) + \varkappa_{tt}],
$$

(5.34)
$$
k_{\nu t} = \sqrt{\frac{a}{\bar{a}}} \left[\bar{a}_t (\tau_t + \varkappa_{\nu t}) + \frac{1}{\bar{a}_t} 2\gamma_{\nu t} (\sigma_t - \varkappa_{tt}) \right] - \tau_t,
$$

$$
k_{nt} = \varkappa_t \left(1 - \frac{1}{\bar{a}_t^2} \sqrt{\frac{\bar{a}}{a}} \right) - \frac{1}{\bar{a}_t^2} \sqrt{\frac{a}{\bar{a}}} 2\gamma_{\nu t} (\gamma_{tt}' + 2\varkappa_t \gamma_{\nu t}) +
$$

$$
+ \sqrt{\frac{a}{\bar{a}}} [2\gamma_{\nu t}' - (\gamma_{tt})_{,\nu} + 2\varkappa_\nu \gamma_{\nu t} + 2\varkappa_t (\gamma_{\nu\nu} - \gamma_{tt})].
$$

Using Eqs. (5.11) it is easy to express the components of $\mathbf{k}_t$ also in terms of physical components of the relative surface strain measures $\eta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ on $\mathscr{C}$.

During the shell deformation compatible with the K–L constraints, the shell boundary surface $\mathbf{p}(s, \zeta) = \mathbf{r}(s) + \zeta\mathbf{n}(s)$ deforms itself into the surface $\bar{\mathbf{p}}(s, \zeta) = \bar{\mathbf{r}}(s) + \zeta\bar{\mathbf{n}}(s)$. According to the discussion presented in Section 4.5, the boundary surface $\bar{\mathbf{p}}(s, \zeta)$ may be entirely described by assuming $\bar{\mathbf{r}} = \bar{\mathbf{r}}^*$ and $\varphi = \varphi^*$ along $\mathscr{C}_u$. These conditions constitute the basic (displacement) version of geometric boundary conditions for the nonlinear theory of shells.

The deformed boundary surface may also be described by the following differential equations [178]:

$$\bar{\mathbf{p}}_{,s} = \bar{\mathbf{r}}' + \zeta\bar{\mathbf{n}}', \quad \bar{\mathbf{p}}_{,\zeta} = \bar{\mathbf{n}}, \quad \bar{\mathbf{r}}' = \bar{\mathbf{a}}_t, \tag{5.35}$$

$$\bar{\mathbf{p}}_{,ss} = \bar{\mathbf{r}}'' + \zeta\bar{\mathbf{n}}'', \quad \bar{\mathbf{p}}_{,\zeta s} = \bar{\mathbf{n}}', \quad \bar{\mathbf{r}}'' = \bar{\mathbf{a}}_t'. \tag{5.36}$$

The set of equations (5.35) describes the surface $\bar{\mathbf{p}}(s, \zeta)$ implicitly, with accuracy up to a translation in the space. According to the relations $(5.30)_2$, the right-hand sides of Eqs. (5.35) are established if γ_{tt} and $\mathbf{R}_t$ are given along $\mathscr{C}_u$. The geometric conditions $\gamma_{tt} = \gamma_{tt}^*$, $\mathbf{R}_t = \mathbf{R}_t^*$ on $\mathscr{C}_u$ are called the kinematic boundary conditions of the nonlinear theory of shells.

Also the set of equations (5.36) describes the surface $\bar{\mathbf{p}}(s, \zeta)$ implicitly, with accuracy up to a translation and rotation in the space. According to (5.32) the right-hand sides of Eqs. (5.36) are established if γ_{tt}, $\mathbf{k}_t$ and $\mathbf{R}_t$ are given along $\mathscr{C}_u$. However, since $\mathbf{R}_t$ can always be included into the description of an arbitrary rotation in the space, it is enough to assume only γ_{tt} and $\mathbf{k}_t$ to be given on $\mathscr{C}_u$. The geometric conditions $\gamma_{tt} = \gamma_{tt}^*$, $\mathbf{k}_t = \mathbf{k}_t^*$ on $\mathscr{C}_u$ are called the deformational boundary conditions of the nonlinear theory of shells.

In the case of the geometrically nonlinear theory of shells, we can simplify the components of $\mathbf{k}_t$ given in the expressions (5.34) by omitting small strains with respect to the unity, what leads to [190, 192]

$$\begin{aligned} k_{tt} &\approx \varkappa_{tt} + \sigma_t\gamma_{tt}, \\ k_{vt} &\approx \varkappa_{vt} + 2(\sigma_t - \varkappa_{tt})\gamma_{vt} - \tau_t\gamma_{vv}, \\ k_{nt} &\approx 2\gamma_{vt}' - \gamma_{tt,v} + 2\varkappa_v\gamma_{vt} - \varkappa_t(\gamma_{tt} - \gamma_{vv}). \end{aligned} \tag{5.37}$$

In terms of the relative strain measures, these approximate relations are

$$\begin{aligned} k_{tt} &\approx \varrho_{tt} + (\tau_t + \varrho_{vt})\eta_{vt}, \\ k_{vt} &\approx \varrho_{vt} + \frac{1}{2}\tau_t(\eta_{tt} - \eta_{vv}) + \frac{3}{2}(\sigma_t - \varrho_{tt})\eta_{vt} - \frac{1}{2}(\sigma_v - \varrho_{vv})\eta_{vt}, \\ k_{nt} &\approx 2\eta_{vt}' - \eta_{tt,v} + 2\varkappa_v\eta_{vt} - \varkappa_t(\eta_{tt} - \eta_{vv}). \end{aligned} \tag{5.38}$$

These results were extended recently [41] to the large-strain theory of shells.

5.3. Shell equations in the rotated basis

Let us introduce the expressions (5.11) into the principle of virtual displacements (3.12), what gives

$$(5.39)\qquad \iint_{\mathcal{M}} (S^{\alpha\beta}\delta\eta_{\alpha\beta}+G^{\alpha\beta}\delta\varrho_{\alpha\beta})dA = \iint_{\mathcal{M}} (\mathbf{p}\cdot\delta\mathbf{u}+\mathbf{h}\cdot\delta\boldsymbol{\beta})dA + \int_{\mathcal{C}_f} (\mathbf{T}\cdot\delta\mathbf{u}+\mathbf{H}\cdot\delta\boldsymbol{\beta})ds,$$

where the following stress and strain measures have been used

$$(5.40)\qquad \begin{aligned} S^{\alpha\beta} &= N^{\alpha\beta}+\frac{1}{2}(\eta^{\alpha}_{\lambda}N^{\lambda\beta}+\eta^{\beta}_{\lambda}N^{\lambda\alpha})+\frac{1}{2}[(\mu^{\alpha\lambda}-b^{\alpha\lambda})M^{\beta}_{\lambda}+(\mu^{\beta\lambda}-b^{\beta\lambda})M^{\alpha}_{\lambda}],\\ G^{\alpha\beta} &= M^{\alpha\beta}+\frac{1}{2}(\eta^{\alpha}_{\lambda}M^{\lambda\beta}+\eta^{\beta}_{\lambda}M^{\lambda\alpha}), \end{aligned}$$

$$(5.41)\qquad \mu_{\alpha\beta} = \varrho_{\alpha\beta}+\frac{1}{2}\varepsilon_{\alpha\beta}\varrho, \qquad \varrho_{\alpha\beta} = \frac{1}{2}(\mu_{\alpha\beta}+\mu_{\beta\alpha}), \qquad \varrho = \varepsilon^{\alpha\beta}\mu_{\alpha\beta}.$$

Note that both surface stress measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and both surface strain measures $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$ are symmetric here by definition. They have been introduced first by Alumäe [8] and independently by Simmonds and Danielson [248].

Since $\delta\mathbf{R}\mathbf{R}^T$ is skew-symmetric, we express it in terms of its axial vector $\delta\boldsymbol{\omega}$ by [199, 137]

$$(5.42)\qquad \delta\mathbf{R}\mathbf{R}^T = -\mathbf{R}\delta\mathbf{R}^T = \delta\boldsymbol{\omega}\times\mathbf{1}, \qquad \delta\boldsymbol{\omega} = \frac{1}{t}\left(\delta\boldsymbol{\theta}-\frac{1}{2}\delta\boldsymbol{\theta}\times\boldsymbol{\theta}\right),$$

which, together with the relations $(5.3)_1$ and $(5.6)_3$ leads to

$$(5.43)\qquad \delta\mathbf{r}_{\alpha} = \delta\boldsymbol{\omega}\times\mathbf{r}_{\alpha}, \qquad \delta\boldsymbol{\beta} = \delta\bar{\mathbf{n}} = \delta\boldsymbol{\omega}\times\bar{\mathbf{n}}.$$

Taking variations of $\boldsymbol{\varepsilon}_{\beta}$ and $\boldsymbol{\lambda}_{\beta}$ given in the relations $(5.7)_2$ and $(5.9)_2$ and using Eqs. (2.1), (5.42) and (5.43), after transformations we also obtain

$$(5.44)\qquad \delta\eta_{\alpha\beta}\mathbf{r}^{\alpha} = \delta\mathbf{u}_{,\beta}+\bar{\mathbf{a}}_{\beta}\times\delta\boldsymbol{\omega}, \qquad \delta\mu_{\alpha\beta}\mathbf{r}^{\alpha} = \delta\boldsymbol{\omega}_{,\beta}\times\bar{\mathbf{n}}.$$

If we take variations of the constraint conditions (5.27) and use Eqs. (5.44), the following relations for $\delta\boldsymbol{\omega}$ in the rotated basis are established:

$$(5.45)\qquad \begin{aligned} \mathbf{r}^{\lambda}\cdot\delta\boldsymbol{\omega} &= \frac{1}{d}\varepsilon^{\alpha\beta}(\delta^{\lambda}_{\alpha}+\eta^{\lambda}_{\alpha})\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\beta},\\ \bar{\mathbf{n}}\cdot\delta\boldsymbol{\omega} &= \frac{1}{2+\eta^{\varkappa}_{\varkappa}}\varepsilon^{\beta\alpha}\mathbf{r}_{\alpha}\cdot\delta\mathbf{u}_{,\beta}. \end{aligned}$$

With the help of the relations (5.43)–(5.45), the principle of virtual displacements can be transformed into

$$(5.46)\quad -\iint_{\mathcal{M}} (\hat{\mathbf{N}}^{\beta}|_{\beta}+\mathbf{p})\cdot\delta\mathbf{u}\,dA + \sum_j (\hat{\mathbf{F}}_j-\hat{\mathbf{F}}_j^{*})\cdot\delta\mathbf{u}_j + \int_{\mathscr{C}_f} [(\hat{\mathbf{P}}-\hat{\mathbf{P}}^{*})\cdot\delta\mathbf{u}-(\hat{M}-\hat{M}^{*})\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}]\,ds = 0,$$

where

$$\hat{\mathbf{N}}^{\beta} = \left(S^{\alpha\beta}+\frac{1}{2}\varepsilon^{\alpha\beta}S\right)\mathbf{r}_{\alpha}+Q^{\beta}\bar{\mathbf{n}},$$

$$(5.47)\quad S = \frac{2}{2+\eta^{\alpha}_{\alpha}}\varepsilon_{\lambda\varkappa}[\eta^{\lambda}_{\varrho}S^{\varrho\varkappa}-(b^{\lambda}_{\varrho}-\mu^{\lambda}_{\cdot\varrho})G^{\varrho\varkappa}],$$

$$Q^{\beta} = \frac{1}{d}\varepsilon^{\beta\varrho}(\delta^{\lambda}_{\varrho}+\eta^{\lambda}_{\varrho})[\varepsilon_{\alpha\lambda}(G^{\alpha\varkappa}|_{\varkappa}+\mathbf{h}\cdot\mathbf{r}^{\alpha})-G^{\varkappa}_{\lambda}k_{\varkappa}],$$

$$\hat{\mathbf{P}} = \hat{\mathbf{N}}^{\beta}\nu_{\beta}+\hat{\mathbf{F}}', \qquad \hat{\mathbf{P}}^{*} = \mathbf{T}+\hat{\mathbf{F}}^{*\prime},$$

$$(5.48)\quad \hat{\mathbf{F}} = \frac{1}{d}\nu^{\alpha}(\delta^{\lambda}_{\alpha}+\eta^{\lambda}_{\alpha})\varepsilon_{\lambda\varkappa}G^{\varkappa\beta}\nu_{\beta}\bar{\mathbf{n}}, \qquad \hat{\mathbf{F}}^{*} = \frac{1}{\bar{a}_t}\left(K_t-\frac{1}{d}2\gamma_{\nu t}K_{\nu}\right)\bar{\mathbf{n}},$$

$$\hat{M} = -\frac{1}{d}t^{\alpha}(\delta^{\lambda}_{\alpha}+\eta^{\lambda}_{\alpha})\varepsilon_{\lambda\varkappa}G^{\varkappa\beta}\nu_{\beta}, \qquad \hat{M}^{*} = \bar{a}_t\frac{1}{d}K_{\nu}.$$

For arbitrary $\delta\mathbf{u}$, $\delta\mathbf{u}_j$ and $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$ from the form (5.46) follow the equilibrium equations and corresponding static boundary conditions

$$(5.49)\quad \hat{\mathbf{N}}^{\beta}|_{\beta}+\mathbf{p} = \mathbf{0} \quad \text{in } \mathcal{M}, \qquad \hat{\mathbf{P}} = \hat{\mathbf{P}}^{*}, \quad \hat{M} = \hat{M}^{*} \quad \text{on } \mathscr{C}_f,$$

$$(5.50)\quad \hat{\mathbf{F}}_j = \hat{\mathbf{F}}^{*}_j \quad \text{at each corner } M_j\in\mathscr{C}_f.$$

It was shown in [142] that $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$ is not integrable in terms of displacement derivatives on $\mathscr{C}$, i.e. there exists no displacement boundary conditions which would be work-conjugate to the relations (5.50). In this chapter we shall use the relations to derive the set of shell equations in terms of rotations and other field variables as independent variables. Therefore, there will be no need to use displacement boundary conditions. However, if one would like to discuss such work-conjugate static and geometric (displacement) boundary conditions, one should apply the general formula (4.30) to transform the corresponding boundary terms in the principle (5.39). Then some modified static boundary parameters $\hat{\mathbf{P}}$, $\hat{M}$, $\hat{\mathbf{F}}$ could be calculated to which there would correspond some work-conjugate displacement parameters $\mathbf{u}$, φ. In this way one could construct an alternative form of the Lagrangian shell equations written in terms of $S^{\alpha\beta}$, $G^{\alpha\beta}$, $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$, k_{β} as given functions of displacements and their derivatives. Here we are not interested in such alternative displacement shell equations.

The equilibrium equations (5.49) can be presented in component form in the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$, what gives

$$S^{\alpha\beta}|_\beta - \varepsilon^{\alpha\lambda} S^\beta_\lambda k_\beta + \frac{1}{2}\varepsilon^{\alpha\beta} S_{,\beta} + \frac{1}{2} a^{\alpha\beta} S k_\beta - Q^\beta \left(b^\alpha_\beta - \varrho^\alpha_\beta - \frac{1}{2} a^{\alpha\lambda} \varepsilon_{\lambda\beta} \varrho \right) + \hat{p}^\alpha = 0, \tag{5.51}$$

$$S^{\alpha\beta}(b_{\alpha\beta} - \varrho_{\alpha\beta}) - \frac{1}{2} S\varrho + Q^\beta|_\beta + q = 0,$$

where $\hat{p}^\alpha = \mathbf{p}\cdot\mathbf{r}^\alpha$ and S, Q^β are functions of $S^{\alpha\beta}$, $G^{\alpha\beta}$ given in Eqs. (5.47).

The dependence of rotations upon displacements has been explicitly taken here into account by applying the relations (5.45) in the transformation of the principle (5.39) into the form (5.46) and using in the form (5.46) the variational expression $\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\nu}$. However, when we intend to use rotations as independent variables, the dependence of $\mathbf{R}$ upon $\mathbf{u}$ should be implicitly taken into account. According to [19] this implicit dependence can be given by three constraint conditions (5.27) for the relative stretch vector $\boldsymbol{\varepsilon}_\beta = \eta_{\alpha\beta}\mathbf{r}^\alpha$. In terms of variations these constraints in $\mathcal{M}$ are

$$\varepsilon^{\alpha\beta}\mathbf{r}_\alpha\cdot\delta\eta_{\lambda\beta}\mathbf{r}^\lambda = 0, \qquad \bar{\mathbf{n}}\cdot\delta\eta_{\alpha\beta}\mathbf{r}^\alpha = 0. \tag{5.52}$$

Let $\frac{1}{2}S$ and Q^β be Lagrange multipliers associated with the respective constraints (5.27) and (5.52). Then the left-hand side of Eq. (5.39) can be presented in an alternative form

$$\iint_{\mathcal{M}} \left\{ \left[\left(S^{\alpha\beta} + \frac{1}{2}\varepsilon^{\alpha\beta} S \right) \mathbf{r}_\alpha + Q^\beta \bar{\mathbf{n}} \right] \cdot \delta\eta_{\lambda\beta}\mathbf{r}^\lambda + G^{\alpha\beta}\delta\varrho_{\alpha\beta} \right\} dA, \tag{5.53}$$

where $\delta\eta_{\alpha\beta}$ and $\delta\varrho_{\alpha\beta}$ are given by Eqs. (5.44) in terms of now independent $\delta\mathbf{u}$ and $\delta\boldsymbol{\omega}$.

Similar constraint conditions (5.52) should also be applied at the shell boundary, only then the constraint $(5.52)_2$ should be multiplied by t^β, what corresponds to the constraint $(2.15)_1$. If now A and B are Lagrange multipliers associated with the respective constraints (5.52) on $\mathscr{C}$, then we should add to the right-hand side of Eq. (5.39) the following line integral:

$$\int_{\mathscr{C}_f} (A\varepsilon^{\alpha\beta}\mathbf{r}_\alpha\cdot\delta\eta_{\lambda\beta}\mathbf{r}^\lambda + B\bar{\mathbf{n}}\cdot\delta\eta_{\alpha\beta}\mathbf{r}^\alpha t^\beta)\, ds. \tag{5.54}$$

Now the principle of virtual displacements (5.39), with the form (5.53) as the left-hand side of Eq. (5.39) and the integral (5.54) added to the right-hand side of Eq. (5.39), can be transformed with the help of the relations (5.44) into

$$\begin{aligned}
&-\iint_{\mathcal{M}} [(\hat{\mathbf{N}}^\beta|_\beta + \mathbf{p})\cdot\delta\mathbf{u} + (\hat{\mathbf{M}}^\beta|_\beta + \bar{\mathbf{a}}_\beta \times \hat{\mathbf{N}}^\beta + \bar{\mathbf{n}} \times \mathbf{h})\cdot\delta\boldsymbol{\omega}]\, dA + \\
&+ \int_{\mathscr{C}_f} \{[\hat{\mathbf{N}}^\beta \nu_\beta - \mathbf{T} + (A\mathbf{r}_\nu + B\bar{\mathbf{n}})']\cdot\delta\mathbf{u} + [\hat{\mathbf{M}}^\beta \nu_\beta - \bar{\mathbf{n}} \times \mathbf{H} + B\bar{\mathbf{a}}_\nu - \\
&- A(\mathbf{r}_\nu \times \bar{\mathbf{a}}_t - \mathbf{r}_t \times \bar{\mathbf{a}}_\beta \nu^\beta)]\cdot\delta\boldsymbol{\omega} + A\mathbf{r}_t\cdot\delta\mathbf{u}_{,\nu}\}\, ds + \sum_j (A\mathbf{r}_\nu + B\bar{\mathbf{n}})_j\cdot\delta\mathbf{u}_j = 0,
\end{aligned} \tag{5.55}$$

where $\hat{\mathbf{N}}^\beta$ is given by Eq. (5.47)$_1$ in terms of $S^{\alpha\beta}$, S and Q^β as independent variables and $\hat{\mathbf{M}}^\beta = \bar{\mathbf{n}} \times G^{\alpha\beta}\mathbf{r}_\alpha$.

It follows from the form (5.55) that for an arbitrary $\mathbf{r}_t \cdot \delta\mathbf{u}_{,\nu}$ on $\mathscr{C}_f$ we always have $A \equiv 0$, i.e. the constraint condition (5.27)$_2$ is always satisfied on $\mathscr{C}$. Taking this into account the line integral of the form (5.55) is reduced to

$$(5.56)\quad \int_{\mathscr{C}_f} \{[\hat{\mathbf{N}}^\beta \nu_\beta - \mathbf{T} + (B\bar{\mathbf{n}})']\cdot\delta\mathbf{u} + [\hat{\mathbf{M}}^\beta \nu_\beta - \bar{\mathbf{n}} \times \mathbf{H} + B\bar{\mathbf{a}}_\nu]\cdot\delta\boldsymbol{\omega}\}\, ds + \sum_j (B\bar{\mathbf{n}})_j \cdot \delta\mathbf{u}_j .$$

Since $\delta\mathbf{u}$ and $\delta\boldsymbol{\omega} \times \bar{\mathbf{n}}$ are now independent, from the relations (5.55) and (5.56) follow vector equilibrium equations and corresponding static boundary conditions:

$$(5.57)\qquad \hat{\mathbf{N}}^\beta|_\beta + \mathbf{p} = \mathbf{0}, \quad \hat{\mathbf{M}}^\beta|_\beta + \bar{\mathbf{a}}_\beta \times \hat{\mathbf{N}}^\beta + \bar{\mathbf{n}} \times \mathbf{h} = \mathbf{0} \quad \text{in } \mathscr{M},$$

$$(5.58)\qquad \hat{\mathbf{N}}^\beta \nu_\beta - \mathbf{T} + (B\bar{\mathbf{n}})' = \mathbf{0}, \quad G^{\alpha\beta}\mathbf{r}_\alpha \nu_\beta - \mathbf{H} - B\bar{\mathbf{a}}_t = \mathbf{0} \quad \text{on } \mathscr{C}_f,$$

$$(5.59)\qquad (B\bar{\mathbf{n}})_j = \mathbf{0} \quad \text{at each corner } M_j \in \mathscr{C}_f .$$

Corresponding work-conjugate geometric boundary conditions follow from the kinematic constraints $\delta\mathbf{u} = \mathbf{0}$, $\delta\boldsymbol{\omega} \times \bar{\mathbf{n}} = \delta\bar{\mathbf{n}} = \mathbf{0}$ on $\mathscr{C}_u$ and $\delta\mathbf{u}_i = \mathbf{0}$ on $M_i \in \mathscr{C}_u$. For independent displacements and rotations these constraints have the solutions

$$(5.60)\qquad \mathbf{u} = \mathbf{u}^*, \quad \mathbf{Rn} = \mathbf{R}^*\mathbf{n} \quad \text{on } \mathscr{C}_u,$$

$$(5.61)\qquad \mathbf{u}_i = \mathbf{u}_i^* \quad \text{at each corner } M_i \in \mathscr{C}_u .$$

The second of the solutions (5.60) is still subjected to the two constraints: (5.27)$_1$ multiplied by t^β and (5.27)$_2$. Therefore, in fact the solution (5.60)$_2$ describes implicitly only one scalar condition.

