1. BULLETIN DE L'ACADÉMIE
POLONAISE DES SCIENCES
Série des sciences techniques
Volume XV, No. 5 — 1967

APPLIED MECHANICS

On a Solving Equation for Shallow Shells

by
W. PIETRASZKIEWICZ

Presented by R. SZEWALSKI on January 6, 1967

1. Introduction

In [1] some completion and extension of the linear theory of shallow shells, based on the approach suggested by Green and Zerna [2], have been given. In this approach vector e_3 normal to shell projection is constant, and its direction is independent of the function describing the middle surface of the shell. Resulting equations are written in the system of basic vectors e_i of shell projection (basis e_i) and expressed by components of unknown vector quantities, also in basis e_i. In this way simple geometric links are possible to establish between the relations pertaining to shells and some of the known relations of the theory of plates and plane elasticity. This may also be useful in designing multivalued solutions for shallow shells with two- or multi-connected regions [3].

In the present Note three methods of reducing the system of fundamental equations for shallow shells obtained in [1], to a single solving equation, are considered. These are the solutions: in terms of displacements, in terms of stress functions, and by a mixed method. With the additional assumption of small variation of the shell curvature, all the three methods lead to solving of the same complex partial differential equation of fourth order, particular integrals of which are determined in various ways.

The problem has also been discussed in [1].

2. Assumptions

Basic notations are those used in [2].

Relations for shallow shells have been obtained in [1] with the following assumptions:

a) Relations of linear shell theory are based on Kirchhoff — Love's hypothesis, [4], [5].

b) For the shell geometry there is

$$|z_{,\alpha}| < 1, \quad |z_{|\alpha\beta}| < 1. \tag{2.1}$$

c) For the state of stress and displacements there is

$$|q^\alpha| < |n^{\alpha\beta}|, \quad |v_\alpha| < |v_3|. \tag{2.2}$$

In the reduction of equations in terms of displacements and stress functions to a single solving equation, the additional assumption is made:

d) The variation of shell curvature is small, so that

$$(z|^\alpha_\beta \cdot A^{\varrho\cdot\cdot}_{\lambda\cdot\cdot})|_\delta \simeq z|^\alpha_\beta \cdot A^{\varrho\cdot\cdot}_{\lambda\cdot\cdot}|_\delta. \tag{2.3}$$

3. Fundamental equations

The fundamental equations of the linear theory of shallow shells have been derived in [1] by representing all the vector relations of the linear shell theory in basis $\boldsymbol{e}_i$. Using then the simplifications resulting from (2.1) and (2.2) the equations can be written as [2]:

equation of equilibrium:

$$\begin{aligned} &k^{\alpha\delta}|_\alpha + s^\delta = 0, \\ &(k^{\alpha\delta} z_{,\alpha})|_\delta + h^{\alpha\delta}|_{\alpha\delta} + s^3 = 0 \end{aligned} \tag{3.1}$$

compatibility equations:

$$\begin{aligned} &(\epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\mu_{\gamma\beta})|_\alpha = 0, \\ &(\epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\mu_{\gamma\beta}\cdot z_{,\alpha})|_\delta - (\epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\alpha_{\gamma\beta})|_{\alpha\delta} = 0, \end{aligned} \tag{3.2}$$

where

$$\begin{aligned} \alpha_{\gamma\beta} &= \frac{1}{2}(u_\gamma|_\beta + u_\beta|_\gamma + z_{,\gamma}\cdot u_{3,\beta} + z_{,\beta}\cdot u_{3,\gamma}), \\ \mu_{\gamma\beta} &= -u_3|_{\gamma\beta} \end{aligned} \tag{3.3}$$

constitutive equations:

$$\begin{aligned} k^{\alpha\beta} &= D\cdot H^{\alpha\beta\varrho\lambda}\alpha_{\varrho\lambda}, \quad & \alpha_{\alpha\beta} &= \frac{1}{E\lambda L} H^{-1}_{\alpha\beta\varrho\lambda}\cdot k^{\varrho\lambda}, \\ h^{\alpha\beta} &= B\cdot H^{\alpha\beta\varrho\lambda}\mu_{\varrho\lambda}, \quad & \mu_{\alpha\beta} &= \frac{12}{E\lambda^3 L} H^{-1}_{\alpha\beta\varrho\lambda}\cdot h^{\varrho\lambda}. \end{aligned} \tag{3.4}$$

where [1]

