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Thermomechanics of Shells with Singular Curves

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Contents

Introduction

We formulate rigorously the global and local laws of mechanics and thermodynamics for shells with singularities at some stationary or moving curves in the shell base surface (itself not necessarily smooth). The laws are derived in an exact manner from underlying laws of continuum thermomechanics written for the shell-like body. Our formulation is sufficiently general to include not only traditional applications to reversible problems of regular shells, but also those modeling irreversible and non-smooth processes in irregular shells. Discontinuities at stationary singular curves model geometric and material irregularities such as folds, abrupt changes in the shell thickness or in material properties, etc. Moving singular curves can model one-dimensional physical phenomena in shells such as wave propagation, motion of coherent or incoherent phase boundaries, strain localization or cracks.

Different ways of interpreting the non-linear theory of shells, and distinct approaches to formulate complete set of the shell governing relations are known in the literature. Generally, the existing approaches can be grouped into two broader classes: a) the so-called *direct formulation*, and b) the *derived* or *deductive formulation*, the latter one having many faces. Naturally then, there exists a variety of shell theories differing if not in the basic concepts then in the form of resulting basic relations. Indeed, virtually each paper in the field contains some sort of derivation of the governing relations. Being of its own interest, the derived relations are obtained by methods convenient to the author within the context of intended specific applications. As a result, the shell relations and physical interpretations of their ingredients vary substantially throughout the literature.

Fortunately, there also exists an extensive literature discussing in detail the two approaches and different versions of shell theory, their advantages and disadvantages, their limitation and generality, their ranges of applicability and usefulness in application to specific problems. We refer to monographs by E. AND F. COSSERAT [1909], MUSHTARI AND GALIMOV [1957], WOŹNIAK [1966], NAGHDI [1972], PIETRASZKIEWICZ [1979], ANTMAN [1995], and LIBAI AND SIMMONDS [1988,1998], to survey papers by KOITER [1966], LIBAI AND SIMMONDS [1983] and PIETRASZKIEWICZ [1989,2001], as well as to collection

of 620 books, conference proceedings and survey papers compiled by NOOR [1990] and PIETRASZKIEWICZ [1992], where extensive references to original papers are given.

The extent to which the non-linear theory of shells has been understood is reflected in its mathematical foundations, clarity of its formal structure, and precision of definitions, statements and results associated with it. It seems to be clear that each approach to formulate shell relations contains a considerable amount of anticipation of the results to be obtained. Therefore, only the very basic concepts underlying different versions of shell theory are worth of a serious discussion. Details of formulation and derivation may then be treated as a purely technical matter, possibly quite involved and requiring much effort. Naturally, the choice of the basic concepts may be a matter of taste.

Among many possible ways leading to formulation of the general shell relations, the approach proposed by SIMMONDS [1984] within the general thermomechanics of shell-like bodies deserves a special attention. This approach is so natural and straightforward that it has a good chance to be generally accepted in the future. The principal features of this formulation distinguishing it from other ones are:

- Dynamics, not kinematics what is more common in shell theories, is taken as the basic concept underlying the non-linear shell theory.
- Dynamic balance laws and the principle of irreversibility for shells are obtained as exact specification of the principles of thermomechanics of the Cauchy continuum for the three-dimensional shell-like body. This specification is made by direct through-the-thickness integration of appropriate 3D fields, and involves no approximations or postulates whatsoever in nature.
- Shell kinematics is not assumed, but is derived exactly through the 2D virtual work identity written for the shell base surface. This results in the displacement vector and the rotation tensor fields as the only independent kinematic variables describing the gross shell motion. Associated shell strain measures are introduced exactly as well, as implied 2D fields work-conjugate to the exact resultant shell stress measures.
- All approximations are made by SIMMONDS [1984] only in the assumed form of the equation of energy balance and in the associated constitutive equations.

This approach, within the purely mechanical setting, was developed further and applied to specific equilibrium problems of regular shells by LIBAI AND SIMMONDS [1983,1998], MAKOWSKI AND STUMPF [1990], CHRÓŚCIELEWSKI *et*

al. [1992], and to irregular shells by MAKOWSKI AND STUMPF [1994], CHRÓŚCIELEWSKI [1996], CHRÓŚCIELEWSKI *et al.* [1997], and PIETRASZKIEWICZ [2001], as well as to non-linear shell dynamics of regular and multifold shells by CHRÓŚCIELEWSKI *et al.* [2000,2002].

Several attempts to develop thermodynamic theory of shells with the regular base surface are known in the literature. Apart of the one by SIMMONDS [1984], other early versions of shell thermomechanics were proposed by KRÄTZIG [1971], GREEN *et al.* [1965], NAGHDI [1965], GREEN AND NAGHDI [1970,1979], HOFFMANN AND CAHN [1972,1979], and MURDOCH [1976]. In all of the versions either some simplifying kinematic assumptions were used when reducing the 3D continuum thermomechanics to the 2D shell thermomechanics, or the shell theory was constructed directly as for the material surface with an additionally assumed internal structure, without any relation to 3D theory.

This report is an attempt to further extend basic results in general thermomechanics of shells, which were obtained by SIMMONDS [1984] using the reduction procedure outlined above and applied recently by SIMMONDS [2001] within one-dimensional shell problems. It seems to us that the most important conclusion following from the Simmonds papers was that any properly formulated general shell theory, whether the direct or derived approach is applied, may be based on only five postulated or derived 2D laws: balance laws of mass, linear momentum, angular momentum and energy, together with the principle of irreversibility. No additional postulates are needed.[1] However, the structure of these five 2D laws, which may follow either from some rational reduction of 3D laws or through an appropriate direct representation of the shell-like body by a material surface, becomes far from the standard one generally accepted in continuum thermomechanics of 3D bodies. On the other hand, various generalised forms of continuum thermomechanics, often introduced with a little physical justification, appear quite natural within the resulting 2D shell thermomechanics, when rigorous reduction of the rational thermomechanics of the 3D Cauchy continuum to the two-dimensional form appropriate for shell-like bodies is performed.

We generalise here the thermomechanics of shells in three principal aspects:

[1]It may be noted that most shell theories are usually using some additional postulates such as, for example, the balance equation of director forces (cf. e.g. NAGHDI [1972]).

1. We modify the two-dimensional law of energy balance for shells by introducing additional terms representing the interstitial working and responsible for spatial interactions of longer range.
2. We allow the shell base surface and various fields defined on it to be non-smooth along specified stationary or moving curves.
3. We develop the general structure and several specific forms of the constitutive equations for regular shell parts.

When thermomechanics of the Cauchy continuum is consistently reduced to the corresponding shell thermomechanics, as in SIMMONDS [1984], the stress tensor field gives rise to the stress resultant and the resultant couple tensor fields naturally defined on the shell base surface. Thus, according to TOUPIN [1962], the resulting shell theory becomes a kind of theory of higher-grade 2D continuum. It is well known that higher-grade models of continua are incompatible, in general, with the usual laws of the rational continuum thermomechanics described in TRUESDELL AND TOUPIN [1960], TRUESDELL AND NOLL [1965] and TRUESDELL [1984]. To allow for spatial interactions of longer range, several broader structures of continuum thermodynamics were proposed in BRIDGMAN [1961], KESTIN [1988], JOU *et al.* [1988,2001], and MÜLLER AND RUGGERI [1998], for example, where references to other papers are given.

We shall use here the particularly simple and attractive concept of the extension of continuum thermomechanics which was proposed by DUNN AND SERRIN [1985] and DUNN [1986]. Adjusting the concept to our 2D shell theory, we propose here, after MAKOWSKI [2000], to preserve without any change the two-dimensional purely mechanical balance laws of linear and angular momenta as well as the purely thermal principle of irreversibility. Only to the 2D balance law of energy we introduce additional fields analogous to those suggested for 3D theory in DUNN AND SERRIN [1985]. These fields are responsible for an additional so-called interstitial working which is lost by the reduction procedure applied by SIMMONDS [1984]. With these additional fields the energy balance for shells may also be regarded as an exact consequence of the 3D energy balance for the shell-like body. Introduction of these fields provides a convenient mechanism for the surface stress measures, energy and entropy fields to depend on higher gradients of the shell deformation and temperature fields. Such a dependence is necessary in the general discussion of the shell constitutive equations.

The second aspect of shell thermomechanics discussed in this report is related to suitable regularity assumptions to be used for the shell base surface and for various fields entering the thermomechanical processes in shells. The initial integral-impulse statements of the laws of thermomechanics for the shell-like body only requires that the reference configuration of the shell base surface be regular enough for the surface and curvilinear integrals to be meaningfully defined. Such statements of basic laws provide a very general background on which a general thermomechanical theory of shells can be founded. Corresponding local field equations and side conditions of shell thermomechanics should then follow from their global statements upon application of the generalised surface gradient-divergence theorem. Several forms of this theorem are derived here, because their standard forms presented in many books and papers are restricted to smooth surfaces with smooth boundary curves and to smooth surface fields. Such regularity restrictions are too strong for many shell shapes and many thermomechanical processes of physical or engineering importance.

Real shell structures often contain folds, branches, self-intersections, stiffeners, stepwise thickness changes, parts made of different materials, technological connections etc. The base surface of such shell-like bodies cannot be regarded as smooth or regular one. Some mechanical problems of irregular shells and their numerical analysis were discussed by MAKOWSKI AND STUMPF [1994], CHRÓŚCIELEWSKI [1996], CHRÓŚCIELEWSKI *et al.* [1997,2002] and MAKOWSKI *et al.* [1999], where other existing results are referred as well.

Equally important for possible future applications are thermomechanical shell problems with discontinuous fields along specified singular curves. We are not aware of any general formulation of the non-linear theory of shells which would take into account discontinuities of the fields describing thermomechanical processes. Therefore, let us note that within the continuum thermomechanics several physical mechanisms are known which develop weak and strong discontinuities in thermomechanical processes at some stationary and moving singular surfaces within the body. As examples of such processes let us mention wave propagation, motion of coherent and incoherent phase boundaries, continuum theory of dislocations and disclinations, strain localisation in plasticity and damage mechanics, and fracture of solids. In some of these problems not only smoothness, but even continuity of the relevant fields cannot be taken for granted.

For description of such singular or discontinuous processes in continuum thermomechanics we refer to monographs by TRUESDELL AND TOUPIN [1960],

TRUESDELL AND NOLL [1965], MAUGIN [1993,1995], ŠILHAVÝ [1997], and GURTIN [2000], where many references to original papers are given. It seems that a 2D theory describing similar problems in shells, with singularities or discontinuities reduced to some stationary and/or moving curves, should be of great interest to the scientific community. It would allow one to model and analyse some of the complex discontinuous 3D processes described above within a simplified 2D setting with hope for better insight into physical understanding of the phenomena.

The contents of the report can concisely be characterised as follows. In Chapter 1 we remind basic laws of the rational continuum thermomechanics expressed in the integral-impulse form. These are balance laws of mass, linear momentum, angular momentum, and energy, as well as the principle of irreversibility. By representing these laws consistently on the shell base surface the appropriate laws of thermomechanics for shells are derived in terms of the fields defined on the base surface.

In Chapter 2 we discuss various regularity assumptions associated with the reference base surface and with various tensor fields defined on it. The base surface is regarded to be only piecewise smooth, in general, with an almost smooth boundary curve. We describe singular curves moving relative to the base surface and piecewise continuous tensor fields having various jump discontinuities at the singular curves. For such discontinuous fields we develop several forms of the generalised surface gradient-divergence theorem and the surface transport theorem, which are to be valid on the piecewise smooth surface. These theorems are the main driving vehicles allowing us to reduce the integral - impulse statements of the laws of shell thermomechanics to their corresponding local forms.

Local field equations, continuity conditions at the singular curves and several side conditions for shell thermomechanics are derived and analysed in Chapter 3, with a special attention focused on those results which follow from the relaxed regularity assumptions used here. In particular, we derive several original mechanical and thermal continuity conditions and inequalities to be satisfied at the singular curves.

In our approach to shell thermomechanics the shell kinematics is implied exactly by the local resultant balance laws of linear and angular momenta. In Chapter 4 we discuss in more detail how to construct such an exact shell kinematics from the weak form of the momenta balance laws. It is proved, in particular, that any shell configuration is completely determined by the position in space of the base surface and by the non-singular second-order tensor field of

shell structure. Relative to the reference configuration, the shell position is established by the displacement vector of its base surface and the rotation field describing the gross mean rotation of shell cross sections.

It is noted in Chapter 5 that the 2D effective mechanical power of the shell, following from corresponding 2D representation on the shell base surface, may not be an exact consequence of the mechanical power of the 3D shell-like body. We propose here to fill this gap by introducing additional fields responsible for an interstitial working, which allow us to restore the lost part of the mechanical power. Then the influence of these fields on all the shell relations are discussed.

The shell strain and bending measures which are work-conjugate to the respective shell stress and couple measures are constructed in Chapter 6 and several related kinematic results are derived.

In Chapter 7 the roles of various thermomechanical fields are discussed and general functional forms of 2D constitutive equations are proposed. We also discuss several additional assumptions about the constitutive nature of heat fluxes and temperatures assigned to the shell faces as well as about the interstitial working, extra entropy source and flux.

Thermodynamic consistency of the shell constitutive equations is analysed in Chapter 8. By introducing thermodynamic potentials, we discuss the structure of kinetic constitutive equations. Shell constitutive equations in the material representation for the class of "simple" shells are developed.

Chapter 9 deals with general and several specific forms of constitutive equations appropriate for heat conduction and thermo-visco-elasticity in shells. In particular, the structure of constitutive equations appropriate for thermoelastic, isothermal, and higher-grad shells is proposed.

We believe that the results reported here provide a good introduction to description of a variety of problems of shell thermomechanics which can model complex irreversible and non-smooth 1D processes in irregular shell structures. Please note, however, that some of the non-smooth or singular 1D phenomena may require additional configuration forces or energy densities to be associated with the singular curves. Within 3D continuum thermomechanics, discussion of such 2D singular processes is given by MAUGIN [1993,1995] and GURTIN [2000]. Modelling of analogous 1D singular processes within 2D shell thermomechanics is still waiting for a sufficiently general and satisfactory description. We believe that the results presented in this report provide a good introduction to the field and will allow one to model such 1D singular processes in shells already in the near future.

Chapter 1
Integral-impulse laws of thermomechanics

1.1 Basic notation

In this report the physical space $\mathcal{E}$ is the three-dimensional Euclidean point space with elements $x, y, o, \ldots \in \mathcal{E}$ called points. Its translation space E is the three-dimensional inner-product vector space with elements $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{u}, \boldsymbol{\omega}, \ldots \in E$ called vectors. Second-order tensors, as linear transformations of E into itself, are denoted by $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{S}, \boldsymbol{T}, \boldsymbol{H}, \ldots \in E \otimes E$, where $\otimes$ is the tensor product. We shall use a direct multilinear algebra and analysis as given, for example, in HALMOS [1958]. Notation conventions of geometric and physical fields are similar as in TRUESDELL AND NOLL [1965], GURTIN [1981], TRUESDELL [1984], and ŠILHAVÝ [1997].

Many fields in this report are defined only on points $x \in M$ of a surface $M \in \mathcal{E}$. If $T_x M \subset \mathcal{E}$ is a tangent space to M at x (a two-dimensional inner-product vector space), then the surface vectors will be distinguished by different fonts: $\boldsymbol{u}, \boldsymbol{w}, \boldsymbol{p}, \boldsymbol{q}, \boldsymbol{j}, \ldots \in T_x M$. On the surface M we apply coordinate-free algebra and analysis developed by GURTIN AND MURDOCH [1975] AND MURDOCH [1990]. The standard concepts of spatial and surface differential operators as defined in GURTIN [1981,2000] will be applied throughout, with appropriate generalizations developed in MAKOWSKI AND STUMPF [1994].

1.2 Framework of continual theories

In a synthetic approach to non-linear thermomechanics of continuous bodies, one begins by choosing a system of fundamental postulates, referred to as physical principles, which are assumed to be valid for all bodies within a considered class.

As is standard in modern theories of continua, the subsequent considerations are based on the idea that all material bodies possess mass, sustain forces and torques, convert energy, as well as basic laws of mechanics and thermodynamics together with axioms of constitution are valid for every

part of the body regardless of its size. In the following discussion $\mathcal{P}$ denotes a part of the body - the shell-like body in this report.

For description of mechanical behavior of the body the following quantities are assumed to be meaningful:

$\mathfrak{m}(\mathcal{P},t)$ – ***mass*** (scalar),

$\mathfrak{c}(\mathcal{P},t)$ – ***mass production*** (scalar),

$\mathfrak{p}(\mathcal{P},t)$ – ***linear momentum*** (vector),

$\mathfrak{f}(\mathcal{P},t)$ – ***total force*** (vector),

$\mathfrak{m}(\mathcal{P},t)$ –***angular momentum*** (axial vector),

$\mathfrak{t}(\mathcal{P},t)$ – ***total torque*** (axial vector).

According to modern interpretation of continuum mechanics, all these entities are primitive objects not defined in terms of other quantities. They must be given *a priori* for each class of particular theories. In general, they must be consistent with the following, global in space and time, balance laws of mechanics:

balance of mass

$$\left[\mathfrak{m}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathfrak{c}(\mathcal{P},t)dt\,, \tag{1.1}$$

balance of linear momentum (inertial frames)

$$\left[\mathfrak{p}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathfrak{f}(\mathcal{P},t)dt\,, \tag{1.2}$$

balance of angular momentum (inertial frames)

$$\left[\mathfrak{m}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \mathfrak{t}(\mathcal{P},t)dt\,. \tag{1.3}$$

When a theory is designed to account for thermal effects, the balance laws of mechanics alone do not suffice to formulate the complete set of governing relations. These laws must be supplemented by additional postulates needed to account for non-mechanical effects associated with heat and temperature. Such additional postulates are commonly called the principles of thermodynamics. It is generally accepted that there are no more but two such additional principles. However, even today there is no general agreement as to which specific forms these principles should take.

In contemporary thermodynamics of irreversible processes of continua several directions are being developed, for example: a) classical irreversible

thermodynamics, developed by ONSAGER [1931] and summarized by PRIGOGINE [1961], and DEGROOT AND MAZUR [1963]; b) extended irreversible thermodynamics, summarized in JOU *et al.* [2001] , and MÜLLER AND RUGGERI [1998]; c) internal variables thermodynamics, developed among others by BRIDGMAN [1961] and KESTIN [1988]; and d) rational thermodynamics, developed by TRUESDELL AND TOUPIN [1960], TRUESDELL AND NOLL [1965], TRUESDELL [1984] and ŠILHAVÝ [1997]. In this report we use the rational thermodynamics based on the Clausius-Duhem inequality, see TRUESDELL [1984].

In general, the first law of thermodynamics grants the primacy of energy, and states that the time rate of energy of a body is balanced by the rate of work of forces (and possibly couples) acting on a body together with the rate of heat addition. There are several possible interpretations of this law, depending on the meaning assigned to the terms "energy", "work", and 'heat". If the term "energy" refers to the total energy and the term "work" is associated with the mechanical power, then the formal approach to continuum thermodynamics may be based on the assumption that for every part of the body – here the shell-like body - the following quantities must be presumed:

$\mathfrak{u}(\mathcal{P},t)$ – ***total energy***,
$\mathfrak{p}(\mathcal{P},t)$ – ***mechanical power***,
$\mathfrak{q}(\mathcal{P},t)$ – ***heating.***

All quantities described above are scalars, with $\mathfrak{q}(\mathcal{P},t)$ having the same physical dimension as $\mathfrak{p}(\mathcal{P},t)$, namely force times distance per unit time. They are assumed to satisfy the first law of thermodynamics

balance of energy (inertial frames)

$$\left[\mathfrak{u}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \left\{\mathfrak{p}(\mathcal{P},t) + \mathfrak{q}(\mathcal{P},t)\right\} dt\,. \tag{1.4}$$

Just as the first law of thermodynamics is intimately related to the concept of energy, the second law of thermodynamics deals with another concept not present in mechanics, namely with the one of entropy. Entropy is inextricably linked to energy, and like energy is more of a concept than a thing. Like energy, the entropy is a relative concept, and thus only the changes of entropy that are significant. A formal approach to the second law of thermodynamics may be based on the assumption that every part of the shell is assigned two scalar quantities

$\mathfrak{h}(\mathcal{P},t)$ – ***entropy***,

$\mathfrak{j}(\mathcal{P},t)$ – ***entropy flux***.

These two fields are assumed to satisfy the inequality

principle of irreversibility

$$\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt \equiv \left[\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{j}(\mathcal{P},t)dt \geq 0. \tag{1.5}$$

This inequality is regarded as the second law of thermodynamics. It states that the ***total entropy production*** $\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt$ over the time interval $[t_1,t_2]$ is never negative. The entropy production is a result of increase of entropy of $\mathcal{P}$ equal to $\left[\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2}$, and the exchange of entropy between parts of the body as well as between the body and the environment expressed by $\int_{t_1}^{t_2} \mathfrak{j}(\mathcal{P},t)dt$. Thus the quantity $\mathfrak{g}(\mathcal{P},t)$ defined in terms of $\mathfrak{h}(\mathcal{P},t)$ and $\mathfrak{j}(\mathcal{P},t)$ is the ***total rate of entropy production***.

The above considerations provide merely the framework upon which a particular thermomechanical theory of continuum, in our case the shell thermomechanics, can be built up. These are the specific forms of all the entities appearing in the laws of mechanics and thermodynamics, which distinguish one theory from another.

1.3 Inertial objects in mechanics

While the principle of balance of mass (1.1) and the principle of irreversibility (1.5) are frame-indifferent, the principles of balance of linear momentum (1.2), balance of angular momentum (1.3), and balance of energy (1.4) are formulated relative to the particular class of frames, called the ***inertial frames*** of reference.

Let $\{\mathrm{o},\boldsymbol{e}_i\}$ be an inertial frame of reference at time t, with $\mathrm{o}\in\mathcal{E}$ and an orthonormal basis $\boldsymbol{e}_i, i=1,2,3$ of E. Let also $\{\mathrm{o}^*,\boldsymbol{e}_i^*\}$ be an arbitrary frame of reference at time $t^*=t-a$, such that $\boldsymbol{e}_i^*=\boldsymbol{O}(t)\boldsymbol{e}_i$, where $\boldsymbol{O}(t)$ is any time-dependent orthogonal transformation, and a is a time shift. When motion of the body is described relative to an arbitrary frame of reference, the balance laws of momenta and the balance law of energy no longer retain their forms (1.2), (1.3) and (1.4).

In continuum thermomechanics the mass, the heating, the entropy and the entropy flux of any part of the body $\mathcal{P}$ are assumed to be unaffected by any change of frame of reference, and we have

$$\begin{aligned} \mathfrak{m}^*(\mathcal{P},t^*) &= \mathfrak{m}(\mathcal{P},t), \qquad \mathfrak{q}^*(\mathcal{P},t^*) = \mathfrak{q}(\mathcal{P},t), \\ \mathfrak{h}^*(\mathcal{P},t^*) &= \mathfrak{h}(\mathcal{P},t), \qquad \mathfrak{j}^*(\mathcal{P},t^*) = \mathfrak{j}(\mathcal{P},t) \ . \end{aligned} \tag{1.6}$$

As part of characterization of forces and torques we must indicate their transformation rules under the changes of frame of reference, regardless of whether these changes can be directly observed or measured. The physical interpretation of the force and torque requires that they be frame-indifferent, that is

$$\mathfrak{f}^*(\mathcal{P},t^*) = \boldsymbol{O}(t)\mathfrak{f}(\mathcal{P},t), \qquad \mathfrak{t}^*(\mathcal{P},t^*) = \boldsymbol{O}(t)\mathfrak{t}(\mathcal{P},t). \tag{1.7}$$

Although the expressions of the momenta are not declared yet, it seems legitimate to infer from (1.2) and (1.3) that a standard additive splitting has been accepted here; namely, that the total force and the total torque are sums of a non-inertial part (left-hand sides) and an inertial part (right-hand sides), a sort of D'Alembert force and torque. Without such a splitting, the balance laws of mechanics (1.2) and (1.3) should be replaced by

$$\int_{t_1}^{t_2} \mathfrak{f}'(\mathcal{P},t)dt = \mathbf{0}, \qquad \int_{t_1}^{t_2} \mathfrak{t}'(\mathcal{P},t)dt = \mathbf{0}, \tag{1.8}$$

where $\mathfrak{f}'(\mathcal{P},t)$ and $\mathfrak{t}'(\mathcal{P},t)$ are the ***total force*** and the ***total torque*** acting on $\mathcal{P}$, respectively. These two forms of the balance laws of mechanics are equivalent if the linear and angular momenta are expressed in terms of the inertial force and the inertial couple, i.e. if and only if the inertial force and torque are determined in terms of the respective linear and angular momenta.

If $\mathfrak{p}(\mathcal{P},t)$ and $\mathfrak{m}(\mathcal{P},t)$ are differentiable with respect to time, then

$$\left[\mathfrak{p}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \dot{\mathfrak{p}}(\mathcal{P},t)dt, \qquad \left[\mathfrak{m}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \dot{\mathfrak{m}}(\mathcal{P},t)dt. \tag{1.9}$$

Let us introduce explicitly the inertia force $\mathfrak{f}^{in}(\mathcal{P},t)$ and the inertia torque $\mathfrak{t}^{in}(\mathcal{P},t)$ as negative rates of change of the linear and angular momenta, respectively,

$$\mathfrak{f}^{in}(\mathcal{P},t) \equiv -\dot{\mathfrak{p}}(\mathcal{P},t), \qquad \mathfrak{t}^{in}(\mathcal{P},t) \equiv -\dot{\mathfrak{m}}(\mathcal{P},t). \tag{1.10}$$

Then the total force and the total torque are defined by

$$\mathfrak{f}'(\mathcal{P},t) \equiv \mathfrak{f}(\mathcal{P},t) + \mathfrak{f}^{in}(\mathcal{P},t), \qquad \mathfrak{t}'(\mathcal{P},t) \equiv \mathfrak{t}(\mathcal{P},t) + \mathfrak{t}^{in}(\mathcal{P},t), \tag{1.11}$$

where the non-inertia force $\mathfrak{f}(\mathcal{P},t)$ and the non-inertia torque $\mathfrak{t}(\mathcal{P},t)$ are explicitly shown. Now (1.8) with (1.11) are general statements of the balance laws of mechanics, which are equivalent to those (1.2) and (1.3) formulated in inertial frames.

Unlike the balance of linear and angular momenta, the balance of energy expressed by (1.4) does not split the total energy into non-inertial and inertial parts. The concept of the mechanical power $\mathfrak{p}(\mathcal{P},t)$ includes the implicit assumption that the total force and torque may be split into the sum of non-inertial and inertial parts, the latter one being defined in terms of the linear and angular momenta. This assumption underlines the statement of the basic laws of mechanics in the form (1.4) and (1.8). Without this assumption, the term "work" should be associated with the ***total mechanical power*** denoted by $\mathfrak{w}(\mathcal{P},t)$, which is generally defined as the rate of work of all forces and couples, thus including the rate of work of the inertial forces and couples. Consistently, the term "energy" should then be understood as the ***internal energy*** $\mathfrak{e}(\mathcal{P},t)$, and the balance of energy should take the form

$$\left[\mathfrak{e}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \left\{\mathfrak{w}(\mathcal{P},t) + \mathfrak{q}(\mathcal{P},t)\right\} dt. \tag{1.12}$$

If the total mechanical power is written as the sum of inertial and non-inertial parts,

$$\mathfrak{w}(\mathcal{P},t) = \mathfrak{p}(\mathcal{P},t) + \mathfrak{w}^{in}(\mathcal{P},t), \tag{1.13}$$

then the balance of energy (1.12) may rewritten as

$$\left[\mathfrak{e}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{w}^{in}(\mathcal{P},t)dt = \int_{t_1}^{t_2} \left\{\mathfrak{p}(\mathcal{P},t) + \mathfrak{q}(\mathcal{P},t)\right\} dt. \tag{1.14}$$

It follows that (1.14) and (1.4) are equivalent statements of the balance of energy provided that the total energy $\mathfrak{u}(\mathcal{P},t)$ is assumed to be determined by the equation

$$\left[\mathfrak{u}(\mathcal{P},t)\right]_{t_1}^{t_2} = \left[\mathfrak{e}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{w}^{in}(\mathcal{P},t)dt. \tag{1.15}$$

The total energy $\mathfrak{u}(\mathcal{P},t)$ is often additively decomposed into the ***internal energy*** $\mathfrak{e}(\mathcal{P},t)$ and the ***kinetic energy*** $\mathfrak{k}(\mathcal{P},t)$

$$\mathfrak{u}(\mathcal{P},t) = \mathfrak{e}(\mathcal{P},t) + \mathfrak{k}(\mathcal{P},t). \tag{1.16}$$

It follows from (1.15) that the splitting (1.16) is equivalent to requiring that

$$\int_{t_1}^{t_2} \mathfrak{w}^{in}(\mathcal{P},t)dt - \left[\mathfrak{k}(\mathcal{P},t)\right]_{t_1}^{t_2} = 0. \tag{1.17}$$

1.4 Isothermal processes

In an isothermal process, that is when temperature is regarded to be constant, the entropy flux $\mathfrak{j}(\mathcal{P},t)$ is assumed to be inversely proportional to the heating $\mathfrak{q}(\mathcal{P},t)$

$$\mathfrak{j}(\mathcal{P},t) = \theta^{-1}\mathfrak{q}(\mathcal{P},t), \tag{1.18}$$

where the strictly positive constant scalar θ, $\theta > 0$, is called the ***absolute temperature***. In this case, the total entropy production over the time interval $[t_1,t_2]$ can be written in the form

$$\begin{aligned}\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt &= \left[\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \theta^{-1}\mathfrak{q}(\mathcal{P},t)dt \\ &= \theta^{-1}\left\{\left[\theta\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{q}(\mathcal{P},t)dt\right\}.\end{aligned} \tag{1.19}$$

With the use of the balance law of energy (1.4), the impulse of heating

$$\int_{t_1}^{t_2} \mathfrak{q}(\mathcal{P},t)dt = \left[\mathfrak{u}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{p}(\mathcal{P},t)dt \tag{1.20}$$

may be eliminated from (1.19) to yield

$$\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt = \theta^{-1}\left\{-\left[\mathfrak{u}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathfrak{p}(\mathcal{P},t)dt\right\}. \tag{1.21}$$

If the total energy can be written as in (1.16), then

$$\mathfrak{u}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t) = \mathfrak{e}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t) + \mathfrak{k}(\mathcal{P},t). \tag{1.22}$$

The ***free energy for an isothermal process*** can be defined by

$$\mathfrak{f}(\mathcal{P},t) = \mathfrak{e}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t). \tag{1.23}$$

Therefore,

$$\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt = \theta^{-1}\left\{-\left[\mathfrak{u}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathfrak{p}(\mathcal{P},t)dt\right\}. \tag{1.24}$$

in which case the total entropy production is given by

$$\int_{t_1}^{t_2} \mathfrak{g}(\mathcal{P},t)dt = \theta^{-1}\left\{-\left[\mathfrak{f}(\mathcal{P},t)\right]_{t_1}^{t_2} + \int_{t_1}^{t_2} \mathfrak{w}(\mathcal{P},t)dt\right\}. \tag{1.25}$$

Since temperature is strictly positive,

$$\left[\mathfrak{u}(\mathcal{P},t) - \theta\mathfrak{h}(\mathcal{P},t)\right]_{t_1}^{t_2} \leq \int_{t_1}^{t_2} \mathfrak{p}(\mathcal{P},t)dt, \tag{1.26}$$

the principle of irreversibility (1.5) implies that

$$\left[\mathfrak{f}(\mathcal{P},t)\right]_{t_1}^{t_2} - \int_{t_1}^{t_2} \mathfrak{w}(\mathcal{P},t)dt \leq 0 \tag{1.27}$$

in every isothermal process.

1.5 Referential description of shell motion

All primitive objects introduced in Section 1.2 – mass, force, momentum, energy, heating, etc. – are defined globally over a part $\mathcal{P}$ of the shell-like body $\mathcal{B}$. In continuum thermomechanics each spatial configuration $P(t) \subset B(t)$ of $\mathcal{P} \subset \mathcal{B}$ at time $t \in T$ is the 3D region of the physical space $\mathcal{E}$, and $P \in B$ is the region occupied by $\mathcal{P}$ in the reference configuration B usually associated with $t = 0$. Body particles $Z \in \mathcal{B}$ are identified by their places $\mathsf{Z} \in \mathcal{E}$ in the reference configuration, see TRUESDELL AND NOLL [1965].

In this report each spatial configuration of the shell-like body $\mathcal{B}$ at time $t \in T$ is represented by a base surface $M(t)$ in the physical space $\mathcal{E}$. Thus, the shell is regarded as a kind of 2D continuum consisting of generalized point-like particles, possibly with an additionally assigned internal structure, which are smoothly distributed over $M(t)$ at each time instant $t \in T$. The additional structure of the shell particles will be discussed in Section 4.5 and in Chapter 6.