In components in the rotated basis Eq. (5.57)$_2$ takes the form

$$(5.62)\qquad \begin{aligned} &G^{\alpha\beta}|_\beta - \varepsilon^{\alpha\lambda} G^\beta_\lambda k_\beta - (\delta^\alpha_\beta + \eta^\alpha_\beta) Q^\beta + \hat{h}^\alpha = 0, \\ &S\left(1 + \frac{1}{2}\eta^\alpha_\alpha\right) - \varepsilon_{\alpha\lambda}\eta^\alpha_\beta S^{\lambda\beta} - \varepsilon_{\alpha\lambda} G^{\alpha\beta}(b^\lambda_\beta - \varrho^\lambda_\beta) - \frac{1}{2}G^\alpha_\alpha \varrho = 0, \end{aligned}$$

while the boundary conditions (5.58)$_2$ written relative to $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$ are

$$(5.63)\qquad \begin{aligned} &\frac{1}{\bar{a}_t} t_\varkappa (\delta^\varkappa_\alpha + \eta^\varkappa_\alpha) G^{\alpha\beta}\nu_\beta - K_t - B\bar{a}_t = 0, \\ &\frac{1}{\bar{a}_t} t^\varkappa (\delta^\lambda_\varkappa + \eta^\lambda_\varkappa)\varepsilon_{\alpha\lambda} G^{\alpha\beta}\nu_\beta - K_\nu = 0. \end{aligned}$$

Note that only two components appear in the conditions (5.63), since A has been eliminated.

The equilibrium equations (5.51) and (5.62) were derived by Alumäe [8]. Equivalent forms of equilibrium equations are given in [5, 248, 240, 19]. Boundary and corner conditions were not discussed in [5, 8], while the four static boundary conditions derived in [248, 19] would follow from our relation (5.58) after elimination of the Lagrange multiplier B with the help of the condition $(5.63)_1$. But then it is not apparent how to construct the work-conjugate geometric boundary condition corresponding to the constraint $\bar{\mathbf{v}}\cdot\delta\bar{\mathbf{n}} = 0$ on $\mathscr{C}_u$ used in [248] and to the equivalent constraint $\bar{\mathbf{t}}\cdot\delta\boldsymbol{\omega} = 0$ used in [19]. Therefore, the work-conjugate geometric boundary conditions were not discussed in [248, 19]. On the other hand, the kinematic parameter $\delta[\bar{\mathbf{t}}\cdot(\omega\mathbf{e})]$ used by Shkutin [240] cannot be regarded as to be equivalent to the one which would appear during the elimination of B from the relation $(5.58)_2$. It seems that the choice of such a parameter in [240] resulted from an identification of the axial vector $\delta\boldsymbol{\omega}$ defined by the relations (5.42) with the variation of the finite rotation vector $\omega\mathbf{e}$, what is correct only for infinitesimal rotations.

5.4. Alternative shell equations in the undeformed basis

Sometimes it may be more convenient to use an alternative form of nonlinear shell equations discussed in Section 5.3, which is referred entirely to the undeformed basis of $\mathscr{M}$. Having this in mind, let us introduce the axial vector $\delta\mathbf{w}$ of the skew-symmetric tensor $\mathbf{R}^T\delta\mathbf{R}$ in analogy to the relations (5.42) by [199]

$$\mathbf{R}^T\delta\mathbf{R} = -\delta\mathbf{R}^T\mathbf{R} = \delta\mathbf{w}\times\mathbf{1}, \qquad \delta\mathbf{w} = \frac{1}{t}\left(\delta\boldsymbol{\theta}+\frac{1}{2}\delta\boldsymbol{\theta}\times\boldsymbol{\theta}\right) \tag{5.64}$$

in terms of which variations of the relative strain measures $(5.7)_1$ and $(5.9)_1$ are given by

$$\begin{aligned} \delta\boldsymbol{\eta}_\beta &= \delta\eta_{\alpha\beta}\mathbf{a}^\alpha = \delta\mathbf{v}_{,\beta}+\mathbf{k}_\beta\times\delta\mathbf{v}+\mathbf{s}_\beta\times\delta\mathbf{w},\\ \delta\boldsymbol{\mu}_\beta &= \delta\mu_{\alpha\beta}\mathbf{a}^\alpha = \delta\mathbf{w}_{,\beta}\times\mathbf{n}+(\mathbf{k}_\beta\times\delta\mathbf{w})\times\mathbf{n}, \end{aligned} \tag{5.65}$$

$$\delta\mathbf{w} = \mathbf{R}^T\delta\boldsymbol{\omega}, \qquad \delta\mathbf{v} = \mathbf{R}^T\delta\mathbf{u}. \tag{5.66}$$

If the rotations are to be regarded as independent variables then the constraint conditions (5.52) are replaced by

$$\begin{aligned} &\varepsilon^{\alpha\beta}\mathbf{a}_\alpha\cdot\delta\boldsymbol{\eta}_\beta = 0, \qquad \mathbf{n}\cdot\delta\boldsymbol{\eta}_\beta = 0 \qquad \text{in } \mathscr{M},\\ &\varepsilon^{\alpha\beta}\mathbf{a}_\alpha\cdot\delta\boldsymbol{\eta}_\beta = 0, \qquad \mathbf{n}\cdot\delta\boldsymbol{\eta}_\beta t^\beta = 0 \qquad \text{on } \mathscr{C}. \end{aligned} \tag{5.67}$$

Let again $\frac{1}{2}S$, Q^β be Lagrange multipliers associated with the respective constraints $(5.67)_1$ in $\mathscr{M}$ and A, B be Lagrange multipliers associated with the respective constraints $(5.67)_2$ on $\mathscr{C}$. The constraint conditions $(5.67)_1$ multiplied by $\frac{1}{2}S$ and Q^β, respectively, may be introduced into the surface integral of the left-hand side of Eq. (5.39). Similarly, the constraint conditions $(5.67)_2$ multiplied by A and B, respectively, may be introduced into the line integral of the right-hand side of Eq. (5.39). Then, the so modified principle of virtual displacements can be transformed with the help of Eqs. (5.65) and (5.66) into

$$-\iint_{M} [(\tilde{\mathbf{N}}^{\beta}|_{\beta}+\mathbf{k}_{\beta}\times\tilde{\mathbf{N}}^{\beta}+\mathbf{R}^{T}\mathbf{p})\cdot\delta\mathbf{v}+(\tilde{\mathbf{M}}^{\beta}|_{\beta}+\mathbf{k}_{\beta}\times\tilde{\mathbf{M}}^{\beta}+\mathbf{s}_{\beta}\times\tilde{\mathbf{N}}^{\beta}+\mathbf{n}\times\mathbf{R}^{T}\mathbf{h})\cdot\delta\mathbf{w}]dA+$$

$$+\int_{\mathscr{C}_f}[\tilde{\mathbf{N}}^{\beta}\nu_{\beta}-\mathbf{R}_t^{T}\mathbf{T}+(B\mathbf{n})'-B\mathbf{n}\times\mathbf{k}_{\beta}t^{\beta}]\cdot\delta\mathbf{v}ds+ \tag{5.68}$$

$$+\int_{\mathscr{C}_f}[\tilde{\mathbf{M}}^{\beta}\nu_{\beta}-\mathbf{n}\times\mathbf{R}_t^{T}\mathbf{H}-B\mathbf{n}\times\mathbf{s}_{\beta}t^{\beta}]\cdot\delta\mathbf{w}_t ds+\sum_j (B\mathbf{n})_j\cdot\delta\mathbf{v}_j=0,$$

where now

$$\tilde{\mathbf{N}}^{\beta}=\left(S^{\alpha\beta}+\frac{1}{2}\varepsilon^{\alpha\beta}S\right)\mathbf{a}_{\alpha}+Q^{\beta}\mathbf{n}=\mathbf{R}^{T}\hat{\mathbf{N}}^{\beta},$$

$$\tilde{\mathbf{M}}^{\beta}=\mathbf{n}\times G^{\alpha\beta}\mathbf{a}_{\alpha}=\mathbf{R}^{T}\hat{\mathbf{M}}^{\beta}, \tag{5.69}$$

$$\delta\mathbf{w}_t=\mathbf{R}_t^{T}\delta\boldsymbol{\omega}_t=\frac{1}{\bar{a}_t}\mathbf{t}\times(\delta\mathbf{v}'+\mathbf{k}_{\beta}t^{\beta}\times\delta\mathbf{v})+(\mathbf{t}\cdot\delta\mathbf{w}_t)\mathbf{t}.$$

In the transformations leading to the principle (5.68) we have taken into account that $A\equiv 0$ on $\mathscr{C}_f$ for an arbitrary $\mathbf{t}\cdot\delta\mathbf{v}_{,\nu}$, in analogy to the reduction of the relations (5.55) to (5.56).

Since $\delta\mathbf{v}$ and $\delta\mathbf{w}$ are independent, from the form (5.68) follow vector equilibrium equations and corresponding static boundary conditions:

$$\left.\begin{aligned}\tilde{\mathbf{N}}^{\beta}|_{\beta}+\mathbf{k}_{\beta}\times\tilde{\mathbf{N}}^{\beta}+\mathbf{R}^{T}\mathbf{p}=\mathbf{0},\\ \tilde{\mathbf{M}}^{\beta}|_{\beta}+\mathbf{k}_{\beta}\times\tilde{\mathbf{M}}^{\beta}+\mathbf{s}_{\beta}\times\tilde{\mathbf{N}}^{\beta}+\mathbf{n}\times\mathbf{R}^{T}\mathbf{h}=\mathbf{0},\end{aligned}\right\}\text{ in } \mathscr{M}, \tag{5.70}$$

$$\left.\begin{aligned}\tilde{\mathbf{N}}^{\beta}\nu_{\beta}-\mathbf{R}_t^{T}\mathbf{T}+(B\mathbf{n})'-B\mathbf{n}\times\mathbf{k}_{\beta}t^{\beta}=\mathbf{0},\\ G^{\alpha\beta}\mathbf{a}_{\alpha}\nu_{\beta}-\mathbf{R}_t^{T}\mathbf{H}-B\mathbf{s}_t=\mathbf{0},\end{aligned}\right\}\text{ on } \mathscr{C}_f, \tag{5.71}$$

$$(B\mathbf{n})_j=\mathbf{0}\quad\text{at each corner } M_j\in\mathscr{C}_f. \tag{5.72}$$

The component form of the relations (5.70) in the undeformed basis $\mathbf{a}_{\alpha}$, $\mathbf{n}$ coincides with Eqs. (5.51) and (5.62) while the components of the relations (5.71) in the basis $\mathbf{v}$, $\mathbf{t}$ are equivalent to those given in the conditions (5.63).

Alternative forms of equilibrium equations written in the stretched basis $\mathbf{s}_{\alpha}$, $\mathbf{n}$ are given by the author [185, 190, 193] and in the rotated basis by Kayuk and Sakhatzkii [109].

5.5. Static-geometric analogy

Between the equilibrium equations (5.57) and the compatibility conditions (5.24) (or, equivalently, between the relations (5.70) and (5.23)) an interesting static-geometric analogy can be established. In order to show this analogy, let us express the compatibility conditions (5.24) in component form in the rotated basis, what gives

$$
(5.73)\quad
\begin{aligned}
&\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta}+\varepsilon^{\alpha\beta}(\delta^{\varkappa}_{\alpha}+\eta^{\varkappa}_{\alpha})\varepsilon_{\varkappa\lambda}k_{\beta}=0,\\
&\varrho\left(1+\frac{1}{2}\eta^{\varkappa}_{\varkappa}\right)-\varepsilon^{\alpha\beta}\eta^{\varkappa}_{\alpha}(b_{\varkappa\beta}-\varrho_{\varkappa\beta})=0,\\
&\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\varrho_{\lambda\alpha|\beta}+\frac{1}{2}\varepsilon^{\varkappa\beta}\varrho_{,\beta}-\varepsilon^{\alpha\beta}k_{\alpha}(b^{\varkappa}_{\beta}-\varrho^{\varkappa}_{\beta})+\frac{1}{2}a^{\alpha\varkappa}k_{\alpha}\varrho=0,\\
&\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\left(b_{\lambda\alpha}-\frac{1}{2}\varrho_{\lambda\alpha}\right)\varrho_{\varkappa\beta}-\frac{1}{4}\varrho^{2}+\varepsilon^{\alpha\beta}k_{\alpha|\beta}=0.
\end{aligned}
$$

Let us introduce modified measures by the formal relations

$$
(5.74)\quad
\begin{aligned}
&\tilde{\varrho}^{\alpha\beta}=-\varepsilon^{\alpha\sigma}\varepsilon^{\beta\tau}\varrho_{\sigma\tau}, && \tilde{\eta}^{\alpha\beta}=+\varepsilon^{\alpha\sigma}\varepsilon^{\beta\tau}\eta_{\sigma\tau}, && \tilde{k}^{\alpha}=+\varepsilon^{\alpha\sigma}k_{\sigma}, && \tilde{\varrho}=-\varrho,\\
&\varrho_{\alpha\beta}=-\varepsilon_{\alpha\sigma}\varepsilon_{\beta\tau}\tilde{\varrho}^{\sigma\tau}, && \eta_{\alpha\beta}=+\varepsilon_{\alpha\sigma}\varepsilon_{\beta\tau}\tilde{\eta}^{\sigma\tau}, && k_{\alpha}=-\varepsilon_{\alpha\sigma}\tilde{k}^{\sigma}, && \varrho=-\tilde{\varrho}.
\end{aligned}
$$

In terms of those modified measures the compatibility conditions (5.73) can be written in the form

$$
(5.75)\quad
\begin{aligned}
&\tilde{\eta}^{\alpha\beta}|_{\beta}-\frac{1}{2}\varepsilon^{\alpha\lambda}\tilde{\eta}^{\beta}_{\lambda}k_{\beta}-\left(\delta^{\alpha}_{\beta}+\frac{1}{2}\eta^{\alpha}_{\beta}\right)\tilde{k}^{\beta}=0,\\
&\tilde{\varrho}\left(1+\frac{1}{4}\eta^{\alpha}_{\alpha}\right)-\frac{1}{2}\varepsilon_{\alpha\lambda}\eta^{\alpha}_{\beta}\tilde{\varrho}^{\lambda\beta}-\varepsilon_{\alpha\lambda}\tilde{\eta}^{\alpha\beta}\left(b^{\lambda}_{\beta}-\frac{1}{2}\varrho^{\lambda}_{\beta}\right)-\frac{1}{4}\tilde{\eta}^{\alpha}_{\alpha}\varrho=0,\\
&\tilde{\varrho}^{\alpha\beta}|_{\beta}-\frac{1}{2}\varepsilon^{\alpha\lambda}\tilde{\varrho}^{\beta}_{\lambda}k_{\beta}+\frac{1}{2}\varepsilon^{\alpha\beta}\tilde{\varrho}_{,\beta}+\frac{1}{4}a^{\alpha\beta}\tilde{\varrho}k_{\beta}-\tilde{k}^{\beta}\left(b^{\alpha}_{\beta}-\frac{1}{2}\varrho^{\alpha}_{\beta}-\frac{1}{4}a^{\alpha\lambda}\varepsilon_{\lambda\beta}\varrho\right)=0,\\
&\tilde{\varrho}^{\alpha\beta}\left(b_{\alpha\beta}-\frac{1}{2}\varrho_{\alpha\beta}\right)-\frac{1}{4}\tilde{\varrho}\varrho+\tilde{k}^{\beta}|_{\beta}=0.
\end{aligned}
$$

If we compare Eqs. (5.75) with the equilibrium equations (5.62) and (5.51), we note that the homogeneous equilibrium equations can be transformed into the modified compatibility conditions (5.75) if $S^{\alpha\beta}$, $G^{\alpha\beta}$, Q^{β}, S are replaced by $\tilde{\varrho}^{\alpha\beta}$, $\tilde{\eta}^{\alpha\beta}$, $\tilde{k}^{\beta}$, $\tilde{\varrho}$, respectively, and all nonlinear (quadratic) terms are multiplied by 1/2. This static-geometric analogy was noted by Alumäe [7, 8]. It extends to the nonlinear theory of shells the static-geometric analogy of the linear theory of shells which was formulated in tensor form by Goldenveizer [76].

The compatibility conditions $(5.73)_{1,2}$ can always be solved for k_{α} and ϱ, what gives

$$
(5.76)\quad
\begin{aligned}
&k_{\alpha}=-\sqrt{\frac{a}{\bar{a}}}\varepsilon^{\varrho\varkappa}(\delta^{\lambda}_{\alpha}+\eta^{\lambda}_{\alpha})\eta_{\lambda\varrho|\varkappa},\\
&\varrho=\frac{1}{1+\frac{1}{2}\eta^{\varkappa}_{\varkappa}}\varepsilon^{\alpha\beta}\eta^{\lambda}_{\alpha}(b_{\lambda\beta}-\varrho_{\lambda\beta}).
\end{aligned}
$$

Similarly, the equilibrium equations (5.62) can be solved for Q^β, S what gives the formulae $(5.47)_{2,3}$. Then k_α, ϱ, Q^β and S may be eliminated from the remaining equilibrium equations (5.51) and the compatibility conditions $(5.73)_{3,4}$, which then are expressed entirely in terms of symmetric measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$. Unfortunately, for such a transformed set of 3+3 equations the static-geometric analogy formulated above does not hold.

5.6. Shell equations in terms of rotations, displacements and Lagrange multipliers

Various nonlinear shell relations discussed in the preceding sections allow for some freedom in choosing independent field variables of an appropriate boundary value problem.

An interesting version of the nonlinear theory of shells can be given in terms of finite rotations $\boldsymbol{\theta}$, displacements $\mathbf{u}$ and Lagrange multipliers S, Q^β as independent field variables [19].

In terms of corresponding stress and strain measures $S^{\alpha\beta}$, $G^{\alpha\beta}$ and $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$, the strain energy density $(3.32)_1$ and the constitutive equations (3.34) are

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}\left(\eta_{\alpha\beta}\eta_{\lambda\mu} + \frac{h^2}{12}\varrho_{\alpha\beta}\varrho_{\lambda\mu}\right) + O(Eh\eta^2\theta^2), \tag{5.77}$$

$$\begin{aligned} S^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\varrho_{\alpha\beta}} = C[(1-\nu)\eta^{\alpha\beta} + \nu a^{\alpha\beta}\eta^\varkappa_\varkappa] + O(Eh\eta\theta^2), \qquad C = \frac{Eh}{1-\nu^2}, \\ G^{\alpha\beta} &= \frac{\partial\Sigma}{\partial\varrho_{\alpha\beta}} = D[(1-\nu)\varrho^{\alpha\beta} + \nu a^{\alpha\beta}\varrho^\varkappa_\varkappa] + O(Eh^2\eta\theta^2), \qquad D = \frac{Eh^3}{12(1-\nu^2)}. \end{aligned} \tag{5.78}$$

Let the constitutive equations (5.78) together with (5.76) be introduced into the equilibrium equations (5.51), (5.62) and then $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$ be expressed in terms of $\mathbf{u}$, $\boldsymbol{\theta}$ with the help of the geometric relations (5.19) and (5.20). As a result, the problem is reduced to nine partial differential equations: six equilibrium equations (5.19), (5.20) expressed in terms of $\boldsymbol{\theta}$, $\mathbf{u}$, S, Q^β and three constraint conditions (5.27) containing only $\boldsymbol{\theta}$, $\mathbf{u}$. Corresponding work-conjugate static and geometric boundary and corner conditions are given by the relations (5.58), (5.60) and (5.59), (5.61), respectively, together with one constraint condition $(5.27)_1$ multiplied by t^β.

The structure of such final set of nine equations is relatively simple. The equilibrium equations (5.51), (5.62) are linear in S, Q^β and their first derivatives, are quadratic in $\mathbf{u}_{,\beta}$ but linear in $\mathbf{u}_{,\alpha\beta}$ while rotations appear in them as polynomials which are quadratic in $\boldsymbol{\theta}_{,\beta}$ but again only linear in $\boldsymbol{\theta}_{,\alpha\beta}$. The constraint conditions (5.27) are polynomials in rotations but linear in $\mathbf{u}_{,\beta}$.

