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APPLIED MECHANICS

On the Multivaluedness of Solutions of Shallow Shells

by

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1. Introduction

Displacement vector $\mathbf{v}$ of the middle surface of the shell and the stress function vector $\mathbf{v}^*$ associated with the vector $\mathbf{v}$ by a static-geometric analogy, are the general solutions of the set of equations of the linear bending theory of thin elastic shells [1]–[4].

The basic relations for the multivalued solutions of the shells with multiconnected region, have been given by Chernykh [5], [6] and by the author [7], [8]. By using the general considerations of papers [5]–[8] it is only possible to ascertain whether the multivaluedness of $\mathbf{v}$ and $\mathbf{v}^*$ exists. The problem of constructing the multivalued part of the solutions in a general case is not solved as yet.

In this Note the problem of multivaluedness of the solutions of shallow shells is considered. The way of constructing the multivalued part of $\mathbf{v}$ and $\mathbf{v}^*$ for the shells is indicated.

A more extensive treatment of the problem will be published in [12].

2. Geometric relations

Basic notations are those used in papers [8]–[11].
The quantities $\bar{\theta}^a$, $\bar{\mathbf{r}}(\bar{\theta}^a)$, $\bar{\mathbf{r}}^i$, $\mathbf{a}_a$, $\mathbf{a}_3$, $a_{\alpha\beta}$, $b_{\alpha\beta}$, $\bar{\in}_{\alpha\beta}$, $(\ )||_a$, $\bar{\Gamma}^a_{\beta\gamma}$, $\bar{\Gamma}_i$, $\bar{\Gamma}$, $\bar{s}$, $\bar{\mathbf{t}}$, $\bar{\boldsymbol{\nu}}$ etc. are associated with the middle surface M of the shell, (Fig. 1). The quantities θ^a, $\mathbf{r}(\theta^a)$, $\mathbf{r}^i$, $\mathbf{e}_a$, $\mathbf{e}_3$, $e_{\alpha\beta}$, $z|_{\alpha\beta}$, $\in_{\alpha\beta}$, $(\ )|_a$, $\Gamma^a_{\beta\gamma}$, Γ_i, Γ, s, t, ν etc, are associated, respectively, with the plane Π.

Using the approximate relations between the quantities associated with M and Π [9], [11], we have

$$(2.1)\qquad \begin{aligned} \bar{\mathbf{t}} &\simeq \mathbf{t}+z,_a\, t^a\, \mathbf{e}_3, & \bar{\boldsymbol{\nu}} &\simeq \boldsymbol{\nu}+z,_a\, \boldsymbol{\nu}^a\, \mathbf{e}_3, \\ \bar{t}^a &\simeq t^a, & \bar{\boldsymbol{\nu}}^a &\simeq \boldsymbol{\nu}^a. \end{aligned}$$

For a space point i [8] associated with the internal closed boundary contour $\bar{\Gamma}_i$, we have (Fig. 1)

$$(2.2)\qquad \begin{aligned} &\bar{\mathbf{r}} - \mathbf{R}^i = \boldsymbol{\rho}^i + (z - z^i)\cdot \mathbf{e}_3, \\ &\boldsymbol{\rho}^i = \mathbf{r} - \mathbf{r}^i, \quad z^i = \mathbf{R}^i \cdot \mathbf{e}_3, \end{aligned}$$

and for an arbitrary point k on surface M, we have

$$\bar{\mathbf{r}} - \mathbf{r}^k = \boldsymbol{\rho}^k + [z - z(\theta^a_k)]\,\mathbf{e}_3. \tag{2.3}$$

3. Multivaluedness of displacements

The multivalued part of the displacements vector $\mathbf{v}$ on each internal closed boundary contour $\bar{\Gamma}_i$ (Fig. 1) has the form [5]—[7]

$$\overset{d}{\mathbf{v}}{}^i = [\mathbf{v}^i + \boldsymbol{\Omega}^i \times (\bar{\mathbf{r}} - \mathbf{R}^i)] \cdot \Phi_i(\bar{\theta}^a), \tag{3.1}$$

where the parameters of dislocation of displacements $\mathbf{v}^i$, $\boldsymbol{\Omega}^i$ have been expressed in [5], [6] by a vector $\boldsymbol{\varkappa}_t$ of the contour curvature variation during deformation, by the following relations

$$\begin{aligned} \mathbf{v}^i &= \oint_{\bar{\Gamma}_i} [(\bar{\mathbf{r}} - \mathbf{R}^i) \times \boldsymbol{\varkappa}_t + (a_{\alpha\beta}\,\bar{t}^\alpha\,\bar{t}^\beta)\,\bar{\mathbf{t}}]\,d\bar{s}, \\ \boldsymbol{\Omega}^i &= \oint_{\bar{\Gamma}_i} \boldsymbol{\varkappa}_t\,d\bar{s}. \end{aligned} \tag{3.2}$$

