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APPLIED MECHANICS

On the Multivaluedness of Stress Functions in the Linear Theory of Shells

by

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1. Introduction

Two general methods of solving the equations of the linear bending theory of thin elastic shells are known — by using displacement vector $\mathbf{v}$ or by using stress function vector $\mathbf{v}^*$. Both these methods are connected by the static-geometric analogy, [1]—[4].

The multivaluedness of $\mathbf{v}$ with the assumption of singlevalued strains, and the multivaluedness of $\mathbf{v}^*$ with the assumption of singlevaluedness of the expressions called "strains of stress function" connected with strains by static-geometric analogy, were considered by Chernykh [5], [6]. The review of the multivalued solutions was published in [7].

In this Note the problem of multivaluedness of the stress function vector in a general case is considered. The shell is assumed to have multivalued region and to be loaded by surface forces, concentrated forces and couples, and non-selfbalanced loads acting on internal closed boundary contours.

The full text of this paper including the detailed conclusions will be published in [8].

2. Fundamental relations

Basic notations are those used by Green and Zerna [9], and by Chernykh [5], [6].

According to the Kirchhoff-Love hypothesis the displacement vector has the form [9]

$$\mathbf{V} = L\,\{\mathbf{v}\,(\theta^\alpha) + \lambda\theta^3\,\mathbf{\Omega}\times\mathbf{a}_3\},$$

where [6]

$$\mathbf{v} = v^\alpha\,\mathbf{a}_\alpha + v^3\,\mathbf{a}_3, \qquad \mathbf{\Omega} = \Omega^\alpha\,\mathbf{a}_\alpha + \Omega^3\,\mathbf{a}_3,$$

(2.1)

$$\Omega^\alpha = \epsilon^{\alpha\beta}\,(v_{3,\beta} + b^\varrho_\beta\,v_\varrho), \qquad \Omega^3 = \frac{1}{2}\,\epsilon^{\beta\alpha}\,v_{\alpha|\beta}.$$

Tensors of strain can be written as [6], [10]

$$(2.2)\qquad \begin{aligned} \alpha_{\varrho\lambda} &= \frac{1}{2}(v_{\varrho|\lambda}+v_{\lambda|\varrho})-b_{\varrho\lambda}\,v_3\,,\\ \mu_{\varrho\lambda} &= -\,v_{3|\varrho\lambda}-(b^{\alpha}_{\lambda}\,v_{\alpha})|_{\varrho}-\frac{1}{2}\,\delta^{\alpha\beta}_{\gamma\varrho}\,b^{\gamma}_{\lambda}\,v_{\alpha|\beta}\,. \end{aligned}$$

As follows from the static-geometric analogy [1], [2] the general solution of the equations of equilibrium may be put in the form [4], [6]

$$(2.3)\qquad \begin{aligned} n^{\alpha\delta} &= \overset{s}{n}{}^{\alpha\delta}+k\cdot\in^{\alpha\gamma}\in^{\delta\beta}\overset{*}{\mu}_{\gamma\beta}\,,\\ m^{\alpha\delta} &= \overset{s}{m}{}^{\alpha\delta}-k\cdot\in^{\alpha\gamma}\in^{\delta\beta}\overset{*}{\alpha}_{\gamma\beta}\,, \end{aligned}$$

where

$\overset{s}{n}{}^{\alpha\delta}$, $\overset{s}{m}{}^{\alpha\delta}$ — particular solution of the equation of equilibrium,
$\overset{*}{\alpha}_{\varrho\lambda}$, $\overset{*}{\mu}_{\varrho\lambda}$ — expressions analogous to (2.2) called "strains of stress function", constructed on components ψ^{α}, ψ^3 of the stress function vector

$$(2.4)\qquad \mathbf{v}^* = \psi^{\alpha}\,\mathbf{a}_{\alpha}+\psi^3\,\mathbf{a}_3\,.$$

All expressions constructed on components ψ^{α}, ψ^3, analogous to the expressions constructed on v^{α}, v^3, are marked with *, [4].