This system of nine nonlinear equations may be considerably simplified in the case of small strains, when additionally we assume that the strains caused by stretching and bending of the reference surface are of comparable order, i.e. $\eta_{\alpha\beta} \sim h\varrho_{\alpha\beta}$. Within the accuracy of the first-approximation theory from the compatibility conditions (5.73) we obtain the estimates

$$k_\beta = O(\eta/\lambda), \qquad \varrho = O\left(\frac{\eta\theta^2}{h}\right), \qquad \varrho^\alpha_{\alpha|\beta} - \varrho^\alpha_{\beta|\alpha} = O\left(\frac{\eta\theta^2}{h\lambda}\right) \tag{5.79}$$

which, introduced together with Eqs. (5.78) into Eqs. (5.51) and (5.62), lead to the following consistently approximated equilibrium equations

$$\begin{aligned} &C[(1-\nu)\eta^\beta_\alpha|_\beta + \nu\eta^\beta_{\beta|\alpha}] + \hat{p}^\alpha = O\left(\mathrm{E}h\frac{\eta\theta^2}{\lambda}\right),\\ &C(b^\alpha_\beta - \varrho^\alpha_\beta)[(1-\nu)\eta^\beta_\alpha + \nu\delta^\beta_\alpha\eta^\lambda_\lambda] + Q^\beta|_\beta + q = O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right),\\ &D\varrho^\beta_\beta|^\alpha - Q^\alpha + \hat{h}^\alpha = O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda}\right),\\ &S - \varepsilon_{\lambda\beta}\{C\eta^\lambda_\alpha[(1-\nu)\eta^{\alpha\beta} + \nu a^{\alpha\beta}\eta^\varkappa_\varkappa] - D(b^\lambda_\alpha - \varrho^\lambda_\alpha)[(1-\nu)\varrho^{\alpha\beta} + \nu a^{\alpha\beta}\varrho^\varkappa_\varkappa] = O(\mathrm{E}h^2\eta^2\theta^2). \end{aligned} \tag{5.80}$$

Within this approximation S apears only in the last algebraic equation $(5.80)_4$ and can be evaluated separately. Equation $(5.80)_3$ can also be solved for Q^α and introduced into Eq. $(5.80)_2$, which then takes the form

$$D\varrho^\alpha_\alpha|^\beta_\beta + C(b^\alpha_\beta - \varrho^\alpha_\beta)[(1-\nu)\eta^\beta_\alpha + \nu\delta^\beta_\alpha\eta^\varkappa_\varkappa] + q + \hat{h}^\alpha|_\alpha = O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right). \tag{5.81}$$

If now the expressions (5.19) and (5.20) are introduced into Eqs. $(5.80)_1$ and (5.81), then we obtain

$$\begin{aligned} &Ca^{\beta\varrho}\left[\frac{1}{2}(1-\nu)(r_{\lambda\alpha}l^\lambda_{\cdot\varrho} + r_{\lambda\varrho}l^\lambda_{\cdot\alpha} + r_\alpha\varphi_\varrho + r_\varrho\varphi_\alpha)|_\beta + \nu(r_{\lambda\beta}l^\lambda_{\cdot\varrho} + r_\beta\varphi_\varrho)|_\alpha\right] + \hat{p}_\alpha = O\left(\mathrm{E}h\frac{\eta\theta^2}{\lambda}\right),\\ &D\left\{\frac{1}{t}\left[\varepsilon^{\alpha\lambda}\left(\psi_{\lambda\alpha} + \frac{1}{2}\theta_\lambda\psi_\alpha\right) - \frac{1}{2}\psi^\alpha_{\cdot\alpha}\theta_3\right]\right\}\Big|^\beta_\beta +\\ &+ C\left\{b^\alpha_\beta - \frac{1}{2t}a^{\alpha\varkappa}\left[\varepsilon_{\varkappa\gamma}\left(\psi^\gamma_{\cdot\beta} + \frac{1}{2}\theta^\gamma\psi_\beta\right) + \varepsilon_{\beta\gamma}\left(\psi^\gamma_{\cdot\varkappa} + \frac{1}{2}\theta^\gamma\psi_\varkappa\right) - \frac{1}{2}(\psi_{\varkappa\beta} + \psi_{\beta\varkappa})\theta_3\right]\right\} \times\\ &\times [(1-\nu)a^{\beta\varrho}(r_{\lambda\alpha}l^\lambda_{\cdot\varrho} + r_\alpha\varphi_\varrho) + \nu\delta^\beta_\alpha a^{\varkappa\varrho}(r_{\lambda\varkappa}l^\lambda_{\cdot\varrho} + r_\varkappa\varphi_\varrho)] + q + \hat{h}^\alpha|_\alpha = O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right). \end{aligned} \tag{5.82}$$

Equations (5.82) together with the conditions (5.27) give us six partial differential equations for six components of $\boldsymbol{\theta}$, $\mathbf{u}$ to be solved.

Simplified static boundary and corner conditions on $\mathscr{C}_f$ follow from the corresponding reduction of the relations (5.58) and (5.59), with the help of Eqs. (5.78), (5.32) and (5.38). In the right-hand sides only the principal terms, which have the same structure as those in the approximate left-hand sides, are taken into account.

As a result, we obtain on $\mathscr{C}_f$

$$C(\eta_{\nu\nu}+\nu\eta_{tt}) = Q_\nu + O(Eh\eta\theta^2), \qquad C(1-\nu)\eta_{\nu t} = Q_t + O(Eh\eta\theta^2),$$

(5.83)
$$D[(\varrho_{\nu\nu}+\nu\varrho_{tt})_{,\nu}+2(1-\nu)(\varrho'_{\nu t}+\varkappa_\nu\varrho_{\nu t})+(1-\nu)\varkappa_t(\varrho_{\nu\nu}-\varrho_{tt})]+\hat{h}_\nu =$$
$$= Q+K'_t+O\left(Eh^2\frac{\eta\theta^2}{\lambda}\right), \qquad D(\varrho_{\nu\nu}+\nu\varrho_{tt}) = K_\nu + O(Eh^2\eta\theta^2)$$

and

(5.84)
$$D(1-\nu)[\varrho_{\nu t}\bar{\mathbf{n}}]_j = [K_t\bar{\mathbf{n}}]_j + O(Eh^2\eta\theta^2) \quad \text{on } M_j\in\mathscr{C}_f,$$

where the relative strain measures still have to be expressed in terms of components of $\boldsymbol{\theta}$, $\mathbf{u}$ by the expressions (5.19) and (5.20). Corresponding work-conjugate geometric boundary and corner conditions are given in the solutions (5.60) and (5.61), with the condition $(5.27)_1$ multiplied by t^β as the constraint.

Let us assume that the external loads $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$ are derivable from the potential functions $\Phi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})]$ and $\Psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})]$ by the relations (4.4). Note that now $\mathbf{u}$ and $\mathbf{R}$ may be treated as independent variables, what allows for some flexibility in the definition of the conservative loads. If the external loads are conservative, the total potential energy of the shell is given by the functional

(5.85)
$$I = \iint_{\mathscr{M}}\left\{\Sigma[\eta_{\alpha\beta}(\mathbf{u},\mathbf{R}),\varrho_{\alpha\beta}(\mathbf{R})]+\frac{1}{2}\varepsilon^{\alpha\beta}S\eta_{\alpha\beta}(\mathbf{u},\mathbf{R})+Q^\beta\mathbf{n}\mathbf{R}^T\boldsymbol{\varepsilon}_\beta(\mathbf{u},\mathbf{R})+\Phi[\mathbf{u},\boldsymbol{\beta}(\mathbf{R})]\right\}dA+$$
$$+\int_{\mathscr{C}_f}\{\Psi[\mathbf{u},\boldsymbol{\beta}(\mathbf{R})]-B\mathbf{n}\mathbf{R}^T\boldsymbol{\varepsilon}_\beta(\mathbf{u},\mathbf{R})t^\beta\}\,ds$$

with the geometric boundary (5.60) and corner (5.61) conditions and the constraints (5.27) on $\mathscr{C}_u$ as subsidiary conditions. The variational principle $\delta I = 0$ states that among all possible values of independent fields $\mathbf{u}$, $\mathbf{R}$, S, Q^β and B, which are subjected to the conditions (5.60), (5.61) and (5.27) on $\mathscr{C}_u$, the actual solution renders the functional I stationary. The stationarity conditions of I are: the equilibrium equations (5.57) in $\mathscr{M}$, the constraint conditions (5.27) in $\mathscr{M}$, the static boundary and corner conditions (5.58), (5.59) on $\mathscr{C}_f$ and the constraint condition $\bar{\mathbf{n}}\cdot\boldsymbol{\varepsilon}_\beta t^\beta = 0$ on $\mathscr{C}_f$.

Note that the functional I defined by Eq. (5.85) is linear in S, Q^β, B and is rational in $\mathbf{u}$, $\mathbf{R}$ and their only **first** surface derivatives. The latter property is important for the computerized numerical analysis of the flexible shells based on direct discretization of the functional (5.85). It allows to apply the simplest shape functions in the finite-element analysis or the simplest difference schemes in the finite-difference analysis, which assure high efficiency of numerical algorithms and better convergence to the accurate final results.

In some applications it may be convenient to apply the more general free functional

$$I_1 = \iint_{\mathcal{M}} \Big\{ \Sigma(\eta_{\alpha\beta}, \varrho_{\alpha\beta}) + \frac{1}{2}\varepsilon^{\alpha\beta} S\eta_{\alpha\beta} + Q^\beta \bar{\mathbf{n}}\cdot\boldsymbol{\varepsilon}_\beta -$$

$$- S^{\alpha\beta}[\eta_{\alpha\beta} - \eta_{\alpha\beta}(\mathbf{u}, \mathbf{R})] - G^{\alpha\beta}[\varrho_{\alpha\beta} - \varrho_{\alpha\beta}(\mathbf{R})] + \Phi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})] \Big\} dA +$$

(5.86)

$$+ \int_{\mathscr{C}_f} \Psi[\mathbf{u}, \boldsymbol{\beta}(\mathbf{R})] ds - \int_{\mathscr{C}} B\mathbf{n}\mathbf{R}^T\cdot\boldsymbol{\varepsilon}_\beta(\mathbf{u}, \mathbf{R}) t^\beta ds - \sum_i (B\bar{\mathbf{n}})_i\cdot(\mathbf{u}_i - \mathbf{u}_i^*) -$$

$$- \int_{\mathscr{C}_u} \{[\hat{\mathbf{N}}^\beta \nu_\beta + (B\bar{\mathbf{n}})']\cdot(\mathbf{u} - \mathbf{u}^*) + [G^{\alpha\beta}\mathbf{r}_\alpha \nu_\beta - B\bar{\mathbf{a}}_t]\cdot(\mathbf{R}\mathbf{n} - \mathbf{R}^*\mathbf{n})\} ds.$$

This free functional follows from the functional (5.85) if we introduce into it the strain-displacement-rotation relations (5.19), (5.20), the geometric boundary conditions (5.60) and the geometric corner conditions (5.61) multiplied by the respective Lagrange multipliers $P^{\alpha\beta}$, $K^{\alpha\beta}$, $\mathbf{P}$, $\mathbf{K}$, $\mathbf{S}_i$. Then some stationarity conditions of so defined I_1 allow to identify the Lagrange multipliers to be $S^{\alpha\beta}$, $G^{\alpha\beta}$, $\hat{\mathbf{N}}^\beta \nu_\beta + (B\bar{\mathbf{n}})'$, $G^{\alpha\beta}\mathbf{r}_\alpha \nu_\beta - B\bar{\mathbf{a}}_t$ and $(B\bar{\mathbf{n}})_i$, respectively, which have already been used in Eq. (5.86). The functional I_1 in Eq. (5.86) is defined on the following free fields subject to variation: $\mathbf{u}$, $\mathbf{R}$ in $\mathcal{M}$, $\mathbf{u}$, $\mathbf{R}$ on $\mathscr{C}$, $\mathbf{u}_i$ at each $M_i \in \mathscr{C}_u$, $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$, $S^{\alpha\beta}$, $G^{\alpha\beta}$, S, Q^β in $\mathcal{M}$, $S^{\alpha\beta}$, $G^{\alpha\beta}$, S, Q^β, B on $\mathscr{C}_u$, B on $\mathscr{C}_f$ and B_i on $M_i \in \mathscr{C}_u$. The variational principle $\delta I_1 = 0$ is equivalent to the complete set of nonlinear shell equations: (5.57), (5.58), (5.59), (5.27), (5.19), (5.20), (5.60), (5.61) and (5.78).

5.7. Shell equations in terms of rotations and stress functions

If all the external forces are functions of the finite rotations alone, the set of nonlinear shell equations can be expressed in terms of the finite rotation vector $\boldsymbol{\theta}$ and the stress function vector $\mathbf{F}$. Such equations were first proposed by Simmonds and Danielson [247, 248].

When rotations are taken as independent variables, the rotational compatibility conditions $(5.24)_2$ or $(5.73)_{3,4}$ are identically satisfied. The force equilibrium equations $(5.57)_1$ can also be satisfied if we introduce the stress function vector $\mathbf{F} = F^\alpha \mathbf{r}_\alpha + F\bar{\mathbf{n}}$ such that

$$\hat{\mathbf{N}}^\beta = \varepsilon^{\beta\alpha}\mathbf{F}_{,\alpha} + \mathbf{P}^\beta, \qquad \mathbf{P}^\beta = P^{\alpha\beta}\mathbf{r}_\alpha + P^\beta \bar{\mathbf{n}}, \tag{5.87}$$

where $\mathbf{P}^\beta$ is a particular solution of Eq. (5.57). Now it follows from Eqs. $(5.87)_1$ and $(5.47)_1$ that $S^{\alpha\beta}$, S and Q^β are prescribed functions of $\boldsymbol{\theta}$, $\mathbf{F}$. It remains to satisfy the moment equilibrium equations $(5.57)_2$, the tangential compatibility conditions $(5.24)_1$ (since here $\mathbf{u}$ will not be regarded as an independent variable) and to eliminate $\mathbf{u}$ and B from the boundary conditions.

Let conservative surface and boundary loads be defined in terms of potentials

$\Phi[\bar{\mathbf{r}}, \bar{\mathbf{n}}(\boldsymbol{\theta})]$, $\Psi[\bar{\mathbf{r}}, \bar{\mathbf{n}}(\boldsymbol{\theta})]$ by

$$\mathbf{p} = -\frac{\partial\Phi}{\partial\bar{\mathbf{r}}}, \quad \mathbf{h} = -\frac{\partial\Phi}{\partial\bar{\mathbf{n}}}, \quad \mathbf{T} = -\frac{\partial\Psi}{\partial\bar{\mathbf{r}}}, \quad \mathbf{H} = -\frac{\partial\Psi}{\partial\bar{\mathbf{n}}}. \tag{5.88}$$

Let us apply the Legendre transformation (3.36) only to the first part Σ_η of the strain energy density given in Eq. (5.77), which contains squares of $\eta_{\alpha\beta}$. Let us also introduce the tangential compatibility conditions $(5.24)_1$ into the functional (5.86) with the help of the Lagrange multiplier $\mathbf{F}$. If the relation (5.19) is also a priori satisfied, then the functional (5.86) can be written in the form

$$\begin{aligned} J_1 = \iint_{\mathscr{M}} \Big\{ & S^{\alpha\beta}\eta_{\alpha\beta} - \Sigma_S^C(S^{\alpha\beta}) + \Sigma_\varrho(\varrho_{\alpha\beta}) + \frac{1}{2}S\varepsilon^{\alpha\beta}\eta_{\alpha\beta} + Q^\beta\bar{\mathbf{n}}\cdot\eta_{\alpha\beta}\mathbf{r}^\alpha - \\ & - G^{\alpha\beta}[\varrho_{\alpha\beta} - \varrho_{\alpha\beta}(\boldsymbol{\theta})] + \varepsilon^{\alpha\beta}(\boldsymbol{\varepsilon}_{\alpha|\beta} + \mathbf{l}_\beta \times \mathbf{r}_\alpha)\cdot\mathbf{F} + \Phi[\bar{\mathbf{r}}, \bar{\mathbf{n}}(\boldsymbol{\theta})]\Big\} dA + \\ & + \int_{\mathscr{C}_f} \Psi[\bar{\mathbf{r}}, \bar{\mathbf{n}}(\boldsymbol{\theta})]\,ds - \int_{\mathscr{C}} (A\varepsilon^{\alpha\beta}\mathbf{r}_\alpha\cdot\eta_{\lambda\beta}\mathbf{r}^\lambda + B^\beta\bar{\mathbf{n}}\cdot\eta_{\lambda\beta}\mathbf{r}^\lambda)\,ds - \\ & - \int_{\mathscr{C}_u} \{\mathbf{L}\cdot(\bar{\mathbf{r}} - \bar{\mathbf{r}}^*) + \mathbf{M}\cdot[\bar{\mathbf{n}}(\boldsymbol{\theta}) - \bar{\mathbf{n}}(\boldsymbol{\theta}^*)]\}\,ds - \sum_i \mathbf{K}_i\cdot(\bar{\mathbf{r}}_i - \bar{\mathbf{r}}_i^*), \end{aligned} \tag{5.89}$$

where A, B^β and $\mathbf{L}$, $\mathbf{M}$, $\mathbf{K}_i$ are corresponding Lagrange multipliers associated with the constraints (5.27) on $\mathscr{C}$ and with the geometric boundary and corner conditions (5.60) and (5.61).

The variational principle $\delta J_1 = 0$ allows us to find various stationarity conditions of J_1, among which are relations that identify the Lagrange multipliers A, B_ν, $\mathbf{L}$, $\mathbf{M}$, $\mathbf{K}_i$ to be

$$\begin{gathered} A = 0, \quad B_\nu = 0, \\ \mathbf{L} = \mathbf{P}^\beta\nu_\beta + \mathbf{F}' + (B_t\bar{\mathbf{n}})', \quad \mathbf{M} = G^{\alpha\beta}\mathbf{r}_\alpha\nu_\beta - B_t\bar{\mathbf{a}}_t, \\ \mathbf{K}_i = \mathbf{F}_i + (B_t\bar{\mathbf{n}})_i. \end{gathered} \tag{5.90}$$

In order to eliminate the free field variables $\eta_{\alpha\beta}$, $G^{\alpha\beta}$, $\varrho_{\alpha\beta}$, $S^{\alpha\beta}$, S, Q^β in $\mathscr{M}$, $\bar{\mathbf{r}}$ on $\mathscr{C}_u$ and $\bar{\mathbf{r}}_i$ at each $M_i \in \mathscr{C}_u$, let us assume that the following stationarity conditions of J_1 are a priori satisfied:

$$\begin{gathered} \eta_{\alpha\beta} = \frac{\partial\Sigma_S^C}{\partial S^{\alpha\beta}}, \quad G^{\alpha\beta} = \frac{\partial\Sigma_\varrho}{\partial\varrho_{\alpha\beta}}, \quad \varrho_{\alpha\beta} = \varrho_{\alpha\beta}(\boldsymbol{\theta}), \quad \hat{\mathbf{N}}^\beta = \varepsilon^{\beta\alpha}\mathbf{F}_{,\alpha} + \mathbf{P}^\beta \quad \text{in } \mathscr{M}, \\ \bar{\mathbf{r}} = \bar{\mathbf{r}}^* \quad \text{on } \mathscr{C}_u, \quad \bar{\mathbf{r}}_i = \bar{\mathbf{r}}_i^* \quad \text{at each } M_i \in \mathscr{C}_u. \end{gathered} \tag{5.91}$$

If now the relations (5.90) and (5.91) are used, then the functional (5.89) is reduced to

$$
\begin{aligned}
J_2 = \iint_{\mathscr{M}} \{ & (\varepsilon^{\beta\alpha}\mathbf{F}_{,\alpha}+\mathbf{P}^{\beta})\cdot\boldsymbol{\varepsilon}_{\beta}(\mathbf{F},\boldsymbol{\theta})-\Sigma_S^C(\mathbf{F},\boldsymbol{\theta})+\Sigma_{\varrho}(\boldsymbol{\theta})+ \\
& +\varepsilon^{\alpha\beta}[\boldsymbol{\varepsilon}_{\alpha|\beta}(\mathbf{F},\boldsymbol{\theta})+\mathbf{l}_{\beta}(\boldsymbol{\theta})\times\mathbf{r}_{\alpha}(\boldsymbol{\theta})]\cdot\mathbf{F}+\mathbf{P}^{\beta}|_{\beta}\cdot\bar{\mathbf{r}}+f[\bar{\mathbf{n}}(\boldsymbol{\theta})]\}\,dA+ \\
& +\int_{\mathscr{C}_f}\{-\mathbf{T}\cdot\bar{\mathbf{r}}+g[\bar{\mathbf{n}}(\boldsymbol{\theta})]\}\,ds-\int_{\mathscr{C}} B_t\bar{\mathbf{n}}(\boldsymbol{\theta})\cdot\boldsymbol{\varepsilon}_{\beta}(\mathbf{F},\boldsymbol{\theta})t^{\beta}\,ds- \\
& -\int_{\mathscr{C}_u}[G^{\alpha\beta}(\boldsymbol{\theta})\mathbf{r}_{\alpha}(\boldsymbol{\theta})\nu_{\beta}-B_t\bar{\mathbf{r}}^{*\prime}]\cdot[\bar{\mathbf{n}}(\boldsymbol{\theta})-\bar{\mathbf{n}}(\boldsymbol{\theta}^*)]\,ds,
\end{aligned}
\tag{5.92}
$$

where $\Phi = \mathbf{P}^{\beta}|_{\beta}\cdot\bar{\mathbf{r}}+f[\bar{\mathbf{n}}(\boldsymbol{\theta})]$ and $\Psi = -\mathbf{T}\cdot\bar{\mathbf{r}}+g[\bar{\mathbf{n}}(\boldsymbol{\theta})]$ have been used.

Since $\bar{\mathbf{a}}_{\alpha|\beta} = \mathbf{l}_{\beta}\times\mathbf{r}_{\alpha}+b_{\alpha\beta}\bar{\mathbf{n}}+\boldsymbol{\varepsilon}_{\alpha|\beta}$ in $\mathscr{M}$, the second line of Eq. (5.92) can be transformed further to the form

$$
\iint_{\mathscr{M}}[-(\varepsilon^{\beta\alpha}\mathbf{F}_{,\alpha}+\mathbf{P}^{\beta})\cdot\bar{\mathbf{a}}_{\beta}+f]\,dA+\int_{\mathscr{C}_f}(\mathbf{P}^{\beta}\nu_{\beta}\cdot\bar{\mathbf{r}}-\mathbf{F}\cdot\bar{\mathbf{r}}')\,ds+\int_{\mathscr{C}_u}(\mathbf{P}^{\beta}\nu_{\beta}\cdot\bar{\mathbf{r}}^*-\mathbf{F}\cdot\bar{\mathbf{r}}^{*\prime})\,ds.
\tag{5.93}
$$

It follows from the relations (5.92) and (5.93) that on $\mathscr{C}_f$ we still have to eliminate $\bar{\mathbf{r}}$ from the following line integral:

$$
\begin{aligned}
&\int_{\mathscr{C}_f}(\mathbf{P}^{\beta}\nu_{\beta}\cdot\bar{\mathbf{r}}-\mathbf{F}\cdot\bar{\mathbf{r}}'-\mathbf{T}\cdot\bar{\mathbf{r}})\,ds = \\
&\qquad = \int_{\mathscr{C}_f}(\mathbf{P}^{\beta}\nu_{\beta}+\mathbf{F}'-\mathbf{T})\cdot\bar{\mathbf{r}}-\sum_j[\mathbf{F}(s_{j+1}-0)\cdot\bar{\mathbf{r}}_{j+1}-\mathbf{F}(s_j+0)\cdot\bar{\mathbf{r}}_j].
\end{aligned}
\tag{5.94}
$$

It is easy to see that the values of $\bar{\mathbf{r}}$ on each $\mathscr{M}_j\in\mathscr{C}_f$ are not known, in general, and the out-of-integral terms in $(5.94)_2$ can not be evaluated only in terms of $\mathbf{F}$ and $\boldsymbol{\theta}$. However, there are two special cases of the boundary conditions when those terms are given. The first obvious case is when the boundary contour $\mathscr{C}$ has no corner points. In this case those terms do not appear at all. The second special case is when on $\mathscr{C}$ only (displacement) geometric boundary conditions are prescribed, i.e. $\mathscr{C}\equiv\mathscr{C}_u$, or $\mathscr{C}$ is divided by the corner points into an even number m of intervals, on which alternately only static $(5.58)_1$ or only geometric $\bar{\mathbf{r}}=\bar{\mathbf{r}}^*$ boundary conditions are prescribed. In the last case all the corner points belong simultaneously to $\mathscr{C}_f$ and to $\mathscr{C}_u$. Intervals $(M_j, M_{j+i})\in\mathscr{C}_f$, $(M_i, M_{i+1})\in\mathscr{C}_u$, where $j=1, 3, 5, \ldots, m-1$, $i=j+1$. Since deformation of $\mathscr{C}$ is continuous and $\bar{\mathbf{r}}_i=\bar{\mathbf{r}}_i^*$ on each $M_i\in\mathscr{C}_u$ we indeed obtain $\bar{\mathbf{r}}_j=\bar{\mathbf{r}}_{i-1}^*$ for any $M_j\in\mathscr{C}_f$.