In general, an ***admissible reference configuration*** of the shell base surface is any geometric surface $M \subset \mathcal{E}$ having the chosen degree of regularity. Naturally, different regularity assumptions will be needed for different classes of problems, and this question will be discussed in more detail in Chapter 2. The reference configuration serves to identify the shell particles. Thus $\mathsf{X} \in M$ is a place occupied by a typical shell particle X in the reference configuration. The position vector of $\mathsf{X} \in M$ relative to an origin $\mathsf{O} \in \mathcal{E}$ of the frame of reference is given by $\boldsymbol{x} = \boldsymbol{x}(\mathsf{X}) = \mathsf{X} - \mathsf{O}$ and may be considered as a vector-valued mapping $\boldsymbol{x} : M \to E$.

Relative to M, motion of the shell base surface over the time interval $[t_1,t_2]\in T$ is described by a mapping

$$\chi : M\times T\to \mathcal{E},\qquad (\mathsf{x},t)\to \mathsf{y}=\chi(\mathsf{x},t). \tag{1.28}$$

In this description, $\mathsf{y}\in M(t)$ is the spatial place at present time t of the shell particle X whose reference place was $\mathsf{x}\in M$. Moreover, $M(t)=\chi(M,t)$ is the spatial configuration of the base surface at present time t. The position vector of $\mathsf{y}\in M(t)$ relative to the same point $\mathsf{o}\in\mathcal{E}$ is $\boldsymbol{y}=\boldsymbol{y}(\mathsf{y},t)=\mathsf{y}-\mathsf{o}$.

With the motion described by (1.28), the position vector of $M(t)$ may be regarded as a vector-valued field on M

$$\boldsymbol{y} : M\times T\to E,\qquad (\mathsf{x},t)\to \boldsymbol{y}=\boldsymbol{y}(\mathsf{y},t)=\chi(\mathsf{x},t)-\mathsf{o}. \tag{1.29}$$

Thus, (1.28) and (1.29) are two equivalent descriptions of the shell motion relative to an arbitrarily fixed reference configuration. In general, M need not be smooth. In this report M is assumed to be connected but not necessarily simply connected surface.

1.6 Surface representation of mechanical objects

In the following considerations, $\Pi\subset M$ will denote the part of M representing in the physical space the part $\mathcal{P}$ of the shell-like body.

The mass of a body is commonly defined as the quantity of matter or substance, or as the amount of material in the body. This quantity is a function of internal structure of the substance and of its dimensions. The balance law of mass embodies two basic assumptions: a) the mass is permanent (it can be created but not destroyed), and b) the balance of mass is invariant with respect to the state of motion. By an additional assumption the mass is a strictly positive scalar, $\mathfrak{m}(\mathcal{P},t)>0$. In what follows we assume that the mass $\mathfrak{m}(\mathcal{P},t)$ and the mass production $\mathfrak{c}(\mathcal{P},t)$ of any part $\mathcal{P}$ of the shell are given by

$$\mathfrak{m}(\mathcal{P},t)=\iint_{\Pi} m_0 da,\qquad \mathfrak{c}(\mathcal{P},t)=\iint_{\Pi} c_0 da, \tag{1.30}$$

where $m_0(\mathsf{x},t)$ and $c_0(\mathsf{x},t)$ are the surface mass and mass production densities per unit area of M, respectively.

The relations (1.30) indicate that the surface densities m_0 and c_0, as well as other surface fields representing primitive objects discussing below, are defined *directly* as representing on M the corresponding primitive objects $\mathfrak{m}(\mathcal{P},t)$ and $\mathfrak{c}(\mathcal{P},t)$, respectively. The representation of $\mathfrak{m}(\mathcal{P},t)$ by the integral

of m_0 over Π requires the assumption that the mass measure m_0 be absolutely continuous function with respect to the area measure. This assumption implies that when the area of Π approaches zero, the mass $\mathfrak{m}(\mathcal{P},t)$ must also approach zero. Hence, any concentrated masses are ruled out by this assumption. The reason for accepting this assumption is that the theory of shells, like continuum mechanics, deals with distributed force fields acting on spatial configurations. Any concentrated mass point would require a concentrated force in the description of its motion, and this would cause singularity in the force field not acceptable in such a shell theory. The same arguments apply to the representation of $\mathfrak{c}(\mathcal{P},t)$ in (1.30).

The linear momentum $\mathfrak{p}(\mathcal{P},t)$ and the angular momentum $\mathfrak{m}(\mathcal{P},t)$ are the basic measures of motion. For extended bodies, these fields are defined in the particular - inertial - frames of reference. Like the mass, the linear and angular momenta are extensive quantities and can be expressed directly by their densities

$$\mathfrak{p}(\mathcal{P},t) = \iint_{\Pi} \boldsymbol{p}\,da, \qquad \mathfrak{m}(\mathcal{P},t) = \iint_{\Pi} (\boldsymbol{s} + \boldsymbol{y} \times \boldsymbol{p})da, \tag{1.31}$$

where $\boldsymbol{p}(\mathsf{x},t)$ is the (surface) linear momentum vector, and $\boldsymbol{s}(\mathsf{x},t)$ is the (surface) angular momentum vector.

Once the mass and momenta has been specified for every part of the shell, there remains to specify mechanical interactions between parts of the shell and between the shell and its environment. Simplest forms of such interactions are described by forces. However, the experience with an elementary analysis of beams makes it plain that for shells the mechanical interactions must also take into account couples.

Forces and couples acting on the shell are of two kind: a) the body forces and couples, and b) the contact forces and couples. The resultant force $\mathfrak{f}(\mathcal{P},t)$ and the resultant torque $\mathfrak{t}(\mathcal{P},t)$ of the shell-like body can therefore be represented in the physical space by

$$\begin{aligned}
\mathfrak{f}(\mathcal{P},t) &= \iint_{\Pi} \boldsymbol{b}da + \int_{\partial\Pi\setminus\partial M_f} \boldsymbol{n}_{\nu}dl + \int_{\partial\Pi\cap\partial M_f} \boldsymbol{n}^{*}dl\,, \\
\mathfrak{t}(\mathcal{P},t) &= \iint_{\Pi} (\boldsymbol{c} + \boldsymbol{y}\times\boldsymbol{b})da + \int_{\partial\Pi\setminus\partial M_f} (\boldsymbol{m}_{\nu} + \boldsymbol{y}\times\boldsymbol{n}_{\nu})dl \\
&\quad + \int_{\partial\Pi\cap\partial M_f} (\boldsymbol{m}^{*} + \boldsymbol{y}\times\boldsymbol{n}^{*})dl\,,
\end{aligned} \tag{1.32}$$

where $\boldsymbol{b}(\mathsf{x},t)$ and $\boldsymbol{c}(\mathsf{x},t)$ are the (surface) force and couple vectors, $\boldsymbol{n}_{\nu}(\mathsf{x},t)$ and $\boldsymbol{m}_{\nu}(\mathsf{x},t)$ are the (surface) contact stress and couple vectors describing internal

mechanical interactions between the shell parts, while ∂M_f is a part of boundary of M along which the external boundary force $\boldsymbol{n}^*(\mathsf{x},t)$ and couple $\boldsymbol{m}^*(\mathsf{x},t)$ vectors are assigned as a result of mechanical interactions between the shell and its environment.

1.7 Surface representation of thermodynamical objects

The balance of energy (1.4) expresses convertibility of mechanical and thermal energies. The global energetic quantities consist of the total energy $\mathfrak{u}(\mathcal{P},t)$ and the heating $\mathfrak{q}(\mathcal{P},t)$. The total energy $\mathfrak{u}(\mathcal{P},t)$, like mass and momenta, is an extensive quantity and may be expressed in terms of the (surface) total energy density $u(\mathsf{x},t)$ per unit surface mass

$$\mathfrak{u}(\mathcal{P},t) = \iint_{\Pi} m_0 u da. \tag{1.33}$$

In general, it represents the rate of increase of energy not necessarily accompanied by the mechanical working alone.

A body is said to absorb or emit heat according to whether $\mathfrak{q}(\mathcal{P},t) > 0$ or $\mathfrak{q}(\mathcal{P},t) < 0$, respectively. Thus, the integral $\int_{t_1}^{t_2} \mathfrak{q}(\mathcal{P},t)dt$ indicates the heat gained by the shell part $\mathcal{P}$ over the time interval $[t_1,t_2]$. The shell may also absorb heat from external environment through the upper M_+ and lower M_- shell faces, and through the part of the shell lateral boundary represented by ∂M_h. To an arbitrary part Π of the reference shell base surface M the heat can also be supplied by internal heating and through the boundary $\partial\Pi$ from other shell parts. Thus, the general 2D expression for heating of the part $\mathcal{P} \subset \mathcal{B}$ represented by $\Pi \subset B$ takes the form

$$\mathfrak{q}(\mathcal{P},t) = \iint_{\Pi} \{m_0 r - (q_+ - q_-)\} da - \int_{\partial\Pi \setminus \partial M_h} q_\nu dl - \int_{\partial\Pi \cap \partial M_h} q^* dl, \tag{1.34}$$

where r is the surface heat supply, $q_\pm$ the heat influxes on shell faces M_+ and M_-, q^* the heat supply through the external boundary ∂M_h, and q_ν the heat supply through the internal boundary $\partial\Pi$.

The concept of expended power is the keystone in the constructing a mechanical theory of shells, because it makes explicit our prejudices about the basic mechanical duality between kinematic and dynamic variables. The mechanical power $\mathfrak{p}(\mathcal{P},t)$ is defined as the rate of work of non-inertial forces

and couples. In general, $\mathfrak{p}(\mathcal{P},t)$ is an arbitrary real-valued function to be specified by means of mechanical theory. The forces and couples are specified by the assumed form (1.32) of the resultant force and torque. However, until this point there has been no need to mention an underlying kinematics of the shell. Hence, there is still no indication what specific form of $\mathfrak{p}(\mathcal{P},t)$ should be postulated.

We can take again that to an arbitrary shell part Π the mechanical power is supplied by the surface power density p, per unit area of M, and by the internal contact p_ν and the external boundary p^* power supplies, per unit length of the corresponding boundary, respectively. As a result, the mechanical power of the part $\mathcal{P} \subset \mathcal{B}$ of a shell-like body can be represented in the physical space as

$$\mathfrak{p}(\mathcal{P},t) = \iint_{\Pi} p da + \int_{\partial\Pi\setminus\partial M} p_\nu dl + \int_{\partial\Pi\cap\partial M} p^* dl . \tag{1.35}$$

Entropy is a quantity which can be created in irreversible processes but cannot be destroyed. Entropy of the whole shell is the sum of entropies of its parts. Thus entropy, like mass and internal energy, is an additive set function, the set being the shell material particles. Entropy is an extensive quantity and in the physical space may be represented by the (surface) specific entropy density $\eta(\mathsf{x},t)$ measured per unit surface mass, so that

$$\mathfrak{h}(\mathcal{P},t) = \iint_{\Pi} m_0 \eta da . \tag{1.36}$$

Entropy can be changed either by interaction with surrounding or by changes within the shell. Central question in continuum thermodynamics is of how to describe difference between the heat flux through a body and the heat supply from the external world. Postulating that one can distinguish between these two mechanisms of the heat transfer, in the statement of the second law of thermodynamics one must then allow separate entropy flows due to the two mechanisms, each of which separately tends to zero as the corresponding heat flux tends to zero. Hence, one is led to the form (1.5) of the second law of thermodynamics with $\mathfrak{h}(\mathcal{P},t)$ being given by (1.36) and $\mathfrak{j}(\mathcal{P},t)$ being given by

$$\mathfrak{j}(\mathcal{P},t) = \iint_{\Pi} \{m_0 s - (j_+ - j_-)\} da - \int_{\partial\Pi} j_\nu dl , \tag{1.37}$$

where s is the (surface) specific entropy supply, $j_{\pm}$ the entropy influxes on shell faces, and j_{ν} the entropy influx through the internal shell boundary represented by $\partial\Pi$.

Chapter 2
General regularity assumptions

2.1 Reference configuration of the base surface

In order to derive local laws of mechanics and thermodynamics for shells from their integral - impulse statements given in Chapter 1, certain regularity assumptions must be introduced. Naturally, different regularity assumptions may be needed for different classes of problems intended to be analyzed. In general, it is desirable to consider the possibly weakest regularity assumptions under which we can still formulate the mathematical initial-boundary value problem. Unfortunately, to the best knowledge of the authors such assumptions have not been stated yet within the general thermomechanics of shells. The main difficulty lies in the fact that various fields appearing in the laws of shell mechanics and thermodynamics are defined over Riemannian domains of the shell base surface, but not over domains of the Euclidean space as in the three-dimensional theory of continuous media. Accordingly, we should first introduce appropriate regularity assumptions to the description of the reference shell base surface M, of motion of the base surface and of all the fields which are defined on M.

The only general assumptions which may be stated about the reference configuration M of the base surface are:

1. M is a (topologically) open and connected (but not necessarily simply connected) surface;
2. a boundary ∂M of M is either an empty set, or has sufficient regularity properties for a surface gradient-divergence theorem to be applicable.

A (topologically) open smooth surface M in the Euclidean space $\mathcal{E}$ is said to have an ***almost smooth boundary*** if:

(a) M is regularly open, i.e. $M = \operatorname{int} \operatorname{cl} M$ (in the relative topology);
(b) for each exterior unit normal $\nu(\mathsf{x}) \in T_{\mathsf{x}}M$ of ∂M, M is locally on one side of ∂M at x;
(3) $L(\partial M) < \infty$ and $L(\partial M \setminus \partial' M) = 0$.

Here ∂M denotes the topological boundary of M, $\partial' M \subset \partial M$ denotes the set of all regular points of ∂M, i.e. points of ∂M at which the outer unit normal vector $\boldsymbol{\nu}(\mathsf{x})$ is defined, and L is the one-dimensional Hausdorff measure on $\mathfrak{S}$ (the "line" measure). It follows that the exterior unit normal vector $\boldsymbol{\nu}(\mathsf{x}) \in T_{\mathsf{x}} M$ is defined for L-a.e. points $\mathsf{x} \in \partial M$.

In general, the surface M need not be smooth but merely Lipschitz continuous with almost smooth boundary ∂M as defined above. Surfaces having this property are close to being piecewise smooth. Roughly speaking, a piecewise smooth surface consists of portions of smooth surfaces joined together. Such a surface may have sharp edges and corners. In what follows the term ***edge*** will refer to one of the finite number of regular arcs comprising the boundary of a regular surface element. The term ***vertex*** will refer to a point, at which two edges meet. If all the edges of a surface belong each to two of its surface elements, then the surface is the closed regular surface. A regular surface (and hence a closed regular surface) is necessarily both connected and bounded.

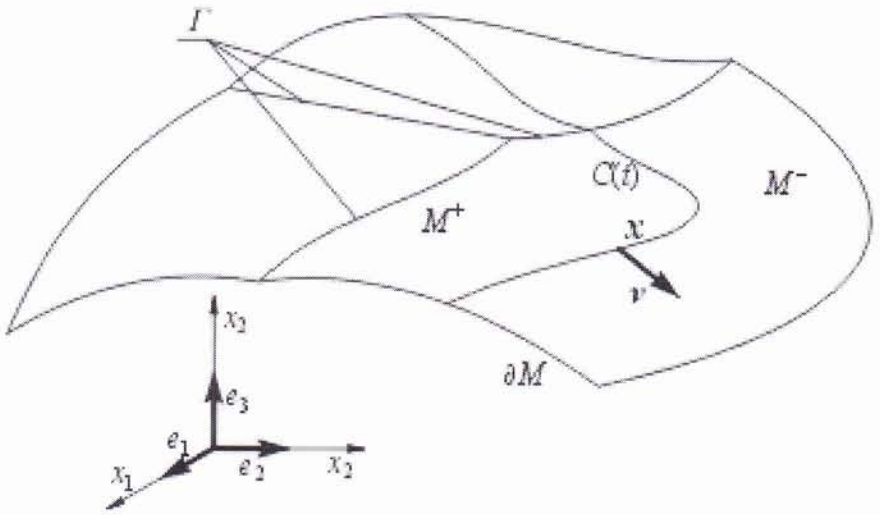

Fig. 2.1 Moving singular curve on a piecewise smooth surface

The position vector $\boldsymbol{x}(\mathsf{x})$ specifies the location of each point $\mathsf{x} \in M$ relative to the origin of the frame of reference, and the field $\boldsymbol{n}(\mathsf{x})$ determines the orientation of M.

The assumption that M is Lipschitz continuous is equivalent to the statement that the position vector of M considered as the mapping $\boldsymbol{x} : M \to E$ is Lipschitz continuous. This assumption implies that the position vector $\boldsymbol{x}(\mathsf{x})$ is differentiable almost everywhere on M, i.e. except of subsets of M whose area measure is zero. Points at which $\boldsymbol{x}(\mathsf{x})$ is differentiable are the regular points of M and at such points the unit normal vector $\boldsymbol{n}(\mathsf{x})$ is well defined.

At every regular point $\mathsf{x} \in M$ there is the well defined vector space $E \cong T_{\mathsf{x}}\mathcal{E}$. If $\mathsf{x} \in M$ is a regular point, then $E \cong T_{\mathsf{x}}\mathcal{E}$ has natural decomposition into the direct sum of the ***tangent space*** $T_{\mathsf{x}}M$ to M and its ***orthogonal complement*** $T_{\mathsf{x}}M^{\perp}$,

$$E \cong T_{\mathsf{x}}\mathcal{E} = T_{\mathsf{x}}M \oplus T_{\mathsf{x}}M^{\perp}. \tag{2.1}$$

With such a decomposition (2.1), the ***unit tensor*** $\mathbf{1}$ of the tangent space, the ***inclusion*** operator $\boldsymbol{I}$ and the ***projection*** operator $\boldsymbol{P}$,

$$\mathbf{1}(\mathsf{x}) : T_{\mathsf{x}}M \to T_{\mathsf{x}}M, \qquad \boldsymbol{I}(\mathsf{x}) : T_{\mathsf{x}}M \to E, \qquad \boldsymbol{P}(\mathsf{x}) : E \to T_{\mathsf{x}}M, \tag{2.2}$$

are well defined and satisfy the following relations:

$$\boldsymbol{I}^{T} = \boldsymbol{P}, \qquad \boldsymbol{P}\boldsymbol{I} = \mathbf{1}, \qquad \boldsymbol{I}\boldsymbol{P} = \mathbf{1} - \boldsymbol{n} \otimes \boldsymbol{n}. \tag{2.3}$$

Here $\mathbf{1}$ is the unit tensor on E, i.e. the identity linear map.

2.2 Time and surface differential operators

To prevent ambiguity in notation, the following rules will be consistently used throughout this report. For fields defined on the current configuration $M(t)$ of the shell base surface, partial differentiation with respect to time will be denoted by ∂_t, while d/dt or overdot will be reserved for the material time derivative, i.e. time derivative following the motion of the shell base surface. The surface gradient and the surface divergence operators on $M(t)$, whenever meaningfully defined, will be denoted by $grad \equiv grad_s$ and $div \equiv div_s$. The corresponding differential operators on the reference base surface M will be denoted by $\nabla \equiv Grad_s$, $Div \equiv Div_s$, respectively.

2.3 Singular curves for surface fields

In the referential description, local laws of shell thermomechanics are described by various fields defined on M. It may happen that values of some fields may change extremely rapidly from one point of M to another as well as within two close time instants. In order to model such rapidly varying processes within phenomenological theories, we assume that such fields suffer ***jump discontinuities*** at certain, possibly time-dependent, subsets of M. Admitting these kinds of discontinuities, in the following considerations it will be assumed that they are localized along stationary or moving curves in M.

A ***moving curve*** in M over the time interval $[t_1,t_2]$ is a one-parameter family $\{C(t)\}$, $t\in[t_1,t_2]$, of piecewise smooth curves in M. ***Velocity*** relative to M of the moving curve is necessarily a tangential vector field on M denoted by $\boldsymbol{u}(\mathsf{x},t)\in T_{\mathsf{x}}M$ and defined only at regular points of M. The normal component V of $\boldsymbol{u}$ is called the ***speed of propagation*** of the singular curve; it is a measure of speed with which the moving curve transverses the stationary surface M. The tangential component T of $\boldsymbol{u}$ is a measure of intrinsic motion of $C(t)$ within itself; its value depends on the way in which $C(t)$ has been parameterized.

For various fields appearing in the integral laws of shell mechanics and thermodynamics, it will suffice to adopt the following definition. A time-dependent field $\boldsymbol{\eta}=\boldsymbol{\eta}(\mathsf{x},t)$ on M having $C(t)$ as the singular curve is any mapping $\boldsymbol{\eta}(.,t):M\setminus C(t)\to F$ defined for almost all t, with F being any finite-dimensional vector space.

For the most part of this report it will suffice to assume that there exists a finite partition $\{M_i(t)\}_{i=1,\ldots,n}$ of M and a set $R(t)\subset M$ such that:

1) $M_i(t)$ is a smooth surface element for any i;
2) $L(R(t))=0$;
3) $\boldsymbol{\eta}(\mathsf{x},t)$ is of class C^1 on $M_i(t)$ and it has extension of the same class to the closure of $M_i(t)$;
4) at all points $\mathsf{x}\in I_{i,j}(t)\setminus R(t)$, where the curve

$$I_{i,j}(t)\equiv \partial M_i(t)\cap\partial M_j(t)\cap M, \qquad 1\le i<j\le n, \tag{2.4}$$

is a smooth curve in M, and

$$A(I_{i,j}(t)\cap I_{k,l}(t))=0 \quad \text{if} \quad (i,j)\neq(k,l). \tag{2.5}$$

Here and throughout this report $L(\Omega)$, $A(\Omega)$ and $V(\Omega)$ indicate the one-, two- and three-dimensional Hausdorff measures of Ω, respectively. We call any of $I_{i,j}(t)$ the singular curve for the field $\boldsymbol{\eta}(\mathsf{x},t)$. In most cases of interest, the set $R(t)$ reduces to a finite union of isolated points in M. In particular, when the singular curve divides the surface M into two parts, the subscripts i and j may be dropped and we may write

$$I(t) \equiv I_{i,j}(t), \qquad M_i(t) \equiv M^-(t), \qquad M_j(t) \equiv M^+(t). \tag{2.6}$$

The field $\boldsymbol{\eta} = \boldsymbol{\eta}(\mathsf{x},t)$ with a singular curve $C(t)$ is said to be r-times piecewise continuously differentiable on M if for every t there is a partition of M such that the restriction of $\boldsymbol{\eta}$ to $M_i(t)$ is continuous with continuous extension to the closure of $M_i(t)$ for each i. If $r = 0$, then $\boldsymbol{\eta}$ is called the ***piecewise continuous*** field. If $r = 1$, then $\boldsymbol{\eta}$ is called the ***piecewise continuously differentiable*** or ***piecewise smooth*** field. These definitions and implied consequences apply to fields with values in any finite-dimensional inner-product vector space, such as $\mathbb{R}$, E or $E \otimes E$, as well as to tangential vector and tensor fields. However, less clear is the question of appropriate restrictions which should be put on the sets $M_i(t)$. There appears to be no formal treatment of the piecewise continuously differentiable fields in the literature, except of the one-dimensional case.

If $\boldsymbol{\eta}$ is piecewise continuous on M, then it is bounded on $M \setminus C(t)$; if it is piecewise continuously differentiable on M, then both $\boldsymbol{\eta}$ and $\nabla\boldsymbol{\eta}$ are bounded on $M \setminus C(t)$. A piecewise smooth field need not be continuous. If it is piecewise continuous and piecewise smooth, then it is locally Lipschitz continuous provided that M is a Lipschitz continuous surface.

If M is a smooth surface, then for any field $\boldsymbol{\eta}$ on M having $C(t)$ as the singular curve, the ***jump*** and the ***mean value*** are defined by

$$[\![\boldsymbol{\eta}]\!](\mathsf{x}) = \boldsymbol{\eta}^+(\mathsf{x}) - \boldsymbol{\eta}^-(\mathsf{x}), \qquad \langle\!\langle \boldsymbol{\eta} \rangle\!\rangle(\mathsf{x}) = \frac{1}{2}\left(\boldsymbol{\eta}^+(\mathsf{x}) + \boldsymbol{\eta}^-(\mathsf{x})\right), \tag{2.7}$$

where $\boldsymbol{\eta}^+$ and $\boldsymbol{\eta}^-$ are one-sided finite limits of $\boldsymbol{\eta}$ at the regular point $\mathsf{x} \in C(t)$.

If the field $\boldsymbol{\eta}$ is continuous across $C(t)$ then $[\![\boldsymbol{\eta}]\!] = \mathbf{0}$ and $\langle\!\langle \boldsymbol{\eta} \rangle\!\rangle = \boldsymbol{\eta}$, since continuity asserts that $\boldsymbol{\eta}^+ = \boldsymbol{\eta}^-$.

If $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$ are fields on M for which $C(t)$ is the common singular curve, then

$$[\![\boldsymbol{\eta}\otimes\boldsymbol{\varphi}]\!]=[\![\boldsymbol{\eta}]\!]\otimes\langle\!\langle\boldsymbol{\eta}\rangle\!\rangle+\langle\!\langle\boldsymbol{\eta}\rangle\!\rangle\otimes[\![\boldsymbol{\varphi}]\!] \tag{2.8}$$

everywhere along $C(t)$.

Let the field $\boldsymbol{\eta}: M\to F$ be differentiable on M, except possibly of points belonging to the curve $C(t)$. Then the surface gradient $\nabla\boldsymbol{\eta}$ exists at all interior points of $M^+(t)$ and $M^-(t)$. Moreover, if one-sided finite limits of $\nabla\boldsymbol{\eta}$ exist at all points of $C(t)$, then the jump of $\nabla\boldsymbol{\eta}$ is defined in the same manner as for the field itself,

$$[\![\nabla\boldsymbol{\eta}]\!](\mathsf{x})=(\nabla\boldsymbol{\eta})^+(\mathsf{x})-(\nabla\boldsymbol{\eta})^-(\mathsf{x}). \tag{2.9}$$

If the field $\boldsymbol{\eta}$ together with its surface gradients up to order $k-1$ suffer no jumps along the curve $C(t)$, but the gradient of order k or higher is discontinuous across $C(t)$, then $C(t)$ is called the ***singular curve of order k*** for the field $\boldsymbol{\eta}$. This means that $[\![\boldsymbol{\eta}]\!]=\mathbf{0}$ and $[\![\nabla^{(l)}\boldsymbol{\eta}]\!]=\mathbf{0}$ for $l=1,2,\ldots,k-1$, while $[\![\nabla^{(k)}\boldsymbol{\eta}]\!]\neq\mathbf{0}$ along $C(t)$. The strongest singularity is of order zero, when the field $\boldsymbol{\eta}$ itself is discontinuous along the curve $C(t)$.

If M is merely the Lipschitz continuous (piecewise smooth) surface, then the curves forming the edge set Γ of M are natural sources of singularities for surface fields. For example, a tangential vector field $\boldsymbol{w}$ on a piecewise smooth surface M cannot be defined in a continuous manner at points $\mathsf{x}\in\Gamma$, since tangent spaces to M are not defined at points of Γ. However, if $\mathsf{x}\in\Gamma$ is a regular point of the edge set, then the one-sided tangent spaces $T_{\mathsf{x}}M^+$ and $T_{\mathsf{x}}M^-$ are well defined and the one-sided limits $\boldsymbol{w}^{\pm}(\mathsf{x})\in T_{\mathsf{x}}M^{\pm}$ of $\boldsymbol{w}$ may exist at $\mathsf{x}\in\Gamma$. If the limits $\boldsymbol{w}^{\pm}(\mathsf{x})$ are finite then, strictly speaking, the definitions (2.7) of the jump and the mean value do not apply to $\boldsymbol{w}$ because $\boldsymbol{w}^+(\mathsf{x})\in T_{\mathsf{x}}M^+$ and $\boldsymbol{w}^-(\mathsf{x})\in T_{\mathsf{x}}M^-$ belong to different tangent spaces. However, $\boldsymbol{I}^{\pm}(\mathsf{x})\boldsymbol{w}^{\pm}(\mathsf{x})$ are elements of the vector space $E\cong T_{\mathsf{x}}\mathcal{E}$ and the following definitions make sense:

$$[\![\boldsymbol{w}]\!]=\boldsymbol{I}^+\boldsymbol{w}^+-\boldsymbol{I}^-\boldsymbol{w}^-,\qquad \langle\!\langle\boldsymbol{w}\rangle\!\rangle=\tfrac{1}{2}(\boldsymbol{I}^+\boldsymbol{w}^++\boldsymbol{I}^-\boldsymbol{w}^-). \tag{2.10}$$

For a tensor field $\boldsymbol{S}$ on a piecewise smooth surface M, such that $\boldsymbol{S}(\mathsf{x}) \in E \otimes T_{\mathsf{x}}M$ at regular points of M, we set

$$\boldsymbol{s}_{\nu}^{\pm}(\mathsf{x}) \equiv \boldsymbol{S}^{\pm}(\mathsf{x})\boldsymbol{\nu}^{\pm}(\mathsf{x}), \tag{2.11}$$

where $\boldsymbol{\nu}^{+}(\mathsf{x})$ and $\boldsymbol{\nu}^{-}(\mathsf{x})$ are the outward unit normals to ∂M^{+} and ∂M^{-} along the edge set Γ of the two parts M^{+} and M^{-}, respectively. Now we can define the generalized jump and the generalized mean value by

$$[\![\boldsymbol{s}_{\nu}]\!] \equiv [\![\boldsymbol{S}\boldsymbol{\nu}]\!] = \boldsymbol{s}_{\nu}^{+} + \boldsymbol{s}_{\nu}^{-}, \qquad \langle \boldsymbol{s}_{\nu} \rangle \equiv \langle \boldsymbol{S}\boldsymbol{\nu} \rangle = \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} - \boldsymbol{s}_{\nu}^{-}), \tag{2.12}$$

at any point of non-smoothness of M.

The definitions (2.12) make sense for fields on Lipschitz continuous (piecewise smooth) surfaces. The relations (2.12) reduce to more common ones (2.7) whenever M is a smooth surface.

2.4 Generalized surface gradient-divergence theorems

Let M be a piecewise regular surface in the sense of Sect. 2.1, i.e. a Lipschitz continuous surface with almost smooth boundary, and let $\boldsymbol{w} \in C(M, TM)$ be a piecewise smooth tangential vector field on M. Then

$$\int_{\partial \Pi} \boldsymbol{w} \cdot \boldsymbol{\nu} dl = \iint_{\Pi \setminus C} Div\boldsymbol{w} da - \int_{C \cap \Pi} [\![\boldsymbol{w} \cdot \boldsymbol{\nu}]\!] dl, \tag{2.13}$$

where the jump at regular points $\mathsf{x} \in C$ is defined by

$$[\![\boldsymbol{w} \cdot \boldsymbol{\nu}]\!] = \boldsymbol{w}^{+} \cdot \boldsymbol{\nu}^{+} + \boldsymbol{w}^{-} \cdot \boldsymbol{\nu}^{-}. \tag{2.14}$$

If M is a smooth surface, and orientation of C is assumed to coincide with orientation of the boundary of M^{-}, then $\boldsymbol{\nu} = \boldsymbol{\nu}^{-} = -\boldsymbol{\nu}^{+}$ and the jump (2.14) can be written as

$$[\![\boldsymbol{w} \cdot \boldsymbol{\nu}]\!] = -\langle\!\langle \boldsymbol{w} \cdot \boldsymbol{\nu} \rangle\!\rangle \boldsymbol{\nu}, \qquad \langle\!\langle \boldsymbol{w} \rangle\!\rangle \equiv \boldsymbol{w}^{+} - \boldsymbol{w}^{-}. \tag{2.15}$$

In this case the theorem (2.13) may be rewritten in the simpler form

$$\int_{\partial \Pi} \boldsymbol{w} \cdot \boldsymbol{\nu} dl = \iint_{\Pi \setminus C} Div\boldsymbol{w} da + \int_{C \cap \Pi} \langle\!\langle \boldsymbol{w} \cdot \boldsymbol{\nu} \rangle\!\rangle \boldsymbol{\nu} dl. \tag{2.16}$$

Let us apply the theorem (2.13) to the tensor field $\boldsymbol{w} \otimes \boldsymbol{S}$ composed of a spatial vector field $\boldsymbol{w} \in C(M, E)$ and a spatial tensor field $\boldsymbol{S} \in C(M, E \otimes TM)$, both piecewise regular on M, which after transformations leads to

$$\int_{\partial\Pi} \boldsymbol{w} \otimes \boldsymbol{S}\boldsymbol{v}dl = \iint_{\Pi\setminus C}\{\boldsymbol{w} \otimes (Div\boldsymbol{S}) + (\nabla \boldsymbol{w})\boldsymbol{S}^T\}da - \int_{C\cap\Pi} [\boldsymbol{w} \otimes \boldsymbol{S}\boldsymbol{v}]dl, \qquad (2.17)$$

where the jump at regular points $x \in C$ is defined by

$$[\boldsymbol{w} \otimes \boldsymbol{S}\boldsymbol{v}] = \boldsymbol{w}^+ \otimes \boldsymbol{S}^+\boldsymbol{v}^+ + \boldsymbol{w}^- \otimes \boldsymbol{S}^-\boldsymbol{v}^-. \qquad (2.18)$$

As an immediate implication of (2.17) we also have

$$\int_{\partial\Pi} \boldsymbol{S}\boldsymbol{v}dl = \iint_{\Pi\setminus C} Div\boldsymbol{S}da - \int_{C\cap\Pi} [\boldsymbol{S}\boldsymbol{v}]dl, \qquad (2.19)$$

and

$$\begin{aligned}
&\int_{\partial\Pi} \boldsymbol{w} \cdot \boldsymbol{S}\boldsymbol{v}dl = \iint_{\Pi\setminus C}\{(Div\boldsymbol{S}) \cdot \boldsymbol{w} + \boldsymbol{S} \cdot \nabla \boldsymbol{w}\}da - \int_{C\cap\Pi}[\boldsymbol{w} \cdot \boldsymbol{S}\boldsymbol{v}]dl,\\
&\int_{\partial\Pi} \boldsymbol{w} \wedge \boldsymbol{S}\boldsymbol{v}dl = \iint_{\Pi\setminus C}\{\boldsymbol{w} \wedge (Div\boldsymbol{S}) + (\nabla \boldsymbol{w})\boldsymbol{S}^T - \boldsymbol{S}(\nabla \boldsymbol{w})^T\}da\\
&\qquad\qquad - \int_{C\cap\Pi}[\boldsymbol{w} \wedge \boldsymbol{S}\boldsymbol{v}]dl, \qquad (2.20)\\
&\int_{\partial\Pi} \boldsymbol{w} \times \boldsymbol{S}\boldsymbol{v}dl = \iint_{\Pi\setminus C}\{\boldsymbol{w} \times (Div\boldsymbol{S}) - ad^{-1}((\nabla \boldsymbol{w})\boldsymbol{S}^T - \boldsymbol{S}(\nabla \boldsymbol{w})^T)\}da\\
&\qquad\qquad - \int_{C\cap\Pi}[\boldsymbol{w} \times \boldsymbol{S}\boldsymbol{v}]dl.
\end{aligned}$$

The jumps in (2.20) are defined as in (2.18), with the tensor product to be replaced by the inner product, the exterior product and the cross product, respectively.

