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MULTIVALUED STRESS FUNCTIONS IN THE LINEAR THEORY OF SHELLS

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1. Introduction

There are two general methods of constructing solutions of problems of the linear theory of shells, which are obtained

1) in displacements,

2) in stress functions.

These two methods are interrelated by the static-geometric analogy [2, 3, 12].

In the equations of the first method, the displacements are introduced by means of the strains. Similarly, the stress functions in the equations of the second method are introduced by means of combinations analogous to the strains and called "strains the stress functions".

The type of the possible multivaluedness of displacements of the middle surface of a shell, assuming unique strains, as also the type of multivaluedness of the stress functions assuming unique "strains of the stress functions" was studied by ČERNYKH [4] (cf., also [5]).

It was shown in [4] that the existence of multivaluedness of stress function vector, after going round a closed contour Γ on the shell, is connected with the difference between the total vectors of the force and the couple produced by the outer loads in the region of the shell bounded by the contour Γ, and those produced by the section forces round the contour Γ, calculated from a particular integral of the equations of equilibrium. A survey of multivalued solutions in the linear theory of shells has been given in the present author's paper [7].

The present work contains a general discussion of the problem of multivaluedness of the stress functions for shells with multiply connected region loaded by a distributed loads as well as concentrated loads and a non-self-equilibrated load acting on closed inner edges. Some conclusions will be drawn concerning the way, in which the particular integral of the equations of equilibrium should be assumed in order that the stress functions should be unique in the entire region of the shell.

2. Fundamental Relations

The basic geometrical notations have been taken from [6]. The position vector of points of the shell is

$$\mathbf{R} = L\{\mathbf{r}(\theta^\alpha)+\lambda\theta^3\mathbf{a}_3(\theta^\alpha)\}. \tag{2.1}$$

With the middle surface of the shell determined by the vector $\mathbf{r}(\theta^\alpha)$ there are associated the following geometrical quantities: θ^α, $\mathbf{r}(\theta^\alpha)$, $\mathbf{a}_\alpha$, $\mathbf{a}_3$, $\mathrm{e}_{\alpha\beta}$, $a_{\alpha\beta}$, $b_{\alpha\beta}$, $()|_\alpha$, $\Gamma^\alpha_{\beta\gamma}$ etc. [6,1].

Taking into consideration the Kirchhoff-Love hypothesis [1], the displacement of points of the shell is determined by the vector

$$\mathbf{V} = L\{\mathbf{v}(\theta^\alpha)+\lambda\theta^3\mathbf{\Omega}\times\mathbf{a}_3\},$$

where [5,6]

$$\mathbf{v} = v^\alpha\mathbf{a}_\alpha+v^3\mathbf{a}_3; \qquad \mathbf{\Omega} = \Omega^\alpha\mathbf{a}_\alpha+\Omega^3\mathbf{a}_3;$$
$$\Omega^\alpha = \mathrm{e}^{\alpha\beta}(v_{3,\beta}+b^\varrho_\beta v_\varrho); \quad \Omega^3 = \frac{1}{2}\mathrm{e}^{\beta\alpha}v_\alpha|_\beta. \tag{2.2}$$

The equations of equilibrium are, [6],

$$n^{\alpha\delta}|_\alpha-b^\delta_\alpha m^{\varrho\alpha}|_\varrho+p^\delta = 0, \qquad m^{\alpha\beta}|_{\alpha\delta}+b_{\alpha\delta}n^{\alpha\delta}+p^3 = 0. \tag{2.3}$$

The compatibility equations are, [1,5],

$$(\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\mu_{\gamma\beta})|_\alpha+b^\delta_\alpha(\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\varrho\beta}\alpha_{\gamma\beta})|_\varrho = 0,$$
$$-(\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\alpha_{\gamma\beta})|_{\alpha\delta}+b_{\alpha\delta}(\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\mu_{\gamma\beta}) = 0, \tag{2.4}$$

where

$$\alpha_{\varrho\lambda} = \frac{1}{2}(v_\varrho|_\lambda+v_\lambda|_\varrho)-b_{\varrho\lambda}v_3,$$
$$\mu_{\varrho\lambda} = -v_3|_{\varrho\lambda}-(b^\alpha_\lambda v_\alpha)|_\varrho-\frac{1}{2}\delta^{\alpha\beta}_{\gamma\varrho}b^\gamma_\lambda v_{\alpha|\beta}. \tag{2.5}$$