Let us assume that $\mathscr{C}$ is divided indeed into an even number of intervals on which alternately only respective static or geometric quantities are prescribed, as discussed above. Then, in order to eliminate $\bar{\mathbf{r}}$ from the line integral of $(5.94)_2$ the following functions are introduced on $\mathscr{C}_f$:

$$
\mathbf{G}_j(s)=\int_{s_j}^{s}(\mathbf{T}-\mathbf{P}^{\beta}\nu_{\beta})\,ds,\qquad \mathbf{T}-\mathbf{P}^{\beta}\nu_{\beta}=(\mathbf{G}_j+\mathbf{C}_j)',
\tag{5.95}
$$

where $\mathbf{C}_j$ are constant vectors which should ensure $\mathbf{F}$ to approach $\mathscr{C}_f$ continuously. Taking further into account that $\mathbf{G}_j(s_j) \equiv \mathbf{0}$, $\bar{\mathbf{r}}(s_j) = \bar{\mathbf{r}}^*_{i-1} = \bar{\mathbf{r}}^*_j$, $\bar{\mathbf{r}}(s_{j+1}) = \bar{\mathbf{r}}^*_i = \bar{\mathbf{r}}^*_{j+1}$ we can transform $(5.94)_2$ into

$$\sum_{j=1,3,\dots}^{m-1} \left\{ \int_{s_j}^{s_{j+1}} [(\mathbf{G}_j+\mathbf{C}_j-\mathbf{F})\cdot\bar{\mathbf{r}}']\,ds - [\mathbf{G}_j(s_{j+1})\cdot\bar{\mathbf{r}}^*_{j+1} + \mathbf{C}_j\cdot(\bar{\mathbf{r}}^*_{j+1}-\bar{\mathbf{r}}^*_j)]\right\} \tag{5.96}$$

where $\bar{\mathbf{r}}'$ on $\mathscr{C}_f$ is understood to be expressed in terms of $\mathbf{F}$, $\boldsymbol{\theta}$ by the tensor $(5.7)_2$, the inverse of Eqs. $(5.78)_1$, $(5.47)_1$ and $(5.87)_1$. As a result, the functional J_2 in Eq. (5.92) can be transformed into the form

$$\begin{aligned} J_2(\mathbf{F}, \boldsymbol{\theta}, B_t, \mathbf{C}_j) = \iint_{\mathscr{M}} \{\Sigma_\varrho(\boldsymbol{\theta}) - \Sigma^C_S(\mathbf{F}, \boldsymbol{\theta}) - (\varepsilon^{\beta\alpha}\mathbf{F}_{,\alpha} + \mathbf{P}^\beta)\cdot\mathbf{r}_\beta(\boldsymbol{\theta}) + f[\bar{\mathbf{n}}(\boldsymbol{\theta})]\}\,dA\, + \\ + \sum_{j=1,3,\dots}^{m-1} \Big(\int_{s_j}^{s_{j+1}} \{(\mathbf{G}_j+\mathbf{C}_j-\mathbf{F})\cdot\bar{\mathbf{r}}'(\mathbf{F}, \boldsymbol{\theta}) + g[\bar{\mathbf{n}}(\boldsymbol{\theta})]\}\,ds\, - \\ - [\mathbf{G}_j(s_{j+1})\cdot\bar{\mathbf{r}}^*_{j+1} + \mathbf{C}_j\cdot(\bar{\mathbf{r}}^*_{j+1} - \bar{\mathbf{r}}^*_j)]\Big) - \int_{\mathscr{C}} B_t\bar{\mathbf{n}}(\boldsymbol{\theta})\cdot\boldsymbol{\varepsilon}_\beta(\mathbf{F}, \boldsymbol{\theta})t^\beta\,ds\, - \\ - \int_{\mathscr{C}_u} \{\mathbf{F}\cdot\bar{\mathbf{r}}^{*\prime} - \mathbf{P}^\beta\nu_\beta\cdot\bar{\mathbf{r}}^* + [G^{\alpha\beta}(\boldsymbol{\theta})\mathbf{r}_\alpha(\boldsymbol{\theta})\nu_\beta - B_t\bar{\mathbf{r}}^{*\prime}]\cdot[\bar{\mathbf{n}}(\boldsymbol{\theta}) - \bar{\mathbf{n}}(\boldsymbol{\theta}^*)]\}\,ds. \end{aligned} \tag{5.97}$$

The variational principle $\delta J_2 = 0$ is equivalent to three tangential compatibility conditions $(5.24)_1$ in $\mathscr{M}$, three moment equilibrium equations $(5.57)_2$ in $\mathscr{M}$, three constraint conditions (5.27) in $\mathscr{M}$, three force static boundary conditions $\mathbf{G}_j + \mathbf{C}_j - \mathbf{F} - B_t\bar{\mathbf{n}} = \mathbf{0}$ on each interval $(M_j, M_{j+1}) \in \mathscr{C}_f$, $j = 1, 3, \dots, m-1$, two relations (5.63) on $\mathscr{C}_f$ (the first identifies B_t and the second is the static boundary condition for the couple), one constraint condition $\bar{\mathbf{n}}\cdot\boldsymbol{\varepsilon}_\beta t^\beta = 0$ on $\mathscr{C}$ and two geometric boundary conditions $(5.60)_2$ on $\mathscr{C}_u$ for the rotations.

If we compare the functional (5.97) with the analogous functional given by Simmonds and Danielson [248, f. (76)], we note that, apart from some unimportant constant terms and the extended potentials f and g which are included into (5.97), also the line integral over $\mathscr{C}$ and the last term in the line integral over $\mathscr{C}_u$ in (5.97) do not appear in the corresponding functional of [248]. Even if B_t is eliminated from the functional (5.97) with the help of the condition $(5.63)_1$, those two line integrals do not reduce themselves and have to be taken into account in the consistent nonlinear theory of shells, which is expressed in terms of stress functions and finite rotations as independent variables.

Several functionals in terms of finite rotations were discussed also by Atluri [14] who used the undeformed as well as the rotated basis as a reference basis. In the reduced forms of the functionals of [14] also the force static boundary conditions were supposed to be a priori satisfied. This means that corresponding $\mathbf{C}_j$ should be constructed separately outside the variational problem, what makes the solution even more difficult. A term analogous to the last one in the functional (5.97) is taken into account in [14], but the line integral over $\mathscr{C}$ of (5.97) still does not appear in the corresponding functional of [14]. In the functional proposed recently by Başar

[24, 25], the rotation vector has been defined as $\boldsymbol{\omega} = \mathbf{n} \times \bar{\mathbf{n}}$, cf. [23]. The so-defined vector has different geometric meaning than the finite rotation vector used here and, therefore, the functional of [25] can-not be compared with the functionals discussed here. If the rotation vector is expressed through displacements, i.e. $\boldsymbol{\omega} = \varepsilon^{\alpha\beta} n_\alpha \mathbf{a}_\beta$, the functional of [25] becomes a particular case of the functional (4.8) of the displacement shell theory developed in [201, 197].

In the literature on computerized FE analysis of flexible shells, rotations are utilized explicitly and implicitly, exactly and approximately, on the level of an element and in the global matrices. As a result, it is not apparent how to compare the theoretical shell model discussed here with the numerical shell models. Let us only note that rotations were used in the numerical shell models proposed, among others, by Ramm [203], Argyris et al. [12], Parisch [181], Hughes and Liu [95], Surana [265], Oliver and Oñate [180], Bergan and Nygard [29], Recke and Wunderlich [204] and Recke [205] where further references are given.

6. Intrinsic shell equations

In some special problems of flexible shells, under particular types of boundary conditions, the basic set of nonlinear shell equations may be expressed entirely in terms of two-dimensional strain and/or stress measures. Such intrinsic shell equations and their approximate versions for the geometrically nonlinear bending theory of thin elastic shells were derived already by Chien [44] in terms of the strain measures. Alternative sets of intrinsic shell equations and/or alternative schemes of their approximation were proposed by Mushtari [152], Alumäe [5], Koiter [115], John [101–103], Westbrook [294], Axelrad [15, 17] and Valid [283]. Intrinsic formulations of thin shell dynamics were discussed in [84, 130, 131, 304]. Danielson [49] selected stress resultants and changes of curvatures as basic independent variables, what allowed him to derive the refined set of intrinsic shell equations. Those equations were then modified slightly by Koiter an Simmonds [120] with the help of John's [101] error estimates. Alternative formulations and special cases of the refined intrinsic shell equations were discussed by Pietraszkiewicz [185, 190], Simmonds [244], Libai and Simmonds [133] and Koiter [119].

The simplicity of the intrinsic shell equations is remarkable. Their solution leads directly to the determination of stress and strain measures in the shell, without necessity to calculate displacements. However, displacements and/or rotations may be calculated, if necessary, by an additional integration of the kinematic relations (2.7), (2.10) or (5.19), (5.20) and (5.27).

6.1. Intrinsic bending shell equations

Let us note that the component form (3.31) of the mixed shell equations $(3.30)_1$ in the deformed basis $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ is already expressed entirely in terms of two-dimensional strain and stress measures. Corresponding four static boundary and three corner conditions $(3.30)_{2,3}$, when written in components along $\bar{\mathbf{v}}, \bar{\mathbf{t}}, \bar{\mathbf{n}}$, are also expressed in terms of the strain and stress measures. Appropriate boundary conditions on $\mathscr{C}_u$ can also be expressed entirely in terms of the strain measures by assuming functions (5.34) and γ_{tt} to be given on $\mathscr{C}_u$. Therefore, the equilibrium equations (3.31) and the compatibility conditions (2.12) constitute the basic set of six nonlinear equations with respect to arbitrarily chosen six components of strain and/or stress measures which are connected by the constitutive equations (3.34) and (3.35).

Let us now assume that the small strains in the shell caused by stretching and bending of its reference surface are of comparable order in the whole shell, i.e. $\gamma_{\alpha\beta} \sim h\varkappa_{\alpha\beta}$. Then, within the error of the first approximation to the strain energy density $(3.32)_1$, the equilibrium equations (3.31) and the compatibility conditions (2.12) can be essentially reduced [185, 193] to the form

$$C[(1-\nu)\gamma^{\beta}_{\alpha}|_{\beta} + \nu\gamma^{\beta}_{\beta|\alpha}] + q_{\alpha} = O\left(\mathrm{E}h\frac{\eta\theta^2}{\lambda}\right),$$

$$D\varkappa^{\alpha}_{\alpha}|^{\beta}_{\beta} + C(b^{\alpha}_{\beta} - \varkappa^{\alpha}_{\beta})[(1-\nu)\gamma^{\beta}_{\alpha} + \nu\delta^{\beta}_{\alpha}\gamma^{\lambda}_{\lambda}] + q + k^{\alpha}|_{\alpha} = O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right),$$

(6.1)

$$\varkappa^{\beta}_{\alpha|\beta} - \varkappa^{\beta}_{\beta|\alpha} = O\left(\frac{\eta\theta^2}{h\lambda}\right),$$

$$\gamma^{\beta}_{\alpha}|^{\alpha}_{\beta} - \gamma^{\alpha}_{\alpha}|^{\beta}_{\beta} - (b^{\beta}_{\alpha}\varkappa^{\alpha}_{\beta} - b^{\alpha}_{\alpha}\varkappa^{\beta}_{\beta}) + \frac{1}{2}(\varkappa^{\beta}_{\alpha}\varkappa^{\alpha}_{\beta} - \varkappa^{\alpha}_{\alpha}\varkappa^{\beta}_{\beta}) = O\left(\frac{\eta\theta^2}{\lambda^2}\right).$$

Corresponding static boundary conditions on $\mathscr{C}_f$ reduce to [193]

$$C(\gamma_{\nu\nu} + \nu\gamma_{tt}) + O(\mathrm{E}h\eta\theta^2) = Q_{\nu},$$

$$C(1-\nu)\gamma_{\nu t} + O(\mathrm{E}h\eta\theta^2) = Q_t,$$

(6.2)

$$D\{\varkappa_{\nu\nu,\nu} + \nu\varkappa_{tt,\nu} + 2(1-\nu)\varkappa'_{\nu t} + (1-\nu)[\varkappa_t(\varkappa_{\nu\nu} - \varkappa_{tt}) + 2\varkappa_{\nu}\varkappa_{\nu t}]\} + k_{\nu} + O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda}\right) = Q + K'_t,$$

$$D(\varkappa_{\nu\nu} + \nu\varkappa_{tt}) + O(\mathrm{E}h^2\eta\theta^2) = K_{\nu},$$

while at each corner point $M_j \in \mathscr{C}_f$ we should assume

(6.3)
$$D(1-\nu)(\varkappa_{\nu t})_j\bar{\mathbf{n}}_j + O(\mathrm{E}h^2\eta\theta^2) = (K_t)_j\bar{\mathbf{n}}_j.$$

Corresponding deformational quantities (5.34) can also be reduced in accordance with the error already introduced into the reduced compatibility conditions $(6.1)_{3,4}$.

This gives us the deformational boundary conditions [193] on $\mathscr{C}_u$:

$$\varkappa_{tt}+O\left(\frac{\eta\theta^2}{h}\right)=k^*_{tt},\qquad \varkappa_{vt}+O\left(\frac{\eta\theta^2}{h}\right)=k^*_{vt},$$

$$2\gamma'_{vt}-\gamma_{tt,v}+2\varkappa_v\gamma_{vt}+\varkappa_t(\gamma_{vv}-\gamma_{tt})+O\left(\frac{\eta\theta^3}{h}\right)=k^*_{nt},\tag{6.4}$$

$$\gamma_{tt}=\gamma^*_{tt}.$$

The resulting set of bending intrinsic shell relations (6.1)–(6.4) is very simple. Four field equations $(6.1)_{1,3}$ are linear while two remaining ones $(6.1)_{2,4}$ are quadratic in terms of $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$. All boundary conditions are linear in the strain measures.

6.3. Refined intrinsic shell equations

In many problems of flexible shells the small strains caused by membrane force resultants may be of essentially different order (higher or smaller by the factor θ^2) from those caused by the couple resultants. In those cases the reduced bending shell equations $(6.1)_{1,3}$ should be approximated with a greater accuracy, since within the accuracy indicated in Eqs. $(6.1)_{1,3}$ they contain only terms of one kind: membrane strains or changes of curvatures, respectively.

The refinement of Eqs. $(6.1)_{1,3}$ may be performed by selecting membrane stress resultants $N^{\alpha\beta}$ and changes of curvatures $\varkappa_{\alpha\beta}$ as the basic independent variables of the shell theory. The estimation procedure presented in detail in [49, 120, 185, 193] leads then to the following refined intrinsic shell equations

$$\begin{aligned}
&N^{\beta}_{\alpha|\beta}+2A(N^{\lambda}_{\alpha}N^{\beta}_{\lambda})_{|\beta}-\frac{1}{2}A[(1-\nu)N^{\lambda}_{\beta}N^{\beta}_{\lambda}+\nu N^{\lambda}_{\lambda}N^{\beta}_{\beta}]|_{\alpha}-\\
&\qquad -D\{(b^{\lambda}_{\alpha}-\varkappa^{\lambda}_{\alpha})[(1-\nu)\varkappa^{\beta}_{\lambda}+\nu\delta^{\beta}_{\lambda}\varkappa^{\varkappa}_{\varkappa}]\}_{|\beta}-(b^{\beta}_{\alpha}-\varkappa^{\beta}_{\alpha})(D\varkappa^{\lambda}_{\lambda|\beta}+k_{\beta})+\\
&\qquad +2A[(1+\nu)N^{\beta}_{\alpha}q_{\beta}-\nu N^{\lambda}_{\lambda}q_{\alpha}]+q_{\alpha}=O\left(\mathrm{E}h\frac{\eta\theta^4}{\lambda}\right),\\
&D\varkappa^{\alpha}_{\alpha}|^{\beta}_{\beta}+(b^{\beta}_{\alpha}-\varkappa^{\beta}_{\alpha})N^{\alpha}_{\beta}+q+k^{\alpha}|_{\alpha}=O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right),\\
&\varkappa^{\beta}_{\alpha|\beta}-\varkappa^{\beta}_{\beta|\alpha}-A(1+\nu)[(b^{\lambda}_{\beta}-\varkappa^{\lambda}_{\beta})N^{\beta}_{\lambda|\alpha}+(b^{\beta}_{\alpha}-\varkappa^{\beta}_{\alpha})N^{\lambda}_{\lambda|\beta}]+\\
&\qquad +A\nu(b^{\beta}_{\beta}-\varkappa^{\beta}_{\beta})N^{\lambda}_{\lambda|\alpha}-2A(1+\nu)(b^{\beta}_{\alpha}-\varkappa^{\beta}_{\alpha})q_{\beta}=O\left(\frac{\eta\theta^4}{h\lambda}\right),\\
&AN^{\alpha}_{\alpha}|^{\beta}_{\beta}+\left(b^{\beta}_{\alpha}-\frac{1}{2}\varkappa^{\beta}_{\alpha}\right)\varkappa^{\alpha}_{\beta}-\left(b^{\alpha}_{\alpha}-\frac{1}{2}\varkappa^{\alpha}_{\alpha}\right)\varkappa^{\beta}_{\beta}+A(1+\nu)q^{\beta}|_{\beta}=O\left(\frac{\eta\theta^2}{\lambda^2}\right),
\end{aligned}\tag{6.5}$$

$$A=\frac{1}{\mathrm{E}h},\qquad D=\frac{\mathrm{E}h^3}{12(1-\nu^2)}.\tag{6.6}$$

The refined intrinsic shell equations (6.5) expressed in terms of $N^{\alpha\beta}$, $\varkappa_{\alpha\beta}$ as independent variables and with all the external surface forces q_α, q taken into account were given in [185]. Here we have additionally supplemented them with the external surface moments k_α. Danielson [49] derived Eqs. (6.5) in terms of $-\varkappa_{\alpha\beta}$ and a modified stress resultant tensor $n^{\alpha\beta}$, with only q taken into account. Koiter and Simmonds [120] expressed Eqs. (6.5) in terms of $n^{\alpha\beta}$, $-\hat{\varrho}_{\alpha\beta}$ in the absence of surface forces, while q_α, q were taken into account in [190].

The boundary and corner conditions associated with Eqs. (6.5) should also be refined. Note that only the tangential static boundary conditions $(6.2)_{1,2}$ and the tangential deformational boundary conditions $(6.4)_1$ need to be refined, since the other boundary conditions $(6.2)_{3,4}$, (6.3) and $(6.4)_{2,3}$ of the bending shell theory are accurate enough for the use with the intrinsic equations (6.5).

Let us multiply the conditions $(3.30)_2$ by $\bar{\mathbf{v}}$ or $\bar{\mathbf{t}}$, apply the transformation rules (3.9) to express $\bar{v}_\alpha$ or $\bar{t}_\alpha$ in terms of v_α or t_α, respectively, and use the constitutive equations $(3.34)_2$. Then within the error $O(\mathrm{E}h\eta\theta^4)$ the tangential static boundary conditions $(3.30)_2$ on $\mathscr{C}_f$ can be reduced to the consistently approximated form

$$(6.7)\quad \begin{aligned} &[1+A(N_{vv}-\nu N_{tt})]N_{vv}-D(\sigma_v-\varkappa_{vv})(\varkappa_{vv}+\nu\varkappa_{tt})+2D(1-\nu)(\tau_t+\varkappa_{vt})\varkappa_{vt} = \\ &\qquad = Q_v+(\tau_t+\varkappa_{vt})K_t+O(\mathrm{E}h\eta\theta^4), \\ &[1+A(N_{tt}-\nu N_{vv})]N_{vt}+2A(1+\nu)N_{vv}N_{vt}+D(\tau_t+\varkappa_{vt})(\varkappa_{vv}+\nu\varkappa_{tt})- \\ &\qquad -2D(1-\nu)(\sigma_t-\varkappa_{tt})\varkappa_{vt} = Q_t-(\sigma_t-\varkappa_{tt})K_t+O(\mathrm{E}h\eta\theta^4). \end{aligned}$$

Corresponding deformational boundary conditions on $\mathscr{C}_u$ can be constructed by the consistent reduction, to within the error $O\left(\frac{\eta\theta^4}{h}\right)$, of the parameters k_{tt} and k_{vt} given by the expressions $(5.34)_{1,2}$ with the subsequent elimination of $\gamma_{\alpha\beta}$ with the help of the constitutive equations $(3.35)_1$. As a result, we obtain the following consistently approximated deformational boundary conditions:

$$(6.8)\quad \begin{aligned} &\varkappa_{tt}+A(\sigma_t-\varkappa_{tt})(N_{tt}-\nu N_{vv}) = k_{tt}^*+O\left(\frac{\eta\theta^4}{h}\right), \\ &\varkappa_{vt}+2A(1+\nu)(\sigma_t-\varkappa_{tt})N_{vt}-A(\tau_t+\varkappa_{vt})(N_{vv}-\nu N_{tt}) = k_{vt}^*+O\left(\frac{\eta\theta^4}{h}\right), \\ &2A(1+\nu)N'_{vt}-A(N_{tt,v}-\nu N_{vv,v})+2A(1+\nu)\varkappa_v N_{vt}+ \\ &\qquad +A(1+\nu)\varkappa_t(N_{vv}-N_{tt}) = k_{nt}^*+O\left(\frac{\eta\theta^3}{h}\right), \\ &A(N_{tt}-\nu N_{vv}) = \gamma_{tt}^*. \end{aligned}$$

The static boundary conditions (6.7) and $(6.2)_{3,4}$ are equivalent to those given by Danielson [49] in terms of $n^{\alpha\beta}$, $-\varkappa_{\alpha\beta}$. The refined form (6.8) of deformational boundary conditions has not been discussed in the literature.