Here and in the sequel we shall often use notation $ad : E \to E \wedge E$ for the linear and invertible map ad which associates with every spatial vector $\boldsymbol{w} \in E$ the uniquely defined skew-symmetric spatial tensor $\boldsymbol{W} \equiv ad\boldsymbol{w} \in E \wedge E$ such that

$$\boldsymbol{W}\boldsymbol{u} = \boldsymbol{w} \times \boldsymbol{u}, \qquad \forall \boldsymbol{u} \in E. \qquad (2.21)$$

We shall also use the operations on the tensor space $E \otimes E$ which deliver the transpose, the skew part, and the trace to any tensor $\boldsymbol{A} \in E \otimes E$, i.e. a linear map of E into itself. These operations are linear and continuous, and when applied to the tensor product $\boldsymbol{a} \otimes \boldsymbol{b}$ of any two vectors are defined by

$$(\boldsymbol{a} \otimes \boldsymbol{b})^T = \boldsymbol{b} \otimes \boldsymbol{a}, \qquad \boldsymbol{a} \wedge \boldsymbol{b} = \boldsymbol{a} \otimes \boldsymbol{b} - \boldsymbol{b} \otimes \boldsymbol{a}, \qquad \mathrm{tr}(\boldsymbol{a} \otimes \boldsymbol{b}) = \boldsymbol{a} \cdot \boldsymbol{b}. \qquad (2.22)$$

Moreover, the inverse of the map $ad : E \to E \wedge E$ is also linear, continuous and is given by

$$ad^{-1}(\boldsymbol{a} \wedge \boldsymbol{b}) = -\boldsymbol{a} \times \boldsymbol{b}. \qquad (2.23)$$

These theorems apply also to vector and tensor fields with moving curves $C(t)$ in M.

2.5 Surface transport theorems

If sufficient regularity assumptions hold to justify the interchange of differentiation and integration operations at the reference surface M, then

$$\frac{d}{dt}\iint_{\Pi}\boldsymbol{\eta}da=\frac{\partial}{\partial t}\iint_{\Pi}\boldsymbol{\eta}da=\iint_{\Pi}\partial_t\boldsymbol{\eta}da=\iint_{\Pi}\dot{\boldsymbol{\eta}}da \tag{2.24}$$

for any domain $\Pi\subset M$ and every time-dependent field $\boldsymbol{\eta}(\mathsf{x},t)$ on M. Naturally, $\boldsymbol{\eta}(\mathsf{x},t)$ must be differentiable almost everywhere and its time derivative must be integrable for each time instant.

Let $N(t)\subset M$ be an evolving region whose boundary $\partial N(t)$ moves with velocity $\boldsymbol{u}(\mathsf{x},t)$. The theorem (2.24) applied to the integral over the evolving region $N(t)$ gives

$$\frac{\partial}{\partial t}\iint_{N(t)}\boldsymbol{\eta}da=\iint_{N(t)}\partial_t\boldsymbol{\eta}da=\iint_{N(t)}\dot{\boldsymbol{\eta}}da, \tag{2.25}$$

where $\dot{\boldsymbol{\eta}}(\mathsf{x},t)=\partial_t\boldsymbol{\eta}(\mathsf{x},t)$ at the interior of $N(t)$. Relative to this velocity, the rate of change of a surface integral of a given field $\boldsymbol{\eta}(\mathsf{x},t)$ over $N(t)$ is determined by the transport theorem

$$\begin{aligned}\frac{d}{dt}\iint_{N(t)}\boldsymbol{\eta}da&=\frac{\partial}{\partial t}\iint_{N(t)}\boldsymbol{\eta}da+\int_{\partial N(t)}(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{\eta}dl\\&=\iint_{N(t)}\dot{\boldsymbol{\eta}}da+\int_{\partial N(t)}(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{\eta}dl.\end{aligned} \tag{2.26}$$

If a moving curve $C(t)\subset M$ intersects a fixed region $\Pi\subset M$ and $\boldsymbol{u}(\mathsf{x},t)$ is the velocity of $C(t)$, then

$$\partial\Pi^{\pm}(t)=(\partial\Pi^{\pm}(t)\setminus C(t))\cup(C(t)\cap\partial\Pi^{\pm}(t)). \tag{2.27}$$

Then the use of theorem (2.26) applied separately on $N^{-}(t)$ and $N^{+}(t)$ makes it possible to show that

$$\frac{d}{dt}\iint_{\Pi\setminus C(t)}\boldsymbol{\eta}da=\iint_{\Pi\setminus C(t)}\dot{\boldsymbol{\eta}}da+\int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{\eta}]\!]dl, \tag{2.28}$$

where the jump is defined by

$$[(\boldsymbol{u} \bullet \boldsymbol{v})\eta] = (\boldsymbol{u}^+ \bullet \boldsymbol{v}^+)\eta^+ + (\boldsymbol{u}^- \bullet \boldsymbol{v}^-)\eta^- . \tag{2.29}$$

This theorem holds even if $C(t)$ instantaneously coincides with any curve belonging to the edge set Γ. If $C(t)$ is a singular curve for $\eta(\mathrm{x},t)$ on a smooth surface M, then at each regular point $\mathrm{x} \in C(t)$ we have

$$\boldsymbol{v} = \boldsymbol{v}^- = -\boldsymbol{v}^+, \qquad \boldsymbol{u} = \boldsymbol{u}^- = \boldsymbol{u}^+, \tag{2.30}$$

and the jump (2.29) may be rewritten as

$$[(\boldsymbol{u} \bullet \boldsymbol{v})\eta] = (\boldsymbol{u} \bullet \boldsymbol{v})\eta^- - (\boldsymbol{u} \bullet \boldsymbol{v})\eta^+ = -(\boldsymbol{u} \bullet \boldsymbol{v})[\![\eta]\!], \qquad [\![\eta]\!] = \eta^+ - \eta^- . \tag{2.31}$$

Then the theorem (2.28) may be expressed in the form

$$\frac{d}{dt}\iint_{\Pi} \eta da = \iint_{\Pi \setminus C(t)} \dot{\eta} da - \int_{C(t)\cap\Pi} V[\![\eta]\!] dl , \tag{2.32}$$

where $V = \boldsymbol{u} \bullet \boldsymbol{v}$ denotes the normal velocity of the curve $C(t)$.

Chapter 3
Thermomechanical local field equations and side conditions

3.1 Regularity of motion

Sometime it may be assumed that motion of the shell base surface is continuous and piecewise continuously differentiable with regard to time. Then χ is unequivocally defined even for points x belonging to the singular curve $C(t)$ and it has a well defined boundary value $\chi^*(\mathsf{x},t)$ for all $\mathsf{x} \in \partial M$. However, the assumption of the motion continuity excludes many singularities of physical importance and cannot be taken for granted.

The discontinuous motion of the shell base surface which does not produce a macroscopic hole may be described as follows. One can consider a piecewise continuously differentiable but discontinuous motion χ with a singular curve $C(t)$ having the property, that for every $t \in [t_1, t_2]$ there exists a diffeomorphism $\varphi(.,t) : C(t) \to C(t)$ such that

$$\chi^+(\mathsf{x},t) = \chi^-(\varphi(\mathsf{x},t),t), \qquad \forall\, \mathsf{x} \in C(t), \tag{3.1}$$

where $\chi^{\pm}(\mathsf{x},t)$ are one-sided limits of χ at $C(t)$. Kinematically, at every time instant t the surface M is cut along $C(t)$, then points of the two sides of the cut are allowed to slip along $C(t)$ one with respect to another, and finally the two sides are glued together.

3.2 Continuity equations

If the mass $\mathrm{m}(\mathcal{P},t)$ of $\mathcal{P}$ is differentiable for almost all $t \in T$, then

$$\left[\mathrm{m}(\mathcal{P},t)\right]_{t_1}^{t_2} = \int_{t_1}^{t_2} \dot{\mathrm{m}}(\mathcal{P},t)\, dt\,. \tag{3.2}$$

By the classical theorem of analysis and application of the transport theorem (2.28) to (3.2) we obtain the rate of change of mass in the form

$$\dot{\mathfrak{m}}(\mathcal{P},t)=\frac{d}{dt}\iint_{\Pi\setminus C(t)} m_0 da=\iint_{\Pi\setminus C(t)}\dot{m}_0 da+\int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{v})m_0]\!]dl\,. \tag{3.3}$$

With this result the balance of mass reads

$$\iint_{\Pi\setminus C(t)}(\dot{m}_0-c_0)da+\int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{v})m_0]\!]dl=0\,, \tag{3.4}$$

from which we read off that locally the balance of mass takes the form

$$\dot{m}_0=c_0\,. \tag{3.5}$$

This equation is trivially satisfied at points of M away of the singular curve $C(t)$, if the referential mass density is time-independent, $m_0=m_0(\mathsf{x})$ with $c_0\equiv 0$. The principle (3.4) requires that at regular points $\mathsf{x}\in C(t)$ of the moving singular curve the following continuity conditions be satisfied:

$$[\![(\boldsymbol{u}\bullet\boldsymbol{v})m_0]\!]=0. \tag{3.6}$$

If $\mathsf{x}\in C(t)$ is a point of smoothness of M, then (3.5) reduces to the form

$$V[\![m_0]\!]=0. \tag{3.7}$$

The balance of mass imposes no restrictions on continuity of the mass density at stationary singular curves such as the ridges Γ, for example.

3.3 Integral laws of mechanics

If time derivatives of $\mathfrak{p}(\mathcal{P},t)$ and $\mathfrak{m}(\mathcal{P},t)$ exist for almost all $t\in T$, and if the inertia force $\mathfrak{f}^{in}(\mathcal{P},t)$ and the inertia torque $\mathfrak{t}^{in}(\mathcal{P},t)$ of $\mathcal{P}$ are defined by (1.10), then application of the transport theorem (2.28) to the laws of mechanics with representations (1.31) yields

$$\begin{aligned}
\dot{\mathfrak{p}}(\mathcal{P},t)&=\frac{d}{dt}\iint_{\Pi}\boldsymbol{p}da\\
&=\iint_{\Pi\setminus C(t)}\dot{\boldsymbol{p}}da+\int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\!]dl\,,\\
\dot{\mathfrak{m}}(\mathcal{P},t)&=\frac{d}{dt}\iint_{\Pi}(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})da\\
&=\iint_{\Pi\setminus C(t)}(\dot{\boldsymbol{s}}+\dot{\boldsymbol{y}}\times\boldsymbol{p}+\boldsymbol{y}\times\dot{\boldsymbol{p}})da+\int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})]\!]dl\,.
\end{aligned} \tag{3.8}$$

At regular points of the base surface the inertia surface force $\boldsymbol{b}^{in}(\mathsf{x},t)$ and the inertia surface couple $\boldsymbol{c}^{in}(\mathsf{x},t)$ are defined by

$$\boldsymbol{b}^{in} \equiv -\dot{\boldsymbol{p}}, \qquad \boldsymbol{c}^{in} \equiv -(\dot{\boldsymbol{s}} + \dot{\boldsymbol{y}} \times \boldsymbol{p}). \tag{3.9}$$

Then the resultant inertia force and torque as defined by (1.10) are obtained in the form

$$\begin{aligned} \tilde{\mathbf{f}}^{in}(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} \boldsymbol{b}^{in} da - \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{p}]\!] dl, \\ \mathbf{t}^{in}(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} (\boldsymbol{c}^{in} + \boldsymbol{y}\times\boldsymbol{c}^{in}) da - \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})]\!] dl. \end{aligned} \tag{3.10}$$

With the boundary $\partial\Pi$ written as

$$\partial\Pi = (\partial\Pi\cap\partial M)\cup(\partial\Pi\setminus\partial M), \tag{3.11}$$

the total force and torque (1.11) take the form

$$\begin{aligned} \tilde{\mathbf{f}}'(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} (\boldsymbol{b}+\boldsymbol{b}^{in}) da - \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{p}]\!] dl \\ &\quad + \int_{\partial\Pi} \boldsymbol{n}_{\nu} dl + \int_{\partial\Pi\cap\partial M} (\boldsymbol{n}^{*} - \boldsymbol{n}_{\nu}) dl, \\ \mathbf{t}'(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} \{\boldsymbol{c}+\boldsymbol{c}^{in}+\boldsymbol{y}\times(\boldsymbol{b}+\boldsymbol{b}^{in})\} da - \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})]\!] da \\ &\quad + \int_{\partial\Pi} (\boldsymbol{m}_{\nu} + \boldsymbol{y}\times\boldsymbol{n}_{\nu}) dl + \int_{\partial\Pi\cap\partial M} \{(\boldsymbol{m}^{*} - \boldsymbol{m}_{\nu}) + \boldsymbol{y}\times(\boldsymbol{n}^{*} - \boldsymbol{n}_{\nu})\} dl. \end{aligned} \tag{3.12}$$

The next step is to convert the line integrals along $\partial\Pi$ in (3.12) into surface integrals, so that the arbitrariness of Π may be exploited to obtain local forms of the balance laws of mechanics. This requires the introduction of certain additional assumptions concerning the nature of the contact force $\boldsymbol{n}_{\nu}(\mathsf{x},t)$ and the contact couple $\boldsymbol{m}_{\nu}(\mathsf{x},t)$. Before we come to this point, one property of the torque should be explained.

In (1.29), $\boldsymbol{y}(\mathsf{x},t)$ is the position vector, relative to an arbitrarily fixed point $\mathsf{o}\in\mathcal{E}$, of the place $\mathsf{y}\in M(t)$ of a surface particle whose reference place was $\mathsf{x}\in M$. If $\mathsf{o}_{*}\in\mathcal{E}$ is another point in the same frame of reference, then the position vector of $\mathsf{y}\in M(t)$ relative to o_{*} is $\boldsymbol{y}_{*}(\mathsf{x},t) = \boldsymbol{y}(\mathsf{x},t) + \boldsymbol{w}_{*}$, where $\boldsymbol{w}_{*} = \mathsf{o}_{*} - \mathsf{o}$ is a constant vector. Hence the total torque is

$$\begin{aligned} \mathbf{t}'_{*}(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} \{\boldsymbol{c}+\boldsymbol{c}^{in}+(\boldsymbol{y}+\boldsymbol{w}_{*})\times(\boldsymbol{b}+\boldsymbol{b}^{in})\} da \\ &\quad - \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})\{\boldsymbol{m}+(\boldsymbol{y}+\boldsymbol{w}_{*})\times\boldsymbol{p}\}]\!] dl \\ &\quad + \int_{\partial\Pi} \{\boldsymbol{m}_{\nu} + (\boldsymbol{y}+\boldsymbol{w}_{*})\times\boldsymbol{n}_{\nu}\} dl \\ &\quad + \int_{\partial\Pi\cap\partial M} \{(\boldsymbol{m}^{*} - \boldsymbol{m}_{\nu}) + (\boldsymbol{y}+\boldsymbol{w}_{*})\times(\boldsymbol{n}^{*} - \boldsymbol{n}_{\nu})\} dl. \end{aligned} \tag{3.13}$$

Subtracting (3.12) from (3.13) we obtain

$$\begin{aligned}\mathbf{t}'_*(\mathcal{P},t)-\mathbf{t}'(\mathcal{P},t)=\iint_{\Pi\setminus C(t)}\boldsymbol{w}_*\times(\boldsymbol{b}+\boldsymbol{b}^{in})da-\int_{C(t)\cap\Pi}[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{w}_*\times\boldsymbol{p}\}]dl\\+\int_{\partial\Pi}\boldsymbol{w}_*\times\boldsymbol{n}_v dl+\int_{\partial\Pi\cap\partial M}\boldsymbol{w}_*\times(\boldsymbol{n}^*-\boldsymbol{n}_v)dl.\end{aligned}\tag{3.14}$$

Noting that $\boldsymbol{w}_*$ is a constant vector, we obtain the transformation

$$\mathbf{t}'_*(\mathcal{P},t)-\mathbf{t}'(\mathcal{P},t)=\boldsymbol{w}_*\times\mathbf{f}'(\mathcal{P},t)\tag{3.15}$$

upon the use of the expression (3.12) for the total force acting on any part of the shell.

This result expresses the property that the total torque is independent of the origin of frame of reference if and only if the total force vanishes. Moreover, the balance of angular momentum implies the balance of linear momentum. The converse is not true. For extended bodies the balance of angular momentum cannot be derived from the balance of linear momentum and is an independent postulate of mechanics.

3.4 Cauchy hypothesis and surface stress and couple tensors

The contact surface stress and couple vectors $\boldsymbol{n}_v(\mathsf{x},t)$ and $\boldsymbol{m}_v(\mathsf{x},t)$ are defined at the internal boundary curve $\partial\Pi$. When two oriented surface curves intersect each other at a point $\mathsf{x}\in M$, the contact vectors associated with each of them need not be the same at that point. In other words, the vectors $\boldsymbol{n}_v(\mathsf{x},t)$ and $\boldsymbol{m}_v(\mathsf{x},t)$ are not just the functions of the point (x,t) but also of orientation of the curve $\partial\Pi$ across which they act. The fundamental postulate – the Cauchy postulate - requires that the contact force and couple vectors be given in terms of the surface stress and couple tensors,

$$\boldsymbol{n}_v(\mathsf{x},t)=\boldsymbol{N}(\mathsf{x},t)\boldsymbol{v}(\mathsf{x}),\qquad \boldsymbol{m}_v(\mathsf{x},t)=\boldsymbol{M}(\mathsf{x},t)\boldsymbol{v}(\mathsf{x}).\tag{3.16}$$

Since $\boldsymbol{v}(\mathsf{x})$ is an element of the tangent space $T_{\mathsf{x}}M$, while $\boldsymbol{n}_v(\mathsf{x},t)$ and $\boldsymbol{m}_v(\mathsf{x},t)$ are spatial vectors, it is apparent that both $\boldsymbol{N}(\mathsf{x},t)$ and $\boldsymbol{M}(\mathsf{x},t)$ are linear maps of $T_{\mathsf{x}}M$ into E.

With the contact vectors defined by (3.16), application of the generalized surface divergence theorem (2.13) yields

$$
\begin{aligned}
\int_{\partial\Pi} \boldsymbol{N}\boldsymbol{v}dl &= \iint_{\Pi\setminus C(t)} Div\boldsymbol{N}da - \int_{C(t)\cap\Pi} [\![\boldsymbol{N}\boldsymbol{v}]\!]dl,\\
\int_{\partial\Pi} \boldsymbol{M}\boldsymbol{v}dl &= \iint_{\Pi\setminus C(t)} Div\boldsymbol{M}da - \int_{C(t)\cap\Pi} [\![\boldsymbol{M}\boldsymbol{v}]\!]dl,\\
\int_{\partial\Pi} \boldsymbol{y}\times\boldsymbol{N}\boldsymbol{v}dl &= \iint_{\Pi\setminus C(t)} \{\boldsymbol{y}\times(Div\boldsymbol{N}) - ad^{-1}(\boldsymbol{F}\boldsymbol{N}^T - \boldsymbol{N}\boldsymbol{F}^T)\}da\\
&\quad - \int_{C(t)\cap\Pi} [\![\boldsymbol{y}\times\boldsymbol{N}\boldsymbol{v}]\!]dl,
\end{aligned}
\tag{3.17}
$$

where the jumps at regular points of the singular curve $C(t)$ are defined by

$$
\begin{aligned}
[\![\boldsymbol{N}\boldsymbol{v}]\!] &\equiv \boldsymbol{N}^+\boldsymbol{v}^+ + \boldsymbol{N}^-\boldsymbol{v}^-,\\
[\![\boldsymbol{M}\boldsymbol{v}]\!] &\equiv \boldsymbol{M}^+\boldsymbol{v}^+ + \boldsymbol{M}^-\boldsymbol{v}^-,\\
[\![\boldsymbol{y}\times\boldsymbol{N}\boldsymbol{v}]\!] &\equiv \boldsymbol{y}^+\times\boldsymbol{N}^+\boldsymbol{v}^+ + \boldsymbol{y}^-\times\boldsymbol{N}^-\boldsymbol{v}^-.
\end{aligned}
\tag{3.18}
$$

Here $\boldsymbol{F}(\mathrm{x},t) = \nabla\boldsymbol{y}(\mathrm{x},t)$ denotes the surface deformation gradient, and $ad: E\to E\wedge E$ is the linear and invertible map associating with every vector $\boldsymbol{w}\in E$ the uniquely defined skew-symmetric tensor $ad\boldsymbol{w}\in E\wedge E$. It should be noted that $\boldsymbol{F}(\mathrm{x},t)\in E\otimes T_{\mathrm{x}}M$ and $\boldsymbol{T}(\mathrm{x},t)\in E\otimes T_{\mathrm{x}}M$. Therefore, $\boldsymbol{F}\boldsymbol{T}^T$ and $\boldsymbol{N}\boldsymbol{F}^T$ are elements of $E\otimes E$, and $\boldsymbol{F}\boldsymbol{N}^T - \boldsymbol{N}\boldsymbol{F}^T$ is the skew-symmetric tensor whose axial vector is well defined.

Substituting (3.17) into (3.12), the total force and the total torque are obtained in the form

$$
\begin{aligned}
\boldsymbol{f}'(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} (Div\boldsymbol{N} + \boldsymbol{b} + \boldsymbol{b}^{in})da\\
&\quad - \int_{C(t)\cap\Pi} \{[\![\boldsymbol{N}\boldsymbol{v}]\!] + [\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\!]\}da + \int_{\partial\Pi\cap\partial M} (\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{v})dl,\\
\boldsymbol{t}'(\mathcal{P},t) &= \iint_{\Pi\setminus C(t)} \Big\{Div\boldsymbol{M} + ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\\
&\qquad + \boldsymbol{c} + \boldsymbol{c}^{in} + \boldsymbol{y}\times(Div\boldsymbol{N} + \boldsymbol{b} + \boldsymbol{b}^{in})\Big\}da\\
&\quad - \int_{C(t)\cap\Pi} \{[\![\boldsymbol{M}\boldsymbol{v}]\!] + [\![\boldsymbol{y}\times\boldsymbol{N}\boldsymbol{v}]\!] + [\![(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{s} + \boldsymbol{y}\times\boldsymbol{p})]\!]\}dl\\
&\quad + \int_{\partial\Pi\cap\partial M} \{(\boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{v}) + \boldsymbol{y}\times(\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{v})\}dl.
\end{aligned}
\tag{3.19}
$$

3.5 Local laws of mechanics

Since both balance laws of mechanics (1.8) are required to hold simultaneously for every part of the shell, the integrands in each integral of (3.19) must vanish separately by virtue of arbitrariness of Π. The dynamic

field equations are the local ***equations of motion*** of the shell and they take the form

$$Div\boldsymbol{N}+\boldsymbol{b}+\boldsymbol{b}^{in}=\boldsymbol{0}, \qquad Div\boldsymbol{M}+ad^{-1}(\boldsymbol{NF}^{T}-\boldsymbol{FN}^{T})+\boldsymbol{c}+\boldsymbol{c}^{in}=\boldsymbol{0} \tag{3.20}$$

at each regular point of $M \setminus C(t)$.

In addition, the ***dynamic boundary conditions***

$$\boldsymbol{n}^{*}-\boldsymbol{N}\boldsymbol{v}=\boldsymbol{0}, \qquad \boldsymbol{m}^{*}-\boldsymbol{M}\boldsymbol{v}=\boldsymbol{0} \tag{3.21}$$

should be satisfied at every regular point $\mathrm{x}\in\partial M_{f}$ with the complementary part of the boundary $\partial M_{d}=\partial M\setminus\partial M_{f}$.

It also follows from (3.19) and (1.8) that the ***dynamic continuity conditions*** at regular points of a singular curve $C(t)$ are

$$[\![\boldsymbol{N}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\!]=\boldsymbol{0}, \qquad [\![\boldsymbol{M}\boldsymbol{v}]\!]+[\![\boldsymbol{y}\times\boldsymbol{N}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})]\!]=\boldsymbol{0}. \tag{3.22}$$

With the inertia force and couple vectors given by (3.9), the equations of motion (3.20) may be rewritten as

$$Div\boldsymbol{N}+\boldsymbol{b}=\dot{\boldsymbol{p}}, \qquad Div\boldsymbol{M}+ad^{-1}(\boldsymbol{NF}^{T}-\boldsymbol{FN}^{T})+\boldsymbol{c}=\dot{\boldsymbol{s}}+\dot{\boldsymbol{y}}\times\boldsymbol{p}. \tag{3.23}$$

In classical three-dimensional continuum mechanics, the balance of angular momentum is satisfied if and only if the balance of linear momentum holds and the Cauchy stress tensor is symmetric. No similar theorem holds in the general theory of shells discussed here.

Specific forms of the dynamic continuity conditions (3.22) depend on the nature of singular curves and certain additional regularity assumptions. For example, if motion of the base surface is continuous across singular curves, i.e. the position vector $\boldsymbol{y}(\mathrm{x},t)$ is continuous over the entire M, then the second of the continuity conditions (3.22) takes the form

$$[\![\boldsymbol{M}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}]\!]+\boldsymbol{y}\times([\![\boldsymbol{N}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\!])=\boldsymbol{0}. \tag{3.24}$$

In view of the first of the continuity conditions (3.22), the assumption that the motion be continuous over the entire M yields the ***reduced dynamic continuity conditions*** at regular points of $C(t)$

$$[\![\boldsymbol{N}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\!]=\boldsymbol{0}, \qquad [\![\boldsymbol{M}\boldsymbol{v}]\!]+[\![(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}]\!]=\boldsymbol{0}. \tag{3.25}$$

If in addition $\mathrm{x}\in C(t)$ is a point of smoothness of M, and we take that $\boldsymbol{v}^{-}=\boldsymbol{v}$ and $\boldsymbol{v}^{+}=-\boldsymbol{v}$, so that

$$[\![\boldsymbol{N}\boldsymbol{\nu}]\!] = -[\![\boldsymbol{N}]\!]\boldsymbol{\nu}, \qquad [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})\boldsymbol{p}]\!] = -V[\![\boldsymbol{p}]\!], \tag{3.26}$$

then the conditions (3.25) can be replaced by

$$[\![\boldsymbol{N}]\!]\boldsymbol{\nu} + V[\![\boldsymbol{p}]\!] = \boldsymbol{0}, \qquad [\![\boldsymbol{M}]\!]\boldsymbol{\nu} + V[\![\boldsymbol{s}]\!] = \boldsymbol{0}, \tag{3.27}$$

where $V = \boldsymbol{u}\bullet\boldsymbol{\nu}$ is the speed of propagation of the singular curve.

At regular points of stationary singular curves, such as the ridges Γ of M, $\boldsymbol{u} = \boldsymbol{0}$ and the ***static continuity conditions*** take the form

$$[\boldsymbol{N}\boldsymbol{\nu}] = \boldsymbol{0}, \qquad [\boldsymbol{M}\boldsymbol{\nu}] + [\boldsymbol{y}\times\boldsymbol{N}\boldsymbol{\nu}] = \boldsymbol{0}. \tag{3.28}$$

Under the additional assumption that the motion of the base surface be continuous, these continuity conditions reduce to

$$[\boldsymbol{N}\boldsymbol{\nu}] \equiv \boldsymbol{N}^{+}\boldsymbol{\nu}^{+} + \boldsymbol{N}^{-}\boldsymbol{\nu}^{-} = \boldsymbol{0}, \qquad [\boldsymbol{M}\boldsymbol{\nu}] \equiv \boldsymbol{M}^{+}\boldsymbol{\nu}^{+} + \boldsymbol{M}^{-}\boldsymbol{\nu}^{-} = \boldsymbol{0}. \tag{3.29}$$

Thus the surface stress vector $\boldsymbol{n}_{\nu} = \boldsymbol{N}\boldsymbol{\nu}$ and the surface couple vector $\boldsymbol{m}_{\nu} = \boldsymbol{M}\boldsymbol{\nu}$ must be continuous across such stationary curves.

3.6 Local laws of energy balance

Derivation of the local equations expressing the energy balance proceeds in the standard way. First, we write the balance of energy (1.4) in the form

$$\dot{\mathfrak{u}}(\mathcal{P},t) = \mathfrak{p}(\mathcal{P},t) + \mathfrak{q}(\mathcal{P},t) \tag{3.30}$$

by assuming that the total energy be differentiable with respect to time. Then, the transport theorem (2.28) applied to (3.30) yields the rate of change of the total energy in the form

$$\begin{aligned}\dot{\mathfrak{u}}(\mathcal{P},t) &= \frac{d}{dt}\iint_{\Pi} m_0 u\,da \\ &= \iint_{\Pi\setminus C(t)} m_0\dot{u}\,da + \int_{C(t)\cap\Pi}[\![(\boldsymbol{u}\bullet\boldsymbol{\nu})m_0 u]\!]dl\,.\end{aligned} \tag{3.31}$$

With the internal heat supply $q_{\nu}(\mathsf{x},t)$ and the internal power supply $p_{\nu}(\mathsf{x},t)$ given in terms of the surface heat influx vector $\boldsymbol{q}(\mathsf{x},t)$ and the internal power flux vector $\boldsymbol{p}(\mathsf{x},t)$,

$$q_{\nu}(\mathsf{x},t) = \boldsymbol{q}(\mathsf{x},t)\bullet\boldsymbol{\nu}(\mathsf{x}), \qquad p_{\nu}(\mathsf{x},t) = \boldsymbol{p}(\mathsf{x},t)\bullet\boldsymbol{\nu}(\mathsf{x}), \tag{3.32}$$

both necessarily the tangential surface vectors on M, the heating is defined by

$$\begin{aligned}\mathfrak{q}(\mathcal{P},t) &= \iint_{\Pi}\{m_0 r-(q_+-q_-)\}da-\int_{\partial\Pi}\boldsymbol{q}\bullet\boldsymbol{v}dl-\int_{\partial\Pi\cap\partial M_h}(q^*-\boldsymbol{q}\bullet\boldsymbol{v})dl\\ &= \iint_{\Pi\setminus C(t)}\{m_0 r-Div\boldsymbol{q}-(q_+-q_-)\}da\\ &\quad+\int_{C(t)\cap\Pi}[\boldsymbol{q}\bullet\boldsymbol{v}]dl-\int_{\partial\Pi\cap\partial M_h}(q^*-\boldsymbol{q}\bullet\boldsymbol{v})dl.\end{aligned} \tag{3.33}$$

Now the ***mechanical power*** $\mathfrak{p}(\mathcal{P},t)$ of $\mathcal{P}$ can be rewritten in the form

$$\begin{aligned}\mathfrak{p}(\mathcal{P},t) &= \iint_{\Pi}pda+\int_{\partial\Pi}\boldsymbol{p}\bullet\boldsymbol{v}dl+\int_{\partial\Pi\cap\partial M}(p^*-\boldsymbol{p}\bullet\boldsymbol{v})dl\\ &= \iint_{\Pi\setminus C(t)}(p+Div\boldsymbol{p})da-\int_{C(t)\cap\Pi}[\boldsymbol{p}\bullet\boldsymbol{v}]dl+\int_{\partial\Pi\cap\partial M}(p^*-\boldsymbol{p}\bullet\boldsymbol{v})dl.\end{aligned} \tag{3.34}$$

In (3.33) and (3.34) the jumps at regular points of the singular curve $C(t)$ are defined by

$$[\boldsymbol{q}\bullet\boldsymbol{v}]\equiv\boldsymbol{q}^+\bullet\boldsymbol{v}^+-\boldsymbol{q}^-\bullet\boldsymbol{v}^-,\qquad[\boldsymbol{p}\bullet\boldsymbol{v}]\equiv\boldsymbol{p}^+\bullet\boldsymbol{v}^+-\boldsymbol{p}^-\bullet\boldsymbol{v}^-. \tag{3.35}$$

As a result, the equation of energy balance reads

$$\begin{aligned}&\iint_{\Pi\setminus C(t)}\{m_0\dot{u}-(p+Div\boldsymbol{p})-m_0 r+Div\boldsymbol{q}+(q_+-q_-)\}da\\ &\quad+\int_{C(t)\cap\Pi}\{[\boldsymbol{p}\bullet\boldsymbol{v}]-[\boldsymbol{q}\bullet\boldsymbol{v}]+[(\boldsymbol{u}\bullet\boldsymbol{v})m_0\boldsymbol{u}]\}dl\\ &\quad+\int_{\partial\Pi\cap\partial M}(\boldsymbol{p}\bullet\boldsymbol{v}-p^*)dl+\int_{\partial\Pi\cap\partial M_h}(q^*-\boldsymbol{q}\bullet\boldsymbol{v})dl=0.\end{aligned} \tag{3.36}$$

From (3.36) we read off the local ***equation of energy balance***

$$m_0\dot{u}-(p+Div\boldsymbol{p})-\{m_0 r-Div\boldsymbol{q}-(q_+-q_-)\}=0 \tag{3.37}$$

at every regular point of M, the local ***continuity condition of energy balance***

$$[(\boldsymbol{u}\bullet\boldsymbol{v})m_0 u]+[\boldsymbol{p}\bullet\boldsymbol{v}]-[\boldsymbol{q}\bullet\boldsymbol{v}]=0, \tag{3.38}$$

at regular points of the singular curves $C(t)$, and the local ***energetic boundary condition***

$$q^*-p^*+(\boldsymbol{p}-\boldsymbol{q})\bullet\boldsymbol{v}=0 \tag{3.39}$$

at boundary points $x\in\partial M_h$.