$$H^{\alpha\beta\varrho\lambda} = \frac{1}{2}\{e^{\alpha\varrho}e^{\beta\lambda} + e^{\alpha\lambda}e^{\beta\varrho} + \nu(\epsilon^{\alpha\varrho}\epsilon^{\beta\lambda} + \epsilon^{\alpha\lambda}\epsilon^{\beta\varrho})\},$$

$$H^{-1}_{\alpha\beta\varrho\lambda} = \frac{1}{2}\{e_{\alpha\varrho}e_{\beta\lambda} + e_{\alpha\lambda}e_{\beta\varrho} - \nu(\epsilon_{\alpha\varrho}\epsilon_{\beta\lambda} + \epsilon_{\alpha\lambda}\epsilon_{\beta\varrho})\},$$

$$A = \frac{12(1-\nu^2)}{\lambda^2}, \quad B = \frac{E\lambda L}{A}, \quad D = \frac{E\lambda L}{1-\nu^2}, \quad k = \frac{E\lambda L}{\sqrt{A}}.$$

The first static-geometric analogy follows from (3.1) and (3.2)

$$k^{\alpha\delta} \leftrightarrow \epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\mu_{\gamma\beta}, \quad h^{\alpha\delta} \leftrightarrow -\epsilon^{\alpha\gamma}\epsilon^{\delta\beta}\alpha_{\gamma\beta}. \tag{3.5}$$

Symbols u_a, u_3 appearing in (3.3) denote the components of displacement vector $\boldsymbol{v}$ in basis $\boldsymbol{e}_i$

$$\boldsymbol{v} = v_\alpha \boldsymbol{a}^\alpha + v_3 \boldsymbol{a}^3 = u_\alpha \boldsymbol{e}^\alpha + u_3 \boldsymbol{e}^3 . \tag{3.6}$$

The expressions (3.1) to (3.5) contain only the components in basis $\boldsymbol{e}_i$.

4. Three ways of solving the fundamental equations

The set of Eqs. (3.1), (3.2) and (3.4) can be solved in three different ways:

a) in terms of displacements u_α, u_3,

b) in terms of stress functions φ_α, φ_3,

c) by a mixed method, with u_3 and φ_3 taken as unknowns.

A similar method has been used by Duddeck [6] in his investigation of the homogeneous equations of edge effect in the linear shell theory, with v_α, v_3 and ψ_α, ψ_3 as unknowns, and with quite different initial simplifying assumptions.

5. Solution in terms of displacements

By using (3.4) and (3.3) Eqs. (3.1) are expressed in terms of displacements u_α, u_3, and take on the following form [1], [5]

$$\begin{aligned} &H^{\alpha\delta\mu\nu} a_{\mu\nu|\alpha} + \frac{1}{D} s^\delta = 0 , \\ &H^{\alpha\delta\mu\nu} \{(a_{\mu\nu} z_{,\alpha})|_\delta + \frac{\lambda^2}{12} \mu_{\mu\nu|\alpha\delta}\} + \frac{1}{D} s^3 = 0 . \end{aligned} \tag{5.1}$$

A system analogous to (5.1) and written in terms of displacements v_α, v_3 and in lines of curvature coordinates, has been considered by Vlasov [7] and other authors. They solve it in a specific coordinate system using the method of operator determinants or the elimination of v_α. An alternate way is to represent v_α, v_3 by a displacement function chosen appropriately to the coordinate system and shell shape.

There exists the possibility of invariant reduction of the system (5.1) to a single solving equation, without representing it in a specific coordinate system, provided the additional assumption of small variation of shell curvature (2.3) is made. This assumption is met for shell shapes encountered most frequently.

With the assumption (2.3) the set of Eqs. (5.1) may be fulfilled identically by introducing the displacement function F in the form [6]

$$\begin{aligned} &u_\mu = 2z|^\lambda_\mu \cdot F|^\varrho_{\varrho\lambda} - z|^\lambda_\varrho \cdot F|^\varrho_{\lambda\mu} + \nu\delta^{\sigma\tau}_{\lambda\varrho}\, z|^\lambda_\sigma \cdot F|^\varrho_{\tau\mu} - z_{,\mu}\, F|^{\varrho\lambda}_{\varrho\lambda} + \bar{u}_\mu , \\ &u_3 = F|^{\varrho\lambda}_{\varrho\lambda} , \end{aligned} \tag{5.2}$$