6.3. Work-conjugate static boundary conditions

The consistently simplified static and deformational boundary conditions given in Sections 6.1 and 6.2 are not work-conjugate to each other since the static parameters in the line integral (3.28) work on virtual displacements and not on variations of the deformational parameters k_{tt}, k_{vt}, k_{nt}, γ_{tt}. In order to derive work-conjugate boundary conditions, the line integral of (3.28) should be transformed as it was suggested in [192, 193].

According to the relations $(2.17)_1$ and $(5.30)_2$, $\bar{\mathbf{a}}_t = \mathbf{t} + \mathbf{u}' = \bar{a}_t \mathbf{R}_t \mathbf{t}$. Taking the variation of this expression with the help of relations (4.27) and using the identity $\delta\mathbf{u}' = (\delta\mathbf{u})'$, we obtain

(6.9)
$$(\delta\bar{\mathbf{u}})' = \frac{1}{\bar{a}_t^2}\bar{\mathbf{a}}_t \delta\gamma_{tt} + \delta\boldsymbol{\omega}_t \times \bar{\mathbf{a}}_t,$$

where by $\delta\bar{\mathbf{u}}$ we understand the variation of the displacement field on $\mathscr{C}$, which is referred then to the deformed basis $\bar{\mathbf{v}}, \bar{\mathbf{t}}, \bar{\mathbf{n}}$, i.e. the virtual displacement field appearing in the principle (3.28).

Let $\mathbf{c}$ be an arbitrary constant vector and $\bar{\mathbf{c}} = \mathbf{R}_t \mathbf{c}$. Then $\delta\bar{\mathbf{c}} = \delta\boldsymbol{\omega}_t \times \mathbf{c}$ and $\bar{\mathbf{c}}' = \mathbf{l}_t \times \mathbf{c}$, according to the relations (4.27) and (5.31). Since again $(\delta\bar{\mathbf{c}})' = \delta(\bar{\mathbf{c}}')$ this leads to

(6.10)
$$(\delta\boldsymbol{\omega}_t)' = \delta\mathbf{l}_t - \delta\boldsymbol{\omega}_t \times \mathbf{l}_t.$$

Using the relations (5.31) and $(5.33)_2$, the relation (6.10) can also be presented in the alternative form

(6.11)
$$(\delta\boldsymbol{\omega}_t)' = -\delta k_{tt}\bar{\mathbf{v}} + \delta k_{vt}\bar{\mathbf{t}} - \delta k_{nt}\bar{\mathbf{n}}.$$

Let $\mathbf{A}$ and $\mathbf{B}(O)$ be the vectors of the total force and the total couple with respect to the origin $O \in \mathscr{E}$ of all the internal force and couple resultants acting on a part of the deformed boundary $\bar{\mathscr{C}}$. With the help of the transformation rule $(3.9)_1$, these vectors are defined by

(6.12)
$$\mathbf{A} = \mathbf{A}_0 + \int_{s_0}^{s} \mathbf{P}ds, \qquad \mathbf{P} = \mathbf{N}^\beta v_\beta - (H\bar{\mathbf{n}})',$$
$$\mathbf{B}(O) = \mathbf{B}_0(O) + \int_{s_0}^{s} (\bar{\mathbf{r}} \times \mathbf{P} + G\bar{\mathbf{t}})ds,$$

where $\mathbf{N}^\beta$, H and G are given in Eqs. (3.29) and $\mathbf{A}_0$, $\mathbf{B}_0(O)$ are initial values of $\mathbf{A}$, $\mathbf{B}(O)$.

The total couple vector $\mathbf{B} = \mathbf{B}(\bar{M})$ with respect to a current point $\bar{M}$ of $\mathscr{C}$ is given by

$$\mathbf{B} = \mathbf{B}(O) - \bar{\mathbf{r}} \times \mathbf{A}. \tag{6.13}$$

Differentiating $\mathbf{A}$ and $\mathbf{B}$ along $\mathscr{C}$, we obtain

$$\mathbf{A}' = \mathbf{P}, \qquad \mathbf{B}' = G\bar{\mathbf{t}} - \bar{\mathbf{a}}_t \times \mathbf{A}. \tag{6.14}$$

Differential relations (6.9), (6.11) and (6.14) can be used to transform the boundary terms in the principle (3.28). Indeed, introducing $\mathbf{A}'$ for $\mathbf{P}$ into Eq. (3.28) and integrating by parts, then again introducing $\mathbf{B}'$ for $G\bar{\mathbf{t}} - \bar{\mathbf{a}}_t \times \mathbf{A}$ and again integrating by parts, we obtain

$$\begin{aligned} &\int_{\mathscr{C}_f} (\mathbf{P}\cdot\delta\bar{\mathbf{u}} + G\bar{\mathbf{t}}\cdot\delta\boldsymbol{\omega}_t)\,ds - \sum_j H_j\bar{\mathbf{n}}_j\cdot\delta\bar{\mathbf{u}}_j = \\ &= -\int_{\mathscr{C}_f}\left[\mathbf{B}\cdot(\delta\boldsymbol{\omega}_t)' + \frac{1}{\bar{a}_t^2}\mathbf{A}\cdot\bar{\mathbf{a}}_t\delta\gamma_{tt}\right]ds - \sum_j [(H_j\bar{\mathbf{n}}_j + \mathbf{A}_j)\cdot\delta\bar{\mathbf{u}}_j + \mathbf{B}_j\cdot\delta\boldsymbol{\omega}_{tj}]. \end{aligned} \tag{6.15}$$

Exactly the same transformations hold for the analogous external force and couple resultant vectors, only in this case $\mathbf{T}$, H^*, G^* and H_j^* appear in place of $\mathbf{N}^\beta \nu_\beta$, H, G and H_j, respectively, in analogous definitions of $\mathbf{P}^*$, $\mathbf{A}^*$, $\mathbf{B}^*$, $\mathbf{A}_j^*$ and $\mathbf{B}_j^*$. As a result, with the help of Eqs. (6.11), (6.15) and an analogous transformed integral for the starred quantities, the boundary terms of the principle (3.28) can be transformed into

$$\begin{aligned} &\int_{\mathscr{C}_f}\left[(\mathbf{B}-\mathbf{B}^*)\cdot\bar{\boldsymbol{\nu}}\delta k_{tt} - (\mathbf{B}-\mathbf{B}^*)\cdot\bar{\mathbf{t}}\delta k_{\nu t} + (\mathbf{B}-\mathbf{B}^*)\cdot\bar{\mathbf{n}}\delta k_{nt} - \frac{1}{\bar{a}_t}(\mathbf{A}-\mathbf{A}^*)\cdot\bar{\mathbf{t}}\delta\gamma_{tt}\right]ds - \\ &\qquad - \sum_j \{[(H_j - H_j^*)\bar{\mathbf{n}}_j + \mathbf{A}_j - \mathbf{A}_j^*]\cdot\delta\bar{\mathbf{u}}_j + (\mathbf{B}_j - \mathbf{B}_j^*)\cdot\delta\boldsymbol{\omega}_{tj}\}. \end{aligned} \tag{6.16}$$

It is apparent from the form (6.16) that on $\mathscr{C}_f$ components of $\mathbf{B}-\mathbf{B}^*$ in the basis $\bar{\boldsymbol{\nu}}$, $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$ and the component $-\bar{a}_t^{-1}(\mathbf{A}-\mathbf{A}^*)\cdot\bar{\mathbf{t}}$ work on variations of the deformational parameters. Therefore, the static boundary conditions which are work-conjugate to the deformational ones have the form

$$\mathbf{B} = \mathbf{B}^*, \qquad \frac{1}{\bar{a}_t}\mathbf{A}\cdot\bar{\mathbf{t}} = \frac{1}{\bar{a}_t}\mathbf{A}^*\cdot\bar{\mathbf{t}} \quad \text{on } \mathscr{C}_f. \tag{6.17}$$

It follows from the form (6.16) that terms associated with the virtual work at the corners $M_j \in \mathscr{C}_f$ are not expressed in the intrinsic form, since the static parameters work there on $\delta\bar{\mathbf{u}}$ and $\delta\boldsymbol{\omega}_t$, respectively, but not on variations of deformational quantities. Therefore, in order to make a shell problem solvable in the intrinsic way, entirely in terms of strain and/or stress measures, those out-of-integral terms should identically vanish. It is easy to note that those terms vanish identically in the case of the smooth boundary contour (i.e. without corners) or when only geometric displacement boundary conditions are assumed on the entire $\mathscr{C}$. Another special case is when $\mathscr{C}$ is divided by corners into an even number of intervals, on which alternately only static or only geometric (displacement) boundary conditions are prescribed. In such a case all the corners belong simultaneously to $\mathscr{C}_f$ and to $\mathscr{C}_u$ and, therefore, $\delta\bar{\mathbf{u}}$ and $\delta\boldsymbol{\omega}_t$ vanish identically at each corner $M \in \mathscr{C}$.

When the work-conjugate static boundary conditions (6.17) are used in conjunction with the bending shell equations (6.1) or with the refined ones (6.5), all the vectors **A**, **B**, **A*** and **B*** should be calculated from the consistently reduced components of **P**, G, **P*** and G^* given in the conditions (6.2) and (6.7), respectively.

6.4. Alternative form of refined intrinsic shell equations

An alternative set of intrinsic shell equations was derived in Chapter 5. Indeed, six equilibrium equations (5.51), (5.62) and six compatibility conditions (5.73) are expressed entirely in terms of the stress measures $S^{\alpha\beta}$, $G^{\alpha\beta}$, Q^{β}, S and the strain measures $\eta_{\alpha\beta}$, $\varrho_{\alpha\beta}$, k_{β}, ϱ.

When strains are small everywhere, the equilibrium equations (5.62) and (5.51) can be reduced within the error of the first approximation theory to the form

(6.18)
$$
\begin{gathered}
G^{\alpha\beta}|_{\beta} - Q^{\alpha} + \hat{h}^{\alpha} = O\left(Eh^2\frac{\eta\theta^2}{\lambda}\right),\\
S - \varepsilon_{\alpha\lambda}\eta^{\alpha}_{\beta}S^{\lambda\beta} - \varepsilon_{\alpha\lambda}G^{\alpha\beta}(b^{\lambda}_{\beta} - \varrho^{\lambda}_{\beta}) = O(Eh\eta\theta^4),\\
S^{\alpha\beta}|_{\beta} - \varepsilon^{\alpha\lambda}S^{\beta}_{\lambda}k_{\beta} + \frac{1}{2}\varepsilon^{\alpha\beta}S_{,\beta} - Q^{\beta}(b^{\alpha}_{\beta} - \varrho^{\alpha}_{\beta}) + \hat{p}^{\alpha} = O\left(Eh\frac{\eta\theta^4}{\lambda}\right),\\
S^{\alpha\beta}(b_{\alpha\beta} - \varrho_{\alpha\beta}) + Q^{\beta}|_{\beta} + \hat{p} = O\left(Eh^2\frac{\eta\theta^2}{\lambda^2}\right).
\end{gathered}
$$

Similarly, the compatibility conditions (5.73) can be reduced into

(6.19)
$$
\begin{gathered}
\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta} + k_{\lambda} = O\left(\frac{\eta\theta^2}{\lambda}\right),\\
\varrho - \varepsilon^{\alpha\beta}\eta_{\alpha\varkappa}(b^{\varkappa}_{\beta} - \varrho^{\varkappa}_{\beta}) = O\left(\frac{\eta\theta^4}{h}\right),\\
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\varrho_{\lambda\alpha|\beta} + \frac{1}{2}\varepsilon^{\varkappa\beta}\varrho_{,\beta} - \varepsilon^{\alpha\beta}k_{\alpha}(b^{\varkappa}_{\beta} - \varrho^{\varkappa}_{\beta}) = O\left(\frac{\eta\theta^4}{h\lambda}\right),\\
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\left(b_{\lambda\alpha} - \frac{1}{2}\varrho_{\lambda\alpha}\right)\varrho_{\varkappa\beta} + \varepsilon^{\alpha\beta}k_{\alpha|\beta} = O\left(\frac{\eta\theta^2}{\lambda^2}\right).
\end{gathered}
$$

It should be noted that the static-geometric analogy formulated in Section 5.5 holds also between the reduced sets of Eqs. (6.18) and (6.19).

Let us solve Eqs. $(6.18)_{1,2}$ for Q^α, S and Eqs. $(6.19)_{1,2}$ for k_α, ϱ, respectively, and introduce the result into the remaining equations (6.18) and (6.19), what leads to

(6.20)
$$S^{\alpha\beta}|_\beta+\varepsilon^{\alpha\lambda}\varepsilon^{\varkappa\varrho}S^\beta_\lambda\eta_{\beta\varkappa|\varrho}+\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\varkappa\varrho}[\eta^\varkappa_\lambda S^{\lambda\varrho}+G^{\varkappa\lambda}(b^\varrho_\lambda-\varrho^\varrho_\lambda)]|_\beta-$$
$$-(G^{\beta\lambda}|_\lambda+\hat{h}^\beta)(b^\alpha_\beta-\varrho^\alpha_\beta)+\hat{p}^\alpha=O\left(Eh\frac{\eta\theta^4}{\lambda}\right),$$
$$S^{\alpha\beta}(b_{\alpha\beta}-\varrho_{\alpha\beta})+G^{\alpha\beta}|_{\alpha\beta}+\hat{h}^\beta|_\beta+\hat{p}=O\left(Eh^2\frac{\eta\theta^2}{\lambda^2}\right),$$

(6.21)
$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\varrho_{\lambda\alpha|\beta}+\frac{1}{2}\varepsilon^{\varkappa\beta}\varepsilon^{\alpha\lambda}[\eta_{\alpha\varrho}(b^\varrho_\lambda-\varrho^\varrho_\lambda)]|_\beta+\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varrho}\eta_{\alpha\lambda|\varrho}(b^\varkappa_\beta-\varrho^\varkappa_\beta)=O\left(\frac{\eta\theta^4}{h\lambda}\right),$$
$$\varepsilon^{\alpha\beta}\varepsilon^{\lambda\varkappa}\left[\left(b_{\lambda\alpha}-\frac{1}{2}\varrho_{\lambda\alpha}\right)\varrho_{\varkappa\beta}-\eta_{\lambda\alpha|\varkappa\beta}\right]=O\left(\frac{\eta\theta^2}{\lambda^2}\right).$$

Introducing the transformations (5.74) into the left-hand sides of Eqs. (6.21) and changing signs, we obtain

(6.22)
$$\tilde{\varrho}^{\alpha\beta}|_\beta+\frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\varkappa\varrho}\tilde{\varrho}^\beta_\lambda\eta_{\beta\varkappa|\varrho}+\frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\varkappa\varrho}\left[\frac{1}{2}\eta^\varkappa_\lambda\tilde{\varrho}^{\lambda\varrho}+\right.$$
$$\left.+\tilde{\eta}^{\varkappa\lambda}\left(b^\varrho_\lambda-\frac{1}{2}\varrho^\varrho_\lambda\right)\right]\Bigg|_\beta-\tilde{\eta}^{\beta\lambda}|_\lambda\left(b^\alpha_\beta-\frac{1}{2}\varrho^\alpha_\beta\right)=0,$$
$$\tilde{\varrho}^{\alpha\beta}\left(b_{\alpha\beta}-\frac{1}{2}\varrho_{\alpha\beta}\right)+\tilde{\eta}^{\alpha\beta}|_{\alpha\beta}=0.$$

When Eqs. (6.22) are compared with the homogeneous equations (6.20), it is seen that the static-geometric analogy still holds between the reduced equations (6.20) and (6.21), what was not the case for the exact sets 3+3 of the transformed intrinsic shell equations (cf. Section 5.5).

The measures $\eta_{\alpha\beta}$ and $G^{\alpha\beta}$ can still be eliminated from Eqs. (6.20) and (6.21) with the help of the constitutive equations

(6.23)
$$\eta_{\alpha\beta}=A[(1+\nu)S_{\alpha\beta}-\nu a_{\alpha\beta}S^\lambda_\lambda]+O(\eta\theta^2),$$
$$G^{\alpha\beta}=D[(1-\nu)\varrho^{\alpha\beta}+\nu a^{\alpha\beta}\varrho^\lambda_\lambda]+O(Eh^2\eta\theta^2),$$

what, after transformations, gives us the following alternative form of the refined intrinsic shell equations:

$$S^{\beta}_{\alpha|\beta}+A[(1+\nu)S^{\lambda}_{\alpha}-\nu\delta^{\lambda}_{\alpha}S^{\varkappa}_{\varkappa}]|_{\beta}S^{\beta}_{\lambda}-\frac{1}{2}A[(1+\nu)S^{\beta}_{\lambda}S^{\lambda}_{\beta}-\nu S^{\lambda}_{\lambda}S^{\beta}_{\beta}]|_{\alpha}-$$
$$-\frac{1}{2}D(1-\nu)(b^{\lambda}_{\alpha}\varrho^{\beta}_{\lambda}-b^{\beta}_{\lambda}\varrho^{\lambda}_{\alpha})|_{\beta}-D(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\varrho^{\lambda}_{\lambda|\beta}+\hat{p}_{\alpha}-(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\hat{h}_{\beta}=O\left(\mathrm{E}h\frac{\eta\theta^4}{\lambda}\right),$$

(6.24)
$$D\varrho^{\alpha}_{\alpha}|^{\beta}_{\beta}+(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})S^{\alpha}_{\beta}+\hat{p}+\hat{h}^{\alpha}|_{\alpha}=O\left(\mathrm{E}h^2\frac{\eta\theta^2}{\lambda^2}\right),$$

$$\varrho^{\beta}_{\alpha|\beta}-\varrho^{\beta}_{\beta|\alpha}+\frac{1}{2}A(1+\nu)[(b^{\lambda}_{\alpha}-\varrho^{\lambda}_{\alpha})S^{\beta}_{\lambda}-(b^{\beta}_{\lambda}-\varrho^{\beta}_{\lambda})S^{\lambda}_{\alpha}]|_{\beta}-$$
$$-A(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})S^{\lambda}_{\lambda|\beta}-A(1+\nu)(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\hat{p}_{\beta}=O\left(\frac{\eta\theta^4}{h\lambda}\right),$$

$$AS^{\alpha}_{\alpha}|^{\beta}_{\beta}+\left(b^{\beta}_{\alpha}-\frac{1}{2}\varrho^{\beta}_{\alpha}\right)\varrho^{\alpha}_{\beta}-\left(b^{\alpha}_{\alpha}-\frac{1}{2}\varrho^{\alpha}_{\alpha}\right)\varrho^{\beta}_{\beta}+A(1+\nu)\hat{p}^{\alpha}|_{\alpha}=O\left(\frac{\eta\theta^2}{\lambda^2}\right).$$

If we apply the identity [49, 185]

(6.25)
$$(S^{\lambda}_{\alpha}S^{\beta}_{\lambda}-S^{\beta}_{\alpha}S^{\lambda}_{\lambda})|_{\beta}=\frac{1}{2}(S^{\lambda}_{\beta}S^{\beta}_{\lambda}-S^{\beta}_{\beta}S^{\lambda}_{\lambda})|_{\alpha}$$

the first two of the equilibrium equations $(6.24)_1$ can be put into another equivalent forms:

$$S^{\beta}_{\alpha|\beta}+A[(1-\nu)S^{\lambda}_{\alpha}+\nu\delta^{\lambda}_{\alpha}S^{\varkappa}_{\varkappa}]|_{\beta}S^{\beta}_{\lambda}-\frac{1}{2}A[(1-\nu)S^{\lambda}_{\beta}S^{\beta}_{\lambda}+\nu S^{\lambda}_{\lambda}S^{\beta}_{\beta}]|_{\alpha}+$$

(6.26)
$$+\frac{1}{2}D(1-\nu)(b^{\beta}_{\lambda}\varrho^{\lambda}_{\alpha}-b^{\lambda}_{\alpha}\varrho^{\beta}_{\lambda})|_{\beta}-D(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\varrho^{\lambda}_{\lambda|\beta}+$$
$$+2A\nu(S^{\beta}_{\alpha}\hat{p}_{\beta}-S^{\beta}_{\beta}\hat{p}_{\alpha})+\hat{p}_{\alpha}-(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\hat{h}_{\beta}=O\left(\mathrm{E}h\frac{\eta\theta^4}{\lambda}\right),$$

$$S^{\beta}_{\alpha|\beta}+AS^{\lambda}_{\alpha|\beta}S^{\beta}_{\lambda}-\frac{1}{2}A(S^{\beta}_{\lambda}S^{\lambda}_{\beta})|_{\alpha}+\frac{1}{2}D(1-\nu)(b^{\beta}_{\lambda}\varrho^{\lambda}_{\alpha}-b^{\lambda}_{\alpha}\varrho^{\beta}_{\lambda})|_{\beta}-$$

(6.27)
$$-D(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\varrho^{\lambda}_{\lambda|\beta}+A\nu(S^{\beta}_{\alpha}\hat{p}_{\beta}-S^{\beta}_{\beta}\hat{p}_{\alpha})+\hat{p}_{\alpha}-(b^{\beta}_{\alpha}-\varrho^{\beta}_{\alpha})\hat{h}_{\beta}=O\left(\mathrm{E}h\frac{\eta\theta^4}{\lambda}\right).$$

The refined intrinsic shell equations (6.24) are fully equivalent to those given by Eqs. (6.5). This can be shown directly if we take into account the transformation rules

(5.11) and (5.40) written here in the appropriately approximated forms:

$$
\begin{aligned}
\varrho^{\beta}_{\alpha} &= \varkappa^{\beta}_{\alpha} + \frac{1}{2}A(1+\nu)[(b^{\lambda}_{\alpha} - \varkappa^{\lambda}_{\alpha})N^{\beta}_{\lambda} + (b^{\beta}_{\lambda} - \varkappa^{\beta}_{\lambda})N^{\lambda}_{\alpha}] - \\
&\quad - A\nu(b^{\beta}_{\alpha} - \varkappa^{\beta}_{\alpha})N^{\lambda}_{\lambda} + O\left(\frac{\eta\theta^4}{h}\right) = \varkappa^{\beta}_{\alpha} + O\left(\frac{\eta\theta^2}{h}\right), \\
S^{\beta}_{\alpha} &= N^{\beta}_{\alpha} + A[(1+\nu)N^{\lambda}_{\alpha} - \nu\delta^{\lambda}_{\alpha}N^{\varkappa}_{\varkappa}]N^{\beta}_{\lambda} - \frac{1}{2}D(1-\nu)(b^{\lambda}_{\alpha}\varkappa^{\beta}_{\lambda} + b^{\beta}_{\lambda}\varkappa^{\lambda}_{\alpha}) - \\
&\quad - D\nu(b^{\beta}_{\alpha} - \varkappa^{\beta}_{\alpha})\varkappa^{\lambda}_{\lambda} + D(1-\nu)\varkappa^{\lambda}_{\alpha}\varkappa^{\beta}_{\lambda} + O(\mathrm{E}h\eta\theta^4) = N^{\beta}_{\alpha} + O(\mathrm{E}h\eta\theta^2).
\end{aligned}
\tag{6.28}
$$

In the comparison of Eqs. (6.24) and (6.5), one should also take into account that the tangential equilibrium equations $(6.24)_1$ are derived here from the components (with the subsequently lowered index α) of the vector equations $(5.49)_1$ in the rotated basis $\mathbf{r}_\alpha$, while the corresponding equations $(6.5)_1$ have been derived from the components (again with the subsequently lowered index α) of the vector equations $(3.30)_1$ in the deformed basis $\bar{\mathbf{a}}_\alpha$. Therefore, we should also take into account the following transformation of the bases

$$
\begin{aligned}
\mathbf{r}_\alpha &= \sqrt{\frac{a}{\bar{a}}}\varepsilon^{\beta\varrho}\varepsilon_{\alpha\lambda}(\delta^{\lambda}_{\varrho} + \eta^{\lambda}_{\varrho})\bar{\mathbf{a}}_\beta = [\delta^{\beta}_{\alpha} - \gamma^{\beta}_{\alpha} + O(\eta\theta^2)]\bar{\mathbf{a}}_\beta = \\
&= \{\delta^{\beta}_{\alpha} - A[(1+\nu)N^{\beta}_{\alpha} - \nu\delta^{\beta}_{\alpha}N^{\lambda}_{\lambda}] + O(\eta\theta^2)]\}\bar{\mathbf{a}}_\beta.
\end{aligned}
\tag{6.29}
$$

If now the rules (6.28) are introduced into Eqs. (6.26) and the effect of change of the basis is taken into account according to the transformation (6.29), then, within the indicated accuracy, the Eqs. (6.26) can be transformed into Eqs. $(6.5)_1$. Applying the same arguments, also the remaining equations of the set (6.24) can be transformed into the corresponding equations of the set (6.5). Therefore, the sets of the refined intrinsic shell equations (6.24) and (6.5) are fully equivalent indeed.