Let Γ denote a curve on the middle surface of the shell, and let $\mathbf{t} = t^{\alpha}\,\mathbf{a}_{\alpha}$ denote the unit tangent vector at a point of Γ. Along Γ we have the rotation vector of a normal element associated with Γ [5], [6]

$$(2.5)\qquad \mathbf{\Omega}_t = \mathbf{\Omega}-\alpha_{\alpha\beta}\,t^{\alpha}\,t^{\beta}\cdot\mathbf{a}_3$$

and the total force $\mathbf{F}$ and the total couple $\mathbf{B}_{(j)}$ of internal forces with respect to an arbitrary space point j [5]—[7] (Fig. 1)

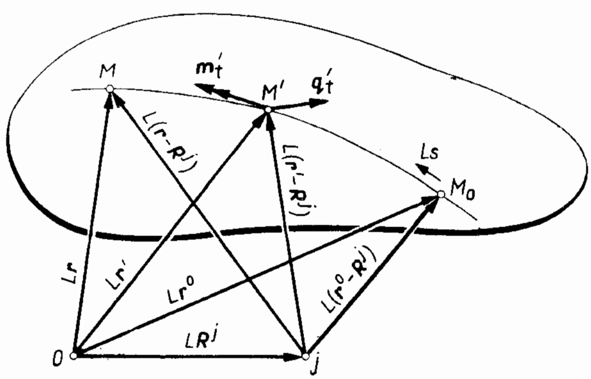

Fig. 1

$$(2.6)\qquad \begin{aligned} \mathbf{F} &= L\int\limits_{M_0}^{M}\mathbf{q}'_t\,ds'\,,\\ \mathbf{B}_{(j)} &= L^2\int\limits_{M_0}^{M}\{\mathbf{m}'_t+(\mathbf{r}'-\mathbf{R}^j)\times\mathbf{q}'_t\}\,ds'\,. \end{aligned}$$

Let us consider the case when the curve Γ forms a closed contour Γ. By using (2.3) we can express the vectors of internal forces $\mathbf{q}_t$ and $\mathbf{m}_t$ in terms of stress functions ψ^α, ψ^3 and, with the help of the static-geometric analogy, after integration we have [5], [6]

$$
(2.7)\qquad
\begin{aligned}
[\boldsymbol{\Omega}_t^*]_\Gamma &= -\frac{1}{kL}\{[\mathbf{F}]_\Gamma - [\overset{s}{\mathbf{F}}]_\Gamma\},\\
[\mathbf{v}^*]_\Gamma &= -\frac{1}{kL^2}\{([\mathbf{B}_{(j)}]_\Gamma - [\overset{s}{\mathbf{B}}_{(j)}]_\Gamma) + ([\mathbf{F}]_\Gamma - [\overset{s}{\mathbf{F}}]_\Gamma)\times L(\mathbf{r}-\mathbf{R}^j)\}
\end{aligned}
$$

where $[W]_\Gamma$ denotes an increment of the quantity W on the closed contour Γ.

3. Multivaluedness of stress functions

Let the shell have the $m+1$-connected region with internal closed boundary contours Γ_i ($i = 1, \ldots, m$), (Fig. 2). Let us assume that the external non-selfbalanced loads acting on each Γ_i have been reduced to the total force $\mathbf{F}^i$ and total couple $\mathbf{B}^i_{(i)}$ with respect to the space point i associated with Γ_i. Let the shell be loaded by surface forces $\frac{1}{L}\,\mathbf{p}$, and by concentrated forces $\mathbf{P}^k$ and couples $\mathbf{M}^k$ acting on points k ($k = 1, \ldots, n$) of the shell.

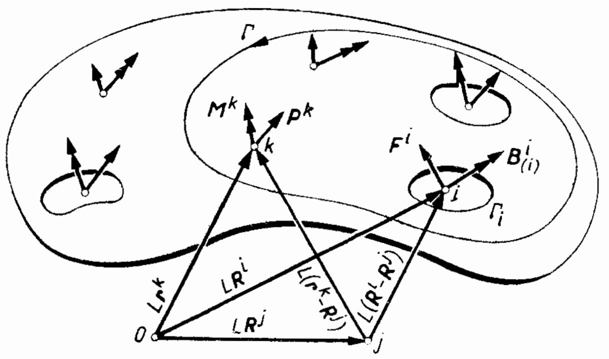

Fig. 2

Let us consider the closed contour Γ (Fig. 2) containing $p \leqslant m$ of the internal closed boundary contours Γ_i and $q \leqslant n$ of points k on which $\mathbf{P}^k$ and $\mathbf{M}^k$ acts.