where $\bar{u}_\mu$ satisfies the set of equations

$$H^{\alpha\delta\mu\nu} \bar{u}_{\mu|\nu\alpha} + \frac{1}{D} s^\delta = 0 .$$

Eq. $(5\ 1)_2$ leads then to the following solving equation

$$\begin{aligned} &F|^{\alpha\beta\varrho\lambda}_{\alpha\beta\varrho\lambda} + A\delta^{\mu\beta}_{\gamma\delta}\, \delta^{\alpha\nu}_{\varrho\lambda} \cdot z|^\gamma_\mu\, z|^\varrho_\alpha \cdot F|^{\delta\lambda}_{\beta\nu} = p , \\ &p = \frac{A}{E\lambda L} \{s^3 + DH^{\alpha\delta\mu\nu}\, \bar{u}_{\mu|\nu}\, z_{|\alpha\delta}\} . \end{aligned} \tag{5.3}$$

Having determined the particular integral $\overset{0}{F}$ of Eq. $(5.3)_1$ we can find a general solution from the relation

$$\overset{0}{F}|^{\alpha\varrho}_{\alpha\varrho} \mp i\sqrt{A}\,\delta^{\alpha\nu}_{\varrho\lambda}\, z|^{\varrho}_{\alpha} \cdot \overset{0}{F}|^{\lambda}_{\nu} = 0. \tag{5.4}$$

6. Solution in terms of stress functions

According to the analogy (3.5) the solution of (3.1) may be put in the form

$$\begin{aligned} k^{\alpha\delta} &= \overset{s}{k}{}^{\alpha\delta} + k\cdot\epsilon^{\alpha\gamma}\,\epsilon^{\delta\beta}\,\mu^{*}_{\gamma\beta}, \\ h^{\alpha\delta} &= \overset{s}{h}{}^{\alpha\delta} - k\cdot\epsilon^{\alpha\gamma}\,\epsilon^{\delta\beta}\,\alpha^{*}_{\gamma\beta}, \end{aligned} \tag{6.1}$$

where $\overset{s}{k}{}^{\alpha\delta}$, $\overset{s}{h}{}^{\alpha\delta}$ — particular integral of (3.1), $\mu^{*}_{.\gamma\beta}$, $\alpha^{*}_{\gamma\beta}$ — expressions analogous to (3.3) constructed on components φ_α, φ_3 of the stress function vector (cf. (3.6))

$$\boldsymbol{v}^{*} = \psi_\alpha\,\boldsymbol{a}^\alpha + \psi_3\,\boldsymbol{a}^3 = \varphi_\alpha\,\boldsymbol{e}^\alpha + \varphi_3\,\boldsymbol{e}^3. \tag{6.2}$$

Eqs. (3.2) written in terms of stress functions φ_α, φ_3 by using (3.4) and (6.1) rearrange themselves to [1], [5]

$$\begin{aligned} & H^{*\alpha\delta\mu\nu}\,\alpha^{*}_{\mu\nu|\alpha} + \frac{1}{D}\,s^{*\delta} = 0, \\ & H^{*\alpha\delta\mu\nu}\,\{(\alpha^{*}_{\mu\nu}\,z_{,\alpha})_{|\delta} + \frac{\lambda^2}{12}\,\mu^{*}_{\mu\nu|\alpha\delta}\} + \frac{1}{D}\,s^{*3} = 0, \end{aligned} \tag{6.3}$$

where

$$H^{*\alpha\delta\mu\nu} = \frac{1}{2}\{e^{\alpha\mu}\,e^{\delta\nu} + e^{\alpha\nu}\,e^{\delta\mu} - \nu(\epsilon^{\alpha\mu}\,\epsilon^{\delta\nu} + \epsilon^{\alpha\nu}\,\epsilon^{\delta\mu})\}.$$

Since the tensors $H^{\alpha\delta\mu\nu}$ and $H^{*\alpha\delta\mu\nu}$ differ only in the sign of the Poisson's coefficient, we can state the validity of the second static-geometric analogy [5]

$$(+\nu)_u \leftrightarrow (-\nu)_\varphi. \tag{6.4}$$

Hence, with the additional assumption (2.3), the system (6.3) may also be reduced invariantly to a single solving equation by introducing stress function Φ in the form (5.2), with φ_μ, $\bar{\varphi}_\mu$, φ_3, $(-\nu)$, Φ written in place of u_μ, $\bar{u}_\mu$, u_3, $(+\nu)$, F, respectively. Final solution consists in the solving of Eq. (5.4) together with the determination of a particular integral of Eq. $(5.3)_1$, with the expression $(5.3)_2$ appropriately constructed on the quantities s^{*3}, $H^{*\alpha\delta\mu\nu}$, and $\bar{\varphi}_\mu$, [1]