Moreover, the balance of energy yields the additional condition

$$p^*-\boldsymbol{p}\bullet\boldsymbol{v}=0, \tag{3.40}$$

which have to be satisfied at the shell boundary $\partial M\setminus\partial M_h$.

3.7 Local forms of entropy inequality

Assuming that time derivative of $\mathfrak{h}(\mathcal{P},t)$ exists (at least for almost all time instants), the principle of irreversibility (1.5) may be expressed in the form

$$\mathfrak{g}(\mathcal{P},t) \geq 0, \tag{3.41}$$

with the ***rate of entropy production*** $\mathfrak{g}(\mathcal{P},t)$ being given by

$$\mathfrak{g}(\mathcal{P},t) = \dot{\mathfrak{h}}(\mathcal{P},t) - \mathfrak{j}(\mathcal{P},t). \tag{3.42}$$

Upon application of the transport theorem (2.26) to the expression (1.6) for entropy, we obtain the rate of change of entropy of the shell part

$$\begin{aligned} \dot{\mathfrak{h}}(\mathcal{P},t) &= \frac{d}{dt}\iint_{\Pi} m_0\eta da \\ &= \iint_{\Pi\setminus C(t)} m_0\dot{\eta} da + \int_{C(t)\cap\Pi} [\![(\boldsymbol{u}\bullet\boldsymbol{\nu})m_0\eta]\!] dl. \end{aligned} \tag{3.43}$$

There are at least two ways of showing that the contact entropy flux may be expressed in the form

$$j_{\nu}(\mathsf{X},t) = \boldsymbol{j}(\mathsf{X},t)\bullet\boldsymbol{\nu}(\mathsf{X}), \tag{3.44}$$

where the ***entropy influx vector*** $\boldsymbol{j}(\mathsf{X},t)$ is necessarily the tangential vector field on M.

With (3.44), the theorem (2.13) can be used to transform the relation (1.37) into

$$\begin{aligned} \mathfrak{j}(\mathcal{P},t) &= \iint_{\Pi}\{m_0 s - (j_+ - j_-)\}da - \int_{\partial\Pi}\boldsymbol{j}\bullet\boldsymbol{\nu}dl \\ &= \iint_{\Pi\setminus C(t)}\{m_0 s - Div\boldsymbol{j} - (j_+ - j_-)\}da + \int_{C(t)\cap\Pi}[\![\boldsymbol{j}\bullet\boldsymbol{\nu}]\!]dl. \end{aligned} \tag{3.45}$$

Substituting (3.43) and (3.45) into (3.42) we obtain

$$\begin{aligned} \mathfrak{g}(\mathcal{P},t) = \iint_{\Pi\setminus C(t)}&\big(m_0\dot{\eta} - \{m_0 s - Div\boldsymbol{j} - (j_+ - j_-)\}\big)da \\ &+ \int_{C(t)\cap\Pi}\big([\![(\boldsymbol{u}\bullet\boldsymbol{\nu})m_0\eta]\!] - [\![\boldsymbol{j}\bullet\boldsymbol{\nu}]\!]\big)dl. \end{aligned} \tag{3.46}$$

It follows that the rate of entropy production $\mathfrak{g}(\mathcal{P},t)$ in any part of the shell may be written as

$$\mathfrak{g}(\mathcal{P},t) = \iint_{\Pi\setminus C(t)} m_0\gamma da + \int_{C(t)\cap\Pi}\gamma_c dl, \tag{3.47}$$

where the ***specific entropy production*** $\gamma(\mathsf{x},t)$ outside of the singular curve $C(t)$ is defined by

$$m_0\gamma = m_0\dot{\eta} - \{m_0 s - Div\boldsymbol{j} - (j_+ - j_-)\}, \tag{3.48}$$

and $\gamma_c(\mathsf{x},t)$ at regular points of the singular curve $C(t)$ is given by

$$\gamma_c = [(\boldsymbol{u}\bullet\boldsymbol{v})m_0\eta] - [\boldsymbol{j}\bullet\boldsymbol{v}]. \tag{3.49}$$

The principle of irreversibility (1.5) requires that $\mathfrak{g}(\mathcal{P},t) \geq 0$ for every part $\mathcal{P}$ of the shell and for (almost) all time instants t. Since the mass density is strictly positive, $m_0(\mathsf{x}) > 0$, this implies the following two local inequalities:

$$\gamma(\mathsf{x},t) \geq 0, \qquad \gamma_c(\mathsf{x},t) \geq 0, \tag{3.50}$$

which have to be satisfied at all regular points $\mathsf{x} \in M \setminus C(t)$ of the shell base surface and at all regular points $\mathsf{x} \in C(t)$ of the singular curve, respectively.

Chapter 4
Kinematics and kinematic side conditions

4.1 Integral virtual work identity

Of the three principles of mechanics, the balance of mass plays solely a supplementary role. Two remaining principles, the balance of linear and angular momenta, are the "true" laws of mechanics. From the assumed integral-impulse form of these two principles, we have derived in Chapter 3, under quite weak regularity assumptions, the local equations of motion (3.10) at regular points $\mathsf{x} \in M \setminus C(t)$, the dynamic boundary conditions (3.20) at points $\mathsf{x} \in \partial M_f$ of the shell boundary, and the dynamic continuity conditions (3.21) at regular points of the singular curve $C(t) \subset M$.

Let $\boldsymbol{v}(\mathsf{x},t)$ and $\boldsymbol{w}(\mathsf{x},t)$ be any two spatial vector fields on M, collectively denoted by $\mathbb{W} \equiv (\boldsymbol{v}, \boldsymbol{w})$. These fields are assumed to be defined at all points of M, except possibly of a singular set of zero area measure. For any part $\mathcal{P}$ of the shell-like body, which in the reference configuration is represented by the region $\Pi \subset M$ of the shell base surface, we set

$$
\begin{aligned}
G(\mathcal{P};\mathbb{W}) \equiv -\iint_{\Pi \setminus C(t)} \Big(& (Div\boldsymbol{N} + \boldsymbol{b} + \boldsymbol{b}^{in}) \cdot \boldsymbol{v} \\
& + \{Div\boldsymbol{M} + ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T) + \boldsymbol{c} + \boldsymbol{c}^{in}\} \cdot \boldsymbol{w} \Big) da \qquad (4.1) \\
& - \int_{\partial\Pi \cap \partial M_f} \{(\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{v}) \cdot \boldsymbol{v} + (\boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{v}) \cdot \boldsymbol{w}\} dl \,,
\end{aligned}
$$

where the inertia force and couple vectors are given by (3.8).

Under the regularity assumptions described in Chapter 3, the two terms in (4.1) containing the surface divergences of $\boldsymbol{N}$ and $\boldsymbol{M}$ can be transformed using the generalized surface divergence theorem (2.20) into

$$\begin{aligned}\int_{\Pi\setminus C(t)}(Div\boldsymbol{N})\cdot\boldsymbol{v}da &= -\int_{\Pi\setminus C(t)}\boldsymbol{N}\cdot A\boldsymbol{v}da+\int_{C(t)\cap\Pi}[\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}]dl\\ &\quad+\int_{\partial\Pi}\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}dl,\\ \int_{\Pi\setminus C(t)}(Div\boldsymbol{M})\cdot\boldsymbol{w}da &= -\int_{\Pi\setminus C(t)}\boldsymbol{M}\cdot A\boldsymbol{w}da+\int_{C(t)\cap\Pi}[\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w}]dl\\ &\quad+\int_{\partial\Pi\cap\partial M}\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w}dl.\end{aligned} \tag{4.2}$$

Substituting (4.2) into (4.1) and rearranging terms , we obtain

$$\begin{aligned}G(\mathcal{P};\mathbb{V})=&-\iint_{\Pi\setminus C(t)}(\boldsymbol{b}^{in}\cdot\boldsymbol{v}+\boldsymbol{c}^{in}\cdot\boldsymbol{w})da\\ &+\iint_{\Pi\setminus C(t)}\Big(\boldsymbol{N}\cdot A\boldsymbol{v}-\{ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)\}\cdot\boldsymbol{w}+\boldsymbol{M}\cdot A\boldsymbol{w}\Big)da\\ &-\iint_{\Pi\setminus C(t)}(\boldsymbol{b}\cdot\boldsymbol{v}+\boldsymbol{c}\cdot\boldsymbol{w})da-\int_{\partial\Pi\setminus\partial M}(\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w})dl\\ &-\int_{\partial\Pi\cap\partial M_f}(\boldsymbol{n}^*\cdot\boldsymbol{v}+\boldsymbol{m}^*\cdot\boldsymbol{w})dl\\ &-\int_{C(t)\cap\Pi}\big([\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}]+[\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w}]\big)dl.\end{aligned} \tag{4.3}$$

Let us introduce the following notation for some integrals in (4.3):

$$G^{in}(\mathcal{P};\mathbb{V})\equiv\iint_{\Pi\setminus C(t)}(\boldsymbol{b}^{in}\cdot\boldsymbol{v}+\boldsymbol{c}^{in}\cdot\boldsymbol{w})da. \tag{4.4}$$

$$G_i(\mathcal{P};\mathbb{V})\equiv\iint_{\Pi\setminus C(t)}\Big(\boldsymbol{N}\cdot\nabla\boldsymbol{v}-\{ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T-\boldsymbol{F}\boldsymbol{N}^T)\}\cdot\boldsymbol{w}+\boldsymbol{M}\cdot\nabla\boldsymbol{w}\Big)da, \tag{4.5}$$

$$\begin{aligned}G_m(\mathcal{P};\mathbb{V})\equiv&\iint_{\Pi\setminus C(t)}(\boldsymbol{b}\cdot\boldsymbol{v}+\boldsymbol{c}\cdot\boldsymbol{w})da\\ &+\int_{\partial\Pi\setminus\partial M}(\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w})dl+\int_{\partial\Pi\cap\partial M_f}(\boldsymbol{n}^*\cdot\boldsymbol{v}+\boldsymbol{m}^*\cdot\boldsymbol{w})dl.\end{aligned} \tag{4.6}$$

Thus (4.3) may be rewritten as

$$\begin{aligned}G(\mathcal{P};\mathbb{V})=&-G^{in}(\mathcal{P};\mathbb{V})+G_i(\mathcal{P};\mathbb{V})-G_m(\mathcal{P};\mathbb{V})\\ &-\int_{C(t)\cap\Pi}\big([\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}]+[\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w}]\big)dl.\end{aligned} \tag{4.7}$$

Let us remind that the inertia force and couple vectors are given through the linear and angular momentum densities by (3.8), that is

$$G^{in}(\mathcal{P};\mathbb{V})=-\iint_{\Pi\setminus C(t)}\{\dot{\boldsymbol{p}}\cdot\boldsymbol{v}+(\dot{\boldsymbol{s}}+\dot{\boldsymbol{y}}\times\boldsymbol{p})\cdot\boldsymbol{w}\}da. \tag{4.8}$$

Let us also note the identity

$$\begin{aligned}\dot{\boldsymbol{p}}\bullet\boldsymbol{v}+(\dot{\boldsymbol{s}}+\dot{\boldsymbol{y}}\times\boldsymbol{p})\bullet\boldsymbol{w}&=\frac{d}{dt}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})-\{\boldsymbol{p}\bullet\dot{\boldsymbol{v}}-(\dot{\boldsymbol{y}}\times\boldsymbol{p})\bullet\boldsymbol{w}+\boldsymbol{s}\bullet\dot{\boldsymbol{w}}\}\\&=\frac{d}{dt}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})-\{\boldsymbol{p}\bullet(\dot{\boldsymbol{v}}-\boldsymbol{w}\times\dot{\boldsymbol{y}})+\boldsymbol{s}\bullet\dot{\boldsymbol{w}}\},\end{aligned}\tag{4.9}$$

from which we have

$$\begin{aligned}G^{in}(\mathcal{P};\mathbb{W})=&\iint_{\Pi\setminus C(t)}\{\boldsymbol{p}\bullet(\dot{\boldsymbol{v}}-\boldsymbol{w}\times\dot{\boldsymbol{y}})+\boldsymbol{s}\bullet\dot{\boldsymbol{w}}\}da\\&-\iint_{\Pi\setminus C(t)}\frac{d}{dt}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da.\end{aligned}\tag{4.10}$$

By the transport theorem (2.26) we obtain

$$\begin{aligned}&\iint_{\Pi\setminus C(t)}\frac{d}{dt}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da\\&=\frac{d}{dt}\iint_{\Pi\setminus C(t)}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da-\int_{C(t)\cap\Pi}[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})]dl,\end{aligned}\tag{4.11}$$

so that

$$\begin{aligned}G^{in}(\mathcal{P};\mathbb{W})=&-\frac{d}{dt}\iint_{\Pi\setminus C(t)}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da\\&+\iint_{\Pi\setminus C(t)}\{\boldsymbol{p}\bullet(\dot{\boldsymbol{v}}-\boldsymbol{w}\times\dot{\boldsymbol{y}})+\boldsymbol{s}\bullet\dot{\boldsymbol{w}}\}da\\&+\int_{C(t)\cap\Pi}[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})]dl,\end{aligned}\tag{4.12}$$

or

$$\begin{aligned}G^{in}(\mathcal{P};\mathbb{W})=&G_d(\mathcal{P};\mathbb{W})-\frac{d}{dt}\iint_{\Pi\setminus C(t)}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da\\&+\int_{C(t)\cap\Pi}[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})]dl,\end{aligned}\tag{4.13}$$

where

$$G_d(\mathcal{P};\mathbb{W})\equiv\iint_{\Pi\setminus C(t)}\{\boldsymbol{p}\bullet(\dot{\boldsymbol{v}}-\boldsymbol{w}\times\dot{\boldsymbol{y}})+\boldsymbol{s}\bullet\dot{\boldsymbol{w}}\}da.\tag{4.14}$$

Substituting (4.13) into (4.7) we finally obtain the expression

$$\begin{aligned}G(\mathcal{P};\mathbb{W})=&\frac{d}{dt}\iint_{\Pi\setminus C(t)}(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})da\\&-G_d(\mathcal{P};\mathbb{W})+G_i(\mathcal{P};\mathbb{W})-G_m(\mathcal{P};\mathbb{W})-\int_{C(t)\cap\Pi}Pdl,\end{aligned}\tag{4.15}$$

where

$$P\equiv[\boldsymbol{N}\boldsymbol{v}\bullet\boldsymbol{v}]+[\boldsymbol{M}\boldsymbol{v}\bullet\boldsymbol{w}]+[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})].\tag{4.16}$$

4.2 More about jumps of surface fields

Let us consider a vector field $\boldsymbol{u}(\mathsf{x},t)$ and a tensor field $\boldsymbol{S}(\mathsf{x},t)$ on a piecewise smooth surface M. At regular points of M we have $\boldsymbol{u}(\mathsf{x},t) \in E \otimes T_{\mathsf{x}}M$ and $\boldsymbol{S}(\mathsf{x},t) \in E \otimes T_{\mathsf{x}}M$. At any regular point $\mathsf{x} \in C(t)$ of a moving curve in M we set

$$\boldsymbol{s}_{\nu}^{\pm}(\mathsf{x}) \equiv \boldsymbol{S}^{\pm}(\mathsf{x})\boldsymbol{\nu}^{\pm}(\mathsf{x}). \tag{4.17}$$

Let us remind that in Section 2.3 we have defined the jump $[\![.]\!]$ and the mean value $\langle\!\langle . \rangle\!\rangle$ by

$$[\![\boldsymbol{u}]\!] \equiv \boldsymbol{u}^{+} - \boldsymbol{u}^{-}, \qquad \langle\!\langle \boldsymbol{u} \rangle\!\rangle \equiv \tfrac{1}{2}(\boldsymbol{u}^{+} + \boldsymbol{u}^{-}), \tag{4.18}$$

and the generalized jump $[\![.]\!]$ and the generalized mean value $\langle . \rangle$ by

$$[\![\boldsymbol{s}_{\nu}]\!] \equiv [\![\boldsymbol{S}\boldsymbol{\nu}]\!] = \boldsymbol{s}_{\nu}^{+} + \boldsymbol{s}_{\nu}^{-}, \qquad \langle \boldsymbol{s}_{\nu} \rangle \equiv \langle \boldsymbol{S}\boldsymbol{\nu} \rangle = \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} - \boldsymbol{s}_{\nu}^{-}). \tag{4.19}$$

Let us now consider the following expressions:

$$\begin{aligned}
[\![\boldsymbol{s}_{\nu}]\!] \otimes \langle\!\langle \boldsymbol{u} \rangle\!\rangle &= \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} + \boldsymbol{s}_{\nu}^{-}) \otimes (\boldsymbol{u}^{+} + \boldsymbol{u}^{-}) \\
&= \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{-} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-}), \\
\langle \boldsymbol{s}_{\nu} \rangle \otimes [\![\boldsymbol{u}]\!] &= \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} - \boldsymbol{s}_{\nu}^{-}) \otimes (\boldsymbol{u}^{+} - \boldsymbol{u}^{-}) \\
&= \tfrac{1}{2}(\boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} - \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{+} - \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{-} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-}).
\end{aligned} \tag{4.20}$$

The sum of these two expressions may be written as

$$\begin{aligned}
[\![\boldsymbol{s}_{\nu}]\!] \otimes \langle\!\langle \boldsymbol{u} \rangle\!\rangle + \langle \boldsymbol{s}_{\nu} \rangle \otimes [\![\boldsymbol{u}]\!] = \tfrac{1}{2}(&\boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{+} - \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{+} \\
&+ \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{-} - \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{-} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-}),
\end{aligned} \tag{4.21}$$

and hence

$$[\![\boldsymbol{s}_{\nu}]\!] \otimes \langle\!\langle \boldsymbol{u} \rangle\!\rangle + \langle \boldsymbol{s}_{\nu} \rangle \otimes [\![\boldsymbol{u}]\!] = \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-}. \tag{4.22}$$

In consistency with (4.19) we define

$$[\![\boldsymbol{s}_{\nu} \otimes \boldsymbol{u}]\!] \equiv \boldsymbol{s}_{\nu}^{+} \otimes \boldsymbol{u}^{+} + \boldsymbol{s}_{\nu}^{-} \otimes \boldsymbol{u}^{-}, \tag{4.23}$$

and from (4.22) it follows that

$$[\![s_\nu \otimes u]\!] = [\![s_\nu]\!] \otimes \langle\!\langle u \rangle\!\rangle + \langle s_\nu \rangle \otimes [\![u]\!], \tag{4.24}$$

or

$$[\![S\nu \otimes u]\!] = [\![S\nu]\!] \otimes \langle\!\langle u \rangle\!\rangle + \langle S\nu \rangle \otimes [\![u]\!]. \tag{4.25}$$

This and many other formulae, which may be derived in the same manner, hold for non-smooth surfaces. For smooth surfaces they reduce to the more transparent forms. In particular, if $C(t)$ is a singular curve on a smooth surface M, then at each regular point of $C(t)$ we can state that $\nu = \nu^- = -\nu^+$ and

$$[\![s_\nu]\!] = -[\![S]\!]\nu, \qquad \langle s_\nu \rangle = -\langle\!\langle S \rangle\!\rangle \nu, \tag{4.26}$$

where

$$[\![S]\!] = S^+ - S^-, \qquad \langle\!\langle S \rangle\!\rangle = \tfrac{1}{2}(S^+ - S^-). \tag{4.27}$$

Thus (4.24) takes the form

$$[\![s_\nu \otimes u]\!] = -[\![S]\!]\nu \otimes \langle\!\langle u \rangle\!\rangle - \langle\!\langle S \rangle\!\rangle \nu \otimes [\![u]\!]. \tag{4.28}$$

Taking the transpose of (4.28) we obtain

$$\begin{aligned} [\![u \otimes s_\nu]\!] &= -\langle\!\langle u \rangle\!\rangle \otimes [\![S]\!]\nu - [\![u]\!] \otimes \langle\!\langle S \rangle\!\rangle \nu \\ &= -\left(\langle\!\langle u \rangle\!\rangle \otimes [\![S]\!] + [\![u]\!] \otimes \langle\!\langle S \rangle\!\rangle\right)\nu. \end{aligned} \tag{4.29}$$

On the other hand, from definition

$$[\![u \otimes s_\nu]\!] = [\![u \otimes S\nu]\!] = u^+ \otimes S^+\nu^+ + u^- \otimes S^-\nu^-, \tag{4.30}$$

so that

$$\begin{aligned} [\![u \otimes S\nu]\!] &= -u^+ \otimes S^+\nu + u^- \otimes S^-\nu \\ &= -\left(u^+ \otimes S^+ - u^- \otimes S^-\right)\nu \\ &= -[\![u \otimes S]\!]\nu. \end{aligned} \tag{4.31}$$

Thus, we finally obtain

$$[\![u \otimes S]\!] = [\![u]\!] \otimes \langle\!\langle S \rangle\!\rangle + \langle\!\langle u \rangle\!\rangle \otimes [\![S]\!], \tag{4.32}$$

which looks like the corresponding formula in the three-dimensional case.

4.3 Implications to terms in dynamic continuity conditions

With the use of identity (4.25), the first two terms in (4.16) may be written as

$$\begin{aligned}[\boldsymbol{Nv}\bullet\boldsymbol{v}]+[\boldsymbol{Mv}\bullet\boldsymbol{w}] &= [\boldsymbol{Nv}]\bullet\langle\!\langle\boldsymbol{v}\rangle\!\rangle+[\boldsymbol{Mv}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle\\ &\quad+\langle\boldsymbol{Nv}\rangle\bullet[\![\boldsymbol{v}]\!]+\langle\boldsymbol{Mv}\rangle\bullet[\![\boldsymbol{w}]\!].\end{aligned}\tag{4.33}$$

From the dynamic continuity conditions (3.24) we have

$$\begin{aligned}[\boldsymbol{Nv}] &= -[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}],\\ [\boldsymbol{Mv}] &= -[\boldsymbol{y}\times\boldsymbol{Nv}]-[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})],\end{aligned}\tag{4.34}$$

and (4.33) may be written as

$$\begin{aligned}[\boldsymbol{Nv}\bullet\boldsymbol{v}]+[\boldsymbol{Mv}\bullet\boldsymbol{w}] &= -[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\bullet\langle\!\langle\boldsymbol{v}\rangle\!\rangle-[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{s}+\boldsymbol{y}\times\boldsymbol{p})]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle\\ &\quad+\langle\boldsymbol{Nv}\rangle\bullet[\![\boldsymbol{v}]\!]+\langle\boldsymbol{Mv}\rangle\bullet[\![\boldsymbol{w}]\!]-[\boldsymbol{y}\times\boldsymbol{Nv}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle.\end{aligned}\tag{4.35}$$

In the same manner we can transform the last term in (4.16) into

$$\begin{aligned}[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})] &= [(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\bullet\langle\!\langle\boldsymbol{v}\rangle\!\rangle+[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle\\ &\quad+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}\rangle\bullet[\![\boldsymbol{v}]\!]+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}\rangle\bullet[\![\boldsymbol{w}]\!].\end{aligned}\tag{4.36}$$

The integrand (4.10) can now be given as

$$\begin{aligned}P &\equiv [\boldsymbol{Nv}\bullet\boldsymbol{v}]+[\boldsymbol{Mv}\bullet\boldsymbol{w}]+[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{p}\bullet\boldsymbol{v}+\boldsymbol{s}\bullet\boldsymbol{w})]\\ &= -[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\bullet\langle\!\langle\boldsymbol{v}\rangle\!\rangle+[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}]\bullet\langle\!\langle\boldsymbol{v}\rangle\!\rangle\\ &\quad-[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle+[(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle\\ &\quad-[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{y}\times\boldsymbol{p})]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle-[\boldsymbol{y}\times\boldsymbol{Nv}]\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle\\ &\quad+\langle\boldsymbol{Nv}\rangle\bullet[\![\boldsymbol{v}]\!]+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}\rangle\bullet[\![\boldsymbol{v}]\!]+\langle\boldsymbol{Mv}\rangle\bullet[\![\boldsymbol{w}]\!]+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}\rangle\bullet[\![\boldsymbol{w}]\!],\end{aligned}\tag{4.37}$$

or after further transformations

$$\begin{aligned}P &= \big(\langle\boldsymbol{Nv}\rangle+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{p}\rangle\big)\bullet[\![\boldsymbol{v}]\!]+\big(\langle\boldsymbol{Mv}\rangle+\langle(\boldsymbol{u}\bullet\boldsymbol{v})\boldsymbol{s}\rangle\big)\bullet[\![\boldsymbol{w}]\!]\\ &\quad-\big([\boldsymbol{y}\times\boldsymbol{Nv}]+[(\boldsymbol{u}\bullet\boldsymbol{v})(\boldsymbol{y}\times\boldsymbol{p})]\big)\bullet\langle\!\langle\boldsymbol{w}\rangle\!\rangle.\end{aligned}\tag{4.38}$$

4.4 Weak form of momentum balance laws

From the definition (4.1) of the expression $G(\mathcal{P};\mathbb{W})$ and the subsequent formal steps leading to (4.7) we see that

$$\int_{t_1}^{t_2} G(\mathcal{P};\mathbb{W})dt = 0, \tag{4.39}$$

for every part $\mathcal{P}$ of the shell-like body and every time interval $[t_1,t_2]$, provided that the balance laws of linear and angular momenta hold. The converse theorem is also true, as it can be shown by reversing the whole process of derivation.

The integral identity (4.39) is often referred to as the weak form of the momentum balance laws. Using (4.15) with (4.16) it may be rewritten in the form

$$\begin{aligned}&\left[\iint_{\Pi\setminus C(t)}(\boldsymbol{p}\cdot\boldsymbol{v}+\boldsymbol{s}\cdot\boldsymbol{w})da\right]_{t_1}^{t_2}+\int_{t_1}^{t_2}\left\{-G_d(\mathcal{P};\mathbb{W})+G_i(\mathcal{P};\mathbb{W})-G_e(\mathcal{P};\mathbb{W})\right\}dt\\&-\int_{t_1}^{t_2}\left\{\int_{C(t)\cap\Pi}Pdl+\int_{\partial\Pi\setminus\partial M}(\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{w})dl\right\}dt=0.\end{aligned} \tag{4.40}$$

The physical meaning of the integral identity (4.40) is actually self-evident: it represents in the weak form the dynamic equilibrium conditions of the entire shell with any kind of irregularities and undergoing possibly non-smooth deformations. In this sense the vector fields $(\boldsymbol{v},\boldsymbol{w})$ may be called the generalized virtual displacements (mathematically, test functions).

Within the purely mechanical theory, the reasoning underlying this derivation of the resultant mechanical power may be replaced by the construction of the virtual work identity. These are two different but closely related concepts.

4.5 Shell kinematics

The integral identity (4.40) must hold in every virtual motion of the shell. In particular, it must hold also in the real motion, in which case the fields $(\boldsymbol{v},\boldsymbol{w})$ should be identified with the real shell velocities. To distinguish the real velocities from the virtual displacements, the former will be denoted by $(\boldsymbol{v},\boldsymbol{\omega})$. From the real velocities it is possible to construct the complete shell kinematics.

In a fixed frame of reference, a spatial place $\mathsf{y}\in M(t)$ instantaneously occupied by a typical shell particle is determined by the position vector $\boldsymbol{y}(\mathsf{x},t)$. The velocity vector $\boldsymbol{v}(\mathsf{x},t)$ is then determined by time derivative of $\boldsymbol{y}(\mathsf{x},t)$:

$$\frac{d}{dt}\boldsymbol{y}(\mathsf{x},t)=\boldsymbol{v}(\mathsf{x},t). \tag{4.41}$$

Here $\boldsymbol{v}(\mathsf{x},t)$ is tangent to trajectory of the shell particle, which in the reference configuration has been placed at $\mathsf{x} \in M$.

Alternatively, we may consider the velocity field $\boldsymbol{v}(\mathsf{x},t)$ as the primitive kinematic variable and define the position $\boldsymbol{y}(\mathsf{x},t)$ vector as the solution to the differential equation (4.41). This solution takes the form

$$\boldsymbol{y}(\mathsf{x},t) = \int_{t_0}^{t} \boldsymbol{v}(\mathsf{x},\tau)d\tau + \boldsymbol{v}(\mathsf{x},t_0), \tag{4.42}$$

where $\boldsymbol{v}(\mathsf{x},t_0)$ is the initial velocity, here taken at time t_0.

In the same manner, let $\boldsymbol{\Omega}(\mathsf{x},t)$ denote the skew-symmetric tensor having the angular velocity $\boldsymbol{\omega}(\mathsf{x},t)$ as its axial vector, both assumed to be known. We then seek the solution to the following first-order ordinary differential equation:

$$\frac{d}{dt}\boldsymbol{Y}(\mathsf{x},t) = \boldsymbol{\Omega}(\mathsf{x},t), \qquad \boldsymbol{\Omega}(\mathsf{x},t) = ad\boldsymbol{\omega}(\mathsf{x},t). \tag{4.43}$$

The general solution to the differential equation (4.43) has the form

$$\boldsymbol{Y}(\mathsf{x},t) = \boldsymbol{Q}(\mathsf{x},t)\boldsymbol{X}(\mathsf{x}), \tag{4.44}$$

where $\boldsymbol{Y}(\mathsf{x},t)$ and $\boldsymbol{X}(\mathsf{x})$ are non-singular tensors, while $\boldsymbol{Q}(\mathsf{x},t)$ is an orthogonal tensor, $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$. Taking into account that $\dot{\boldsymbol{Y}} = \dot{\boldsymbol{Q}}\boldsymbol{X}$ and $\boldsymbol{Y}^{-1} = \boldsymbol{X}^{-1}\boldsymbol{Q}^T$, we obtain

$$\boldsymbol{\Omega} = \dot{\boldsymbol{Y}}\boldsymbol{Y}^{-1} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T. \tag{4.45}$$

It follows from (4.42) and (4.44) that every spatial configuration of the shell is completely determined by the base surface $M(t)$, having $\boldsymbol{y}(\mathsf{y},t)$ as the position vector, and by the non-singular structure tensor $\boldsymbol{Y}(\mathsf{y},t)$. In particular, the reference configuration is determined by the position vector $\boldsymbol{x}(\mathsf{x})$ of the base surface M and by the non-singular structure tensor $\boldsymbol{X}(\mathsf{x})$.

In the referential description the shell motion is determined by two fields

$$\boldsymbol{y} : M \times T \to E, \qquad \boldsymbol{Y} : M \times T \to E \otimes E, \tag{4.46}$$

with the position vector $\boldsymbol{y}$ and the structure tensor $\boldsymbol{Y}$ being given by

$$\boldsymbol{y}(\mathsf{x},t) = \boldsymbol{x}(\mathsf{x}) + \boldsymbol{u}(\mathsf{x},t), \qquad \boldsymbol{Y}(\mathsf{x},t) = \boldsymbol{Q}(\mathsf{x},t)\boldsymbol{X}(\mathsf{x}), \tag{4.47}$$

where $\boldsymbol{u}(\mathsf{x},t)$ denotes the associated displacement vector of the base surface and $\boldsymbol{Q}(\mathsf{x},t)$ is the rotation tensor describing an independent mean rotation of the shell cross sections.