The corresponding set of refined static boundary conditions in terms of $S^{\alpha\beta}$, $\varrho_{\alpha\beta}$ can be derived from the relations (6.7) and $(6.2)_{3,4}$ if we apply there the reversed transformation rules (6.28). Then, after appropriate estimates and transformations, we obtain on $\mathscr{C}_f$

$$
\begin{aligned}
&S_{\nu\nu} - A(1+\nu)S^2_{\nu t} + D(1-\nu)(\tau_t + \varrho_{\nu t})\varrho_{\nu t} = Q_\nu + (\tau_t + \varrho_{\nu t})K_t + O(\mathrm{E}h\eta\theta^4), \\
&S_{\nu t} + A(1+\nu)S_{\nu\nu}S_{\nu t} + \frac{1}{2}D(1-\nu)(\sigma_\nu - \varrho_{\nu\nu})\varrho_{\nu t} - \\
&\quad - \frac{3}{2}D(1-\nu)(\sigma_t - \varrho_{tt})\varrho_{\nu t} + \frac{1}{2}D(1-\nu)(\tau_t + \varrho_{\nu t})\varrho_{\nu\nu} - \\
&\quad - \frac{1}{2}D(1-\nu)(\tau_t + \varrho_{\nu t})\varrho_{tt} = Q_t - (\sigma_t - \varrho_{tt})K_t + O(\mathrm{E}h\eta\theta^4),
\end{aligned}
\tag{6.30}
$$

$$D\{\varrho_{\nu\nu,\nu}+\nu\varrho_{tt,\nu}+2(1-\nu)\varrho'_{\nu t}+(1-\nu)[\varkappa_t(\varrho_{\nu\nu}-\varrho_{tt})+2\varkappa_\nu\varrho_{\nu t}]\}+\hat{h}_\nu = \\ = Q+K'_t+O\left(Eh^2\frac{\eta\theta^2}{\lambda}\right),$$

$$D(\varrho_{\nu\nu}+\nu\varrho_{tt}) = K_\nu+O(Eh^2\eta\theta^2),$$

while at each corner point $M_j\in\mathscr{C}_f$ we have

$$D(1-\nu)(\varrho_{\nu t})_j\bar{\mathbf{n}}_j = (K_t)_j\bar{\mathbf{n}}_j+O(Eh^2\eta\theta^2). \tag{6.31}$$

Similarly, the corresponding set of refined deformational boundary conditions in terms of $S^{\alpha\beta}$, $\varrho_{\alpha\beta}$ can be derived from the conditions (6.8) again by applying the reversed transformation rules (6.28). Then, after appropriate estimates and transformations we obtain on $\mathscr{C}_u$

$$\varrho_{tt}+A(1+\nu)(\tau_t+\varrho_{\nu t})S_{\nu t} = k^*_{tt}+O\left(\frac{\eta\theta^4}{h}\right),$$

$$\varrho_{\nu t}-\frac{1}{2}A(1+\nu)(\sigma_\nu-\varrho_{\nu\nu})S_{\nu t}+\frac{3}{2}A(1+\nu)(\sigma_t-\varrho_{tt})S_{\nu t}- \\ -\frac{1}{2}A(1+\nu)(\tau_t+\varrho_{\nu t})(S_{\nu\nu}-S_{tt}) = k^*_{\nu t}+O\left(\frac{\eta\theta^4}{h}\right), \tag{6.32}$$

$$2A(1+\nu)S'_{\nu t}-A(S_{tt,\nu}-\nu S_{\nu\nu,\nu})+2A(1+\nu)\varkappa_\nu S_{\nu t}+A(1+\nu)\varkappa_t(S_{\nu\nu}-S_{tt}) = k^*_{nt}+O\left(\frac{\eta\theta^3}{h}\right),$$

$$A(S_{tt}-\nu S_{\nu\nu}) = \gamma^*_{tt}+O(\eta\theta^2).$$

It should be noted that within the indicated error the homogeneous equations (6.24) may be shown to be equivalent to the ones proposed by Koiter and Simmonds [120]. In particular, when linearized, both sets of equations reduce to those of the „best" linear theory of thin shells according to [37]. However, a) our equations (6.24) are expressed in terms of the measures $S^{\alpha\beta}$, $\varrho_{\alpha\beta}$ which appear naturally in the nonlinear theory of shells (cf. Chapter 5) while the corresponding equations of [120] are expressed in terms of some modified measures for which no exact Eulerian counterparts can be defined (cf. discussion in Section 3.4); b) our equations (6.24) take into account all the surface loads $\hat{p}_\alpha \sim Eh\eta/\lambda$, $\hat{p} \sim Eh^2\eta/\lambda^2$, $\hat{h}_\alpha \sim Eh^2\eta/\lambda$, while those of [120] are given for the case of zero surface loads (the loads $\bar{p}^\alpha$ and $\bar{p}$ have been included in [190, 119]); c) our equations (6.24) follow from the set of 3+3 reduced shell equations (6.20), (6.21) which obey the static-geometric analogy in the nonlinear range of deformation, while such an analogy cannot be established between the initial relations of [120]; d) our equations (6.24) are supplemented by appropriately simplified static and deformational boundary conditions, while no such boundary conditions were given in [120].

6.5. Some special cases of intrinsic shell equations

As it was noted in the Introduction, already Chien [44] proposed a formal classification of approximate versions of his intrinsic equations under the assumption of a slowly varying geometry and slowly varying strain states of plates and shells. Mushtari [152] applied a less formal qualitative analysis and constructed approximate versions of intrinsic equations for small and medium bending of shells and plates. In [152] several sets of intrinsic equations of the boundary layer type were also given. Alumäe [5] introduced the notion of wave length of deformation patterns and discussed 12 cases of intrinsic equations for the buckling analysis of shells which are shallow or almost shallow relative to deformation patterns. The solution of the most complete set of such equations was then reduced to the solution of two equations expressed in terms of stress and deformation functions F, W. Similar assumptions were applied independently by Libai [129] and Koiter [115] to derive the equations for quasi-shallow shells. However, no corresponding intrinsic boundary conditions were discussed in the papers referred to above.

The derivation of refined equations [49] provided new possibilities in the proper formulation of intrinsic equations for various types of shell problems. As a starting point for further discussion, three different but equivalent versions of the refined shell equations may be used: the one proposed by Koiter and Simmonds [120] and supplemented by the surface forces in [190], which is expressed in terms of some modified stress and strain measures, the one derived in terms of $N^{\alpha\beta}$, $\varkappa_{\alpha\beta}$ by the author [185, 193] and summarized here as Eqs. (6.5)–(6.8), as well as the one derived in terms of $S^{\alpha\beta}$, $\varrho_{\alpha\beta}$ in this report, Eqs. (6.24), (6.30)–(6.32). Referring to the discussion after Eqs. (6.32), the third version, as the most complete, seems to be preferable.

Let us look more carefully into the structure of Eqs. (6.24). Let $\eta_{\alpha\beta} \sim \eta$ and $h\varrho_{\alpha\beta} \sim h\varrho$ be the maximum extensional and bending strains, respectively. Let also L_η and L_ϱ be the wave lengths of deformation patterns associated with the extensional and bending strains, such that $\eta^{\beta}_{\alpha|\gamma} \sim \eta/L_\eta$ and $\varrho^{\beta}_{\alpha|\gamma} \sim \varrho/L_\varrho$, respectively. Then, dividing Eqs. $(6.24)_{1,2}$ by E and multiplying Eqs. $(6.24)_{3,4}$ by h^2, we obtain the following order estimates for magnitudes of individual terms in the refined intrinsic shell equations (6.24):

(6.33)

$$(6.24)_1:\ \frac{h}{L_\eta}\cdot\eta,\quad \frac{h}{L_\eta}\cdot\eta^2,\quad \frac{h}{R}\cdot\frac{h}{l}\cdot h\varrho,\quad \frac{h}{R}\cdot\frac{h}{L_\varrho}\cdot h\varrho,\quad \frac{h}{L_\varrho}\cdot(h\varrho)^2,$$

$$(6.24)_2:\ \left(\frac{h}{L_\varrho}\right)^2\cdot h\varrho,\quad \frac{h}{R}\cdot\eta,\quad \eta\cdot h\varrho,$$

$$(6.24)_3:\ \frac{h}{L_\varrho}\cdot h\varrho,\quad \frac{h}{R}\cdot\frac{h}{l}\cdot\eta,\quad \frac{h}{R}\cdot\frac{h}{L_\eta}\cdot\eta,\quad \frac{h}{L_\eta}\cdot\eta\cdot h\varrho,\quad \frac{h}{L_\eta}\cdot\eta\cdot h\varrho$$

$$(6.24)_4:\ \left(\frac{h}{L_\eta}\right)^2\cdot\eta,\quad \frac{h}{R}\cdot h\varrho,\quad (h\varrho)^2.$$

Similar estimates can also be given for terms appearing in the static (6.30), (6.31) and deformational (6.32) boundary conditions.

Various small parameters appearing in the estimations (6.33) describe quite different phenomena. The parameters h/R and h/l describe the initial geometry of the shell and its spatial variability, which is supposed to be known in advance. The parameters η, $h\varrho$, h/L_η and h/L_ϱ describe the respective predictions of orders of magnitudes of the extensional and bending strains as well as their predicted spatial variability. These parameters are not known in advance, for they strongly depend on the type of shell problem being solved, i.e. on the geometry, external surface and boundary loads, boundary conditions etc. Within the accuracy of the first-approximation theory of shells it is already assumed that terms of the order of h/R, $(h/l)^2$, $(h/L_\eta)^2$, $(h/L_\varrho)^2$, η and $h\varrho$ can be omitted with respect to the unity. This gives us the upper bounds for estimates of various small parameters. However, in different types of shell problems the real magnitudes of some small terms may be far from their upper bounds.

For some shell problems it is possible to predict in advance the type of solution behaviour in the whole internal shell region. This prediction may then be used to compare the orders of magnitudes of various terms appearing in the set of equations (6.24), (6.30)–(6.32), what allows us to omit some terms which are of the order of error of the first-approximation theory. Then the predicted solution of the shell problem may be obtained from a considerably simplified set of intrinsic shell equations. However, it is always advisable to check at the end whether the solution calculated from the simplified equations represents indeed the predicted type of solution of the shell problem. Note that the type of shell problem is described in the estimates (6.33) by all six small parameters given above, whose orders of magnitude are entirely independent. As a result, a large variety of special cases of intrinsic shell equations may be generated from Eqs. (6.24). In what follows we shortly discuss only few special cases which seem to be most important.

In the limit $b_{\alpha\beta} \to O$ Eqs. (6.24) reduce to intrinsic equations of the geometrically nonlinear **theory of plates** (less error terms):

$$
\begin{aligned}
&\Big\{ S^\beta_\alpha + A[(1+\nu)S^\lambda_\alpha - \nu\delta^\lambda_\alpha S^\varkappa_\varkappa]S^\beta_\lambda - \frac{1}{2}A\delta^\beta_\alpha[(1+\nu)S^\varkappa_\lambda S^\lambda_\varkappa - \nu S^\lambda_\lambda S^\varkappa_\varkappa] + \\
&\qquad + D\Big[\Big(\varrho^\beta_\alpha - \frac{1}{2}\delta^\beta_\alpha\varrho^\lambda_\lambda\Big)\varrho^\varkappa_\varkappa\Big]\Big\}\Big|_\beta + A[(1+\nu)S^\beta_\alpha\hat{p}_\beta - \nu S^\beta_\beta\hat{p}_\alpha] + \hat{p}_\alpha + \varrho^\beta_\alpha\hat{h}_\beta = 0, \\
&D\varrho^\alpha_\alpha|^\beta_\beta - \varrho^\beta_\alpha S^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0, \\
&\varrho^\beta_{\alpha|\beta} - \varrho^\beta_{\beta|\alpha} - \frac{1}{2}A(1+\nu)(\varrho^\lambda_\alpha S^\beta_\lambda - \varrho^\beta_\lambda S^\lambda_\alpha)|_\beta + A\varrho^\beta_\alpha S^\lambda_{\lambda|\beta} + A(1+\nu)\varrho^\beta_\alpha\hat{p}_\beta = 0, \\
&AS^\alpha_\alpha|^\beta_\beta - \frac{1}{2}\varrho^\beta_\alpha\varrho^\alpha_\beta + \frac{1}{2}\varrho^\alpha_\alpha\varrho^\beta_\beta + A(1+\nu)\hat{p}^\alpha|_\alpha = 0.
\end{aligned}
\tag{6.34}
$$

When the plate is loaded by edge forces only, Simmonds [244, 133] managed to reduce the solution of an equivalent to Eqs. (6.34) set of plate equations of [120] into two coupled equations for the stress and deformation functions F, W, except in the case in which rotations are $O(1)$ and simultaneously the variability of deformation is very large. These extended von Kármán equations are [133]

$$A(\Delta\Delta F + W|^{\alpha}_{\beta} P^{\beta}_{\alpha}) + \frac{1}{2}\langle W, W\rangle = 0,$$

(6.35)

$$D\left\{\Delta\Delta W + \left[\frac{1}{2}(\Delta W)^2 - \langle W, W\rangle\right]\Delta W\right\} - \langle W, F\rangle = 0,$$

where

$$\left[P^{\alpha\beta} + \frac{1}{2}(1+\nu)\varepsilon^{\alpha\beta}\varepsilon_{\gamma\mu}F|^{\gamma}_{\lambda}W|^{\lambda\mu}\right]\Big|_{\beta} = \Delta W \Delta F|^{\alpha},$$

(6.36)

$$\langle W, F\rangle = \varepsilon_{\alpha\lambda}\varepsilon^{\beta\varkappa}W|^{\alpha}_{\beta}F|^{\lambda}_{\varkappa}.$$

The (almost) **inextensional bending theory of shells** is usually defined as the one in which the extensional strains $\eta_{\alpha\beta}$ are much smaller than the bending strains $h\varrho_{\alpha\beta}$, i.e. $\eta/h\varrho \ll 1$. Here we assume additionally that the spatial variability of the bending strains is lower than in the general theory, $h/L_{\varrho} \ll 1$. Such slowly variable bending strain states are typical for the inextensional bending deformation of the shell. If also $L_{\eta} \leqslant l$, then within the error of the first-approximation theory Eqs. (6.24) reduce to the following set of intrinsic equations of the geometricaly nonlinear inextensional bending theory of shells (less error terms)

$$\left\{S^{\beta}_{\alpha} - \frac{1}{2}D(1-\nu)(b^{\lambda}_{\alpha}\varrho^{\beta}_{\lambda} - b^{\beta}_{\lambda}\varrho^{\lambda}_{\alpha})\right\}_{|\beta} - D(b^{\beta}_{\alpha} - \varrho^{\beta}_{\alpha})\varrho^{\lambda}_{\lambda|\beta} + \hat{p}_{\alpha} - (b^{\beta}_{\alpha} - \varrho^{\beta}_{\alpha})\hat{h}_{\beta} = 0,$$

$$D\varrho^{\alpha}_{\alpha}|^{\beta}_{\beta} + \underline{(b^{\beta}_{\alpha} - \varrho^{\beta}_{\alpha})S^{\alpha}_{\beta}} + \hat{p} + \hat{h}^{\alpha}|_{\alpha} = 0,$$

(6.37)

$$(\varrho^{\beta}_{\alpha} - \delta^{\beta}_{\alpha}\varrho^{\lambda}_{\lambda})|_{\beta} = 0,$$

$$\left(b^{\beta}_{\alpha} - \frac{1}{2}\varrho^{\beta}_{\alpha}\right)\varrho^{\alpha}_{\beta} - \left(b^{\alpha}_{\alpha} - \frac{1}{2}\varrho^{\alpha}_{\alpha}\right)\varrho^{\beta}_{\beta} = 0.$$

In comparison to our previous inextensional bending shell equations [185, 190] derived from equivalent refined intrinsic equations, the underlined terms in Eq. $(6.37)_2$ are taken here into account, what results from the additional requirement $h/L_{\varrho} \ll 1$ used here. The presence of those terms allows for a smooth transition to the inextensional bending theory of plates if the limit $b_{\alpha\beta} \to 0$ is taken in Eqs. (6.37). The set of equations (6.37) follows also quite formally from Eqs. (6.24) by taking the limit $A \to 0$, cf. [244, 119].

Note that the reduced compatibility conditions $(6.37)_{3,4}$ can be solved with respect to $\varrho_{\alpha\beta}$ independently of the stress state in the shell. In this sense the inextensional bending problems of shells are geometrically determined. When $\varrho_{\alpha\beta}$ are calculated, $S^{\alpha\beta}$ follow from the reduced equilibrium equations $(6.37)_{1,2}$ and then the constitutive equations (6.23) allow to recover $\eta_{\alpha\beta}$ and $G^{\alpha\beta}$.

The (almost) **membrane theory of shells** is usually defined as the one in which the bending strains are much smaller than the extensional strains, $h\varrho/\eta \ll 1$. Here we assume additionally that the spatial variability of the extensional strains is lower than in the general theory, $h/L_\eta \ll 1$, what again is typical for the membrane stress states in the shell. If also $L_\varrho \leqslant l$, then within the error of the first-approximation theory Eqs. (6.24) reduce to the following set of intrinsic equations (less error terms):

$$
\begin{gathered}
S^\beta_{\alpha|\beta} + \hat{p}_\alpha = 0,\\
\underline{D\varrho^\alpha_\alpha|^\beta_\beta} + (b^\beta_\alpha - \underline{\varrho^\beta_\alpha}) S^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0,\\
\left[\varrho^\beta_\alpha - \delta^\beta_\alpha \varrho^\lambda_\lambda + \frac{1}{2}A(1+\nu)(b^\lambda_\alpha S^\beta_\lambda - b^\beta_\lambda S^\lambda_\alpha) - A b^\beta_\alpha S^\lambda_\lambda\right]\Bigg|_\beta + A b^\beta_{\beta|\alpha} S^\lambda_\lambda - A(1+\nu) b^\beta_\alpha \hat{p}_\beta = 0,\\
A S^\alpha_\alpha|^\beta_\beta + \underline{b^\beta_\alpha \varrho^\alpha_\beta - b^\alpha_\alpha \varrho^\beta_\beta} + A(1+\nu)\hat{p}^\alpha|_\alpha = 0.
\end{gathered}
\tag{6.38}
$$

In comparison to our previous membrane shell equations [185, 190], which followed from equivalent refined intrinsic equations, the secondary nonlinear terms are omitted in Eq. $(6.38)_1$ and the underlined terms in Eqs. $(6.38)_{2,4}$ are taken into account, what again results from the additional requirement $h/L_\eta \ll 1$ used here. It should be noted, in particular, that the equilibrium equations $(6.38)_{1,2}$ cannot be solved here independently for $S^{\alpha\beta}$ since in Eq. $(6.38)_2$ we have the underlined terms which provide the coupling between the equilibrium equations and the compatibility conditions. As it was noted in [119], this coupling removes from the nonlinear membrane theory the degeneration prevalent in the linear theory, [254]. In particular, the geometrically nonlinear membrane theory of plates follows from Eqs. (6.38) in the limit $b_{\alpha\beta} \to 0$. At the same time, our equations (6.38) are considerably simpler than those which would follow formally from Eqs. (6.24) by taking the limit $D \to 0$, what was suggested in [244, 119].