7. Solution by a mixed method

As follows from the analogy (3.5), Eqs. $(3.1)_1$ are fulfilled identically by substituting

$$k^{\alpha\delta} = \bar{k}^{\alpha\delta} + k\cdot\epsilon^{\alpha\gamma}\,\epsilon^{\delta\beta}\,\mu^{*}_{.\gamma\beta}, \tag{7.1}$$

where $\bar{k}^{\alpha\delta}$ satisfies the set of equations

$$\bar{k}^{\alpha\delta}|_\alpha + s^\delta = 0.$$

From $(3.1)_2$ and $(3.2)_2$ we obtain the solving equation

$$\begin{aligned} &\Psi|^{\alpha\varrho}_{\alpha\varrho} \mp i\sqrt{A}\,\delta^{\alpha\gamma}_{\delta\beta}\, z|^{\delta}_{\alpha} \cdot \Psi|^{\beta}_{\gamma} = q, \\ (7.2)\qquad &q = \frac{A}{E\lambda L}\left(s^3 - z_{,\alpha} s^{\alpha} + \bar{k}^{\alpha\delta} z|_{\alpha\delta}\right) \pm i\frac{\sqrt{A}}{E\lambda L}\left(\delta^{\alpha\lambda}_{\delta\varrho}\bar{k}^{\varrho}_{\lambda}|^{\delta}_{\alpha} + \nu s^{\alpha}|_{\alpha}\right), \\ &\Psi = u_3 \pm i\varphi_3, \quad A = \frac{12(1-\nu^2)}{\lambda^2}, \quad k = \frac{E\lambda L}{\sqrt{A}}. \end{aligned}$$

The Airy stress function φ introduced by Vlasov [7], Green and Zerna [2] and other authors, has not the geometric sense of the component of $\boldsymbol{v}^*$ vector. It may be represented by the relation

$$\varphi \approx -\frac{1}{k}\psi_3 \approx -\frac{1}{k}\varphi_3$$

giving also the function φ a geometric sense of the component of $-\frac{1}{k}\boldsymbol{v}^*$ vector.

Alternate forms of mixed method solving equation can also be derived by defining, e.g. complex functions $\varphi_3 \pm iu_3$, $\varphi \pm i\frac{1}{k}u_3$, etc.

DEPARTMENT OF DYNAMICS OF MACHINES, INSTITUTE OF FLUID-FLOW MACHINERY, GDAŃSK, POLISH ACADEMY OF SCIENCES
(ZAKŁAD DYNAMIKI MASZYN, INSTYTUT MASZYN PRZEPŁYWOWYCH, GDAŃSK, PAN)

REFERENCES

[1] W. Pietraszkiewicz, *O liniowej teorii powlok o malej wynioslości* [in Polish] [*On the linear theory of shallow shells*] Rozpr. Inż., **16** (1967), No. **2**.

[2] A. E. Green, W. Zerna, *Theoretical elasticity*, Oxford, 1954.

[3] W. Pietraszkiewicz, Ph. D. Dissertation, Technical University, Gdańsk, 1966.

[4] K. F. Chernykh, *Lineynaya tieoria obolochek* [in Russian], [*Linear theory of shells*], Vol. 2, Leningrad, 1964.

[5] P. M. Naghdi, *Foundations of elastic shell theory* [in:] *Progress in solid mechanics*, Vol. 4, North-Holland, 1963.

[6] H. Duddeck, *Das Randstörungsproblem der technischen Biegetheorie dünner Schalen in drei korrespondierenden Darstellungen*, Öster. Ing. — Arch. **1** (1962).

[7] V. Z. Vlasov, *Sobrannye sochinienya* [in Russian], [*Collected papers*], Vol. 1, AN SSSR, Moscow, 1962.

В. ПЕТРАШКЕВИЧ, О РАЗРЕШАЮЩЕМ УРАВНЕНИИ ПОЛОГИХ ОБОЛОЧЕК

Рассматриваются уравнения пологих оболочек выраженные лишь компонентами неизвестных векторных величин в базисе проекции оболочки на плоскость. Уравнения решаются в перемещениях, в функциях напряжений и смешанным методом. При дополнительном условии малой изменяемости кривизны оболочки, все три метода приводят к решению одного и того-же разрешающего комплексного уравнения 4-го порядка при различном вычислении частного решения.