Having described the "real" motion of the shell, the virtual motion can now be described by

$$\boldsymbol{y}_\varepsilon(\mathsf{x},t) = \boldsymbol{y}(\mathsf{x},t) + \varepsilon\boldsymbol{v}(\mathsf{x},t), \qquad \boldsymbol{Y}_\varepsilon(\mathsf{x},t) = \exp(\varepsilon\boldsymbol{W}(\mathsf{x},t))\boldsymbol{Y}(\mathsf{x},t). \tag{4.48}$$

The relation between (4.48) and two vector fields $\mathbb{W} \equiv (\boldsymbol{v}, \boldsymbol{w})$ used to construct the integral identity (4.40) becomes now clear if we note that

$$\frac{d}{d\varepsilon}\boldsymbol{y}_\varepsilon(\mathsf{x},t)_{|\varepsilon=0} = \boldsymbol{v}(\mathsf{x},t), \qquad \frac{d}{d\varepsilon}\boldsymbol{Y}_\varepsilon(\mathsf{x},t)_{|\varepsilon=0} = \boldsymbol{W}(\mathsf{x},t)\boldsymbol{Y}(\mathsf{x},t). \tag{4.49}$$

Recalling the representations (3.16) of the stress and couple vectors, it becomes obvious that both vectors may have all three non-vanishing components with regard to any basis (typically, two tangential components and one normal component). In the component form the equilibrium equations (3.20) constitute the system of six independent scalar equations involving six components of the stress and couple vectors. Also the dynamic boundary conditions (3.21), when expressed in the component form, constitute the systems of six scalar equations. It is apparent that in order to support these six dynamic shell relations we need also six independent kinematic fields to describe the shell motion. And indeed, it follows from discussion above that the general shell kinematics is described by the displacement vector $\boldsymbol{u}(\mathsf{x},t)$ and the rotation tensor $\boldsymbol{Q}(\mathsf{x},t)$ constituting the six independent scalar fields work-conjugate to the six dynamic shell equations.

4.6 Kinematic side conditions

Let us assume that the shell boundary be represented as the sum of two disjoint parts, denoted by ∂M_f and ∂M_d, and the dynamic boundary conditions derived in the previous Chapter be prescribed along ∂M_f. Then from the line integral in (4.6) we may conclude that along the complementary part ∂M_d the following kinematic boundary conditions must be specified:

$$\boldsymbol{y}(\mathsf{x},t) = \boldsymbol{y}^*(\mathsf{x},t), \qquad \boldsymbol{Y}(\mathsf{x},t) = \boldsymbol{Y}^*(\mathsf{x},t). \tag{4.50}$$

Chapter 5

Effective mechanical power and interstitial working

5.1 Effective mechanical power and stress power theorem

In mechanics, the mechanical power is generally defined as the rate of working of all forces and couples acting on a body under considerations. This is true in mechanics of rigid and deformable three-dimensional bodies, as well as in mechanics of two-dimensional shell-like and one-dimensional rod-like bodies.

Within the considered theory of shells, the forces and couples acting on any part Π of the shell base surface are specified by the assumed form (1.32) of the force $\mathbf{f}(\mathcal{P},t)$ and torque $\mathbf{t}(\mathcal{P},t)$. Moreover, from the results of foregoing considerations it may be concluded that the mechanical power of the shell can be given in the form

$$\begin{aligned}\mathfrak{p}_e(\mathcal{P},t) = \iint_{\Pi}(\boldsymbol{b}\bullet\boldsymbol{v}+\boldsymbol{c}\bullet\boldsymbol{\omega})da + \int_{\partial\Pi\setminus\partial M}(\boldsymbol{n}_\nu\bullet\boldsymbol{v}+\boldsymbol{m}_\nu\bullet\boldsymbol{\omega})dl \\ + \int_{\partial\Pi\cap\partial M_f}(\boldsymbol{n}^*\bullet\boldsymbol{v}+\boldsymbol{m}^*\bullet\boldsymbol{\omega})dl,\end{aligned} \tag{5.1}$$

where $\boldsymbol{v}(\mathrm{x},t)$ and $\boldsymbol{\omega}(\mathrm{x},t)$ are the linear and angular velocity fields, respectively.

However, if one requires that the mechanical power for the shell be an exact consequence of the mechanical power of the 3D shell-like body (i.e. the numerical value $\mathfrak{p}_e(\mathcal{P},t)$ be equal to the mechanical power of all forces acting on the corresponding part $\mathcal{P}$ of the 3D spatial region representing the shell-like body), then (5.1) cannot be identified with the mechanical power $\mathfrak{p}(\mathcal{P},t)$ of the 3D shell-like body. Hence the subscript "e" in the effective mechanical power defined by (5.1). That some additional term is needed here is obvious when analyzing derivation of the 2D shell mechanical power from the mechanical power of continuum mechanics, which was given in LIBAI AND SIMMONDS [1998], Chapter 8, using the reduction procedure.

Let $\mathfrak{p}_a(\mathcal{P},t)$ be such an additional part of the shell mechanical power which completes $\mathfrak{p}_e(\mathcal{P},t)$ in the sense that now

$$\mathfrak{p}(\mathcal{P},t) = \mathfrak{p}_e(\mathcal{P},t) + \mathfrak{p}_a(\mathcal{P},t). \tag{5.2}$$

In general, at the shell base surface the additional mechanical power can be represented by the sum of three parts, analogous to those given in (1.35),

$$\mathfrak{p}_a(\mathcal{P},t) = \iint_{\Pi} w da + \int_{\partial\Pi\setminus\partial M} w_v dl + \int_{\partial\Pi\cap\partial M_f} w^* dl, \tag{5.3}$$

where w is the (surface) interstitial power density, w_v the (surface) contact interstitial power supply, and w^* the boundary interstitial power supply. The names "interstitial" are taken here by analogy to terms introduced in extended continuum thermomechanics by DUNN AND SERRIN [1985].

TOUPIN's [1962] theory of n-grade materials has several novel characteristics, among which is a need for couple stresses and for existence of an asymmetric Cauchy stress tensor which does not deliver the surface traction. DUNN AND SERRIN [1985] demonstrated that Toupin's theory is not compatible with continuum thermodynamics unless the latter one reduces to thermoelasticity. They also presented a formalism which does a "minimum of violence" to the thermodynamical structure of such a continuum, modifying only the equation of energy balance by introducing a rate of supply of the mechanical energy across material surfaces of the body. For lower-dimension continua such as shells, plates and rods long-range interactions are naturally expected, and the concept of DUNN AND SERRIN [1985] can be used in such theories as well.

With the surface stress and couple vectors being given by the surface stress and couple tensors, the effective mechanical power (5.2) may be rewritten in the form

$$\begin{aligned}\mathfrak{p}(\mathcal{P},t) = &\iint_{\Pi} (\boldsymbol{b}\bullet\boldsymbol{v} + \boldsymbol{c}\bullet\boldsymbol{\omega} + w) da + \int_{\partial\Pi} (\boldsymbol{N}\boldsymbol{v}\bullet\boldsymbol{v} + \boldsymbol{M}\boldsymbol{v}\bullet\boldsymbol{\omega} + w_v) dl \\ &+ \int_{\partial\Pi\cap\partial M_f} \{(\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{v})\bullet\boldsymbol{v} + (\boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{v})\bullet\boldsymbol{\omega} + (w^* - w_v)\} dl.\end{aligned} \tag{5.4}$$

Applying the generalized surface divergence theorem (2.20) to the respective boundary terms in (5.4) we can present them as

$$\begin{aligned}\int_{\partial\Pi} \boldsymbol{N}\boldsymbol{v}\bullet\boldsymbol{v} dl &= \iint_{\Pi\setminus C(t)} \{(Div\boldsymbol{N})\bullet\boldsymbol{v} + \boldsymbol{N}\bullet\nabla\boldsymbol{v}\} da - \int_{C(t)\cap\Pi} [\![\boldsymbol{N}\boldsymbol{v}\bullet\boldsymbol{v}]\!] dl, \\ \int_{\partial\Pi} \boldsymbol{M}\boldsymbol{v}\bullet\boldsymbol{\omega} dl &= \iint_{\Pi\setminus C(t)} \{(Div\boldsymbol{M})\bullet\boldsymbol{\omega} + \boldsymbol{M}\bullet\nabla\boldsymbol{\omega}\} da - \int_{C(t)\cap\Pi} [\![\boldsymbol{M}\boldsymbol{v}\bullet\boldsymbol{\omega}]\!] dl.\end{aligned} \tag{5.5}$$

Substituting the results into (5.4) we obtain

$$\begin{aligned}\mathfrak{p}(\mathcal{P},t) = \iint_{\Pi\setminus C(t)} &\big((Div\boldsymbol{N}+\boldsymbol{b})\cdot\boldsymbol{v} + (Div\boldsymbol{M}+\boldsymbol{c})\cdot\boldsymbol{\omega} \\ &+ \boldsymbol{N}\cdot\nabla\boldsymbol{v} + \boldsymbol{M}\cdot\nabla\boldsymbol{\omega} + w\big)da \\ - \int_{C(t)\cap\Pi} &\big([\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}] + [\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{\omega}]\big)dl \\ + \int_{\partial\Pi} w_v dl + \int_{\partial\Pi\cap\partial M_f} &\Big((\boldsymbol{n}^* - \boldsymbol{N}\boldsymbol{v})\cdot\boldsymbol{v} + (\boldsymbol{m}^* - \boldsymbol{M}\boldsymbol{v})\cdot\boldsymbol{\omega} + (w^* - w_v)\Big)dl.\end{aligned} \tag{5.6}$$

This result follows from (5.4) merely through the use of the generalized surface divergence theorem, without any reference to the underlying laws of mechanics. It can be simplified further if the balance of linear and angular momenta hold. Under this assumption, the first two terms in the last line integral of (5.6) vanish by virtue of the dynamic boundary conditions. Moreover, from the dynamic equations of motion (3.20)

$$Div\boldsymbol{N} + \boldsymbol{b} = -\boldsymbol{b}^{in}, \quad Div\boldsymbol{M} + \boldsymbol{c} = -\boldsymbol{c}^{in} - ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T). \tag{5.7}$$

The left-hand side terms in (5.7) can be eliminated from (5.6), and the shell mechanical power is obtained in the form

$$\begin{aligned}\mathfrak{p}(\mathcal{P},t) = \iint_{\Pi} &\Big(-(\boldsymbol{b}^{in}\cdot\boldsymbol{v} + \boldsymbol{c}^{in}\cdot\boldsymbol{\omega}) \\ &+ \boldsymbol{N}\cdot\nabla\boldsymbol{v} - ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\cdot\boldsymbol{\omega} + \boldsymbol{M}\cdot\nabla\boldsymbol{\omega} + w\Big)da \\ - \int_{C(t)\cap\Pi} &\big([\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}] + [\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{\omega}]\big)dl + \int_{\partial\Pi} w_v dl + \int_{\partial\Pi\cap\partial M_f} (w^* - w_v)dl.\end{aligned} \tag{5.8}$$

It is convenient to introduce explicitly the shell stress power density $\sigma(\mathsf{x},t)$ defined by

$$\sigma \equiv \boldsymbol{N}\cdot\nabla\boldsymbol{v} - \{ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\}\cdot\boldsymbol{\omega} + \boldsymbol{M}\cdot\nabla\boldsymbol{\omega}, \tag{5.9}$$

and the power of the inertial force and couple vectors

$$\sigma^{in} \equiv \boldsymbol{b}^{in}\cdot\boldsymbol{v} + \boldsymbol{c}^{in}\cdot\boldsymbol{\omega}. \tag{5.10}$$

Then the mechanical power (5.8) takes the more readable form

$$\begin{aligned}\mathfrak{p}(\mathcal{P},t) = &\iint_{\Pi}(-\sigma^{in} + \sigma + w)da + \int_{\partial\Pi} w_v dl \\ &- \int_{C(t)\cap\Pi}\big([\boldsymbol{N}\boldsymbol{v}\cdot\boldsymbol{v}] + [\boldsymbol{M}\boldsymbol{v}\cdot\boldsymbol{\omega}]\big)dl + \int_{\partial\Pi\cap\partial M_f}(w^* - w_v)dl.\end{aligned} \tag{5.11}$$

Let us assume that the interstitial flux w_v of the mechanical power obey the Cauchy hypothesis and hence can be expressed in terms of the interstitial power flux vector field $\boldsymbol{w}(\mathsf{x},t)$,

$$w_v(\mathrm{X},t) = \boldsymbol{w}(\mathrm{X},t) \bullet \boldsymbol{v}(\mathrm{X}). \tag{5.12}$$

Then applying the theorem (2.13) we obtain

$$\begin{aligned} \int_{\partial\Pi} w_v dl &= \int_{\partial\Pi} \boldsymbol{w} \bullet \boldsymbol{v} dl \\ &= \iint_{\Pi\setminus C(t)} Div\boldsymbol{w} da - \int_{C(t)\cap\Pi} [\![\boldsymbol{w} \bullet \boldsymbol{v}]\!] dl . \end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.11), the effective mechanical power is obtained in the form

$$\begin{aligned} \mathfrak{p}(\mathcal{P},t) = &- \iint_{\Pi} \sigma^{in} da + \iint_{\Pi} (\sigma + w + Div\boldsymbol{w}) da \\ &- \int_{C(t)\cap\Pi} \sigma_c dl + \int_{\partial\Pi\cap\partial M_f} (w^* - \boldsymbol{w} \bullet \boldsymbol{v}) dl , \end{aligned} \tag{5.14}$$

where

$$\sigma_c \equiv [\![\boldsymbol{N}\boldsymbol{v} \bullet \boldsymbol{v}]\!] + [\![\boldsymbol{M}\boldsymbol{v} \bullet \boldsymbol{\omega}]\!] + [\![\boldsymbol{w} \bullet \boldsymbol{v}]\!] \tag{5.15}$$

represents the mechanical power associated with the moving singular curve $C(t)$.

The first two terms in (5.15) may be written in the form identical to (4.35). Then upon the use of the dynamic continuity conditions (3.21) we obtain

$$\begin{aligned} \sigma_c = &- [\![(\boldsymbol{u} \bullet \boldsymbol{v})\boldsymbol{p}]\!] \bullet \langle\!\langle \boldsymbol{v} \rangle\!\rangle - [\![(\boldsymbol{u} \bullet \boldsymbol{v})(\boldsymbol{s} + \boldsymbol{y} \times \boldsymbol{p})]\!] \bullet \langle\!\langle \boldsymbol{\omega} \rangle\!\rangle \\ &+ \langle \boldsymbol{N}\boldsymbol{v} \rangle \bullet [\![\boldsymbol{v}]\!] + \langle \boldsymbol{M}\boldsymbol{v} \rangle \bullet [\![\boldsymbol{\omega}]\!] - [\![\boldsymbol{y} \times \boldsymbol{N}\boldsymbol{v}]\!] \bullet \langle\!\langle \boldsymbol{\omega} \rangle\!\rangle + [\![\boldsymbol{w} \bullet \boldsymbol{v}]\!]. \end{aligned} \tag{5.16}$$

Thus, under the assumption that the balance of linear and angular momenta holds, the effective mechanical power (5.4) takes the form (5.14).

5.2 Local laws of energy balance

With the mechanical power (5.4) and the heating (3.33), the law of energy balance reads

$$\begin{aligned} &\iint_{\Pi\setminus C(t)} \Big(- m_0 \dot{u} - \sigma^{in} + (\sigma + w + Div\boldsymbol{w}) \\ &\qquad + \{ m_0 r - Div\boldsymbol{q} - (q_+ - q_-) \} \Big) da \\ &+ \int_{C(t)\cap\Pi} \big([\![\boldsymbol{q} \bullet \boldsymbol{v}]\!] - \sigma_c - [\![(\boldsymbol{u} \bullet \boldsymbol{v}) m_0 u]\!] \big) dl \\ &+ \int_{\partial\Pi\cap\partial M_f} (w^* - \boldsymbol{w} \bullet \boldsymbol{v}) dl + \int_{\partial\Pi\cap\partial M_h} (\boldsymbol{q} \bullet \boldsymbol{v} - q^*) dl = 0. \end{aligned} \tag{5.17}$$

At points of M away of the singular curve $C(t)$, the balance of energy (5.17) yields the local equation

$$-m_0\dot{u} - \sigma^{in} + (\sigma + w + Div\boldsymbol{w}) + \{m_0 r - Div\boldsymbol{q} - (q_+ - q_-)\} = 0, \tag{5.18}$$

where the stress power $\sigma(\mathsf{X},t)$ is given by (5.9).

The energy continuity condition is

$$[\boldsymbol{q} \bullet \boldsymbol{v}] - \sigma_c - [(\boldsymbol{u} \bullet \boldsymbol{v})m_0 u] = 0 \tag{5.19}$$

at regular points of the singular curve $C(t)$ in M, with σ_c defined by (5.15).

The boundary condition for the heat flux

$$\boldsymbol{q}(\mathsf{X},t) \bullet \boldsymbol{v}(\mathsf{X}) = q^*(\mathsf{X},t) \tag{5.20}$$

should be satisfied along the part ∂M_h of the external boundary, with the complementary part being $\partial M_\theta = \partial M \setminus \partial M_h$.

Additionally, we have the boundary condition

$$\boldsymbol{w}(\mathsf{X},t) \bullet \boldsymbol{v}(\mathsf{X}) - w^*(\mathsf{X},t) = 0 \tag{5.21}$$

along the part ∂M_f of the shell boundary, where the external resultant forces and couples are prescribed in consistency with the basic laws of mechanics.

Each term in (5.18) has a definite physical meaning following from the integral statement of the energy balance. The first term represents the rate of change of the total energy, and the sum of remaining terms represents the cause for this change. The first of these terms represents the contribution to the rate of change of the total energy due to inertial force and couple vectors, and σ represents the conversion of the mechanical energy into thermal energy due to the action of contact forces and couples. The final term in the equation (5.18) represents the rate at which heat is being added by conduction and radiation from the outside environment.

It should be clear from the above steps that (5.18) is not a direct and exact implication of the integral – impulse statement (1.4) of the energy balance. It is the local form of the balance of energy under the assumption that the balance of linear and angular momenta holds, and an additional mechanical power representing the interstitial working has been taken into account. These assumptions have been used in deriving the integral form (5.11) of the mechanical power for the shell.

Chapter 6

Geometry, local deformation and strains

6.1 Geometry of the reference configuration

Within the considered formulation of the theory of shells, an arbitrarily chosen reference configuration of the shell is completely determined by the base surface M with the field $\boldsymbol{X}$ of structure tensors defined on it. The position vector $\boldsymbol{x}(\mathsf{x})$ specifies the location of each point $\mathsf{x} \in M$ relative to the origin of the frame of reference, and the unit normal vector $\boldsymbol{n}(\mathsf{x})$ determines the orientation of M at that point.

In a fixed frame of reference, the position vector $\boldsymbol{x}$ and the structure tensor $\boldsymbol{X}$ (an $E \otimes E$ –valued tensor field) determine completely the shell reference configuration. The structure tensor $\boldsymbol{X}(\mathsf{x})$ associated with every point of M is non-singular, $|\det \boldsymbol{X}(\mathsf{x})| \neq 0$. Thus the co-domain of $\boldsymbol{X}$ is an open subset $GL(E)$ of the tensor space $E \otimes E$. While M need not be smooth but merely Lipschitz continuous with almost smooth boundary ∂M, the structure tensor $\boldsymbol{X}(\mathsf{x})$ is assumed to be defined at all points $\mathsf{x} \in M$ including boundary points, except possibly the corner points of the edge set Γ.

Let γ be any curve on M given in the parametric form $\boldsymbol{x} = \boldsymbol{x}(\lambda)$, where λ stands for an arbitrary scalar parameter. Then the position vector $\boldsymbol{x} = \boldsymbol{x}(\mathsf{x}(\lambda))$, the unit normal vector $\boldsymbol{n} = \boldsymbol{n}(\mathsf{x}(\lambda))$ and the structure tensor $\boldsymbol{X} = \boldsymbol{X}(\mathsf{x}(\lambda))$ along the curve γ may be considered as functions of the parameter λ. If differentiation with regard to λ is denoted by prime, $(.)'$, infinitesimal changes of these fields along this curve are

$$d\boldsymbol{x} = \boldsymbol{x}' d\lambda = (\nabla \boldsymbol{x}) d\boldsymbol{x}, \quad d\boldsymbol{n} = \boldsymbol{n}' d\lambda = (\nabla \boldsymbol{n}) d\boldsymbol{x}, \quad d\boldsymbol{X} = \boldsymbol{X}' d\lambda = (\nabla \boldsymbol{X}) d\boldsymbol{x}, \qquad (6.1)$$

where $d\boldsymbol{x} \in T_{\mathsf{x}} M$ denotes the tangent vector to the curve γ at the point $\mathsf{x} \in M$.

The surface gradient of the position vector $\boldsymbol{x}$ defines the inclusion operator $\boldsymbol{I}(\mathsf{x})$, and the surface gradient of the unit normal vector $\boldsymbol{n}$ defines the Weingarten operator $\boldsymbol{K}(\mathsf{x})$, both operators being the linear maps of the tangent space $T_{\mathsf{x}} M$ into the 3D translation space E:

$$\nabla \boldsymbol{x} = \boldsymbol{I} = \boldsymbol{I}\mathbf{1}, \qquad \nabla \boldsymbol{n} = -\boldsymbol{K} = -\boldsymbol{I}\mathbf{K}. \tag{6.2}$$

The extra geometry of the shell reference configuration is related to the structure tensor $\boldsymbol{X}(\mathsf{x})$. Noting that $\nabla \boldsymbol{X}(\mathsf{x}) : T_{\mathsf{x}}M \to E \otimes E$, it becomes clear that $d\boldsymbol{X} = (\nabla \boldsymbol{X})d\boldsymbol{x}$ is also the $E \otimes E$-valued tensor.

Since $\boldsymbol{X}$ is non-singular, we may define an associated tensor $\bar{\boldsymbol{X}} = \bar{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$ by

$$\bar{\boldsymbol{X}} \equiv (d\boldsymbol{X})\boldsymbol{X}^{-1} = ((\nabla \boldsymbol{X})d\boldsymbol{x})\boldsymbol{X}^{-1}. \tag{6.3}$$

Let $\tilde{\boldsymbol{X}} = \tilde{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$ and $\hat{\boldsymbol{X}} = \hat{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$ be the symmetric and the skew-symmetric part of $\bar{\boldsymbol{X}}$, respectively, and let $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(\mathsf{x}; d\boldsymbol{x})$ be the axial vector of $\hat{\boldsymbol{X}}$,

$$\bar{\boldsymbol{X}} = \tilde{\boldsymbol{X}} + \hat{\boldsymbol{X}}, \qquad \tilde{\boldsymbol{X}} = \tfrac{1}{2}(\bar{\boldsymbol{X}} + \bar{\boldsymbol{X}}^T), \qquad \hat{\boldsymbol{X}} = \tfrac{1}{2}(\bar{\boldsymbol{X}} - \bar{\boldsymbol{X}}^T) = ad\,\hat{\boldsymbol{x}}. \tag{6.4}$$

All three tensors $\bar{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$, $\tilde{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$ and $\hat{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x})$ as well as the vector $\hat{\boldsymbol{x}}(\mathsf{x}; d\boldsymbol{x})$ depend linearly on the tangent vector $d\boldsymbol{x}$ for every curve passing through the point $\mathsf{x} \in M$. Hence, there exists a linear map (the third-order tensor) $\mathcal{G}(\mathsf{x}) : T_{\mathsf{x}}M \to E \otimes E$ such that

$$\tilde{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x}) = \mathcal{G}(\mathsf{x})d\boldsymbol{x}, \tag{6.5}$$

and there exists a tensor $\boldsymbol{B}(\mathsf{x}) : T_{\mathsf{x}}M \to E$ such that

$$\hat{\boldsymbol{X}}(\mathsf{x}; d\boldsymbol{x}) = ad\,\hat{\boldsymbol{x}}(\mathsf{x}; d\boldsymbol{x}) = ad(\boldsymbol{B}(\mathsf{x})d\boldsymbol{x}). \tag{6.6}$$

The tensor $\bar{\boldsymbol{X}}$ may now be written as

$$\bar{\boldsymbol{X}} = (d\boldsymbol{X})\boldsymbol{X}^{-1} = \mathcal{G}d\boldsymbol{x} + ad(\boldsymbol{B}d\boldsymbol{x}). \tag{6.7}$$

It is obvious that the so defined extrinsic curvature tensor $\boldsymbol{B}$ is independent of the curve γ. Moreover, this tensor should not be confused with the more familiar tensors $\boldsymbol{K}$ and $\mathbf{K}$.

6.2 Geometry of spatial configurations

A spatial configuration of the shell at time t is completely determined by the base surface $M(t)$ with the field $\boldsymbol{Y}$ of structure tensors defined on it. In a chosen frame of reference, the base surface $M(t)$ and its unit normal vector

$\boldsymbol{m}(t)$ are described by the position vector $\boldsymbol{y} = \boldsymbol{y}(\mathsf{y},t)$, and $\boldsymbol{Y} = \boldsymbol{Y}(\mathsf{y},t)$ is a structure tensor field on $M(t)$ with co-domain to be an open subset $GL(E)$ of the tensor space $E \otimes E$. Thus, the mappings

$$\boldsymbol{y} : M(t) \times T \to E, \quad \boldsymbol{m} : M(t) \times T \to E, \quad \boldsymbol{Y} : M(t) \times T \to E \otimes E \tag{6.8}$$

determine completely the instantaneous shell configuration, and for differentials we obtain

$$d\boldsymbol{y} = (D\boldsymbol{y})d\boldsymbol{y}, \qquad d\boldsymbol{m} = (D\boldsymbol{m})d\boldsymbol{y}, \qquad d\boldsymbol{Y} = (D\boldsymbol{Y})d\boldsymbol{y}, \tag{6.9}$$

where D denotes the spatial gradient taken at the base surface $M(t)$.

The underlying Euclidean structure of the ambient space makes it possible to decompose the vector space $E \cong T_{\mathsf{y}}\mathcal{E}$ at every point $\mathsf{y} \in M(t)$ into the direct sum of the tangent space $T_{\mathsf{y}}M(t)$ and the orthogonal complement $T^{\perp}M(t)$. Such a decomposition naturally leads to the canonical ***inclusion*** and ***projection*** operators, and to the unit tensor defined on the tangent space $T_{\mathsf{y}}M(t)$:

$$\bar{\boldsymbol{I}}(\mathsf{y},t) : T_{\mathsf{y}}M(t) \to E, \quad \bar{\boldsymbol{P}}(\mathsf{y},t) : E \to T_{\mathsf{y}}M(t), \quad \bar{\mathbf{1}}(\mathsf{y},t) : T_{\mathsf{y}}M(t) \to T_{\mathsf{y}}M(t). \tag{6.10}$$

These operators satisfy the following relations:

$$\bar{\boldsymbol{I}}^{T} = \bar{\boldsymbol{P}}, \qquad \bar{\boldsymbol{P}}\bar{\boldsymbol{I}} = \bar{\mathbf{1}}, \qquad \bar{\boldsymbol{I}}\bar{\boldsymbol{P}} = \mathbf{1} - \boldsymbol{m} \otimes \boldsymbol{m}. \tag{6.11}$$

Here $\mathbf{1} : E \to E$ denotes the usual unit tensor of the Euclidean space E, and by analogy to (6.2) we also have

$$D\boldsymbol{y} = \bar{\boldsymbol{I}} = \bar{\boldsymbol{I}}\bar{\mathbf{1}}, \qquad D\boldsymbol{m} = -\bar{\boldsymbol{K}} = -\bar{\boldsymbol{I}}\bar{\mathbf{K}}. \tag{6.12}$$

The structure tensor $\bar{\boldsymbol{Y}}$, in analogy to (6.4)-(6.7), may be written as

$$\bar{\boldsymbol{Y}} = (d\boldsymbol{Y})\boldsymbol{Y}^{-1} = ((D\boldsymbol{Y})d\boldsymbol{y})\boldsymbol{Y}^{-1}, \tag{6.13}$$

$$\bar{\boldsymbol{Y}}(\mathsf{y},t;d\boldsymbol{y}) = \tilde{\boldsymbol{Y}}(\mathsf{y},t;d\boldsymbol{y}) + \hat{\boldsymbol{Y}}(\mathsf{y},t;d\boldsymbol{y}) = \bar{\mathcal{G}}(\mathsf{y},t)d\boldsymbol{y} + ad(\bar{\boldsymbol{B}}(\mathsf{y},t)d\boldsymbol{y}). \tag{6.14}$$

6.3 Resultant stress measures and work-conjugate deformation rates

The concepts of shell deformation and strain measures, like the concept of motion, are entirely geometric and associated with the shell base surface. They are independent of physical laws which govern the motion and deformation of the shell-like body. As such, these concepts may be discussed without any

relations to mechanical and thermal effects associated with the shell. In seeking appropriate measures for shell strains and bendings within the considered shell theory, it is natural to begin by considering the effective mechanical power $\mathcal{P}_e(\mathcal{P},t)$, which in the referential description is given by (5.1). Provided that the balance laws of linear and angular momenta hold, the stress power theorem asserts that the mechanical power is given by (5.8), with the apparent stress power density $\sigma = \sigma(\mathsf{x},t)$ given by

$$\begin{aligned}\sigma &= \boldsymbol{N}\bullet\nabla\boldsymbol{v} - \{ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\}\bullet\boldsymbol{\omega} + \boldsymbol{M}\bullet\nabla\boldsymbol{\omega}\\ &= \boldsymbol{N}\bullet(\nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}) + \boldsymbol{M}\bullet\nabla\boldsymbol{\omega}.\end{aligned} \tag{6.15}$$

Noting that $\dot{\boldsymbol{F}} = \nabla\boldsymbol{v}$, we also have

$$\sigma = \boldsymbol{T}\bullet(\dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{F}) + \boldsymbol{H}\bullet\nabla\boldsymbol{\omega}. \tag{6.16}$$

Let us introduce notation $\boldsymbol{\Lambda}^\circ(\mathsf{x},t)$ and $\boldsymbol{\Theta}^\circ(\mathsf{x},t)$ for

$$\boldsymbol{\Lambda}^\circ \equiv \dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{F}, \qquad \boldsymbol{\Theta}^\circ \equiv \nabla\boldsymbol{\omega}, \tag{6.17}$$

so that the stress power per unit area of M is

$$\sigma = \boldsymbol{N}\bullet\boldsymbol{\Lambda}^\circ + \boldsymbol{M}\bullet\boldsymbol{\Theta}^\circ. \tag{6.18}$$

6.4 Local deformation of the shell

In terms of the position vectors we obtain

$$\begin{aligned}d\boldsymbol{y} &= (D\boldsymbol{y})d\boldsymbol{y} = \bar{\boldsymbol{I}}d\boldsymbol{y}\\ &= (\nabla\boldsymbol{y})d\boldsymbol{x} = \boldsymbol{F}d\boldsymbol{x} = \bar{\boldsymbol{I}}\mathbf{F}d\boldsymbol{x},\end{aligned} \tag{6.19}$$

where $\boldsymbol{F}(\mathsf{x},t) = \nabla\boldsymbol{y}(\mathsf{x},t)$ denotes the surface deformation gradient and $\mathbf{F}(\mathsf{x},t)$ is the tangential surface deformation gradient. These two tensors are related by

$$\boldsymbol{F} = \bar{\boldsymbol{I}}\mathbf{F}, \qquad \mathbf{F} = \bar{\boldsymbol{P}}\boldsymbol{F}. \tag{6.20}$$

Taking into account that $d\boldsymbol{y} = \mathbf{F}d\boldsymbol{x}$ and $\nabla\boldsymbol{Y} = (D\boldsymbol{Y})\mathbf{F}$, by the chain rule we have

$$d\boldsymbol{Y} = (D\boldsymbol{Y})d\boldsymbol{y} = (D\boldsymbol{Y})\mathbf{F}d\boldsymbol{x} = (\nabla\boldsymbol{Y})d\boldsymbol{x}. \tag{6.21}$$

The tensor defined by (6.13) may be written as

$$\bar{\boldsymbol{Y}} = (d\boldsymbol{Y})\boldsymbol{Y}^{-1} = ((D\boldsymbol{Y})d\boldsymbol{y})\boldsymbol{Y}^{-1} = ((D\boldsymbol{Y})\mathbf{F}d\boldsymbol{x})\boldsymbol{Y}^{-1} = ((\nabla\boldsymbol{Y})d\boldsymbol{x})\boldsymbol{Y}^{-1}. \tag{6.22}$$

On the other hand, for the structure tensor $\boldsymbol{Y}$ we can write

$$\boldsymbol{Y} = \boldsymbol{Q}\boldsymbol{X}, \qquad \boldsymbol{Y}^{-1} = \boldsymbol{X}^{-1}\boldsymbol{Q}^T. \tag{6.23}$$