The complete solution of the shell equations searched for should satisfy the conditions of total equilibrium of the inside part of the shell bounded by the contour Γ

$$
(3.1)\qquad
\begin{aligned}
[\mathbf{F}]_\Gamma &= L\iint_{S_\Gamma}\mathbf{p}'\,dS' + \sum_{k=1}^{q}\mathbf{P}^k + \sum_{i=1}^{p}\mathbf{F}^i,\\
[\mathbf{B}_{(j)}]_\Gamma &= L^2\iint_{S_\Gamma}(\mathbf{r}'-\mathbf{R}^j)\times\mathbf{p}'\,dS' + \sum_{k=1}^{q}\{\mathbf{M}^k + L(\mathbf{r}^k-\mathbf{R}^j)\times\mathbf{P}^k\} +\\
&\quad + \sum_{i=1}^{p}\{\mathbf{B}^i_{(i)} + L(\mathbf{R}^i-\mathbf{R}^j)\times\mathbf{F}^i\}.
\end{aligned}
$$

The particular solution of the equations of equilibrium appearing in (2.3) can be constructed in various manners. Only surface forces $\frac{1}{L}\mathbf{p}$ are involved in this particular solution directly. The construction of the particular solutions in the multivalued region S_Γ loaded by surface forces $\frac{1}{L}\mathbf{p}$ seems to be much easier when the region S_Γ is replaced by the singlevalued region $S_\Gamma + \sum_i S_{\Gamma_i}$ loaded by surface forces

$$\frac{1}{L}\bar{\mathbf{p}} = \begin{cases} \frac{1}{L}\mathbf{p} \text{ on the region } S_\Gamma, \\ \frac{1}{L}\mathbf{p}_i \text{ on the regions } S_{\Gamma_i} (i = 1, \ldots, p), \end{cases}$$

where $\frac{1}{L}\mathbf{p}_i$ — arbitrary surface forces for each extended shell region S_{Γ_i}, taken in a convenient form to get a simple particular solution.

The particular solution can also contain any of the loads $\mathbf{F}^i$, $\mathbf{B}^i_{(i)}$, $\mathbf{P}^k$, $\mathbf{M}^k$. These loads can be introduced in their full or partial value into the particular solution through the boundary conditions.

Thus, the particular solution of the equations of equilibrium should satisfy the following conditions of total equilibrium of the inside part of the shell bounded by the contour Γ

$$[\overset{s}{\mathbf{F}}]_\Gamma = L \iint\limits_{S_\Gamma + \sum_i S_{\Gamma_i}} \bar{\mathbf{p}}'\, dS' + \sum_{k=1}^{q} \overset{s}{\mathbf{P}}{}^k + \sum_{i=1}^{p} \overset{s}{\mathbf{F}}{}^i,$$

(3.2)
$$[\overset{s}{\mathbf{B}}_{(j)}]_\Gamma = L^2 \iint\limits_{S_\Gamma + \sum_i S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^j) \times \bar{\mathbf{p}}'\, dS' + \sum_{k=1}^{q} \{\overset{s}{\mathbf{M}}{}^k + L(\mathbf{r} - \mathbf{R}^j) \times \overset{s}{\mathbf{P}}{}^k\} +$$
$$+ \sum_{i=1}^{p} \{\overset{s}{\mathbf{B}}{}^i_{(i)} + L(\mathbf{R}^i - \mathbf{R}^j) \times \overset{s}{\mathbf{F}}{}^i\},$$

where $\overset{s}{\mathbf{F}}{}^i$, $\overset{s}{\mathbf{B}}{}^i_{(i)}$, $\overset{s}{\mathbf{P}}{}^k$, $\overset{s}{\mathbf{M}}{}^k$ are the vectors corresponding to the constructed particular solution of equations of equilibrium, which, in general, may differ from $\mathbf{F}^i$, $\mathbf{B}^i_{(i)}$, $\mathbf{P}^k$, $\mathbf{M}^k$.