Expressing all the vectors in (3.1) and (3.2) by components in basis $\mathbf{e}_j$, and introducing then the approximations of the shallow shells in accord with [9]—[11], for the components $\overset{d}{\mathbf{v}}{}^i$ in basis $\mathbf{e}_j$ we have

$$\begin{aligned} \overset{d}{u}{}^i_\alpha &\simeq \{u^i_\alpha + \epsilon_{\beta\alpha}\,[\varrho^{i\beta}\,\omega^i_3 - (z - z^i)\,\omega^{i\beta}]\} \cdot \Phi_i(\theta^a), \\ \overset{d}{u}{}^i_3 &\simeq \{u^i_3 + \epsilon^{\alpha\beta}\,\omega^i_\alpha\,\varrho^i_\beta\}\,\Phi_i(\theta^a), \end{aligned} \tag{3.3}$$

where

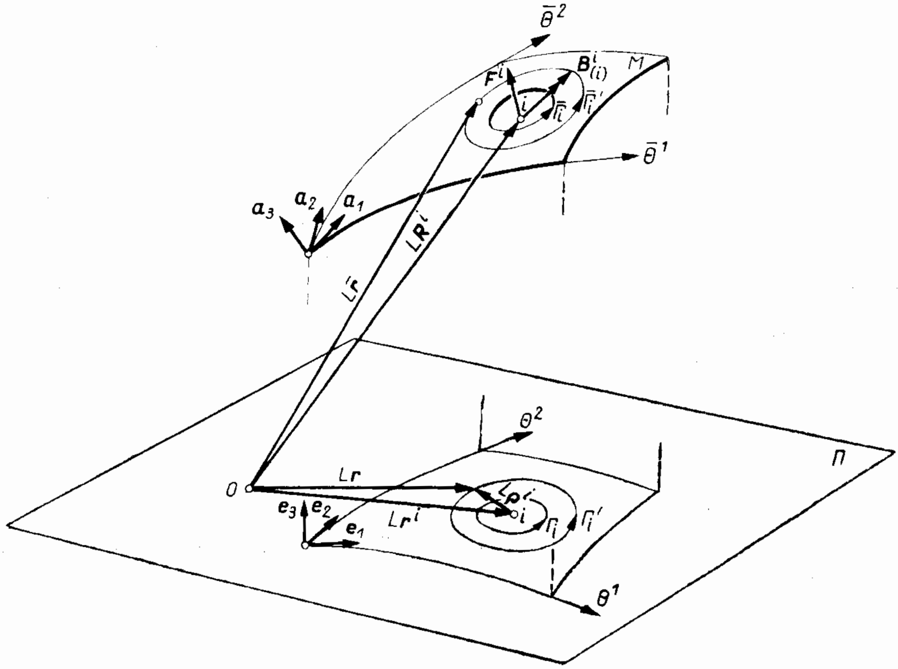
$$\begin{aligned} u^i_3 &\simeq \oint_{\Gamma_i} \epsilon_{\alpha\beta}\,\varrho^{i\alpha}\,\varkappa^\beta_t \cdot ds, \\ \omega^i_\alpha &\simeq \mathbf{e}_\alpha \cdot \oint_{\Gamma_i} \varkappa^\beta_t\,\mathbf{e}_\beta \cdot ds, \\ \varkappa^\beta_t &\simeq -u_{3|\varrho\lambda}\,t^\varrho\,\delta^{\lambda\beta}_{\sigma\tau}\,\nu^\sigma\,t^\tau. \end{aligned} \tag{3.4}$$

The function $z(\theta^a)$ describing the middle surface of the shell does not appear in the relations $(3.3)_2$ and (3.4).