Hence

$$d\boldsymbol{Y} = ((\nabla\boldsymbol{Q})d\boldsymbol{x})\boldsymbol{X} + \boldsymbol{Q}((\nabla\boldsymbol{X})d\boldsymbol{x}), \tag{6.24}$$

$$\begin{aligned}\bar{\boldsymbol{Y}} &= \{((\nabla\boldsymbol{Q})d\boldsymbol{x})\boldsymbol{X} + \boldsymbol{Q}((\nabla\boldsymbol{X})d\boldsymbol{x})\}\boldsymbol{X}^{-1}\boldsymbol{Q}^T \\ &= \{((\nabla\boldsymbol{Q})d\boldsymbol{x})\boldsymbol{X}\boldsymbol{X}^{-1}\boldsymbol{Q}^T + \boldsymbol{Q}((\nabla\boldsymbol{X})d\boldsymbol{x})\boldsymbol{X}^{-1}\boldsymbol{Q}^T\},\end{aligned} \tag{6.25}$$

$$\bar{\boldsymbol{Y}} = ((\nabla\boldsymbol{Q})d\boldsymbol{x})\boldsymbol{Q}^T + \boldsymbol{Q}\bar{\boldsymbol{X}}\boldsymbol{Q}^T. \tag{6.26}$$

With some transformations we also obtain the relations

$$((\nabla\boldsymbol{Q})d\boldsymbol{x})\boldsymbol{Q}^T = ad\,\theta(\mathrm{x},t;d\boldsymbol{x}) = ad(\boldsymbol{\Theta}(\mathrm{x},t)d\boldsymbol{x}), \tag{6.27}$$

$$\bar{\boldsymbol{Y}} = ad(\boldsymbol{\Theta}d\boldsymbol{x}) + \boldsymbol{Q}\bar{\boldsymbol{X}}\boldsymbol{Q}^T, \tag{6.28}$$

$$\bar{\boldsymbol{Y}} = \bar{\mathcal{G}}\,d\boldsymbol{y} + ad(\bar{\boldsymbol{B}}d\boldsymbol{y}) = \bar{\mathcal{G}}\mathbf{F}d\boldsymbol{x} + ad(\bar{\boldsymbol{B}}\mathbf{F}d\boldsymbol{x}), \tag{6.29}$$

$$\bar{\mathcal{G}}\mathbf{F}d\boldsymbol{x} + ad(\bar{\boldsymbol{B}}\mathbf{F}d\boldsymbol{x}) = \boldsymbol{Q}\{\mathcal{G}d\boldsymbol{x} + ad(\boldsymbol{B}d\boldsymbol{x})\}\boldsymbol{Q}^T + ad(\boldsymbol{\Theta}d\boldsymbol{x}), \tag{6.30}$$

$$\bar{\mathcal{G}}\mathbf{F}d\boldsymbol{x} = \boldsymbol{Q}\{\mathcal{G}d\boldsymbol{x}\}\boldsymbol{Q}^T, \qquad ad(\bar{\boldsymbol{B}}\mathbf{F}d\boldsymbol{x}) = \boldsymbol{Q}\{ad(\boldsymbol{B}d\boldsymbol{x})\}\boldsymbol{Q}^T + ad(\boldsymbol{\Theta}d\boldsymbol{x}), \tag{6.31}$$

$$\boldsymbol{Q}\{ad(\boldsymbol{B}d\boldsymbol{x})\}\boldsymbol{Q}^T = ad(\boldsymbol{Q}\boldsymbol{B}d\boldsymbol{x})\}. \tag{6.32}$$

Therefore

$$\bar{\boldsymbol{B}}\mathbf{F} = \boldsymbol{Q}\boldsymbol{B} + \boldsymbol{\Theta}. \tag{6.33}$$

6.5 Natural shell strain measures

The natural strain tensor $\boldsymbol{\Lambda}(\mathrm{x},t)$ and the bending tensor $\boldsymbol{\Theta}(\mathrm{x},t)$ for shells are defined by

$$\boldsymbol{\Lambda} = \boldsymbol{F} - \boldsymbol{Q}\boldsymbol{I} = \bar{\boldsymbol{I}}\mathbf{F} - \boldsymbol{Q}\boldsymbol{I}, \qquad \boldsymbol{\Theta} = \bar{\boldsymbol{B}}\mathbf{F} - \boldsymbol{Q}\boldsymbol{B}, \tag{6.34}$$

so that the strain rates (6.17) can be given as co-rotational rates of the strain measures

$$\boldsymbol{\Lambda}^{\circ} = \boldsymbol{Q}(\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Lambda})), \qquad \boldsymbol{\Theta}^{\circ} = \boldsymbol{Q}(\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Theta})). \tag{6.35}$$

Let us introduce another shell strain measures $\boldsymbol{E}(\mathrm{x},t)$ and $\boldsymbol{K}(\mathrm{x},t)$ by

$$\boldsymbol{E} = \boldsymbol{Q}^T\boldsymbol{\Lambda}, \qquad \boldsymbol{K} = \boldsymbol{Q}^T\boldsymbol{\Theta}. \tag{6.36}$$

A simple calculation shows that

$$\begin{aligned} \boldsymbol{Q}^T\boldsymbol{\Lambda}^{\circ} &= \frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Lambda}) = \frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{E}) = \dot{\boldsymbol{E}}, \\ \boldsymbol{Q}^T\boldsymbol{\Theta}^{\circ} &= \frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Theta}) = \frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{Q}\boldsymbol{K}) = \dot{\boldsymbol{K}}. \end{aligned} \tag{6.37}$$

Hence, we obtain the transformation rules

$$\boldsymbol{\Lambda} = \boldsymbol{Q}\boldsymbol{E}, \qquad \boldsymbol{\Theta} = \boldsymbol{Q}\boldsymbol{K}, \qquad \boldsymbol{\Lambda}^{\circ} = \boldsymbol{Q}\dot{\boldsymbol{E}}, \qquad \boldsymbol{\Theta}^{\circ} = \boldsymbol{Q}\dot{\boldsymbol{K}}. \tag{6.38}$$

Substituting (6.38) into (6.18), the stress power density is obtained in the form

$$\sigma = \boldsymbol{N} \bullet \boldsymbol{Q}\dot{\boldsymbol{E}} + \boldsymbol{M} \bullet \boldsymbol{Q}\dot{\boldsymbol{K}} = \boldsymbol{Q}^T\boldsymbol{N} \bullet \dot{\boldsymbol{E}} + \boldsymbol{Q}^T\boldsymbol{M} \bullet \dot{\boldsymbol{K}}. \tag{6.39}$$

The form (6.39) suggests that there is a merit to introduce the new surface stress and couple tensors $\boldsymbol{S}(\mathrm{x},t)$ and $\boldsymbol{\Sigma}(\mathrm{x},t)$ by

$$\boldsymbol{N} = \boldsymbol{Q}\boldsymbol{S}, \qquad \boldsymbol{M} = \boldsymbol{Q}\boldsymbol{\Sigma}. \tag{6.40}$$

Then from (6.39) we obtain

$$m_0\sigma = \boldsymbol{N} \bullet \boldsymbol{\Lambda}^{\circ} + \boldsymbol{M} \bullet \boldsymbol{\Theta}^{\circ} = \boldsymbol{S} \bullet \dot{\boldsymbol{E}} + \boldsymbol{\Sigma} \bullet \dot{\boldsymbol{K}}. \tag{6.41}$$

The tensors $\boldsymbol{E}$ and $\boldsymbol{K}$ can be directly defined as

$$\boldsymbol{E} = \boldsymbol{Q}^T\boldsymbol{F} - \boldsymbol{I} = \boldsymbol{Q}^T\bar{\boldsymbol{I}}\mathbf{F} - \boldsymbol{I}, \qquad \boldsymbol{K} = \boldsymbol{Q}^T\bar{\boldsymbol{B}}\mathbf{F} - \boldsymbol{B}. \tag{6.42}$$

Please note that the material time derivative of the strain tensor $\boldsymbol{\Lambda}$ is

$$\dot{\boldsymbol{\Lambda}} = \dot{\boldsymbol{F}} - \dot{\boldsymbol{Q}}\boldsymbol{I} = \dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{Q}\boldsymbol{I} = \dot{\boldsymbol{F}} - \boldsymbol{\Omega}(\boldsymbol{F} - \boldsymbol{\Lambda}) = \dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{F} + \boldsymbol{\Omega}\boldsymbol{\Lambda}, \tag{6.43}$$

so that

$$\dot{\boldsymbol{F}} - \boldsymbol{\Omega}\boldsymbol{F} = \dot{\boldsymbol{\Lambda}} - \boldsymbol{\Omega}\boldsymbol{\Lambda}. \tag{6.44}$$

We can also take spatial gradients of the velocity fields

$$\boldsymbol{v} = \boldsymbol{Q}\boldsymbol{v}, \qquad \boldsymbol{\omega} = \boldsymbol{Q}\boldsymbol{w}, \tag{6.45}$$

which leads to

$$\nabla(\boldsymbol{Q}\boldsymbol{v}) = \boldsymbol{Q}(\nabla\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v}), \qquad \nabla(\boldsymbol{Q}\boldsymbol{w}) = \boldsymbol{Q}(\nabla\boldsymbol{w} + \boldsymbol{K}\boldsymbol{w}). \tag{6.46}$$

Then

$$\begin{aligned} \sigma &= \boldsymbol{N} \bullet (\nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}) + \boldsymbol{M} \bullet \nabla\boldsymbol{\omega} \\ &= \boldsymbol{N} \bullet \{\boldsymbol{Q}(\nabla\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v}) - \boldsymbol{Q}\boldsymbol{W}\boldsymbol{F}\} + \boldsymbol{M} \bullet \boldsymbol{Q}(\nabla\boldsymbol{w} + \boldsymbol{K}\boldsymbol{w}) \\ &= \boldsymbol{Q}^T\boldsymbol{N} \bullet (\nabla\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}) + \boldsymbol{Q}^T\boldsymbol{M} \bullet (\nabla\boldsymbol{w} + \boldsymbol{K}\boldsymbol{w}), \end{aligned} \tag{6.47}$$

$$\begin{aligned} \sigma &= \boldsymbol{N} \bullet (\nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}) + \boldsymbol{M} \bullet \nabla\boldsymbol{\omega} \\ &= \boldsymbol{S} \bullet (\nabla\boldsymbol{v} + \boldsymbol{K}\boldsymbol{v} - \boldsymbol{W}\boldsymbol{F}) + \boldsymbol{\Sigma} \bullet (\nabla\boldsymbol{w} + \boldsymbol{K}\boldsymbol{w}). \end{aligned} \tag{6.48}$$

It follows from (6.48), (6.18) and (6.39) that the pairs $\boldsymbol{N},\boldsymbol{\Lambda}$ and $\boldsymbol{M},\boldsymbol{\Theta}$ as well as $\boldsymbol{S},\boldsymbol{E}$ and $\boldsymbol{\Sigma},\boldsymbol{K}$ form two sets of work-conjugate shell stress and strain measures.

Chapter 7
Constitutive equations and assumptions

7.1 Introductory remarks

The aim of this and two remaining Chapters is to complete the results obtained in the previous Chapters by discussing the general structure of appropriate constitutive equations in shells and pointing out some particular forms of the equations. While the general theory of constitutive equations for thermomechanics of the Cauchy continuum is well developed, no such a theory exists for the thermomechanics of shells developed here. Apart of several usual field variables appearing in the shell local relations and requiring a constitutive description, there are also several non-classical terms appearing in the 2D balance law of energy and in the entropy inequality. The role assigned to these terms in shell theory may also be regarded as a part of the constitutive problem itself, with several possibilities of how to treat them.

Contrary to the general formulation of shell thermomechanics presented in previous Chapters, in what follows we restrict our considerations to only regular shell base surfaces, and no kind of singular curves for all the fields is admitted as well. This allows us to concentrate here only on those problems of the constitutive theory which are of primary importance for regular shell parts. Additional constitutive relations associated with irregularities of the shell base surface as well as the ones generated by moving singular curves are not fully developed yet and should be the subject of additional research.

Specific forms of the constitutive equations can be established by two different approaches. The first *direct* one consists of developing, for a restricted class of thin shell-like bodies, a general structure of the constitutive equations satisfying some number of reasonable assumptions. Then we should devise a suitable set of physical experiments from which we can directly establish the appropriate constants or functions entering the constitutive equations. The second approach is based on devising suitable mathematical methods which would allow us to deduce the 2D constitutive equations for shells as an exact, or asymptotic, or otherwise rationally approximate consequence of a given set of corresponding 3D constitutive equations. Whichever of the two approaches

is applied, the 2D constitutive equations must be consistent with the general structure of shell thermomechanics developed here.

In this report we follow the direct approach. We believe that the general formal structure of 2D constitutive equations should be studied first, before any effort is undertaken to formulate specific forms of constitutive equations valid for particular classes of shell problems.

7.2 Referential description of motion

Let us briefly remind that the governing shell relations may be formulated in the spatial or in the referential descriptions. In this report the referential description has been chosen as the basic one. Any shell configuration is completely determined by the position of the shell base surface and by the field of structure tensors. In the chosen reference configuration the base surface M serves to identify shell material particles. The position vector of $\mathsf{x} \in M$ relative to the origin of a fixed frame of reference is denoted by $\boldsymbol{x}(\mathsf{x})$. The structure tensor at the same place is the non-singular second-order tensor $\boldsymbol{X}(\mathsf{x})$. Thus, the reference configuration of the shell is described by two fields given on M,

$$\boldsymbol{x} : M \to E, \qquad \boldsymbol{X} : M \to E \otimes E, \tag{7.1}$$

where E denotes the usual three-dimensional Euclidean vector space (the translational space of the physical point space $\mathcal{E}$). By an additional assumption we require that $\det \boldsymbol{X}(\mathsf{x}) > 0$.

The spatial configuration of the shell at current time t is determined by the base surface $M(t)$ having $\boldsymbol{y}(\mathsf{y},t)$ as the position vector of the place $\mathsf{y} \in M(t)$ instantaneously occupied by a typical shell particle, and $\boldsymbol{Y}(\mathsf{y},t)$ is the structure tensor associated with this place. The motion of the base surface relative to the reference configuration is described by a time dependent mapping $\chi : M \times T \to \mathcal{E}$. Thus, $\mathsf{y} = \chi(\mathsf{x},t)$ is the spatial place of the shell particle whose referential place was $\mathsf{x} \in M$. Accordingly, $\boldsymbol{y}(\chi(\mathsf{x},t),t)$ and $\boldsymbol{Y}(\chi(\mathsf{x},t),t)$ may be considered as time-dependent fields on M,

$$\boldsymbol{y} : M \times T \to E, \qquad \boldsymbol{Y} : M \times T \to E \otimes E. \tag{7.2}$$

The first mapping in (7.2) describes the motion of the base surface throughout the space, and the second one in (7.2) describes an independent time change of the structure tensor. While the structure tensor $\boldsymbol{X}$ is any tensor with positive determinant, the structure tensor $\boldsymbol{Y}$ in the current configuration is

not another arbitrary tensor, but is related to $\boldsymbol{X}$ through the rotation tensor $\boldsymbol{Q}$. In effect, motion of the shell may equivalently be described by the displacement vector $\boldsymbol{u}(\mathsf{x},t)$ and the rotation tensor $\boldsymbol{Q}(\mathsf{x},t)$ fields,

$$\boldsymbol{y}(\mathsf{x},t) = \boldsymbol{x}(\mathsf{x}) + \boldsymbol{u}(\mathsf{x},t), \qquad \boldsymbol{Y}(\mathsf{x},t) = \boldsymbol{Q}(\mathsf{x},t)\boldsymbol{X}(\mathsf{x}). \tag{7.3}$$

With the motion described by (7.3), the linear velocity $\boldsymbol{v}(\mathsf{x},t)$ and the angular velocity $\boldsymbol{\omega}(\mathsf{x},t)$ of the shell are defined by

$$\boldsymbol{v} = \dot{\boldsymbol{y}} = \dot{\boldsymbol{u}}, \qquad \boldsymbol{\Omega} = \dot{\boldsymbol{Y}}\boldsymbol{Y}^{-1} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^{T} = ad\boldsymbol{\omega}. \tag{7.4}$$

Here and throughout this work, the superimposed dot denotes time derivative keeping x fixed. Moreover, the surface gradient and divergence operators on M will be denoted by ∇ and Div, respectively. Definitions of surface differential operators and various differential identities may be found in GURTIN AND COHEN [1975], MURDOCH [1990], and MAKOWSKI AND STUMPF [1994].

7.3 Summary of local laws of shell thermomechanics

In the referential description, the complete set of ***local laws of thermomechanics*** for shells consists of the following equations and one inequality (see Chapter 3):

balance of mass

$$\dot{m}_0 - c_0 = 0, \tag{7.5}$$

balance of linear momentum

$$Div\boldsymbol{N} + \boldsymbol{b} = \dot{\boldsymbol{p}}, \tag{7.6}$$

balance of angular momentum

$$Div\boldsymbol{M} + ad^{-1}(\boldsymbol{N}\boldsymbol{F}^{T} - \boldsymbol{F}\boldsymbol{N}^{T}) + \boldsymbol{c} = \dot{\boldsymbol{s}} + \boldsymbol{v} \times \boldsymbol{p}, \tag{7.7}$$

balance of energy

$$-m_0\dot{u} - \sigma^{in} + \sigma + (w + Div\boldsymbol{w}) + \{m_0 r - Div\boldsymbol{q} - (q_+ - q_-)\} = 0, \tag{7.8}$$

entropy inequality

$$m_0\gamma \equiv m_0\dot{\eta} - \{m_0 s - Div\boldsymbol{j} - (j_+ - j_-)\} \geq 0. \tag{7.9}$$

All the surface fields appearing in (7.5)-(7.9) are functions of $\mathsf{x} \in M$ and $t \in T$, and have the following meaning:

$m_0(\mathsf{x},t)$ – (surface) mass density,
$c_0(\mathsf{x},t)$ – (surface) mass production,
$\boldsymbol{p}(\mathsf{x},t)$ – (surface) linear momentum vector,
$\boldsymbol{s}(\mathsf{x},t)$ – (surface) angular momentum vector,
$\boldsymbol{b}(\mathsf{x},t)$ – surface force vector,
$\boldsymbol{c}(\mathsf{x},t)$ – surface couple vector,
$\boldsymbol{N}(\mathsf{x},t)$ – (surface) stress tensor (first Piola-Kirchhoff type),
$\boldsymbol{M}(\mathsf{x},t)$ – (surface) couple tensor (first Piola-Kirchhoff type),
$u(\mathsf{x},t)$ – (surface) specific total energy density,
$r(\mathsf{x},t)$ – (surface) heat supply,
$q_{\pm}(\mathsf{x},t)$ – heat influxes on shell faces,
$\boldsymbol{q}(\mathsf{x},t)$ – (surface) heat influx vector,
$\eta(\mathsf{x},t)$ – (surface) specific entropy,
$s(\mathsf{x},t)$ – (surface) specific entropy supply,
$j_{\pm}(\mathsf{x},t)$ – entropy influxes on shell faces,
$\boldsymbol{j}(\mathsf{x},t)$ – (surface) entropy influx vector.

It should be noted that: m_0, c_0, $q_{\pm}$ and $j_{\pm}$ are scalar fields measured per unit area of M; u, r, η and s are scalar fields measured per unit surface mass; $\boldsymbol{p}$, $\boldsymbol{s}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ are spatial (E-valued) vector fields measured per unit area of M; $\boldsymbol{N}$ and $\boldsymbol{M}$ are mixed tensor fields (linear maps of the tangent space $T_{\mathsf{x}}M$ into E); $\boldsymbol{q}$ and $\boldsymbol{j}$ are tangential vector fields on M (elements of the tangent space $T_{\mathsf{x}}M$). Moreover, $\boldsymbol{F}(\mathsf{x},t) = \nabla \boldsymbol{y}(\mathsf{x},t)$ denotes the surface deformation gradient (the linear map of $T_{\mathsf{x}}M$ into E),

$$\sigma^{in} = \boldsymbol{b}^{in} \bullet \boldsymbol{v} + \boldsymbol{c}^{in} \bullet \boldsymbol{\omega} = -\{\dot{\boldsymbol{p}} \bullet \boldsymbol{v} + (\dot{\boldsymbol{s}} + \boldsymbol{v} \times \boldsymbol{p}) \bullet \boldsymbol{\omega}\} \tag{7.10}$$

is the power of the inertial surface forces and couples, and

$$\sigma = \boldsymbol{N} \bullet \nabla \boldsymbol{v} - \{ad^{-1}(\boldsymbol{N}\boldsymbol{F}^T - \boldsymbol{F}\boldsymbol{N}^T)\} \bullet \boldsymbol{\omega} + \boldsymbol{M} \bullet \nabla \boldsymbol{\omega} \tag{7.11}$$

is the effective stress power density. The remaining two fields in the equation of energy balance (7.8), the scalar field $w(\mathsf{x},t)$ and the tangential vector field

$\boldsymbol{w}(\mathrm{X},t)$, represent the difference between the "true" and effective mechanical power and are associated with the interstitial working.

If there is no mass production, then $c_0 \equiv 0$, and the referential mass density is time-independent, $m_0 = m_0(\mathrm{X})$. Then the balance law of mass is satisfied identically. This is a typical situation in continuum mechanics, but there are certain problems in the theory of materially inhomogeneous continua and in the theory of shells where the conservation of mass may not be satisfied identically. In the following considerations, the law of balance of mass plays no essential role and no assumption in this respect is needed here.

7.4 Roles of various thermomechanical field variables

The local laws of thermomechanics (7.5)-(7.9) still do not provide the complete formulation of the thermomechanical theory of shells, because the inequality expressing the entropy growth at regular points of M is not related to the remaining field equations. It is then necessary to take additional assumptions which would make it possible to relate the entropy supply s and the entropy influx $\boldsymbol{j}$ to the heat supply r and the heat influx $\boldsymbol{q}$. Following derivation of the entropy inequality (7.9) from three-dimensional laws of continuum thermomechanics, we can conclude that the entropy influxes $j_{\pm}$ and the heat influxes $q_{\pm}$ on the shell faces are related by

$$j_+ = (\theta_+)^{-1} q_+, \qquad j_- = (\theta_-)^{-1} q_-. \tag{7.12}$$

Here $\theta_{\pm}(\mathrm{X},t) > 0$ are the fields of absolute temperature assigned on the shell faces.

In general, the entropy supply s and the entropy influx vector $\boldsymbol{j}$ may be assumed in the form

$$s = \theta^{-1} r + \lambda, \qquad \boldsymbol{j} = \theta^{-1} \boldsymbol{q} + \boldsymbol{\eta}, \tag{7.13}$$

where $\theta(\mathrm{X},t) > 0$ denotes the absolute temperature at the base surface M. The two new field variables, $\lambda(\mathrm{X},t)$ and $\boldsymbol{\eta}(\mathrm{X},t)$, which may be called the extra entropy source and the extra entropy flux, respectively, result from impossibility to reduce *exactly* the Clausius-Duhem inequality written for the 3D shell-like body to a two-dimensional form represented on the base surface. Both $\lambda(\mathrm{X},t)$ and $\boldsymbol{\eta}(\mathrm{X},t)$ vanish if the temperature is constant across the shell thickness, in which case $\theta = \theta_+ = \theta_-$. In the general case, these two fields

should be taken into account and their role in the thermomechanical theory of shells must be specified by additional assumptions.

The role of two fields w and $\boldsymbol{w}$ appearing in the balance of energy (7.8) is also unspecified yet. These two variables have their counterparts neither in the classical theories of continuum thermomechanics, nor in the known formulations of the theory of shells. Any meaning which may be assigned to these fields is actually a part of formulation of the shell theory itself.

Let us note that with (7.12) and (7.13) the complete collection of field variables appearing in the local laws of shell mechanics and thermodynamics consists of

$$(\boldsymbol{y},\boldsymbol{Y},\boldsymbol{p},\boldsymbol{m},\theta,\theta_+,\theta_-,m_0,\boldsymbol{b},\boldsymbol{c},\boldsymbol{N},\boldsymbol{M},u,w,r,q_+,q_-,\boldsymbol{q},\eta,\lambda,\boldsymbol{\eta}), \tag{7.14}$$

provided that $c_0 = 0$. This collection of fields may be split into three groups according to the role they should play in shell theory.

In most problems of shell theory, the fields

$$(m_0,\boldsymbol{b},\boldsymbol{c},r) \tag{7.15}$$

may be assumed to have been specified as part of the problem data.

The fields

$$(\boldsymbol{y},\boldsymbol{Y},\theta) \tag{7.16}$$

constitute the basic thermo-kinematic independent variables of the shell problem, which must be determined as solutions of the initial-boundary value problem for given initial and boundary conditions. Let us note that according to (7.3) the collection (7.16) consists of seven scalar fields: three components of the displacement vector $\boldsymbol{u}$, three independent parameters needed to specify the rotation tensor $\boldsymbol{Q}$, and the absolute temperature θ. That only seven scalar fields can be taken as independent variables of the general shell thermomechanics follows from the fact that there are only seven scalar field equations to determine them: three scalar equations (7.6) representing the balance of forces, three scalar equations (7.7) representing the balance of moments, and one scalar equation (7.8) representing the balance of energy.

With the above in mind, it may be assumed that the fields

$$(\boldsymbol{p},\boldsymbol{m},\boldsymbol{N},\boldsymbol{M},u,\boldsymbol{q},\eta) \tag{7.17}$$

are the basic thermomechanical variables of shell theory, which have to be specified by additional constitutive equations. In effect, the remaining in (7.14) field variables

$$(\theta_+, \theta_-, w, \boldsymbol{w}, q_+, q_-, \lambda, \boldsymbol{\eta}) \tag{7.18}$$

are the supplementary variables, and their role may vary from one particular formulation of shell theory to another.

7.5 General theory of constitutive equations

Within the continuum thermomechanics discussed in TRUESDELL AND NOLL [1965] and WANG AND TRUESDELL [1973], the general theory of the constitutive equations is based on certain assumptions, usually called principles, which are believed to be physically reasonable for all kinds of materials encountered in reality. The most important of them are:

(A) principle of determinism,
(B) principle of material frame-indifference,
(C) principle of local action,
(D) principle of equipresence.

The general procedure of using the balance laws and the entropy inequality in the theory of 3D constitutive equations is based on the following assumptions:

a) Field equations implied by the balance laws of linear momentum and energy are assumed to hold for an arbitrary choice of thermo-kinematic field variables (motion-temperature pair in the classical theories) including, if required, an arbitrary choice of space and time derivatives of these functions.
b) Fields of the stress tensor, the internal energy (or the free energy), the entropy and the heat flux vector are assumed to be specified by constitutive equations.
c) Values of the body force and of the heat source are calculated from the balance of linear momentum and the balance of energy.
d) Equations resulting from the angular momentum balance and from the entropy inequality are regarded as identities for all constitutive equations.

This procedure imposes certain restrictions on possible forms of the constitutive relations. The advances in continuum thermomechanics made over the last decades have been based primarily on the development of various techniques for effective exploitation of the procedure.

While the general meaning of the constitutive equations in the theory of shells differs substantially from their role in continuum mechanics of three-

dimensional bodies, we assume here that the principles (A)-(D) as well as the assumptions (a)-(d) remain applicable also to shell theory. The only exception is that the balance law of angular momentum can no longer be regarded as identity for the shell constitutive equations; in shell thermomechanics it gives additional field equations for determination of independent thermo-kinematic field variables.

7.6 Functional form of constitutive equations

Let X be a point or a vector space. Given a function

$$\varphi : M \times T \to X, \qquad (\mathsf{x}, t) \to \varphi(\mathsf{x}, t), \tag{7.19}$$

the ***history*** $\varphi^{(t)}$ of φ up to present time t is defined by

$$\varphi^{(t)}(\mathsf{x}, s) \equiv \varphi(\mathsf{x}, t - s), \qquad s \in [0, +\infty). \tag{7.20}$$

It follows that $\varphi^{(t)}(\mathsf{x}, 0) \equiv \varphi(\mathsf{x}, t)$ for all $\mathsf{x} \in M$.

If the function (7.19) is differentiable with respect to both arguments, then

$$\nabla \varphi^{(t)}(\mathsf{x}, s) = \nabla \varphi(\mathsf{x}, t - s). \tag{7.21}$$

Let the fields

$$\boldsymbol{y} = \boldsymbol{y}(\mathsf{x}, t), \qquad \boldsymbol{Y} = \boldsymbol{Y}(\mathsf{x}, t), \qquad \theta = \theta(\mathsf{x}, t), \tag{7.22}$$

be collective denoted by

$$\mathscr{e}(\mathsf{x}, t) \equiv (\boldsymbol{y}(\mathsf{x}, t), \boldsymbol{Y}(\mathsf{x}, t), \theta(\mathsf{x}, t)) \tag{7.23}$$

as the independent thermo-kinematic variables of shell theory. Then, according to the principle of determinism, a thermomechanical response of a shell particle is mathematically defined by the constitutive relations

$$\begin{aligned}
\boldsymbol{N}(\mathsf{x}, t) &= \boldsymbol{N}^{\infty}_{s=0}(\mathscr{e}^{(t)}(\mathsf{z}, s); \mathsf{z}, \boldsymbol{X}(\mathsf{z})), \\
\boldsymbol{M}(\mathsf{x}, t) &= \boldsymbol{M}^{\infty}_{s=0}(\mathscr{e}^{(t)}(\mathsf{z}, s); \mathsf{z}, \boldsymbol{X}(\mathsf{z})), \\
u(\mathsf{x}, t) &= u^{\infty}_{s=0}(\mathscr{e}^{(t)}(\mathsf{z}, s); \mathsf{z}, \boldsymbol{X}(\mathsf{z})), \\
\eta(\mathsf{x}, t) &= \eta^{\infty}_{s=0}(\mathscr{e}^{(t)}(\mathsf{z}, s); \mathsf{z}, \boldsymbol{X}(\mathsf{z})), \\
\boldsymbol{q}(\mathsf{x}, t) &= \boldsymbol{q}^{\infty}_{s=0}(\mathscr{e}^{(t)}(\mathsf{z}, s); \mathsf{z}, \boldsymbol{X}(\mathsf{z})).
\end{aligned} \tag{7.24}$$

Here $\mathsf{x} \in M$ denotes the place occupied by the shell particle for which the constitutive equations are specified, and $\mathsf{z} \in M$ denotes any other place in the

reference configuration. Moreover, $\boldsymbol{N}^{\infty}_{s=0}$, $\boldsymbol{M}^{\infty}_{s=0}$, $u^{\infty}_{s=0}$, $\eta^{\infty}_{s=0}$ and $\boldsymbol{q}^{\infty}_{s=0}$ are given ***response functionals*** of the history of motion and temperature in the entire shell.

The explicit dependence of the response functionals on Z and $\boldsymbol{X}(\mathsf{Z})$ indicates that the thermomechanical properties described by the constitutive relations (7.24) may vary with shell particles, thus allowing for material inhomogeneity. The explicit dependence of the response functionals on the structure tensor $\boldsymbol{X}$ indicates that the response of the shell particle may also depend on the geometry of its reference configuration. It must also be noted that the form of the response functionals in (7.24) depends on the choice of the reference configuration M of the shell base surface as well.

As was already noted, the general theory of constitutive equations for shells differs in several aspects from the theory of constitutive equations developed in continuum thermomechanics. In particular, in the theory of shells the constitutive equations are also needed for the surface linear momentum and angular momentum vector fields. For the time being, we can only write formally here that for these fields we should provide the constitutive relations in the similar form as in (7.24),

$$\begin{aligned} \boldsymbol{p}(\mathsf{X},t) &= \boldsymbol{p}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{Z},s);\mathsf{Z},\boldsymbol{X}(\mathsf{Z})), \\ \boldsymbol{s}(\mathsf{X},t) &= \boldsymbol{s}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{Z},s);\mathsf{Z},\boldsymbol{X}(\mathsf{Z})). \end{aligned} \tag{7.25}$$

7.7 Additional constitutive assumptions

In general, in shell theory developed here it is necessary to make more specific assumptions about the constitutive nature of the heat influxes q_+, q_- and the temperature fields θ, θ_+ and θ_-. There are two possibilities: either the fields θ, θ_+ and θ_- are independent of each other, or there exist relationships between them. In the former case, the constitutive equations provided for q_+ and q_- must depend on all the three temperature fields as well as on their spatial gradients and time derivatives of any order. In the latter case, only the surface temperature θ (and its spatial gradients and time derivatives) may be taken as the independent constitutive field variable.

Similarly, there are two possibilities how to treat the field variables $(w,\boldsymbol{w},\lambda,\boldsymbol{\eta})$: either the fields are independent of each other, or there exist relationships between them. If they are independent of each other, additional

relations and constitutive equations are required in order to complete the shell theory. Such additional relations and constitutive equations are needed even if only certain relationships exist between them. Let us note, however, that the fields $(w,\boldsymbol{w},\lambda,\boldsymbol{\eta})$ appear in our theory of shells as a consequence of the exact representation of the laws of continuum thermomechanics at the shell base surface. This suggests some possibilities how to treat these fields.

In certain shell problems it may be assumed that contribution of the term $w+Div\boldsymbol{w}$ to the balance of energy is negligibly small indeed, and hence this term may be omitted in the equation (7.8). This would reduce our balance of energy to the one considered by Simmonds and others (see SIMMONDS [1984, 2001] and reference cited therein).