The substitution of (3.1) and (3.2) into (2.7) shows that the stress functions are, in general, the multivalued ones in the region S_Γ.

Multivalued part of the stress function vector can be separated from the full solution and vectors $\boldsymbol{\Omega}_t^*$ and $\mathbf{v}^*$ can be expressed in the form [5], [6]

(3.3)
$$\boldsymbol{\Omega}_t^* = \boldsymbol{\Omega}_t^{*0} + \sum_{i=1}^{m+n} \boldsymbol{\Omega}^{*i}\, \Phi_i(\theta^\alpha) + \overset{p}{\boldsymbol{\Omega}}{}_t^*,$$
$$\mathbf{v}^* = \mathbf{v}^{*0} + \boldsymbol{\Omega}_t^{*0} \times (\mathbf{r} - \mathbf{r}^0) + \sum_{i=1}^{m+n} \{\mathbf{v}^{*i} + \boldsymbol{\Omega}^{*i} \times (\mathbf{r} - \mathbf{R}^i)\}\, \Phi_i(\theta^\alpha) + \overset{p}{\mathbf{v}}{}^*,$$

where

$\boldsymbol{\Omega}_t^{*0}$, $\mathbf{v}^{*0}$ — the part of the solution which is analogous to the "displacement of the shell as a rigid body",

$\overset{p}{\boldsymbol{\Omega}}{}_t^{*}(\theta^\alpha)$, $\overset{p}{\mathbf{v}}{}^{*}(\theta^\alpha)$ — singlevalued correctional part of the solution,

$\boldsymbol{\Omega}^{*i}$, $\mathbf{v}^{*i}$ — the so called "parameters of dislocation of stress functions" (according to Chernykh [5]).

For each contour Γ_i' containing only one internal closed boundary contour Γ_i, by using (3.2), (3.1) and (2.7) we have

$$\text{(3.4)}\qquad \begin{aligned} \boldsymbol{\Omega}^{*i} &= -\frac{1}{kL}\Big\{\mathbf{F}^i - \overset{s}{\mathbf{F}}{}^i - L\iint_{S_{\Gamma_i}} \mathbf{p}_i'\,dS'\Big\},\\ \mathbf{v}^{*i} &= -\frac{1}{kL^2}\Big\{\mathbf{B}^i_{(i)} - \overset{s}{\mathbf{B}}{}^i_{(i)} - L^2\iint_{S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^i)\times\mathbf{p}_i'\,dS'\Big\}. \end{aligned}$$

When the contour Γ_k' contains only one point k of concentrated loads, in a similar way we have

$$\text{(3.5)}\qquad \begin{aligned} \boldsymbol{\Omega}^{*k} &= -\frac{1}{kL}\{\mathbf{P}^k - \overset{s}{\mathbf{P}}{}^k\},\\ \mathbf{v}^{*k} &= -\frac{1}{kL^2}\{\mathbf{M}^k - \overset{s}{\mathbf{M}}{}^k\}. \end{aligned}$$

Now, it is evident from (3.3) that the stress functions ψ^α, ψ^3 may be regarded as a singlevalued only if the particular solution is constructed in such a manner that the parameters of dislocation calculated from (3.4) and (3.5) are zero.

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В. ПЕТРАШКЕВИЧ, О МНОГОЗНАЧНОСТИ ФУНКЦИЙ НАПРЯЖЕНИЙ В ЛИНЕЙНОЙ ТЕОРИИ ОБОЛОЧЕК

Рассмотрена оболочка многосвязной области нагруженная поверхностной нагрузкой, сосредоточенными силами и моментами, а также неуравновешенной краевой нагрузкой на внутренних краевых контурах. Выведены зависимости которые должно выполнять частное решение уравнений равновесия оболочки для того чтобы функции напряжений были однозначными во всей многосвязной области оболочки.