4. Multivaluedness of stress functions

The multivalued part of the stress function vector $\mathbf{v}^*$ on each internal closed boundary contour $\bar{\Gamma}_i$ (Figure) has, according to the static-geometric analogy, the form analogous to (3.1), [5]—[8]

$$\overset{d}{\mathbf{v}}{}^{*i} = [\mathbf{v}^{*i} + \boldsymbol{\Omega}^{*i} \times (\bar{\mathbf{r}} - \mathbf{R}^i)]\,\Phi_i(\bar{\theta}^a). \tag{4.1}$$



Parameters of dislocation of the stress functions $\mathbf{v}^{*i}$, $\mathbf{\Omega}^{*i}$ has been expressed in [8] by a total force $\mathbf{F}^i$ and a total couple $\mathbf{B}^i_{(i)}$ of the external boundary loading, and by certain arbitrary surface forces $\frac{1}{L}\mathbf{p}_i$ acting on extended shell regions $S_{\bar{\Gamma}_i}$.

Expressing all the vectors in (4.1) (and in (4.5) to (4.7) of [8]) by the component in basis $\mathbf{e}_j$ and introducing then the approximations of the shallow shells according to [9]—[11], for the components of $\overset{d}{\mathbf{v}^i}$ in basis $\mathbf{e}_j$ we have

$$\text{(4.2)}\qquad \begin{aligned} \overset{d}{\varphi^i_\alpha} &\equiv \overset{d}{u^{*i}_\alpha} = \{u^{*i}_\alpha + \epsilon_{\beta\alpha}\,[\varrho^{i\beta}\,\omega^{*i}_3 - (z - z^i)\,\omega^{*i\beta}]\}\Phi_i(\theta^\alpha),\\ \overset{d}{\varphi^i_3} &\equiv \overset{d}{u^{*i}_3} = \{u^{*i}_3 + \epsilon^{\alpha\beta}\,\omega^{*i}_\alpha\,\varrho^i_\beta\}\Phi_i(\theta^\alpha), \end{aligned}$$

where, according to [8], for instance for the loaded internal closed boundary contour we have

$$\text{(4.3)}\qquad \begin{aligned} u^{*i}_3 &= -\frac{1}{kL^2}\{B^i_{(i)3} - \overset{s}{B^i_{(i)3}} - L^2 \iint\limits_{S_{\Gamma_i}} \epsilon^{\alpha\beta}\,\varrho^i_\alpha\, s_{i\beta}\cdot dS\},\\ \omega^{*i}_\alpha &= -\frac{1}{kL}\{F^i_\alpha - \overset{s}{F^i_\alpha} - L\iint\limits_{S_{\Gamma_i}} s_{i\alpha}\cdot dS\}. \end{aligned}$$

As in the par. **3**, here also the function $z(\theta^\alpha)$ describing the middle surface of the shell does not appear in the relations $(4.2)_2$ and (4.3).

5. Conclusions

It results from the relations $(3.3)_2$, (3.4) and $(4.2)_2$, (4.3) that, within the bounds of approximations of the linear theory of shallow shells [9], [10], the following conclusions are valid:

1. The existence of multivaluedness of displacement u_3 (or — stress function φ_3), after integrating round the internal closed boundary contour $\overline{\Gamma}_i$ of the shell, depends only on three components u_3^i, ω_α^i (or — u_3^{*i}, ω_α^{*i}) of the parameters of dislocation of displacements (or — stress functions), and is independent of the remaining three components.

2. The structure of the multivalued part of u_3 (or — φ_3) for a shallow shell is the same as for a plate (or — plane elasticity) with the same shape as shell projection on reference plane Π and with the same parameters of dislocation u_3^i, ω_α^i (or — u_3^{*i}, ω_α^{*i}).

The foregoing conclusions allow to construct easily the multivalued terms of u_3 and φ_3 on the basis of known solutions of plane elasticity for multivalued regions [13]. By using the mixed method [9], [10] of solving the equations of shallow shells, with u_3 and φ_3 as basic unknowns, the multivaluedness of the solutions can be eliminated from further considerations where the problem reduces to a singlevalued boundary value problem.

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В. ПЕТРАШКЕВИЧ, О МНОГОЗНАЧНОСТИ РЕШЕНИЙ ПОЛОГИХ ОБОЛОЧЕК

Рассмотрена проблема многозначности общих решений уравнений линейной теории пологих оболочек: вектора перемещений и вектора функций напряжений. Област оболочки считается многосвязной. Доказывается что, с точностью до основных упрощающих гипотез пологих оболочек, компоненты многозначного члена этих векторов по вектору перпендикулярному к плоскости проекции оболочки зависят только от трех (из шести) компонент т. наз. параметров дислокации. Вид многозначного члена этих компонент не зависит от формы оболочки и можно его строить так как строится для пластины (или плоского напряженного состояния) с граничным контуром проекции оболочки на плоскость и тех-же компонентах параметров дислокации.