Similarly, if it may be assumed that λ and $\boldsymbol{\eta}$ give only a small contribution to the entropy growth, the relations (7.13) can be reduced to the form identical with the corresponding relations in continuum thermomechanics of three-dimensional bodies following from the Clausius-Duhem inequality. As have already been noted earlier, this happens for example if the temperature is constant through the shell thickness. Still other possibilities exist. For example, it may happen or it may be assumed that the collective contribution of all four field variables $(w,\boldsymbol{w},\lambda,\boldsymbol{\eta})$ to the respective equations of thermomechanics are balanced by approximations made in other constitutive equations.

The above considerations indicate only some of approximations which may be made in the constitutive description leading to special shell theories. However, before such approximations are introduced into the general theory, it is desirable to study first the restrictions imposed on the form of constitutive equations by the principle of entropy inequality (7.9), with which all special shell theories must be consistent.

Chapter 8
Thermodynamically consistent constitutive equations

8.1 Reduced dissipation inequality

According to (7.9), the rate of entropy production is never negative, $\gamma \geq 0$. Accepting (7.12) and (7.13) as a part of constitutive characterization of the shell, γ takes the form

$$m_0\gamma = m_0\dot{\eta} - m_0(\theta^{-1}r + \lambda) + Div(\theta^{-1}\boldsymbol{q} + \boldsymbol{\eta}) + \{(\theta_+)^{-1}q_+ - (\theta_-)^{-1}q_-\}. \tag{8.1}$$

The use of differential identity

$$\begin{aligned} Div(\theta^{-1}\boldsymbol{q} + \boldsymbol{\eta}) &= Div(\theta^{-1}\boldsymbol{q}) + Div\boldsymbol{\eta} \\ &= \theta^{-1}Div\boldsymbol{q} - \theta^{-2}\boldsymbol{q}\cdot\nabla\theta + Div\boldsymbol{\eta} \end{aligned} \tag{8.2}$$

allows us to modify (8.1) into

$$\begin{aligned} m_0\gamma = m_0\dot{\eta} - \theta^{-1}(m_0r - Div\boldsymbol{q}) - \theta^{-2}\boldsymbol{q}\cdot\nabla\theta \\ + \{(\theta_+)^{-1}q_+ - (\theta_-)^{-1}q_-\} - (m_0\lambda - Div\boldsymbol{\eta}). \end{aligned} \tag{8.3}$$

From the equation (7.8) representing the local law of energy balance we have

$$m_0r - Div\boldsymbol{q} = m_0\dot{u} + \sigma^{in} - (\sigma + w + Div\boldsymbol{w}) + (q_+ - q_-), \tag{8.4}$$

which allows us to eliminate these two terms from (8.3) and to get the rate of entropy production in the form

$$\begin{aligned} m_0\gamma = m_0\dot{\eta} - \theta^{-1}\{(m_0\dot{u} + \sigma^{in}) - \sigma + \theta^{-1}\boldsymbol{q}\cdot\nabla\theta\} \\ + \{(\theta_+)^{-1}q_+ - (\theta_-)^{-1}q_-\} \\ - \theta^{-1}(q_+ - q_-) + \theta^{-1}(w + Div\boldsymbol{w}) - (m_0\lambda - Div\boldsymbol{\eta}). \end{aligned} \tag{8.5}$$

Since the absolute temperature is strictly positive, $\theta(\mathrm{x},t) \geq 0$, the inequality (7.9) is equivalent to

$$\delta \equiv \theta\gamma \geq 0. \tag{8.6}$$

The scalar $\delta(\mathrm{x},t)$ is often called the specific dissipation. In view of (8.5) it is given by

$$\begin{aligned} m_0\delta = m_0\theta\dot{\eta} - \{(m_0\dot{u} + \sigma^{in}) - \sigma + \theta^{-1}\boldsymbol{q}\bullet\nabla\theta\} \\ + \theta\{(\theta_+)^{-1}q_+ - (\theta_-)^{-1}q_-\} - (q_+ - q_-) \\ + (w + Div\boldsymbol{w}) - \theta(m_0\lambda - Div\boldsymbol{\eta}). \end{aligned} \tag{8.7}$$

Because different groups of terms in (8.7) play distinct roles in shell theory, it is convenient to introduce the following notation:

$$\begin{aligned} Q &\equiv \theta\{(\theta_+)^{-1}q_+ - (\theta_-)^{-1}q_-\} - (q_+ - q_-) \\ &= \{\theta(\theta_+)^{-1} - 1\}q_+ - \{\theta(\theta_-)^{-1} - 1\}q_- \end{aligned} \tag{8.8}$$

$$H \equiv w - m_0\lambda + Div\boldsymbol{w} + \theta Div\boldsymbol{\eta}. \tag{8.9}$$

The quantity Q defined by (8.8) represents the amount of heating through the shell faces, and the quantity H contains all "non-standard" field variables.

The inequality (8.6) with the dissipation δ written as

$$m_0\delta = -(m_0\dot{u} + \sigma^{in} - m_0\theta\dot{\eta}) + \sigma - \theta^{-1}\boldsymbol{q}\bullet\nabla\theta + Q + H \tag{8.10}$$

may be called the ***reduced dissipation inequality***. It represents locally the principle of irreversibility under the assumption that the remaining four balance principles are satisfied. According to the general theory of constitutive equations, the reduced dissipation inequality should be regarded as identity for all thermodynamically admissible forms of the constitutive equations.

8.2 Thermodynamic potentials and kinetic constitutive equations

The aim of this report is neither to study the possible forms of constitutive equations in their full generality, nor to derive their very special forms for narrow classes of shell problems. Our principal aim is to establish a general framework for the constitutive equations, which would be consistent with the 2D laws of thermomechanics and within which special classes of constitutive equations for shells could be discussed. With this in mind we first introduce and study additional variables generally referred to as thermodynamic potentials. The internal energy, free energy and kinetic energy are such related concepts.

The problem associated with the one of the thermodynamically consistent kinetic constitutive equations, i.e. the constitutive equations for surface linear

and angular momenta vectors, is the existence of a kinetic energy. The related problem concerns the possibility of splitting the total energy into the sum of the internal and kinetic energies.

Let us assume that the total energy u may be written as the sum of three parts

$$u = \varepsilon + \kappa + \varrho, \tag{8.11}$$

where $\varepsilon(\mathsf{x},t)$ is the specific internal energy, and $\kappa(\mathsf{x},t)$ is the specific kinetic energy. In (8.11), $\varrho(\mathsf{x},t)$ denotes the additional coupling term introduced here to point it out that in shell thermomechanics the splitting of the total energy into the sum of the internal and kinetic energies can be done only approximately, in general.

Substituting (8.11) into (8.10), the dissipation takes the form

$$m_0\delta = -m_0(\dot{\varepsilon} - \theta\dot{\eta}) - (m_0\dot{\kappa} + \sigma^{in}) + \sigma - \theta^{-1}\boldsymbol{q}\bullet\nabla\theta + Q + H - m_0\dot{\varrho}. \tag{8.12}$$

Let us further recall that the surface inertia force and couple vectors have been defined in terms of the surface linear and angular momenta by

$$\boldsymbol{b}^{in} = -\dot{\boldsymbol{p}}, \qquad \boldsymbol{c}^{in} = -(\dot{\boldsymbol{s}} + \boldsymbol{v}\times\boldsymbol{p}), \tag{8.13}$$

so that the power of inertia forces and couples takes the form

$$\sigma^{in} = -\{\dot{\boldsymbol{p}}\bullet\boldsymbol{v} - (\dot{\boldsymbol{s}} + \boldsymbol{v}\times\boldsymbol{p})\bullet\boldsymbol{\omega}\}. \tag{8.14}$$

Now the splitting (8.11) may be considered as the problem of constructing some specific constitutive equations. Indeed, let us assume that a constitutive equation for the total energy u be such that it may be written in the form (8.11) without *a priori* identification of κ as the kinetic energy.

Let us further assume that the constitutive equations for the linear momentum $\boldsymbol{p}$ and the angular momentum $\boldsymbol{s}$ have the properties that

$$m_0\dot{\kappa} + \sigma^{in} = m_0\dot{\kappa} - \{\dot{\boldsymbol{p}}\bullet\boldsymbol{v} - (\dot{\boldsymbol{s}} + \boldsymbol{v}\times\boldsymbol{p})\bullet\boldsymbol{\omega}\} = 0. \tag{8.15}$$

Then κ may be identified indeed as the kinetic energy.

Conversely, let us assume that all thermodynamically admissible kinetic constitutive equations are such that there exists a scalar field $\kappa(\mathsf{x},t)$ such that (8.15) holds. Then again, the field κ may be identified as the kinetic energy, and the splitting (8.11) is obtained as the consequence, not as the assumption.

Whether the condition (8.15) holds or not, the splitting (8.11) naturally leads to the concept of the specific free energy $\psi(\mathsf{x},t)$ defined by

$$\psi = \varepsilon - \theta\eta . \tag{8.16}$$

Then $\dot{\varepsilon} - \theta\dot{\eta} = \dot{\psi} + \dot{\theta}\eta$ and the dissipation given by (8.12) can be transformed into

$$\begin{aligned} m_0\delta = &- m_0(\dot{\psi} + \dot{\theta}\eta) - (m_0\dot{\kappa} + \sigma^{in}) + \sigma \\ &- \theta^{-1}\boldsymbol{q}\bullet\nabla\theta + Q + H - m_0\dot{\varrho} . \end{aligned} \tag{8.17}$$

In the following considerations the kinetic constitutive equations will be referred to as thermodynamically consistent if the condition (8.6) with (8.12) or (8.17) holds. In this case the splitting (8.11) will be regarded as derived, not as assumed. Within this class of shell theories, the constitutive prescription of the total energy is equivalent to providing the constitutive equation either for the internal energy ε , or for the free energy ψ .

8.3 Local theories of shells

As is usual in continuum thermomechanics, various special cases of the general constitutive equations (7.25) and (7.26) can be obtained approximating the independent field variables by their successive surface gradients. To within the first order, the fields (7.25) may be approximated by their first surface gradients alone. This naturally leads to a particular class of shell behavior defined by the following constitutive equations:

$$\begin{aligned} \boldsymbol{N}(\mathsf{x},t) &= \boldsymbol{N}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{x},s),\nabla\theta(\mathsf{x},t);\mathsf{x},\nabla\boldsymbol{X}(\mathsf{x})), \\ \boldsymbol{M}(\mathsf{x},t) &= \boldsymbol{M}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{x},s),\nabla\theta(\mathsf{x},t);\mathsf{x},\nabla\boldsymbol{X}(\mathsf{x})), \\ u(\mathsf{x},t) &= u^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{x},s),\nabla\theta(\mathsf{x},t);\mathsf{x},\nabla\boldsymbol{X}(\mathsf{x})), \\ \eta(\mathsf{x},t) &= \eta^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{x},s),\nabla\theta(\mathsf{x},t);\mathsf{x},\nabla\boldsymbol{X}(\mathsf{x})), \\ \boldsymbol{q}(\mathsf{x},t) &= \boldsymbol{q}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(\mathsf{x},s),\nabla\theta(\mathsf{x},t);\mathsf{x},\nabla\boldsymbol{X}(\mathsf{x})), \end{aligned} \tag{8.18}$$

together with similarly reduced kinetic constitutive equations (7.26), with $\boldsymbol{e}(\mathsf{x},t)$ being now defined by

$$\boldsymbol{e}(\mathsf{x},t) \equiv (\boldsymbol{F}(\mathsf{x},t),\boldsymbol{Y}(\mathsf{x},t),\nabla\boldsymbol{Y}(\mathsf{x},t),\theta(\mathsf{x},t)). \tag{8.19}$$

In consistency with terminology of continuum thermomechanics, the constitutive equations (8.18) define the class of so-called "simple" shells. Let

us note that the structure tensor $\boldsymbol{Y}$ cannot be omitted in the list of arguments in the constitutive equations (8.19).

An alternative to (8.19) list of arguments for the constitutive equations may be obtained recalling that the effective stress power (7.11) may be written as

$$\sigma = \boldsymbol{N} \bullet (\dot{\boldsymbol{F}} - \boldsymbol{\Omega F}) + \boldsymbol{M} \bullet \nabla \boldsymbol{\omega}, \tag{8.20}$$

where $\boldsymbol{\Omega}(\mathsf{x},t)$ denotes the skew-symmetric tensor associated with the angular velocity vector $\boldsymbol{\omega}(\mathsf{x},t)$ by $\boldsymbol{\Omega} = ad\boldsymbol{\omega}$.

The expression (8.20) suggests in a natural way that we can introduce the surface strain measures $\boldsymbol{\Lambda}(\mathsf{x},t)$ and $\boldsymbol{\Theta}(\mathsf{x},t)$ defined by (see (6.34))

$$\boldsymbol{\Lambda} = \boldsymbol{F} - \boldsymbol{QI} = \bar{\boldsymbol{I}}\mathbf{F} - \boldsymbol{QI}, \qquad \boldsymbol{\Theta} = \bar{\boldsymbol{B}}\mathbf{F} - \boldsymbol{QB}. \tag{8.21}$$

Here $\boldsymbol{B}(\mathsf{x})$ denotes the generalized curvature tensor of the shell reference configuration derived from the surface gradient of the structure tensor $\boldsymbol{X}(\mathsf{x})$, $\bar{\boldsymbol{B}}(\mathsf{y},t)$ is the corresponding generalized curvature tensor of the current configuration, $\mathbf{F}(\mathsf{x},t)$ is the tangential surface deformation gradient, and $\bar{\boldsymbol{I}}(\mathsf{y},t)$ denotes the inclusion operator in the current configuration. With the co-rotational rates of the strain measures (8.21) defined by

$$\boldsymbol{\Lambda}^{\circ} \equiv \boldsymbol{Q}(\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Lambda})) = \dot{\boldsymbol{F}} - \boldsymbol{\Omega F}, \qquad \boldsymbol{\Theta}^{\circ} \equiv \boldsymbol{Q}(\frac{d}{dt}(\boldsymbol{Q}^T\boldsymbol{\Theta})) = \nabla\boldsymbol{\omega}. \tag{8.22}$$

the effective stress power (8.20) may be written concisely as

$$\sigma = \boldsymbol{N} \bullet \boldsymbol{\Lambda}^{\circ} + \boldsymbol{M} \bullet \boldsymbol{\Theta}^{\circ}. \tag{8.23}$$

It is seen that the surface strain measures $\boldsymbol{\Lambda}$ and $\boldsymbol{\Theta}$ are work-conjugate to the surface stress measures $\boldsymbol{N}$ and $\boldsymbol{M}$, respectively.

As a result, the list (8.19) of independent variables in the constitutive equations (8.18) may be replaced by

$$\mathscr{e}(\mathsf{x},t) \equiv (\boldsymbol{\Lambda}(\mathsf{x},t), \boldsymbol{\Theta}(\mathsf{x},t), \theta(\mathsf{x},t)), \tag{8.24}$$

and the gradient $\nabla\boldsymbol{X}$ of the structure tensor in (8.18) may be replaced by the generalized curvature tensor $\boldsymbol{B}$.

In the following consideration we shall often simplify notation and omit the argument x in all the field variables. Thus, for the class of "simple" shells the constitutive equations can be written as

$$
\begin{aligned}
\boldsymbol{N}(t) &= \boldsymbol{N}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(s), \nabla\theta(t); \mathrm{x}, \boldsymbol{B}), \\
\boldsymbol{M}(t) &= \boldsymbol{M}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(s), \nabla\theta(t); \mathrm{x}, \boldsymbol{B}), \\
u(t) &= u^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(s), \nabla\theta(t); \mathrm{x}, \boldsymbol{B}), \\
\eta(t) &= \eta^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(s), \nabla\theta(t); \mathrm{x}, \boldsymbol{B}), \\
\boldsymbol{q}(t) &= \boldsymbol{q}^{\infty}_{s=0}(\boldsymbol{e}^{(t)}(s), \nabla\theta(t); \mathrm{x}, \boldsymbol{B}).
\end{aligned}
\tag{8.25}
$$

These constitutive equations, supplemented by appropriate kinetic constitutive equations and other relations pointed out in the previous Chapter, describe a large class of inelastic shell behavior. We generally call such theories as spatially local shell theories, because no higher gradients than the first one of the independent variables are present as arguments in the constitutive equations.

Possible forms of the constitutive equations (8.25) as well as of other constitutive equations and relations, which are needed in order to complete the initial-boundary value problem of shell theory, are further restricted by the principle of material frame-indifference and the entropy inequality (7.9). The explicit form of such restricted relations can more easily be obtained if the basic shell relations are expressed in the material representation.

8.4 Surface strain and stress measures in material representation

One of the most characteristic features of shell theory discussed here is the appearance of the rotation tensor $\boldsymbol{Q}$ as the independent kinematic field variable. Rotation tensors, being elements of the linear tensor space, belong at the same time to the Lie group $SO(3)$. This purely geometric feature has many important implications. Without going into all details to be found in the mathematical literature, let us only note that the tangent space at any "point" of the Lie group is isomorphic with the tangent space of the group at the identity in two ways (this has direct relation to the left and right invariant vector fields on Lie groups). In the theory of shells, this implies the existence of two entirely equivalent forms of all shell relations and equations. This is well known from the theory of rigid bodies, and has recently been exploited also in the context of rod and shell theories. Below we only list the relations which are needed in the following considerations.

Noting that time derivative of the rotation tensor $\boldsymbol{Q}(\mathrm{x},t)$ may be written in two forms

$$\dot{\boldsymbol{Q}} = \boldsymbol{\Omega Q} = \boldsymbol{QW}, \qquad \boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T = ad\boldsymbol{\omega}, \qquad \boldsymbol{W} = \boldsymbol{Q}^T\dot{\boldsymbol{Q}} = ad\boldsymbol{w}, \tag{8.26}$$

we can introduce the material velocity vector $\boldsymbol{v}(\mathrm{X},t)$ and the material angular velocity vector $\boldsymbol{w}(\mathrm{X},t)$, which are related to $\boldsymbol{v}(\mathrm{X},t)$ and $\boldsymbol{\omega}(\mathrm{X},t)$ by the formulae

$$\boldsymbol{v} = \boldsymbol{Qv}, \qquad \boldsymbol{\omega} = \boldsymbol{Qw}, \qquad \boldsymbol{\Omega} = \boldsymbol{QWQ}^T. \tag{8.27}$$

The definitions (8.21) and (8.22) of the surface train measures and their time rates as well as the expression (8.23) for the effective stress power suggest that it may be convenient to introduce the material surface strain measures $\boldsymbol{E}(\mathrm{X},t)$ and $\boldsymbol{K}(\mathrm{X},t)$, and the material surface stress measures $\boldsymbol{S}(\mathrm{X},t)$ and $\boldsymbol{\Sigma}(\mathrm{X},t)$ defined by

$$\boldsymbol{E} = \boldsymbol{Q}^T\boldsymbol{\Lambda}, \qquad \boldsymbol{K} = \boldsymbol{Q}^T\boldsymbol{\Theta}, \qquad \boldsymbol{S} = \boldsymbol{Q}^T\boldsymbol{N}, \qquad \boldsymbol{\Sigma} = \boldsymbol{Q}^T\boldsymbol{M}. \tag{8.28}$$

Then it immediately follows that the strain rates (8.22) are given by

$$\boldsymbol{\Lambda}^{\circ} = \boldsymbol{Q}\dot{\boldsymbol{E}}, \qquad \boldsymbol{\Theta}^{\circ} = \boldsymbol{Q}\dot{\boldsymbol{K}}, \tag{8.29}$$

and the effective stress power may be written as

$$\sigma = \boldsymbol{N} \bullet \boldsymbol{\Lambda} + \boldsymbol{M} \bullet \boldsymbol{\Theta}^{\circ} = \boldsymbol{S} \bullet \dot{\boldsymbol{E}} + \boldsymbol{\Sigma} \bullet \dot{\boldsymbol{K}}, \tag{8.30}$$

thus showing that the surface stress measures $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$ are work-conjugate to the surface strain measures $\boldsymbol{E}$ and $\boldsymbol{K}$, respectively.

8.5 Momenta and equations of motion in material representation

A closer look at the power of inertia forces given by (8.14) and results of the previous Section suggest that in the material representation the linear and angular momenta densities, $\boldsymbol{q}(\mathrm{X},t)$ and $\boldsymbol{r}(\mathrm{X},t)$, should be defined by

$$\boldsymbol{p} = \boldsymbol{Qq}, \qquad \boldsymbol{s} = \boldsymbol{Qr}. \tag{8.31}$$

Then with the use of vector identities

$$\begin{aligned} \boldsymbol{Q}^T\dot{\boldsymbol{p}} &= \boldsymbol{Q}^T(\boldsymbol{Q}\dot{\boldsymbol{q}} + \dot{\boldsymbol{Q}}\boldsymbol{q}) = \dot{\boldsymbol{q}} + \boldsymbol{Wq} = \dot{\boldsymbol{q}} + \boldsymbol{w} \times \boldsymbol{q}, \\ \boldsymbol{Q}^T(\dot{\boldsymbol{s}} + \boldsymbol{v} \times \boldsymbol{p}) &= \boldsymbol{Q}^T(\boldsymbol{Q}\dot{\boldsymbol{r}} + \dot{\boldsymbol{Q}}\boldsymbol{r}) + \boldsymbol{Q}^T\boldsymbol{v} \times \boldsymbol{Q}^T\boldsymbol{p} = \dot{\boldsymbol{r}} + \boldsymbol{Wr} + \boldsymbol{v} \times \boldsymbol{q}, \end{aligned} \tag{8.32}$$

we easily find that

$$\dot{\boldsymbol{p}} \bullet \boldsymbol{v} + (\dot{\boldsymbol{s}} + \boldsymbol{v} \times \boldsymbol{p}) \bullet \boldsymbol{\omega} = (\dot{\boldsymbol{q}} + \boldsymbol{w} \times \boldsymbol{q}) \bullet \boldsymbol{v} + (\dot{\boldsymbol{r}} + \boldsymbol{w} \times \boldsymbol{r} + \boldsymbol{v} \times \boldsymbol{q}) \bullet \boldsymbol{w}. \tag{8.33}$$

Noting further vector identities

$$\begin{aligned}(\boldsymbol{w}\times\boldsymbol{r})\cdot\boldsymbol{w}&=\boldsymbol{0},\\(\boldsymbol{w}\times\boldsymbol{q})\cdot\boldsymbol{v}+(\boldsymbol{v}\times\boldsymbol{q})\cdot\boldsymbol{w}&=(\boldsymbol{v}\times\boldsymbol{w})\cdot\boldsymbol{q}+(\boldsymbol{w}\times\boldsymbol{v})\cdot\boldsymbol{q}=\boldsymbol{0},\end{aligned}\tag{8.34}$$

we find that the stress power of the inertia forces and couples can be expressed in the form

$$\sigma^{in}=\boldsymbol{b}^{in}\cdot\boldsymbol{v}+\boldsymbol{c}^{in}\cdot\boldsymbol{\omega}=-(\dot{\boldsymbol{q}}\cdot\boldsymbol{v}+\dot{\boldsymbol{r}}\cdot\boldsymbol{w}),\tag{8.35}$$

which is simpler than the form (8.14).

Using the relations (8.28) between the surface stress measures, and making use of definition of the bending tensor $\boldsymbol{K}$ we obtain

$$Div\boldsymbol{N}=Div(\boldsymbol{QS})=\boldsymbol{Q}\{Div\boldsymbol{S}+ad^{-1}(\boldsymbol{KS}^{T}-\boldsymbol{SK}^{T})\},\tag{8.36}$$

with the same formula for the surface divergence of the surface couple tensor $\boldsymbol{M}$.

Noting further that $\boldsymbol{F}=\boldsymbol{Q}(\boldsymbol{I}+\boldsymbol{E})$, we easily find that

$$\begin{aligned}\boldsymbol{NF}^{T}-\boldsymbol{FN}^{T}&=\boldsymbol{QS}(\boldsymbol{I}+\boldsymbol{E})^{T}\boldsymbol{Q}^{T}-\boldsymbol{Q}(\boldsymbol{I}+\boldsymbol{E})\boldsymbol{S}^{T}\boldsymbol{Q}^{T}\\&=\boldsymbol{Q}\{\boldsymbol{S}(\boldsymbol{I}+\boldsymbol{E})^{T}-(\boldsymbol{I}+\boldsymbol{E})\boldsymbol{S}^{T}\}\boldsymbol{Q}^{T}.\end{aligned}\tag{8.37}$$

The term in bracket of (8.37) is necessarily skew-symmetric, and by the fundamental property of the ad map we obtain

$$ad^{-1}\{\boldsymbol{TF}^{T}-\boldsymbol{FT}^{T}\}=\boldsymbol{Q}ad^{-1}\{\boldsymbol{S}(\boldsymbol{I}+\boldsymbol{E})^{T}-(\boldsymbol{I}+\boldsymbol{E})\boldsymbol{S}^{T}\}.\tag{8.38}$$

With these relations and definitions, the dynamic equations of motion (7.6) and (7.7) may be rewritten in the entirely equivalent form

$$Div\boldsymbol{S}+ad^{-1}(\boldsymbol{KS}^{T}-\boldsymbol{SK}^{T})+\boldsymbol{Q}^{T}\boldsymbol{b}=\dot{\boldsymbol{q}}+\boldsymbol{w}\times\boldsymbol{q},\tag{8.39}$$

$$\begin{aligned}&Div\boldsymbol{\Sigma}+ad^{-1}(\boldsymbol{K\Sigma}^{T}-\boldsymbol{\Sigma K}^{T})+ad^{-1}\{\boldsymbol{S}(\boldsymbol{I}+\boldsymbol{E})^{T}-(\boldsymbol{I}+\boldsymbol{E})\boldsymbol{S}^{T}\}+\boldsymbol{Q}^{T}\\&\quad=\dot{\boldsymbol{r}}+\boldsymbol{w}\times\boldsymbol{r}+\boldsymbol{v}\times\boldsymbol{q}.\end{aligned}\tag{8.40}$$

In the material representation of basic shell equations, the local equation of energy balance (7.8), the entropy inequality (7.9), and various inequalities derived from (7.9) preserve their form with the only difference that the inertia stress power and the effective stress power are given now by (8.35) and (8.30), respectively.

It can next be shown that the linear and angular velocity vectors, $\boldsymbol{v}$ and $\boldsymbol{w}$, the linear and angular momenta, $\boldsymbol{q}$ and $\boldsymbol{r}$, the surface strain measures, $\boldsymbol{E}$ and

$\boldsymbol{K}$, as well as the surface stress measures, $\boldsymbol{S}$ and $\boldsymbol{\Sigma}$, remain unchanged under the change of frame of reference. The advantage of using these field variables and the material representation of basic shell equations in formulation of the constitutive equations is now apparent: they satisfy identically the restrictions imposed by the principle of material objectivity. In effect, in what follows we can concentrate our considerations only on restrictions imposed on the form of constitutive equations by the principle of entropy inequality.

Chapter 9
Heat conduction and thermo-visco-elasticity

9.1 Constitutive equations in the material representation

In this Chapter we assume that the kinetic constitutive equations (considered later on) are thermodynamically consistent, i.e. they satisfy the condition (8.6). Thus, the splitting (8.11) of the total energy can be used, and the total energy in the constitutive equations may be replaced either by the internal energy or by the free energy. With these assumptions, the subsequent considerations are restricted to a class of shells defined by the following constitutive equations:

$$\begin{aligned}
\boldsymbol{S}(t) &= \bar{\boldsymbol{S}}(\mathscr{e}(t), \dot{\mathscr{e}}(t), \theta(t), \nabla\theta(t); \mathsf{x}, \boldsymbol{B}),\\
\boldsymbol{\Sigma}(t) &= \bar{\boldsymbol{\Sigma}}(\mathscr{e}(t), \dot{\mathscr{e}}(t), \theta(t), \nabla\theta(t); \mathsf{x}, \boldsymbol{B}),\\
\psi(t) &= \bar{\psi}(\mathscr{e}(t), \dot{\mathscr{e}}(t), \theta(t), \nabla\theta(t); \mathsf{x}, \boldsymbol{B}),\\
\eta(t) &= \bar{\eta}(\mathscr{e}(t), \dot{\mathscr{e}}(t), \theta(t), \nabla\theta(t); \mathsf{x}, \boldsymbol{B}),\\
\boldsymbol{q}(t) &= \bar{\boldsymbol{q}}(\mathscr{e}(t), \dot{\mathscr{e}}(t), \theta(t), \nabla\theta(t); \mathsf{x}, \boldsymbol{B}),
\end{aligned} \tag{9.1}$$

with $\mathscr{e}(\mathsf{x},t)$ and $\dot{\mathscr{e}}(\mathsf{x},t)$ being a concise notation for

$$\mathscr{e} \equiv (\boldsymbol{E}, \boldsymbol{K}), \qquad \dot{\mathscr{e}} \equiv (\dot{\boldsymbol{E}}, \dot{\boldsymbol{K}}). \tag{9.2}$$

In (9.1), the overbar is used to distinguish the respective response functions from the thermomechanical fields themselves determined by the constitutive equations.

The constitutive equations (9.1) are local in space and time. Yet they are general enough to describe not only an elastic shell behavior, but also to take into account heat conduction and viscous effects. In general, the relations (9.1) have to satisfy the reduced dissipation inequality

$$D \equiv m_0\delta = m_0\theta\gamma \geq 0, \tag{9.3}$$

with δ being given by (8.17). Under the assumption that the condition (8.6) holds, the dissipation function D, measured per unit area of the shell base surface M, is given by

$$D = \boldsymbol{S} \bullet \dot{\boldsymbol{E}} + \boldsymbol{\Sigma} \bullet \dot{\boldsymbol{K}} - m_0\dot{\psi} - m_0\dot{\theta}\eta - \theta^{-1}\boldsymbol{q} \bullet \nabla\theta + Q + H - m_0\dot{\varrho}, \tag{9.4}$$

where the expression (8.30) for the effective stress power has been used.

9.2 Special shell theories

In order to derive the restrictions imposed by the dissipation inequality (9.3) on the response functions in (9.1), let us assume the free energy function $\bar{\psi}$ to be differentiable with regard to all its arguments. Then

$$\begin{aligned}\dot{\psi} = {} & (\partial_{\boldsymbol{E}}\bar{\psi}) \bullet \dot{\boldsymbol{E}} + (\partial_{\boldsymbol{K}}\bar{\psi}) \bullet \dot{\boldsymbol{K}} + (\partial_{\dot{\boldsymbol{E}}}\bar{\psi}) \bullet \ddot{\boldsymbol{E}} + (\partial_{\dot{\boldsymbol{K}}}\bar{\psi}) \bullet \ddot{\boldsymbol{K}} \\ & + (\partial_{\theta}\bar{\psi})\dot{\theta} + (\partial_{\nabla\theta}\bar{\psi}) \bullet \nabla\dot{\theta},\end{aligned} \tag{9.5}$$

and the dissipation function (9.4) can be transformed into the form

$$\begin{aligned}D = {} & (\bar{\boldsymbol{S}} - m_0\partial_{\boldsymbol{E}}\bar{\psi}) \bullet \dot{\boldsymbol{E}} + (\bar{\boldsymbol{\Sigma}} - m_0\partial_{\boldsymbol{K}}\bar{\psi}) \bullet \dot{\boldsymbol{K}} - m_0(\bar{\eta} + \partial_{\theta}\bar{\psi})\dot{\theta} \\ & - m_0(\partial_{\dot{\boldsymbol{E}}}\bar{\psi}) \bullet \ddot{\boldsymbol{E}} - m_0(\partial_{\dot{\boldsymbol{K}}}\bar{\psi}) \bullet \ddot{\boldsymbol{K}} - m_0(\partial_{\nabla\theta}\bar{\psi}) \bullet \nabla\dot{\theta} \\ & - \theta^{-1}\bar{\boldsymbol{q}} \bullet \nabla\theta + Q + H - m_0\dot{\varrho}.\end{aligned} \tag{9.6}$$

Recalling now that Q and H in (9.4) are defined by (8.8) and (8.9), respectively, it becomes apparent that any conclusion which may be drawn from the reduced dissipation inequality (9.3) strongly depends on assumptions taken for the field variables appearing in the expressions for Q, H and for the term ϱ. Some possibilities described below, each one leading to a different formulation of shell theory, are worth of being considered.

The simplest class of shell theories, subsequently referred to as the classical theory of shells, is obtained by assuming that

$$H(\mathsf{x},t) - m_0\dot{\varrho}(\mathsf{x},t) = 0, \qquad \forall\,(\mathsf{x},t) \in M \times T. \tag{9.7}$$

In other words, we omit as negligibly small all the "non-standard" terms in the balance of energy and in the principle of entropy inequality, and set to zero the coupling term ϱ in the expression (8.11) for the total energy. These simplifications lead to the version of shell thermomechanics obtained by SIMMONDS [1984] using the reduction procedure. There are advantages and limitations of using the simplifying assumptions. Let us point out below some of these limitations.