The **bending theory of shells** equivalent to the one discussed in Section (6.1) follows from Eqs. (6.24) if

$$
\max\left(\frac{L_\varrho}{l}\frac{h}{R}, \frac{L_\varrho}{L_\eta}\frac{h}{R}\right) \ll \frac{h\varrho}{\eta} \ll \min\left(\frac{l}{L_\eta}\frac{R}{h}, \frac{L_\varrho}{L_\eta}\frac{R}{h}, \frac{L_\varrho}{L_\eta}\frac{1}{h\varrho}\right).
\tag{6.39}
$$

Then Eqs. (6.24) can be reduced to the set of equations (less error terms)

$$
\begin{gathered}
S^\beta_{\alpha|\beta} + \hat{p}_\alpha = 0,\\
D\varrho^\alpha_\alpha|^\beta_\beta + (b^\beta_\alpha - \varrho^\beta_\alpha) S^\alpha_\beta + \hat{p} + \hat{h}^\alpha|_\alpha = 0,\\
(\varrho^\beta_\alpha - \delta^\beta_\alpha \varrho^\lambda_\lambda)|_\beta = 0,\\
A S^\alpha_\alpha|^\beta_\beta + \left(b^\beta_\alpha - \frac{1}{2}\varrho^\beta_\alpha\right)\varrho^\alpha_\beta - \left(b^\alpha_\alpha - \frac{1}{2}\varrho^\alpha_\alpha\right)\varrho^\beta_\beta + A(1+\nu)\hat{p}^\alpha|_\alpha = 0.
\end{gathered}
\tag{6.40}
$$

Let us introduce the stress function F and the deformation function W by

$$(6.41)\qquad \begin{aligned} S^{\beta}_{\alpha} &= \varepsilon_{\alpha\lambda}\varepsilon^{\beta\varrho}(F|^{\lambda}_{\varrho}+\delta^{\lambda}_{\varrho}KF)+P^{\beta}_{\alpha},\\ \varrho^{\beta}_{\alpha} &= +W|^{\beta}_{\alpha}+\delta^{\beta}_{\alpha}KW, \qquad P^{\beta}_{\alpha|\beta}+\hat{p}_{\alpha}=0. \end{aligned}$$

The equilibrium equations $(6.40)_1$ are approximately satisfied by Eq. $(6.41)_1$ and the compatibility conditions $(6.40)_3$ by Eqs. $(6.41)_2$ provided

$$(6.42)\qquad \frac{L_{\eta}}{l}|K|L^2_{\eta} \ll 1, \qquad \frac{L_{\varrho}}{l}|K|L^2_{\varrho} \ll 1,$$

respectively. Then the remaining equations $(6.40)_{2,4}$ in terms of F, W take the form

$$(6.43)\qquad \begin{aligned} &D\Delta(\Delta W+2KW)+\varepsilon^{\alpha\lambda}\varepsilon_{\beta\varrho}(b^{\beta}_{\alpha}-W|^{\beta}_{\alpha}-\delta^{\beta}_{\alpha}KW)(F|^{\varrho}_{\lambda}+\delta^{\varrho}_{\lambda}KF)+\\ &\qquad\qquad +(b^{\beta}_{\alpha}-W|^{\beta}_{\alpha}-\delta^{\beta}_{\alpha}KW)P^{\alpha}_{\beta}+\hat{p}+\hat{h}^{\alpha}|_{\alpha}=0,\\ &A\Delta(\Delta F+2KF)-\varepsilon^{\alpha\lambda}\varepsilon_{\beta\varrho}\left(b^{\beta}_{\alpha}-\frac{1}{2}W|^{\beta}_{\alpha}-\frac{1}{2}\delta^{\beta}_{\alpha}KW\right)(W|^{\varrho}_{\lambda}+\delta^{\varrho}_{\lambda}KW)+\\ &\qquad\qquad +A[\Delta P^{\alpha}_{\alpha}-(1+\nu)P^{\beta}_{\alpha}|^{\alpha}_{\beta}]=0. \end{aligned}$$

These are the nonlinear **bending equations for shells of slowly varying curvature**, which are equivalent to the ones proposed recently by Rychter [213]. Under a more restrictive assumption $|K|L^2 \ll 1$, where $L = \min(L_{\eta}, L_{\varrho}, l)$, we can also omit in Eqs. (6.43) all terms with K, what leads to the nonlinear **equations of quasi-shallow shells**, given by Alumäe [5] and Koiter [115].

The limited space of the paper does not permit to present here the explicit reduced forms of intrinsic boundary conditions to be used with each of the reduced sets of intrinsic equations discussed above. For each particular case those boundary conditions follow immediately from Eqs. (6.30)–(6.32) if corresponding estimates are introduced and appropriate simplifications are made. The reader can easily derive them himself if necessary.

Other special cases of intrinsic shell equations and some of their applications are discussed in [51, 49, 190, 244, 119, 133, 18], where further references are given.

7. Closing remarks

In this report we have reviewed some achievements associated with the derivation, classification and simplification of various sets of equations of the nonlinear first-approximation theory of a thin shell, the deformation of which is expressible entirely by deformation of its reference surface. Basic sets of shell equations, which govern static problems of a thin shell made of a linearly-elastic homogeneous isotropic material undergoing small strains but unrestricted rotations, and associted variational principles have been formulated either in terms of displacements, or in terms of rotations and other fields or in terms of strain and/or stress measures as independent variables. References have been given primarily to those original papers and monographs which deal with general aspects of the nonlinear theory of thin elastic shells and have been written in an invariant tensor notation. Apart from the unification of various partial results which are available in the literature, the report contains also some original results which have not been published elsewhere.

The subject of this report is quite narrow and many important aspects of the nonlinear theory of shells have not been discussed. Among those associated subjects let us mention, for example, stability analysis, dynamic behaviour, large-strain theory, inelastic material behaviour, composite shells, interaction problems, higher-order shell theories, Cosserat-type theories, existence and uniqueness of solutions etc. Beyond the scope of this report there are also specific problems of shells with definite geometries as well as various analytic and numerical methods of analysis of the flexible shells.

Acknowledgements. The main part of this report was written during the author's visit at the Ruhr-Universität Bochum, within the Agreement on Scientific Cooperation between the Institute of Fluid-Flow Machinery of the Polish Academy of Sciences in Gdańsk and the Institut für Mechanik der Ruhr-Universität Bochum. The author would like to thank o. Prof. Dr.-Ing. H. Stumpf for his continuous support.

References

1. H. Abé, Int. J. Nonlinear Mech., **4**, 107–121, 1969.
2. N. P. Abovskii, N. P. Andreev, A. P. Deruga, *Variational principles of elasticity and shell theory* (in Russian), Nauka, Moskva 1978.
3. L. Ya. Aynola, Prikl. Mat. Mekh., **21**, 3, 399–405, 1957.
4. N. A. Alumäe, Trudy Tallinskogo Polit. In-ta, Ser. A, 1948.
5. N. A. Alumäe, Prikl. Mat. Mekh., **13**, 1, 95–106, 1949.
6. N. A. Alumäe, Prikl. Mat. Mekh., **14**, 2, 197–202, 1950.
7. N. A. Alumäe, Izv. AN Est. SSR, **4**, 2, 230–232, 1955.
8. N. A. Alumäe, Prikl. Mat. Mekh., **20**, 1, 136–139, 1956.
9. S. S. Antman, SIAM Rev., **14**, 649–652, 1972.
10. S. S. Antman, ARMA, **61**, 307–351, 1976.
11. J. Arbocz, in: *Collapse*, Ed. by J. M. T. Thompson and G. W. Hunt, 43–74; Cambridge University Press 1983.
12. J. H. Argyris et al., Comp. Meth. Appl. Mech. and Engng., **17**, 1–106, 1979.
13. H. Aron, J. Reine u. Angew. Math., **78**, 136–174, 1874.
14. S. N. Atluri, Computers and Structures, **18**, 1, 93–116, 1984.
15. E. L. Axelrad, *Flexible shells* (in Russian), Nauka, Moskva 1976.
16. E. L. Axelrad, Int. J. Sol. Str., **17**, 3, 301–304, 1981.
17. E. L. Axelrad, *Theory of flexible shells*, North-Holland P. Co., Amsterdam 1987.
18. E. L. Axelrad, F. A. Emmerling, Int. J. Nonlinear Mech., **22**, 2, 139–150, 1987.
19. J. Badur, W. Pietraszkiewicz, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 19–32; Springer–Verlag, Berlin 1986.
20. Y. Başar, Dissertation, Fak. für Bauwesen, TH Hannover 1964.

21. Y. Başar, ZAMM, **52**, 5, 197–211, 1972.
22. Y. Başar, W. B. Krätzig, in: *Stability in the mechanics of continua*, Ed. by F. H. Schroeder, 133–145; Springer–Verlag, Berlin 1982.
23. Y. Başar, Ruhr-Univ., IKB Ber., 38/39, 7–14, Bochum 1981.
24. Y. Başar, ZAMM, **66**, 7, 297–308, 1986.
25. Y. Başar, Int. J. Solids and Structures, **23**, 10, 1401–1415, 1987.
26. Y. Başar, W. B. Krätzig, *Mechanik der Flächentragwerke*, Fr. Vieweg u. Sohn, Braunschweig 1985.
27. A. B. Basset, Phil. Trans. Royal Soc., **181** A, 433, 1890.
28. V. L. Berdichevskiĭ, *Variational principles of continuum mechanics* (in Russian), Nauka, Moskva 1983.
29. P. G. Bergan, M. K. Nygard, in: *FE methods for nonlinear problems*, Proc. Eur.-US Symp., Trondheim 1985.
30. N. Berger, SIAM J. Appl. Math., **24**, 1, 97–120, 1973.
31. M. Bernadou, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 33–46; Springer–Verlag, Berlin 1986.
32. V. V. Bolotin, *Non-conservative problems of the theory of elastic stability* (in Russian), Fizmatgiz, Moskva 1961.
33. V. V. Bolotin, Inzhenernyi Zhurn., MTT, 4, 117–123, 1967.
34. P. Brož, Acta Technica ČSAV, **22**, 3, 346–360, 1977.
35. D. O. Brush, B. O. Almroth, *Buckling of bars, plates and shells*, Mc Graw-Hill B. Co., New York 1975.
36. B. Budiansky, J. Appl. Mech., Trans. ASME, **35**, 2, 393–401, 1968.
37. B. Budiansky, J. L. Sanders, in: *Progress in applied mechanics* (W. Prager Anniv. Vol.), 129–140; Macmillan, New York 1963.
38. H. Bufler, ZAMM, **64**, 7, 287–295, 1984.
39. K. F. Chernykh, *Linear theory of shells* (in Russian), Leningrad State University Press, **1**, 1962; **2**, 1964.
40. K. F. Chernykh, Mechanics of Solids (Izv. AN SSSR, MTT), 2, 148–159, 1980.
41. K. F. Chernykh, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 101–110; Springer–Verlag, Berlin 1986.
42. K. F. Chernykh, *Nonlinear theory of elasticity for mechanical engineering design* (in Russian), Mashinostroenie, Leningrad 1986.
43. K. F. Chernykh, V. A. Shamina, Proc. IX All-Union Conf. on Theory of Shells and Plates (in Russian), 99–103; Sudostroenie, Leningrad 1975.
44. W. Z. Chien, Quart. Appl. Math., **1**, 4, 297–327, 1944; **2**, 2, 120–135, 1944.
45. A. N. Chuyko, Sov. Appl. Mech. (PM), **19**, 11, 66–71, 1983.
46. A. N. Chuyko, Sov. Appl. Mech. (PM), **22**, 9, 61–66, 1986.
47. J. Chróścielewski, R. Schmidt, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 111–120; Springer–Verlag, Berlin 1986.
48. R. Courant, D. Hilbert, *Methods of mathematical physics*, **1**, Wiley-Intersciences, New York 1953.
49. D. A. Danielson, Int. J. Engng. Sci., **8**, 3, 251–259, 1970.
50. D. A. Danielson, Proc. Koninkl. Ned. Akad. Wet., B **74**, 3, 294–300, 1971.
51. D. A. Danielson, J. G. Simmonds, Int. J. Engng. Sci., 7, 5, 459–468, 1969.
52. V. M. Darevskii, Proc. VIth All-Union Conf. on Theory of Shells and Plates (in Russian), 355–368; Nauka, Moskva 1966.
53. M. Dikmen, *Theory of thin elastic shells*, Pitman Publ., Boston 1982.
54. L. H. Donnell, NACA Techn. Rep., 479, 1933.
55. L. H. Donnell, Trans ASME, **56**, 795–806, 1934.
56. M. Duszek, Rozpr. Inż., **20**, 3, 389–407, 1972.
57. M. Duszek, Bull. Acad. Polon. Sci., Série Sci. Techn., **26**, 1, 39–47, 1978.
58. I. K. Ebcioglu, Int. J. Nonlinear Mech., **6**, 8, 469–478, 1971.
59. U. Eckstein, Ruhr-Univ., IKB, TW Mitt., 8–33, Bochum 1983.

60. N. Fukuchi, S. N. Atluri, in: *Nonlinear FE analysis of plates and shells*, Ed. by T. Hughes et al., AMD, **18**, 233–247; ASME, New York 1981.
61. N. Fukuchi, S. N. Atluri (in print).
62. K. Z. Galimov, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 1, 25–46, 1948.
63. K. Z. Galimov, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 2, 3–38, 1950.
64. K. Z. Galimov, Prikl. Mat. Mekh. **15**, 6, 723–742, 1951.
65. K. Z. Galimov, Uchenye zapiski Kazanskogo Univ., **113**, 10, 161–164, 1953.
66. K. Z. Galimov, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 3, 3–17, 1953.
67. K. Z. Galimov, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 10, 3–26, 1956.
68. K. Z. Galimov, Uchenye zapiski Kazanskogo Univ., **116**, 5, 13–18, 1956.
69. K. Z. Galimov, Izv. VUZ, Matematika, **1**, 2, 3–11, 1958.
70. K. Z. Galimov, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 12, 17–27, 1958.
71. K. Z. Galimov, *Foundations of the nonlinear theory of thin shells* (in Russian), Kazan' Univ. Press, Kazan' 1975.
72. K. Z. Galimov, *Developments in the theory of plates and shells* (Issl. po Teorii Pl. i Obol.), **16**, 7–29, Kazan' Univ. Press 1981.
73. K. Z. Galimov, V. N. Paimushin, *Theory of shells of complex geometry* (in Russian), Kazan' Univ. Press 1985.
74. N. Gass, ZAMM, **55**, 9, 515–521, 1975.
75. K. Girkmann, *Flächentragwerke*, 6 Auf., Springer–Verlag, Wien 1963.
76. A. L. Goldenveizer, Prikl. Mat. Mekh, **9**, 6, 463–478, 1945.
77. A. L. Goldenveizer, A. I. Lurie, Prikl. Mat. Mekh., **11**, 5, 565–592, 1947.
78. A. L. Goldenveizer, Trudy Vses. S'ezda po Teor. Prikl. Mekh., 339–357, Izd. AN SSSR, Moskva 1962.
79. A. L. Goldenveizer, *Theory of elastic thin shells* (in Russian), Gostekhteoretizdat, Moskva 1953.
80. A. L. Goldenveizer, Prikl. Mat. Mekh., **29**, 4, 701–715, 1965.
81. A. L. Goldenveizer, Prikl. Mat. Mekh., **32**, 4, 684–695, 1968.
82. A. L. Goldenveizer, Prikl. Mat. Mekh., **33**, 6, 996–1028, 1969.
83. A. L. Goldenveizer, *Theory of thin elastic shells* (in Russian), Nauka, Moskva 1976.
84. A. E. Green, P. M. Naghdi, J. Appl. Mech., Trans. ASME, **41**, 1, 173–176, 1974.
85. A. E. Green, W. Zerna, *Theoretical elasticity*, 2nd ed., Clarendon Press, Oxford 1968.
86. E. I. Grigolyuk, V. I. Mamai, A. N. Frolov, Mechanics of Solids (Izv. AN SSSR, MTT), 5, 154–165, 1972.
87. E. I. Grigolyuk, V. I. Mamai, Sov. Appl. Mech. (PM), **10**, 2, 3–13, 1974.
88. E. I. Grigolyuk, V. I. Mamai, Probl. Prochnosti, 10, 19–30, 1985.
89. E. I. Grigolyuk, V. V. Kabanov, *Stability of shells* (in Russian), Nauka, Moskva 1978.
90. R. Harte, Ruhr-Univ., IKB, TW Mitt., 82–10, Bochum 1982.
91. A. M. A. van der Heijden, *Thesis*, Delft University of Technology, Delft 1976.
92. A. M. A. van der Heijden, *Elastic stability*, Koiter's Course '78–'79, part I–IV, Delft 1979.
93. H.-C. Hu, Scienta Sinica, **4**, 1, 33–54, 1955.
94. N. C. Huang, Int. J. Mech. Sci., **15**, 6, 457–464, 1973.
95. J. R. Hughes, W.-K. Liu, Comp. Meth. Appl. Mech. and Engng., **26**, 331–362, 1981; **27**, 167–181, 1981.
96. M. Iura, M. Hirashima, Proc. JSCE, Struct. Eng./Earthq. Eng., 344/I-1, 153–163, 1984.
97. M. Iura, Int. J. Solids and Str., **22**, 2, 141–154, 1986.
98. R. G. Jeffers, M. A. Brull, Israel J. of Techn., **13**, 1–2, 111–121, 1975.
99. O. Jenssen, P. Papatzakos, Acta Polytech. Scan., Ser. Ci., 58, 1–24, Trondheim 1969.
100. O. Jenssen, C.-J. Lin, Norges Tekniske Høgskole, I. for Mekanikk., Medd., 19, 1–68, 1971.
101. F. John, Comm. Pure and Appl. Math., **18**, 1/2, 235–267, 1965.
102. F. John, in: *Theory of thin shells*, Ed. by F. I. Niordson, 1–14; Springer–Verlag, Berlin 1969.
103. F. John, Comm. Pure and Appl. Math., **24**, 5, 583–615, 1971.

104. V. V. KABANOV, Mechanics of Solids (Izv. AN SSSR, MTT), 4, 155–161, 1979.
105. B. Ya. KANTOR, *Nonlinear problems of non-homogeneous shallow shell theory* (in Russian), Naukova Dumka, Kiev 1971.
106. T. von KÁRMÁN, Encykl. der Math. Wiss., **IV/4**, 311–385; Teubner, Leipzig 1910.
107. T. L. KÁRMÁN, H. S. TSIEN, J. Aeronaut. Sci., **8**, 8, 303–312, 1941.
108. Ya. F. KAYUK, V. G. SAKHATZKII, Sov. Appl. Mech. (PM), **19**, 5, 36–41, 1983.
109. Ya. F. KAYUK, V. G. SAKHATZKII, Sov. Appl. Mech. (PM), **21**, 4, 65–73, 1985.
110. N. A. KILCHEVSKII, *Foundations of analytic mechanics of shells* (in Russian), AN USSR, Kiev 1963.
111. G. KIRCHHOFF, J. f. Math. (Crelle), **40**, 1850.
112. G. KIRCHHOFF, *Vorlesungen über Mathematische Physik*, **1**, Mechanik, 1876.
113. W. T. KOITER, Proc. IUTAM Symp. on Theory of Thin Shells, 12–33; North-Holland P. Co., Amsterdam 1960.
114. W. T. KOITER, *On the stability of elastic equilibrium* (in Dutch), Thesis, Delft; H. J. Paris, Amsterdam 1945.
115. W. T. KOITER, Proc. Koninkl. Ned. Ak. Wet., **B 60**, 1, 1–54, 1966.
116. W. T. KOITER, Proc. Symp. on Theory of Shells, 187–228; Univ. of Houston, Texas 1967.
117. W. T. KOITER, in: *Theory of thin shells*, Ed. by F. I. Niordson, 93–105; Springer–Verlag, Berlin 1969.
118. W. T. KOITER, Actes Congr. Int. Math., **3**, 123–130; Paris 1971.
119. W. T. KOITER, in: *Mechanics today*, **5**, Ed. by S. Nemat-Nasser, 139–154; Pergamon Press, Oxford 1980.
120. W. T. KOITER, J. G. SIMMONDS, in: *Theoretical and applied mechanics*, Ed. by E. Becker and G. K. Mikhailov, 150–175; Springer–Verlag, Berlin 1973.
121. M. S. KORNISHIN, *Nonlinear problems of plate and shallow shell theory and methods of their solutions* (in Russian), Nauka, Moskva 1964.
122. W. B. KRÄTZIG, Ing.-Arch., **40**, 311–326, 1971.
123. W. B. KRÄTZIG, in: *Konstruktiver Ingenieurbau in Forschung und Praxis*, 33–41; Werner–Verlag, Düsseldorf 1976.
124. W. B. KRÄTZIG, Y. BAŞAR, U. WITTEK, in: *Buckling of shells*, **1**, 2.1–2.38; Inst. f. Baustatik, U. Stuttgart 1982.
125. W. B. KRÄTZIG, U. WITTEK, Y. BAŞAR, in: *Collapse*, Ed. by J. M. T. Thompson and G. W. Hunt, 377–394; Cambridge Univ. Press, Cambridge 1983.
126. H. LAMB, Proc. London Math. Soc., **21**, 119–146, 1891.
127. H. L. LANGHAAR, Acta Mech., **19**, 1–2, 109–128, 1974.
128. R. W. LEONARD, *Thesis*, Virginia Polytechnic Institute, 1961.
129. A. LIBAI, J. Aerospace Sci., **29**, 1190–1195, 1962.
130. A. LIBAI, J. Appl. Mech., Trans. ASME, **48**, 4, 909–914, 1981.
131. A. LIBAI, J. Appl. Mech., Trans. ASME, **50**, 3, 531–536, 1983.
132. A. LIBAI, J. G. SIMMONDS, Int. J. Nonlinear Mech., **16**, 2, 91–103, 1981.
133. A. LIBAI, J. G. SIMMONDS, in: *Advances in applied mechanics*, **23**, Ed. by J. W. Hutchinson and T. Y. Wu, 271–371; Academic Press, New York 1983.
134. A. LIBAI, J. G. SIMMONDS, *The nonlinear theory of elastic shells, one spatial dimension*, Academic Press, Boston 1988.
135. A. E. H. LOVE, *A treatise on the mathematical theory of elasticity*, 4th ed., Cambridge Univ. Press, 1927.
136. A. I. LURIE, Prikl. Mat. Mekh., **4**, 2, 7–34, 1940.
137. A. I. LURIE, *Analytical mechanics* (in Russian), Nauka, Moskva 1961.
138. A. I. LURIE, *Nonlinear theory of elasticity* (in Russian), Nauka, Moskva 1980.
139. J. MAKOWSKI, Rozprawa Doktorska, Politechnika Gdańska, IBL, Gdańsk 1981.
140. J. MAKOWSKI, L.-P. NOLTE, Int. J. Nonlinear Mech., **22**, 1, 1–13, 1987.
141. J. MAKOWSKI, L.-P. NOLTE, H. STUMPF, Ruhr-Univ., Inst. f. Mech., Mitt., 46, Bochum 1985.
142. J. MAKOWSKI, W. PIETRASZKIEWICZ, Trans. ASME, J. Appl. Mech., 1989 (in print).