Already NOLL [1958] developed the mathematically rigorous and unified theory of mechanical response of a variety of materials, which was effectively

applied in various studies of continuum mechanics. Advances in continuum thermomechanics made over the last decades have been based primarily on the development of techniques of how to use effectively the restrictions following from the second law of thermodynamics in the form of the Clausius-Duhem inequality.

COLEMAN AND NOLL [1963] succeeded in clarifying and making rigorous the procedure by which the laws of mechanics and the Clausius-Duhem inequality could be used to deduce constitutive restrictions on a vast variety of materials, see also COLEMAN [1964], and COLEMAN AND MIZEL [1968,1968A]. However, it began to be clear almost from the beginning that the procedure of COLEMAN AND NOLL [1963], when applied to the usual laws of thermodynamics (such as those for shells following under the assumption (9.7)), impose extraordinary restrictions on the long-range spatial interactions allowable in the constitutive equations. While in 3D continuum thermomechanics the long-range interactions may often be ignored indeed, they cannot be taken for granted in shell thermomechanics.

If we are not using the assumption (9.7), many possibilities are then open for discussion. Please note that according to (8.9) there are four terms in the definition of H, and each of them may be significant in a particular class of shell problems. In general, it may be assumed that all field variables in the definition of H should be given by appropriate constitutive equations having the same list of arguments as the constitutive equations (9.1). The common feature of all such shell models is that they would allow to incorporate long-range interactions into the description of shell behavior. However, the resulting shell models would apparently become quite complex and they would require additional and thorough studies. Therefore, let us restrict further considerations to the version of shell thermomechanics in which (9.7) is satisfied.

9.3 Reduced forms of constitutive equations

Under the assumption (9.7), the dissipation function (9.6) reduces to the form

$$\begin{aligned} D = {} & (\bar{\boldsymbol{S}} - m_0 \partial_{\boldsymbol{E}} \bar{\psi}) \bullet \dot{\boldsymbol{E}} + (\bar{\boldsymbol{\Sigma}} - m_0 \partial_{\boldsymbol{K}} \bar{\psi}) \bullet \dot{\boldsymbol{K}} - m_0 (\bar{\eta} + \partial_\theta \bar{\psi}) \dot{\theta} \\ & - m_0 (\partial_{\dot{\boldsymbol{E}}} \bar{\psi}) \bullet \ddot{\boldsymbol{E}} - m_0 (\partial_{\dot{\boldsymbol{K}}} \bar{\psi}) \bullet \ddot{\boldsymbol{K}} - m_0 (\partial_{\nabla\theta} \bar{\psi}) \bullet \nabla \dot{\theta} \\ & - \theta^{-1} \bar{\boldsymbol{q}} \bullet \nabla \theta + Q. \end{aligned} \tag{9.8}$$

To proceed further it becomes necessary to make certain assumptions about the constitutive nature of heat influxes q_+, q_- and the temperature fields θ, θ_+ and θ_-. As it was already pointed out in Section 8.7, there are two possibilities here. If θ, θ_+ and θ_- are independent of each other, what is actually the most general case, then it may be assumed that the heat influxes q_+ and q_- are given by the constitutive functions having $e(\mathsf{x},t)$, $\dot{e}(\mathsf{x},t)$ and the temperature fields θ_+ and θ_-, including their first surface gradients, as the arguments. This would lead to Q being given by the constitutive equation of the form

$$Q = \bar{Q}(e,\dot{e},\theta,\theta^+,\theta^-,\nabla\theta,\nabla\theta^+,\nabla\theta^-;\mathsf{x},\boldsymbol{B}). \tag{9.9}$$

Since the list of arguments of the constitutive equations (9.1) and (9.9) does not contain $\ddot{\boldsymbol{E}}$, $\ddot{\boldsymbol{K}}$ or $\nabla\dot{\theta}$, these quantities may be varied independently of each other and independently of all quantities appearing as arguments in the constitutive equations. Accordingly, the reduced dissipation inequality (9.3) with (9.6) will be satisfied if and only if the following conditions hold:

$$\partial_{\dot{\boldsymbol{E}}}\bar{\psi} = \boldsymbol{0}, \qquad \partial_{\dot{\boldsymbol{K}}}\bar{\psi} = \boldsymbol{0}, \qquad \partial_{\nabla\theta}\bar{\psi} = \boldsymbol{0}. \tag{9.10}$$

As a result, the constitutive equation for the free energy function reduces to the form

$$\psi = \bar{\psi}(e,\theta,\nabla\theta) = \bar{\psi}(\boldsymbol{E},\boldsymbol{K},\theta,\nabla\theta). \tag{9.11}$$

Here and in the sequel dependence of the response functions on x and $\boldsymbol{B}$ will not be indicated explicitly for brevity.

Now the response function (9.11) for the free energy serves as the thermodynamical potential for the entropy,

$$\eta = \bar{\eta}(e,\theta,\nabla\theta) = -\,\partial_\theta\bar{\psi}(e,\theta,\nabla\theta). \tag{9.12}$$

Other constitutive equations must still obey the dissipation inequality (9.3), with D being now given by

$$D = (\bar{\boldsymbol{S}} - m_0\partial_{\boldsymbol{E}}\bar{\psi})\bullet\dot{\boldsymbol{E}} + (\bar{\boldsymbol{\Sigma}} - m_0\partial_{\boldsymbol{K}}\bar{\psi})\bullet\dot{\boldsymbol{K}} - \theta^{-1}\bar{\boldsymbol{q}}\bullet\nabla\theta + \bar{Q}. \tag{9.13}$$

The constitutive equations for the surface stress and couple tensors may now be written in the form

$$\begin{aligned} \boldsymbol{S} &= \bar{\boldsymbol{S}}(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta) = m_0 \partial_E \bar{\psi}(\boldsymbol{e}, \theta) + \bar{\boldsymbol{S}}_*(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta), \\ \boldsymbol{\Sigma} &= \bar{\boldsymbol{\Sigma}}(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta) = m_0 \partial_K \bar{\psi}(\boldsymbol{e}, \theta) + \bar{\boldsymbol{\Sigma}}_*(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta), \end{aligned} \tag{9.14}$$

in which case

$$D = \bar{\boldsymbol{S}}_* \bullet \dot{\boldsymbol{E}} + \bar{\boldsymbol{\Sigma}}_* \bullet \dot{\boldsymbol{K}} - \theta^{-1} \bar{\boldsymbol{q}} \bullet \nabla\theta + \bar{Q}. \tag{9.15}$$

Those parts of the surface stress measures that are determined by derivatives of the free energy are called the equilibrium stresses, and the ones determined by $\bar{\boldsymbol{S}}_*$ and $\bar{\boldsymbol{\Sigma}}_*$ are called the dynamic stresses. It is seen that the equilibrium stresses are not of dissipative type, because they do not contribute to the total dissipation given by (9.15). In contrast, the dynamic stresses are responsible for dissipation, but they are not derivable from a thermodynamic potential.

It follows from the above discussion that the constitutive equations for this class of shell theories consist of (9.11), (9.12) and (9.14) together with

$$\boldsymbol{q} = \bar{\boldsymbol{q}}(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta) \tag{9.16}$$

and (9.9). Moreover, the total dissipation (9.15) can be written as the sum of the term

$$D_i \equiv \bar{\boldsymbol{S}}_* \bullet \dot{\boldsymbol{E}} + \bar{\boldsymbol{\Sigma}}_* \bullet \dot{\boldsymbol{K}}, \tag{9.17}$$

which represents the internal dissipation due to viscous parts of the stress measures, and the term

$$D_c \equiv -\theta^{-1} \bar{\boldsymbol{q}} \bullet \nabla\theta + \bar{Q}, \tag{9.18}$$

which takes into account dissipation due to heat conduction.

It is clear that two separate assumptions $D_i \geq 0$ and $D_c \geq 0$ imply that the principle of entropy inequality is satisfied. The converse is not true, in general. Furthermore, even if it may be assumed that $D_c \geq 0$, we cannot conclude from (9.18) that

$$\bar{\boldsymbol{q}} \bullet \nabla\theta \leq 0, \tag{9.19}$$

unless $Q = 0$.

Please note that the relation (9.19) is well known in thermomechanics of three-dimensional bodies. This is still another difference between the theory of shells discussed here and the theory of the 3D Cauchy continuum.

9.4 Thermoelastic shells and other special shell theories

Under the additional assumption that viscous parts of the surface stress measures vanish, the constitutive equations (9.14) reduce to the form

$$\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{e},\theta) = m_0 \partial_{\boldsymbol{E}} \bar{\psi}(\boldsymbol{e},\theta), \qquad \boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}}(\boldsymbol{e},\theta) = m_0 \partial_{\boldsymbol{K}} \bar{\psi}(\boldsymbol{e},\theta). \tag{9.20}$$

These constitutive equations together with (9.11), (9.12), (9.16) and

$$Q = \bar{Q}(\boldsymbol{e},\theta,\theta_+,\theta_-,\nabla\theta,\nabla\theta_+,\nabla\theta_-) \tag{9.21}$$

define the class of thermoelastic shells, in which case the dissipation inequality takes the form

$$D = -\theta^{-1}\tilde{\boldsymbol{q}} \bullet \nabla\theta + \bar{Q} \geq 0. \tag{9.22}$$

Other special classes of shell theories may be derived from the constitutive equations presented in this Chapter under various additional assumptions. For example, if the temperature fields θ, θ_+ and θ_- are not independent of each other but there exist relations between them, the heat influxes q_+ and q_- on the shell faces may be given by the constitutive equations of the form

$$q_\pm = \bar{q}_\pm(\boldsymbol{e},\theta,\nabla\theta), \tag{9.23}$$

and consequently

$$Q = \bar{Q}(\boldsymbol{e},\theta,\nabla\theta). \tag{9.24}$$

This would be the case when the temperature fields on the shell faces are given in terms of the temperature field θ and possibly of its surface gradient, e.g. by the relations of the form $\theta_\pm = \bar{\theta}_\pm(\theta,\nabla\theta)$.

A very special class of shell theories can be obtained if the influence of shell deformation on the constitutive equations is totally disregarded. Then the constitutive equations for the problem of rigid heat conducting shells are obtained.

9.5 Isothermal theory of shells

Returning to the general case, let us consider possible simplifications of the general constitutive equations obtained under the assumption that the temperature field be constant throughout the whole process of motion. Thus, assuming that

$$\theta = \theta_+ = \theta_- = const, \tag{9.25}$$

we find that the extra entropy source and the extra entropy flux vanish,

$$\lambda = 0, \qquad \boldsymbol{\eta} = 0. \tag{9.26}$$

From (8.8) and (8.9) we then obtain

$$Q = 0, \qquad H = w + Div\boldsymbol{w}. \tag{9.27}$$

Moreover, $\nabla\theta = 0$ and we must have $\boldsymbol{q} = \boldsymbol{0}$, so that the dissipation function D given by (9.4) reduces now to the form

$$D = \boldsymbol{S} \bullet \dot{\boldsymbol{E}} + \boldsymbol{\Sigma} \bullet \dot{\boldsymbol{K}} - m_0\dot{\psi} + w + Div\boldsymbol{w} - m_0\dot{\varrho}. \tag{9.28}$$

Under isothermal conditions the constitutive equations for shells may be assumed in the form

$$\begin{aligned} \boldsymbol{S} &= \bar{\boldsymbol{S}}(\boldsymbol{E}, \boldsymbol{K}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{K}}), \\ \boldsymbol{\Sigma} &= \bar{\boldsymbol{\Sigma}}(\boldsymbol{E}, \boldsymbol{K}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{K}}), \\ \psi &= \bar{\psi}(\boldsymbol{E}, \boldsymbol{K}, \dot{\boldsymbol{E}}, \dot{\boldsymbol{K}}). \end{aligned} \tag{9.29}$$

Like in the general case, any definite conclusions which may be derived from the entropy inequality will depend on the assumptions made for the last three terms in (9.28).

In order to illustrate certain possibilities, let us simply assume that all these terms collectively vanish,

$$w + Div\boldsymbol{w} - m_0\dot{\varrho} = 0. \tag{9.30}$$

Then it follows at once that for the entropy inequality to be satisfied by the constitutive equations (9.29) we must have $\partial_{\dot{\boldsymbol{E}}}\bar{\psi} = \boldsymbol{0}$ and $\partial_{\dot{\boldsymbol{K}}}\bar{\psi} = \boldsymbol{0}$, so that the free energy function can depend only on the surface strain measures,

$$\psi = \bar{\psi}(\boldsymbol{E}, \boldsymbol{K}), \tag{9.31}$$

and the constitutive equations for the surface stress measures are given by

$$\begin{aligned} \boldsymbol{S} &= \bar{\boldsymbol{S}}(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta) = m_0\partial_{\boldsymbol{E}}\bar{\psi}(\boldsymbol{e}, \theta) + \bar{\boldsymbol{S}}_*(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta), \\ \boldsymbol{\Sigma} &= \bar{\boldsymbol{\Sigma}}(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta) = m_0\partial_{\boldsymbol{K}}\bar{\psi}(\boldsymbol{e}, \theta) + \bar{\boldsymbol{\Sigma}}_*(\boldsymbol{e}, \dot{\boldsymbol{e}}, \theta, \nabla\theta), \end{aligned} \tag{9.32}$$

with their viscous parts satisfying the inequality

$$D = \bar{\boldsymbol{S}}_* \bullet \dot{\boldsymbol{E}} + \bar{\boldsymbol{\Sigma}}_* \bullet \dot{\boldsymbol{K}} \geq 0. \tag{9.33}$$

With the additional assumption that $\bar{\boldsymbol{S}}_* = \boldsymbol{0}$ and $\bar{\boldsymbol{\Sigma}}_* = \boldsymbol{0}$, the classical theory of non-linearly elastic shells is obtained.

9.6 Higher-grad shell theories

For simplicity, in this Section we restrict considerations to the isothermal case, so that the dissipation function D is given by (9.28). Let us assume that

$$w - m_0\dot{\varrho} = 0, \tag{9.34}$$

so that now

$$D = \boldsymbol{S} \bullet \dot{\boldsymbol{E}} + \boldsymbol{\Sigma} \bullet \dot{\boldsymbol{K}} - m_0\dot{\psi} + Div\boldsymbol{w}. \tag{9.35}$$

With a view for allowing non-local interactions, let us assume the following form of the constitutive equations:

$$\begin{aligned} &\boldsymbol{S} = \bar{\boldsymbol{S}}(\boldsymbol{E}, \boldsymbol{K}, \nabla\boldsymbol{E}, \nabla\boldsymbol{K}), \qquad \boldsymbol{\Sigma} = \bar{\boldsymbol{\Sigma}}(\boldsymbol{E}, \boldsymbol{K}, \nabla\boldsymbol{E}, \nabla\boldsymbol{K}), \\ &\psi = \bar{\psi}(\boldsymbol{E}, \boldsymbol{K}, \nabla\boldsymbol{E}, \nabla\boldsymbol{K}). \end{aligned} \tag{9.36}$$

These constitutive equations define the elastic shells whose mechanical response depends not only on the surface strain measure but also on their spatial gradients. With (9.36), the dissipation function (9.35) takes the form

$$\begin{aligned} D = (\bar{\boldsymbol{S}} - m_0\partial_{\boldsymbol{E}}\bar{\psi}) \bullet \dot{\boldsymbol{E}} + (\bar{\boldsymbol{\Sigma}} - m_0\partial_{\boldsymbol{K}}\bar{\psi}) \bullet \dot{\boldsymbol{K}} \\ - (\partial_{\dot{\boldsymbol{E}}}\bar{\psi}) \bullet \ddot{\boldsymbol{E}} - (\partial_{\dot{\boldsymbol{K}}}\bar{\psi}) \bullet \ddot{\boldsymbol{K}} + Div\boldsymbol{w}. \end{aligned} \tag{9.37}$$

If we additionally assume that $\boldsymbol{w} = \boldsymbol{0}$, then for the entropy inequality to be satisfied we must have $\partial_{\dot{\boldsymbol{E}}}\bar{\psi} = \boldsymbol{0}$ and $\partial_{\dot{\boldsymbol{K}}}\bar{\psi} = \boldsymbol{0}$, and the constitutive equations (9.36) reduce to the former ones of the classical non-linearly elastic shells. However, if we assume that the interstitial flux vector $\boldsymbol{w}$ does not vanish but is prescribed by the constitutive equation of the type (9.36),

$$\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{E}, \boldsymbol{K}, \nabla\boldsymbol{E}, \nabla\boldsymbol{K}), \tag{9.38}$$

then the non-local interactions are not excluded by the entropy inequality.

9.7 Kinetic constitutive equations

As it was already pointed out earlier, the surface linear and angular momenta vectors must also be given by appropriate kinetic constitutive equations. These constitutive equations should be considered in the general setting based on the total energy, without its *a priori* possible splitting of the

kind (8.11). Accordingly, let us assume that the total energy be given by the constitutive equation (for simplicity, all thermal effects will be ignored in the following considerations)

$$u = \bar{u}(\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w}), \qquad \boldsymbol{e} \equiv (\boldsymbol{E}, \boldsymbol{K}), \tag{9.39}$$

where $\boldsymbol{v}(\mathrm{x},t)$ and $\boldsymbol{w}(\mathrm{x},t)$ are the linear and angular velocity vectors in the material representation. Consistently with (9.39), we can take the kinetic constitutive equations in the form

$$\boldsymbol{q} = \bar{\boldsymbol{q}}(\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w}), \qquad \boldsymbol{r} = \bar{\boldsymbol{r}}(\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w}). \tag{9.40}$$

Although the forms of (9.39) and (9.40) are rather special, they suffice to illustrate basic concepts of the kinetic constitutive equations.

The reasonable assumption, which can be made about the form of the response functions in the kinetic constitutive equations (9.40), would be that $\bar{\boldsymbol{q}}$ and $\bar{\boldsymbol{r}}$ are linear with respect to the velocity vectors,

$$\begin{aligned} \boldsymbol{q} &= \bar{\boldsymbol{q}}(\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{A}_1(\boldsymbol{e})\boldsymbol{v} + \boldsymbol{A}_3(\boldsymbol{e})\boldsymbol{w}, \\ \boldsymbol{r} &= \bar{\boldsymbol{r}}(\boldsymbol{e}, \boldsymbol{v}, \boldsymbol{w}) = \boldsymbol{A}_4(\boldsymbol{e})\boldsymbol{v} + \boldsymbol{A}_2(\boldsymbol{e})\boldsymbol{w}, \end{aligned} \tag{9.41}$$

where $\boldsymbol{A}_1(\boldsymbol{e}), \ldots, \boldsymbol{A}_4(\boldsymbol{e})$ are second-order tensor functions of the surface strain measures alone. Special cases of the kinetic constitutive equations (9.41) were derived by LIBAI AND SIMMONDS [1983, 1998] and applied by CHRÓŚCIELEWSKI *et al.* [2000,2002] to analyze highly non-linear dynamic problems of the flexible shell structures.

Further restrictions of the constitutive equations (9.39) and (9.41) may now be derived from the entropy inequality. This principle leads to the dissipation inequality (8.10), and since the considerations are restricted here to the isothermal case with the additional assumption $H = 0$ (introduced here for simplicity), we have

$$D \equiv m_0 \delta = -m_0 \dot{u} + \dot{\boldsymbol{q}} \bullet \boldsymbol{v} + \dot{\boldsymbol{r}} \bullet \boldsymbol{w} + \sigma \geq 0, \tag{9.42}$$

where it was taken into account that the inertia power is given by $(8.35)_2$.

References

ANTMAN, S.S. [1995], *Nonlinear Problems of Elasticity*, Springer-Verlag, New York *et al.*

BRIDGMANN, P.W. [1961], *The Nature of Thermodynamics*, Harvard Univ. Press, Boston.

CAHN, J.W. AND HOFFMAN, D.W. [1974], A vector thermodynamics for anisotropic surfaces. II. Curved and faceted surfaces, *Acta Metall.*, Vol. 22, pp. 1205-1214.

CHRÓŚCIELEWSKI, J. [1996], *Rodzina elementów skończonych klasy C^0 w nieliniowej sześcio-parametrowej teorii powłok*, Zesz. Nauk. Pol. Gdańskiej, Nr 540, pp. 1-291.

CHRÓŚCIELEWSKI, J., MAKOWSKI, J. AND PIETRASZKIEWICZ, W. [2000], Large overall motion of flexible branched shell structures, in *NATO-ARW on Computational Aspects of Nonlinear Structural Systems with Large Rigid Body Motion* (eds. J. Ambrosio and M. Kleiber), pp. 201-218; IDMEC Press, Lisboa.

CHRÓŚCIELEWSKI, J., MAKOWSKI, J. AND PIETRASZKIEWICZ, W. [2002], Non-linear dynamics of flexible shell structures, *Comp. Ass. Mechanics and Engng. Sci.*, Vol. 9, No 3, pp. 341-357.

CHRÓŚCIELEWSKI, J., MAKOWSKI, J. AND STUMPF, H. [1992], Genuinely resultant shell finite elements accounting for geometric and material non-linearity, *Int. J. Num. Meth. Engng.*, Vol. 35, pp. 63-94.

CHRÓŚCIELEWSKI, J., MAKOWSKI, J. AND STUMPF, H. [1997], Finite element analysis of smooth, folded and multi-shell structures, *Comp. Meth. Appl. Mech. Engng.*, Vol. 141. pp.1-46.

COLEMAN, B.D. [1964], Thermodynamics of materials with memory, *Arch. Rat. Mech. Anal.*, Vol. 17, pp. 1-46.

COLEMAN, B.D. AND MIZEL, V.J. [1968], A general theory of dissipation in materials with memory, *Arch. Rat. Mech. Anal.*, Vol. 27, pp. 255-274.

COLEMAN, B.D. AND MIZEL, V.J. [1968A], On the general theory of fading memory, *Arch. Rat. Mech. Anal.*, Vol. 29, pp. 18-31.

COLEMAN, B.D. AND NOLL, W. [1963], The thermodynamics of elastic materials with heat conduction and viscosity, *Arch. Rat. Mech. Anal.*, Vol. 13, pp. 167-178.

COSSERAT, E. AND COSSERAT, F. [1909], *Theorie des corps deformables*, Herman et Flis, Paris (English transl.: NASA TT F-11, 561, NASA, Washington DC, 1968).

DEGROOT, S.R. AND MAZUR, P. [1969], *Non-Equilibrium Thermodynamics*, North-Holland P. Co., Amsterdam.

DUNN, J.E. [1986], Interstitial working and a nonclassical continuum thermodynamics, in *New Perspectives in Thermodynamics* (ed. J. Serrin), pp. 187-222; Springer-Verlag, Berlin.

DUNN, J.E. AND SERRIN, J. [1985], On thermodynamics of interstitial working, *Arch. Rat. Mech. Anal.*, Vol. 88, pp. 95-133.

GREEN, A.E. AND NAGHDI, P.M. [1970], Non-isothermal theory of rods, plates and shells, *Int. J. Solids Structures*, Vol. 6, pp. 209-244.

GREEN, A.E. AND NAGHDI, P.M. [1979], On thermal effects in the theory of shells, *Proc. R. Soc. London*, Vol. 365A, pp. 161-190.

GREEN, A.E., NAGHDI, P.M. AND WAINWRIGHT, W.L. [1965], A general theory of a Cosserat surface, *Arch. Rat. Mech. Anal.*, Vol. 20 (1965), pp. 287-308.

GURTIN, M.E. [1981], *An Introduction to Continuum Mechanics*, Academic Press, New York.

GURTIN, M.E. [2000], *Configurational Forces as Basic Concepts of Continuum Physics*, Springer-Verlag, New York, Inc.

GURTIN, M.E. AND MURDOCH, A.I. [1975], A continuum theory of elastic material surfaces, *Arch. Rat. Mech. Anal.*, Vol. 57, pp. 291.323.

HALMOS, P.R. [1958], *Finite-Dimensional Vector Spaces*, 2nd ed., Van Nostrand, Princeton.

HOFFMANN, D.W. AND CAHN, J.W. [1972], A vector thermodynamics for anisotropic surfaces. I. Fundamentals and application to plane surface junctions, *Surface Sci.*, Vol. 31, pp. 368-388.

JOU, D., CASAS-VÁZQUEZ, J. AND LEBON, G. [1988], Extended irreversible thermodynamics, *Rep. Prog. Phys.*, Vol. 51, pp. 1109-1179.

JOU, D., CASAS-VÁZQUEZ, J. AND LEBON, G. [2001], *Extended Irreversible Thermodynamics*, 3rd ed. Springer-Verlag, Berlin.

KESTIN, J. [1988], *Internal Variables in Thermodynamics and Continuum Mechanics*, Lecture Notes, Int. Centre for Mech. Sci., Udine.

KOITER, W.T. [1966], On the nonlinear theory of thin elastic shells, *Proc. Koninkl. Ned. Akad. Wetensch.*, Ser. B, Vol. 69, pp. 1-54.

KRÄTZIG, W.B. [1971], Allgemeine Schalentheorie beliebiger Werkstoffe und Verformung, *Ing.-Archiv*, Vol. 40, pp. 311-326.

LIBAI, A. AND SIMMONDS, J.G. [1983], Nonlinear elastic shell theory, in *Advances in Applied Mechanics*, Vol. 23, pp. 271-371; Academic Press.

LIBAI, A. AND SIMMONDS, J.G. [1988], *The Nonlinear Theory of Elastic Shells: One Spatial Dimension*, Academic Press.

LIBAI, A. AND SIMMONDS, J.G. [1998], *The Nonlinear Theory of Elastic Shells*, 2nd ed., Cambridge University Press.

MAKOWSKI, J. [2000], Distributed defects and configurational forces in shells, in *Volume of Abstracts of 33rd Solid Mechanics Conference*, pp. 273-274, Zakopane.

MAKOWSKI, J., PIETRASZKIEWICZ, W. AND STUMPF, H. [1999], Jump conditions in the non-linear theory of thin irregular shells, *Journal of Elasticity*, Vol. 54, pp.1-26.

MAKOWSKI, J. AND STUMPF, H. [1990], Buckling equations for elastic shells with rotational degrees of freedom undergoing finite strain deformation, *Int. J. Solids and Structures*, Vol. 26, pp. 353-368.

MAKOWSKI, J. AND STUMPF, H. [1994], *Mechanics of Irregular Shell Structures*, Mitt. Institut für Mechanik Nr 95, Ruhr-Universität Bochum.

MAUGIN, G.A. [1993], *Material Inhomogeneities in Elasticity*, Chapman and Hall, London

MAUGIN, G.A. [1995], Material forces: Concepts and applications, *Appl. Mech. Revs*, Vol. 48, pp. 213-245.

MÜLLER, I. AND RUGGERI, T. [1998], *Rational Extended Thermodynamics*, 2nd ed., Springer-Verlag, Berlin.

MURDOCH, A.I. [1976], A thermodynamical theory of elastic material interfaces, *Quart. J. Mech. Appl. Math.*, Vol. 29, pp. 245-275.

MURDOCH, A.I. [1990], A coordinate-free approach to surface kinematics, *Glasgow Math. J.*, Vol. 32, pp. 299-307.

MUSHTARI, KH.M. AND GALIMOV, K.Z. [1957], *Nonlinear Theory of Elastic Shells* (in Russian), Tatknigoizdat, Kazan' (English transl.: The Israel Program for Sci. Transl., Jerusalem, 1961).

NAGHDI, P.M. [1965], On the non-linear thermoelastic theory of shells, in: *Non-Classical Shell Problems* (eds. W. Olszak and A. Sawczuk), pp. 5-26; North-Holland Pub. Comp., Amsterdam,.

NAGHDI, P.M. [1972], The theory of plates and shells, in: *Handbuch der Physik* (eds. S. Flügge and C. Truesdell), Vol. VIa/2, pp.425-640; Springer-Verlag, Berlin.

NOLL, W. [1958], A mathematical theory of the mechanical behavior of continuous media, *Arch. Rat. Mech. Anal.*, Vol. 2, pp. 197-226.

NOOR, A.K. [1990], Bibliography of monographs and surveys on shells, *Appl. Mech. Revs.*, Vol. 43, pp. 223-234.

PIETRASZKIEWICZ, W. [1979], *Finite Rotations and Lagrangian Description in the Non-linear Theory of Shells*, Polish Sci. Publ., Warszawa-Poznań.

PIETRASZKIEWICZ, W. [1989], Geometrically nonlinear theories of thin elastic shells, *Advances in Mechanics*, Vol. 12, pp. 51-130.

PIETRASZKIEWICZ, W. [1992], Addendum to: Bibliography of monographs and surveys on shells, *App. Mech. Revs.*, Vol. 45, pp. 249-250.

PIETRASZKIEWICZ, W. [2001], Teorie nieliniowe powłok, w *Mechanika Sprężystych Płyt i Powłok* (red. Cz. Woźniak), ss. 424-497; Wydawnictwo Naukowe PWN, Warszawa.

SIMMONDS, J.G. [1984], The nonlinear thermodynamical theory of shells: Descent from 3-dimensions without thickness expansions, in *Flexible Shells, Theory and Applications* (eds. E.L .Axelrad and F.A. Emmerling), pp.1-11; Springer-Verlag, Berlin.

SIMMONDS, J.G. [2001], A simplified, nonlinear thermodynamic theory of beamshells, *Quart. Appl. Math.*, Vol. 59, pp. 401-412.

ŠILHAVÝ, M. [1997], *The Mechanics and Thermodynamics of Continuous Media*, Springer-Verlag, Berlin.

TOUPIN, R.A. [1962], Elastic materials with couple-stresses, *Arch. Rat. Mech. Anal.*, Vol. 11, pp. 385-414.

TRUESDELL, C. [1984], *Rational Thermodynamics*, 2nd ed., Springer-Verlag, New York .

TRUESDELL, C. AND NOLL, W. [1965], The Non-linear Field Theories of Mechanics, in *Handbuch der Physik* (ed. S. Flügge), Vol. III/3, Springer-Verlag, Berlin-Heidelberg-New York.

TRUESDELL, C. AND TOUPIN, R. [1960], The Classical Field Theories, in *Handbuch der Physik* (ed. S. Flügge), Vol. III/1, Springer-Verlag, Berlin.

WANG, C-C AND TRUESDELL, C. [1973], *Introduction to Rational Elasticity*, Noordhoff, Amsterdam.

WOŹNIAK, CZ. [1966], *Nieliniowa Teoria Powłok*, PWN, Warszawa.

Summary

We formulate rigorously the global and local laws of mechanics and thermodynamics for shells with singularities at some stationary or moving curves in the shell base surface (itself not necessarily smooth). The laws are derived in an exact manner from underlying laws of continuum thermomechanics written in the integral – impulse form for the shell-like body. Our formulation is sufficiently general to include not only traditional applications to reversible problems of regular shells, but also those modeling irreversible and non-smooth processes in irregular shells.

We assume that the shell-like body is represented in the physical space by the base surface, which in a reference configuration is only Lipschitz continuous with almost smooth boundary. By a moving singular curve we mean a one-parameter family of piecewise smooth surface curves which transverse the reference configuration of the shell base surface and across which various thermomechanical field variables may suffer jump discontinuities. However, all the fields are assumed to be regular enough for the generalized surface transport and gradient-divergence theorems to be applicable.

As a result of complex transformations presented in the report, at regular points of the reference base surface and for almost all time instants we obtain five local laws of shell thermomechanics: the balance of mass, linear momentum, angular momentum, and energy as well as the entropy inequality. From the transformations we also obtain, corresponding to the laws of shell thermomechanics, five continuity conditions at regular points of every singular surface curve. Additionally, we discuss exact 2D shell kinematics and exact 2D shell strain measures.

The principal features of the derived field equations and side conditions are: 1) the classical expressions for the linear and angular momenta are not assumed from the outset (they must be given by appropriate constitutive equations), 2) there is no classical splitting of the total energy into the sum of internal and kinetic energies (such a splitting is considered as a part of constitutive theory), 3) the entropy source and the entropy influx are not directly related to temperature, 4) there are two additional terms in the equation

of energy balance which represent the interstitial working (they require a suitable constitutive prescription).

We show that within the general shell thermomechanics the constitutive equations are needed for the surface stress tensor, the surface couple tensor, the specific total energy, the specific entropy, and the heat influx vector fields. But additionally we need the constitutive prescription for the linear and angular momenta vectors, as well as possibly for several other supplementary field variables. General expressions for the constitutive equations are given through response functionals of the histories of motion and temperature fields. For spatially first-grad "simple" shells we propose reduced forms of constitutive equations in the spatial and material representations. We also discuss additional constitutive assumptions which would allow us to eliminate temperatures, heats and entropy influxes at the upper and lower shell faces, as well as fields describing the interstitial working, the extra entropy source and the extra entropy flux.

We derive the reduced dissipation inequality for shells and use it to develop thermodynamically consistent constitutive equations appropriate for heat conducting and thermo-visco-elastic shells. Particular forms of constitutive equations for thermoelastic, isothermal or higher-grad shells are proposed. By introducing thermodynamic potentials we also discuss constitutive nature of representing the total shell energy density as the sum of potential, kinetic and interstitial energy densities. Finally, we propose general and some specific forms of the kinetic constitutive equations for the linear and angular momenta.

The results presented in this report may be considered as an introduction to a variety of thermomechanical problems of the regular and irregular shells, which might be formulated and solved already in the near future.