143. J. MAKOWSKI, H. STUMPF, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 175–194; Springer–Verlag, Berlin 1986.
144. K. MARGUERRE, Proc. 5th Int. Congr. of Appl. Mech., 93–101, Wiley and Sons, New York 1939.
145. J. MASON, *Variational, incremental and energy methods in solid mechanics and shell theory*, Elsevier Sci., Amsterdam 1980.
146. Kh. M. MUSHTARI, Trudy Kazanskogo Aviats. In-ta, 2, 1933.
147. Kh. M. MUSHTARI, Trudy Kazanskogo Aviats. In-ta, 4, 1935.
148. Kh. M. MUSHTARI, Izv. Kazanskogo Fiz.-Mat. Ob-va, Ser. 3, **11**, 71–150, 1938.
149. Kh. M. MUSHTARI, Prikl. Mat. Mekh., **2**, 4, 439–456, 1939.
150. Kh. M. MUSHTARI, Trudy Kazanskogo Khim.-Tekh. In-ta, 13, Kazan' 1948.
151. Kh. M. MUSHTARI, Dokl. AN SSSR, **69**, 4, 511–513, 1949.
152. Kh. M. MUSHTARI, Prikl. Mat. Mekh., **13**, 2, 121–134, 1949.
153. Kh. M. MUSHTARI, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 2, 39–52, 1950.
154. Kh. M. MUSHTARI, Prikl. Mat. Mekh., **15**, 6, 743–750, 1951.
155. Kh. M. MUSHTARI, Izv. KF AN SSSR, Ser. Fiz.-Mat. Tekh. Nauk, 12, 5–15, 1958.
156. Kh. M. MUSHTARI, Proc. 2nd All-Union Konf. on Theory of Plates and Shells (in Russian), 7–15, AN USSR, Kiev 1962.
157. Kh. M. MUSHTARI, K. Z. GALIMOV, *Nonlinear theory of elastic shells* (in Russian), Tatknigoizdat, Kazan' 1957.
158. P. M. NAGHDI, in: *Progress in solid mechanics*, **IV**, Ed. by I. N. Sneddon and R. Hill, 1–90, North–Holland, Amsterdam 1963.
159. P. M. NAGHDI, in: *Handbuch der Physik*, Ed. by S. Flügge, **VI/2**, 425–640; Springer–Verlag, Berlin 1972.
160. P. M. NAGHDI, in: *Finite elasticity*, Ed. by R. Rivlin, AMD, **27**, 77–89; ASME, New York 1977.
161. P. M. NAGHDI, in: Proc. IUTAM Symp. on Finite Elasticity, Ed. by D. E. Carlson and R. T. Shield, 47–104; Martinus Nijhoff, The Hague 1982.
162. P. M. NAGHDI, R. P. NORDGREN, Quart. App. Math., **21**, 1, 49–59, 1963.
163. P. M. NAGHDI, L. VONGSARNPIGOON, ARMA, **83**, 3, 245–283, 1983.
164. L.-P. NOLTE, ZAMM, **63**, 4, T79–T83, 1983.
165. L.-P. NOLTE, Dissertation, Ruhr-Univ., Inst. f. Mech., Mitt., 39, Bochum 1983.
166. L.-P. NOLTE, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 224–238; Springer–Verlag, Berlin 1986.
167. L.-P. NOLTE, J. CHRÓŚCIELEWSKI, in: Proc. Int. Conf. on Num. Meth. for Nonlinear Problems, Ed. by C. Taylor et al., 391–404; Pineridge Press, Swansea 1986.
168. L.-P. NOLTE, J. MAKOWSKI, Mech. Res. Comm., **12**, 6, 339–346, 1985.
169. L.-P. NOLTE, J. MAKOWSKI, H. STUMPF, Ing.-Arch., **56**, 2, 145–160, 1986.
170. L.-P. NOLTE, H. STUMPF, Mech. Res. Comm., **10**, 4, 213–221, 1983.
171. B. NOVOTNY, Int. J. Solids and Str., **6**, 4, 433–451, 1970.
172. B. NOVOTNY, ZAMM., **54**, 3, 145–156, 1974.
173. V. V. NOVOZHILOV, Dokl. AN SSSR., **32**, 5, 316–319, 1941.
174. V. V. NOVOZHILOV, *Foundations of the nonlinear theory of elasticity* (in Russian), Gostekhizdat, Moskva 1948.
175. V. V. NOVOZHILOV, *Thin shell theory* (in Russian), 2nd ed., Sudostroenie, Leningrad 1964.
176. V. V. NOVOZHILOV, *Developments in the theory of plates and shells* (in Russian), **VII**, 3–22; Kazan' Univ. Press, 1970.
177. V. V. NOVOZHILOV, R. FINKELSHTEIN, Prikl. Mat. Mekh., 7, 5, 331–340, 1943.
178. V. V. NOVOZHILOV, V. A. SHAMINA, Mechanics of Solids (Izv. AN SSSR, MTT), 5, 63–74, 1975.
179. T. J. ODEN, J. N. REDDY, *Variational methods in theoretical mechanics*, Springer–Verlag, Berlin 1976.
180. J. OLIVER, E. OÑATE, Int. J. Num. Meth. Engng., **20**, 12, 2253–2281, 1984.
181. H. PARISCH, Comp. Meth. in Appl. Mech. and Engng., **14**, 159–178, 1978.
182. W. PIETRASZKIEWICZ, Trans. Inst. Fluid-Flow Mach. PASci (Prace IMP PAN), **64**, 91–103, 1974.

183. W. PIETRASZKIEWICZ, Arch. Mech. Stos., **26**, 2, 221–228, 1974.
184. W. PIETRASZKIEWICZ, Rozprawa Habilitacyjna, Biul. IMP PAN, 172 (880), Gdańsk 1976.
185. W. PIETRASZKIEWICZ, *Introduction to the nonlinear theory of shells*, Ruhr-Univ., Inst. f. Mech., Mitt., 10, Bochum 1977.
186. W. PIETRASZKIEWICZ, Trans. Inst. Fluid-Flow Mach. PASci (Prace IMP PAN), **75**, 165–173, 1978.
187. W. PIETRASZKIEWICZ, in: *Konstrukcje powłokowe, teoria i zastosowania*, Ed. by J. Orkisz, Z. Waszczyszyn, **I**, 27–50; PWN, Warszawa 1978.
188. W. PIETRASZKIEWICZ, Vestnik Leningradskogo Un-ta, Ser. Mat.,Mekh. i Astron., 1, 115–124, 1979.
189. W. PIETRASZKIEWICZ, ZAMM, **59**, 5, T206–T208, 1979.
190. W. PIETRASZKIEWICZ, *Finite rotations and Lagrangean description in the nonlinear theory of shells*, Polish Sci. Publ., Warszawa–Poznań 1979.
191. W. PIETRASZKIEWICZ, Mech. Teor. Stos., **18**, 2, 169–192, 1980.
192. W. PIETRASZKIEWICZ, in: *Theory of shells*, Ed. by W. T. Koiter and G. K. Mikhailov, 445–471; North-Holland, Amsterdam 1980.
193. W. PIETRASZKIEWICZ, in: *Thin shell theory, new trends and applications*, Ed. by W. Olszak, 153–208; Springer-Verlag, Wien 1980.
194. W. PIETRASZKIEWICZ, Trans. Inst. Fluid–Flow Mach. PASci. (Prace IMP PAN), **81**, 79–92, 1981.
195. W. PIETRASZKIEWICZ, Ruhr-Univ., Inst. f. Mech., Mitt., 26, Bochum 1981.
196. W. PIETRASZKIEWICZ, ZAMM, **63**, T200–T202, 1983.
197. W. PIETRASZKIEWICZ, Int. J. Nonlinear Mech., **19**, 2, 115–140, 1984.
198. W. PIETRASZKIEWICZ, in: *Flexible shells*, Ed. by E. L. Axelrad and F. A. Emmerling, 106–123; Springer–Verlag, Berlin 1984.
199. W. PIETRASZKIEWICZ, J. BADUR, Int. J. Engng. Sci., **21**, 9, 1097–1115, 1983.
200. W. PIETRASZKIEWICZ, J. BADUR, in: *Trends in Appl. Pure Math. to Mechanics*, **IV**, Ed. by J. Brilla, 197–227; Pitman Publ., Boston 1983.
201. W. PIETRASZKIEWICZ, M. L. SZWABOWICZ, Arch. Mech., **33**, 2, 273–288, 1981.
202. W. PIETRASZKIEWICZ, M. L. SZWABOWICZ, ZAMM, **62**, T156–T158, 1982.
203. E. RAMM, in: *Formulations and computational algorithms in FE analysis*, Ed. by K. J. Bathe et al., 264–293; MIT Press, Boston 1977.
204. L. RECKE, W. WUNDERLICH, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 239–258; Springer–Verlag, Berlin 1986.
205. L. RECKE, Dissertation, Ruhr-Universität, IKB, TW Mitt., 86–12, Bochum 1986.
206. E. REISSNER, in: *Contributions to applied mechanics*, Ed. by J. W. Edwards, 231–247; Ann Arbor, Mich., 1949.
207. E. REISSNER, in: Proc. Symp. Appl. Math., **III**, 27–52; McGraw-Hill, New York 1950.
208. E. REISSNER, in: *Thin shell structures*, Ed. by Y. C. Fung and E. E. Sechler, 29–44; Prentice Hall, Englewood Cl., N. J., 1974.
209. E. REISSNER, Int. J. Nonlinear Mech., **17**, 3, 217–221, 1982.
210. G. ROMANO, Meccanica, **7**, 3, 141–146, 1972.
211. D. RÜDIGER, ZAMM, **41**, 5, 198–207, 1961.
212. Z. RYCHTER, Rozprawy Inżynierskie, **32**, 1, 115–122, 1984.
213. Z. RYCHTER, Mech. Teor. Stos., **26**, 2, 303–309, 1988.
214. T. SAKURAI, A. HASEGAWA, F. NISHINO, Proc. JSCE, 330, 151–159, 1983.
215. J. L. SANDERS, Quart. Appl. Math., **21**, 21–36, 1963.
216. R. SCHMIDT, Doktorarbeit, Abt. f. BIW, Ruhr-Universität, Bochum 1981.
217. R. SCHMIDT, ZAMM, **62**, T165–T167, 1982.
218. R. SCHMIDT, in: Proc. Int. Conf. on FEM, Ed. by He Guanggian and Y. K. Cheung, 621–626; Gordon and Breach Sci. Publ., New York 1982.
219. R. SCHMIDT, in: *Numerical methods for nonlinear problems*, **2**, Ed. by C. Taylor et al., 170–181; Pineridge Press, Swansea 1984.
220. R. SCHMIDT, in: Proc. SECTAM XII, **I**, 564–569; Auburn Univ., 1984.

221. R. Schmidt, in: *Flexible shells*, Ed. by E. L. Axelrad and F. A. Emmerling, 76–90; Springer–Verlag, Berlin 1984.
222. R. Schmidt, Computers and Structures, **20**, 1–3, 265–275, 1985.
223. R. Schmidt, ZAMM, **65**, 2, 119–221, 1985.
224. R. Schmidt, W. Pietraszkiewicz, Ing.-Arch., **50**, 3, 187–201, 1981.
225. R. Schmidt, H. Stumpf, Mech. Res. Comm., **11**, 2, 105–114, 1984.
226. K. Schweizerhoff, E. Ramm, Computers and Structures, **18**, 6, 1099–1114, 1984.
227. K. Schweizerhoff, E. Ramm, Proc. ASCE, J. Engng. Mech., **113**, 1, 72–88, 1987.
228. P. Seide, *Small elastic deformations of thin shells*, Noordhoff Int. Press, Leyden 1975.
229. P. Seide, Int. J. Nonlinear Mech., **6**, 3, 361–376, 1971.
230. I. P. Semeniuk, N. B. Zhukova, Sov. Appl. Mech. (PM), **22**, 5, 53–59, 1986.
231. M. J. Sewell, ARMA, **23**, 327–351, 1967.
232. V. A. Shamina, Mechanics of Solids (Izv. AN SSSR, MTT), 1, 15–22, 1974.
233. V. A. Shamina, in: *Actual problems of nonlinear continuum mechanics* (in Russian), **1**, 132–147; Leningrad Univ. Press., Leningrad 1977.
234. V. A. Shamina, in: *Actual problems of nonlinear continuum mechanics* (in Russian), **13**, 82–89; Leningrad Univ. Press., Leningrad 1980.
235. L. A. Shapovalov, Inzh. Zhurnal. MTT, 1, 56–62, 1968.
236. L. A. Shapovalov, Mechanics of Solids (Izv. AN SSSR, MTT), 3, 62–71, 1976.
237. D. I. Shilkrut. *Some problems of the nonlinear theory of shells and plates* (in Russian), Izd-vo AN Mold. SSR, Kishinev 1967.
238. D. I. Shilkrut, P. M. Vyrlan, *Stability of nonlinear shells* (in Russian), Shtintza, Kishinev 1977.
239. L. I. Shkutin, in: *Dynamics of continua* (in Russian), **26**, 133–136; Novosibirsk 1976.
240. L. I. Shkutin, in: *Applied problems of strength and plasticity* (in Russian), **7**, 3–9, 1977; **8**, 38–43, 1978; **9**, 19–25, Gorkii 1978.
241. L. I. Shkutin, Zhurnal PMTF, 1, 163–167, 1982.
242. J. P. Shrivastava, P. G. Glockner, Proc. ASCE, J. Engng. Mech. Div., **96**, 10, 547–563, 1970.
243. J. G. Simmonds, in: *Advances in engineering sciences*, Proc. 13th Annual Meet., Soc. Engng. Sci., 617–626; NASA CP-2001, 1976.
244. J. G. Simmonds, in: *Trends in solid mechanics 1979*, Ed. by J. F. Besseling and A. M. A. van der Heijden, 211–223; Sijthoff and Noordhoff, Delft 1979.
245. J. G. Simmonds, Amer. J. Physics, **52**, 9, 851–852, 1984.
246. J. G. Simmonds, Int. J. Solids and Str., **21**, 1, 67–77, 1985.
247. J. G. Simmonds, D. A. Danielson, Proc. Koninkl. Ned. Ak. Wet., **B73**, 5, 460–478, 1970.
248. J. G. Simmonds, D. A. Danielson, J. Appl. Mech., Trans. ASME, **39**, 4, 1085–1090, 1972.
249. I. N. Slezinger, V. I. Olifer, Sov. Appl. Mech. (PM), **17**, 11, 60–67, 1981.
250. L. S. Srubshchik, *Buckling and post-buckling behaviour of shells* (in Russian), Rostov Univ. Press, Rostov 1981.
251. E. Stein, in: *Theory of shells*, Ed. by W. T. Koiter and G. K. Mikhailov, 509–535; North–Holland, Amsterdam 1980.
252. E. Stein, A. Berg, W. Wagner, in: *Buckling of shells*, **2**, 13, 1–46; Inst. f. Baustatik, Univ. Stuttgart 1982.
253. E. Stein, W. Wagner, K. H. Lambertz, in: *Flexible shells*, Ed. by E. L. Axelrad and F. A. Emmerling, 22–43; Springer–Verlag, Berlin 1984.
254. J. J. Stoker, *Nonlinear elasticity*, Gordon and Breach, New York 1968.
255. H. Stumpf, ZAMM, **56**, T153–T155, 1976.
256. H. Stumpf, in: *Applications of methods of functional analysis to problems of mechanics*, Ed. by P. Germain and B. Neyroles, 500–510; Springer–Verlag, Berlin 1976.
257. H. Stumpf, Ing.-Arch., **48**, 221–237, 1979.
258. H. Stumpf, Ing.-Arch., **51**, 195–213, 1981.

259. H. Stumpf, in: *Mechanics of inelastic media and structures*, Ed. by O. Mahrenholtz and A. Sawczuk, 267–281; PWN, Warszawa–Poznań 1982.
260. H. Stumpf, in: *Stability in the mechanics of continua*, Ed. by F. H. Schroeder, 89–100; Springer–Verlag, Berlin 1982.
261. H. Stumpf, Int. J. Nonlinear Mech., **19**, 3, 195–215, 1984.
262. H. Stumpf, in: *Flexible shells*, Ed. by E. L. Axelrad and F. A. Emmerling, 76–90; Springer–Verlag, Berlin 1984.
263. H. Stumpf, ZAMM, **66**, 8, 337–350, 1986.
264. H. Stumpf, J. Makowski, Acta Mechanica, **65**, 1–4, 153–168, 1986.
265. K. S. Surana, Int. J. Num. Meth. Engng., **19**, 4, 581–615, 1983.
266. J. L. Synge, W. Z. Chien, Th. von Kármán Anniv. Vol., 103–120; Calif. Inst. of Techn., Pasadena 1941.
267. M. L. Szwabowicz, Rozprawa Doktorska, Instytut Maszyn Przepływowych PAN, Gdańsk 1982.
268. M. L. Szwabowicz, Zesz. Nauk. WSI Opole, **89**, 18, 267–272, 1982.
269. M. L. Szwabowicz, Trans. Inst. Fluid–Flow Mach. PASci (Prace IMP PAN), **86**, 99–110, 1983.
270. M. L. Szwabowicz, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 306–316; Springer–Verlag, Berlin 1986.
271. M. L. Szwabowicz, Int. J. Solids and Str., **22**, 11, 1161–1175, 1986.
272. D. Talaslidis, Dissertation, Ruhr-Univ., Bochum 1978.
273. I. G. Teregulov, Dokl. AN SSSR, **142**, 3, 568–571, 1962.
274. I. G. Teregulov, *Bending and stability of thin plates and shells in creep* (in Russian), Nauka, Moskva 1969.
275. S. Timoshenko, S. Woinowsky-Krieger, *Theory of plates and shells*, 2nd ed., McGraw-Hill, New York 1959.
276. E. Tonti, Meccanica, **2**, 4, 201–208, 1967.
277. E. Tonti, Atti Acad. Naz. Linzei, Cl. fis. mat., ser. 8, **44**, 75–82, 1, 1968.
278. S. Timoshenko, J. M. Gere, *Theory of elastic stability*, 2nd ed., McGraw-Hill, New York 1961.
279. C. Truesdell, R. Toupin, in: *Handbuch der Physik*, Ed. by S. Flügge, **III/1**, 226–793; Springer–Verlag, Berlin 1960.
280. C. Truesdell, W. Noll, in: *Handbuch der Physik*, Ed. by S. Flügge, **III/3**, 1–602; Springer–Verlag, Berlin 1965.
281. R. Valid, Computers and Structures, **10**, 1–2, 183–194, 1979.
282. R. Valid, ONERA, TP, 1980–86, Chatillon 1980.
283. R. Valid, *Mechanics of continuous media and analysis of structures*, North-Holland, Amsterdam 1981.
284. R. Valid, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 317–332; Springer–Verlag, Berlin 1986.
285. M. M. Vainberg, *Variational methods for the study of nonlinear operators* (Trans. from Russian), Holden-Day, San Francisco 1964.
286. N. V. Valishvili, *Computer methods of solution of shells of revolution* (in Russian), Nauka, Moskva 1976.
287. C. T. Wang, NACA TN 2620, 1952.
288. K. Washizu, MIT Aero. and Str. Res. Lab. TR, 25–18, Cambridge, Mass. 1955.
289. K. Washizu, *Variational methods in elasticity and plasticity*, 2nd ed., Pergamon Press, Oxford 1975.
290. G. Wempner, Int. J. Solids and Str., **5**, 2, 117–153, 1969.
291. G. Wempner, *Mechanics of solids with applications to thin bodies*, Sijthoff & Nordhoff, Alphen aan den Rijn 1981.
292. G. Wempner, J. Appl. Mech., Trans. ASME, **53**, 4, 881–885, 1986.
293. G. Wempner, in: *Finite rotations in structural mechanics*, Ed. by W. Pietraszkiewicz, 357–368; Springer–Verlag, Berlin 1986.
294. D. R. Westbrook, Int. J. Solids and Str., **7**, 8, 901–912, 1971.

295. V. Z. Vlasov, Prikl. Mat. Mekh., **7**, 2, 109–140, 1944.
296. V. Z. Vlasov, *General theory of shells and its applications in technology* (in Russian), Gostekhizdat, Moskva 1949.
297. A. S. Volmir, *Flexible plates and shells* (in Russian), Gostekhizdat, Moskva 1956.
298. A. S. Volmir, *Stability of deformable systems* (in Russian), 2nd ed., Nauka, Moskva 1967.
299. C. Woźniak, *Nieliniowa teoria powłok*, PWN, Warszawa 1966.
300. C. Woźniak, Ruhr-Univ., Inst. f. Mech., Mitt., 20, Bochum 1980.
301. C. Woźniak, in: *Konstrukcje powłokowe, teoria i zastosowania*, **I**, Ed. by J. Orkisz, Z. Waszczyszyn, 13–33; PWN, Warszawa 1978.
302. L. M. Zubov, Proc. Xth All-Union Conf. on Theory of Plates and Shells (in Russian), **I**, 129–134; Metzniereba, Tbilisi 1975.
303. L. M. Zubov, Appl. Mat. Mech. (PMM), **40**, 1, 85–95, 1976.
304. L. M. Zubov, *Methods of nonlinear elasticity in shell theory* (in Russian), Rostov Univ. Press, Rostov 1982.
305. L. M. Zubov, in: *Problems of dynamics and strength* (in Russian), 44, 22–28, 1984.

РЕЗЮМЕ

Геометрически нелинейные теории тонких упругих оболочек

В работе обсуждаются основные зависимости нелинейной теории тонких упругих оболочек. Рассматриваются различные виды уравнений равновесия и условий совместности деформации, а также соответственные энергетически согласованные статические и геометрические краевые условия и условия в угловых точках края оболочки. Эти основные системы зависимостей выражены через перемещения срединной поверхности оболочки или через повороты и другие параметры или же через двумерные меры деформаций и/или напряжений как независимые переменные. Разрешающие системы нелинейных уравнений теории оболочек соответственно упрощаются при предположении что деформации всюду малы. Уравнения оболочек в перемещениях дополнительно упрощаются при ограничении величины поворотов, а уравнения оболочек в мерах деформаций и/или напряжений дополнительно упрощаются при предположении различных соотношений мембранной и изгибной деформации. В случае консервативной поверхностной и краевой нагрузки строятся соответствующие вариационные функционалы для теории оболочек в перемещениях или в поворотах и других параметрах. Кроме обстоятельного обзора достижений в области построения различных вариантов нелинейной теории первого приближения тонких упругих оболочек, в работе представлен также ряд новых результатов полученных автором в этой области.

INSTYTUT MASZYN PRZEPŁYWOWYCH
PAN
UL. FISZERA 14
80-952 GDAŃSK

(Дата поступления статьи — 1988 III 23)