The static-geometric analogy [2, 3, 1],

$$n^{\alpha\delta}\leftrightarrow \mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\mu_{\gamma\beta}, \qquad m^{\alpha\delta}\leftrightarrow -\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\alpha_{\gamma\beta}. \tag{2.6}$$

enables the representation of the general solution of Eqs. (2.3) in the form [5]

$$n^{\alpha\delta} = \overset{s}{n}{}^{\alpha\delta}+k\cdot\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\mu^*_{\gamma\beta},$$
$$m^{\alpha\delta} = \overset{s}{m}{}^{\alpha\delta}-k\cdot\mathrm{e}^{\alpha\gamma}\mathrm{e}^{\delta\beta}\alpha^*_{\gamma\beta},$$
$$k = \frac{E\lambda L}{\sqrt{A}} = \frac{E\lambda^3 L}{\sqrt{12(1-\nu^2)}}, \tag{2.7}$$

where $\overset{s}{n}{}^{\alpha\delta}$, $\overset{s}{m}{}^{\alpha\delta}$ denote particular integrals of Eqs. (2.3), and $\alpha^*_{\varrho\lambda}$, $\mu^*_{\varrho\lambda}$ are the "strains of the stress functions", expressions analogous to (2.5) but constructed on the components of the stress function vector

$$\mathbf{v}^* = \psi^\alpha\mathbf{a}_\alpha+\psi^3\mathbf{a}_3. \tag{2.8}$$

Expressions constructed on ψ^α, ψ^3 in a manner analogous to those constructed on v^α, v^3, will be denoted by an asterisk (cf. [1]).

3. Relations along the Curve Γ

Along the curve Γ on the middle surface of the shell, with which is connected the unit vectors $\mathbf{t}$ (the tangent vector) and $\mathbf{\nu}$ (the vector of outer normal), we have the following vector of rotation of a normal element connected with the curve Γ [5,4]:

$$\mathbf{\Omega}_t = \mathbf{\Omega}-\alpha_{\alpha\beta}t^\alpha t^\beta\cdot\mathbf{a}_3 \tag{3.1}$$

and the vector of curvature variation of the curve Γ during the deformation [5,4]

$$\varkappa_t = \frac{d\mathbf{\Omega}_t}{ds}. \tag{3.2}$$

The section forces per unit length along Γ (Fig. 1) forms a vector of force and a vector of moment [5].

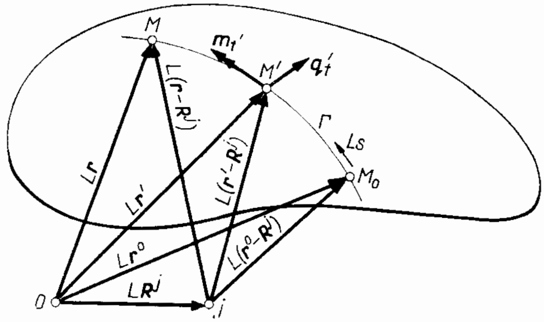

FIG. 1.

$$\begin{aligned} \mathbf{q}_t &= (n^{\alpha\beta}\nu_\beta + \tau_t\cdot m^{\alpha\beta}t_\beta)\nu_\alpha\cdot\boldsymbol{\nu} + (n^{\alpha\beta} - \sigma_t\cdot m^{\alpha\beta})\nu_\alpha t_\beta\cdot\mathbf{t} \\ &\qquad + \left[q^\alpha\nu_\alpha + \frac{d}{ds}(m^{\alpha\beta}\nu_\alpha\nu_\beta)\right]\mathbf{a}_3 \equiv \overset{s}{\mathbf{q}}_t - \boldsymbol{\varkappa}_t^*, \\ \mathbf{m}_t &= Lm^{\alpha\beta}\nu_\alpha\nu_\beta\cdot\mathbf{t} \equiv \overset{s}{\mathbf{m}}_t - kL\cdot\alpha^*_{\alpha\beta}t^\alpha t^\beta\cdot\mathbf{t}, \end{aligned} \tag{3.3}$$

where σ_t and τ_t are, respectively, the normal curvature and the geodesic torsion of the curve Γ [5].

The total vectors of the force $\mathbf{F}$ and the couple $\mathbf{B}_{(j)}$, of the section forces along the curve Γ with respect to any point j in the space, are obtained from (3.3)

$$\mathbf{F} = L\int_{M_0}^{M} \mathbf{q}_t' ds', \quad \mathbf{B}_{(j)} = L^2\int_{M_0}^{M} \{\mathbf{m}_t' + (\mathbf{r}' - \mathbf{R}^j)\times\mathbf{q}_t'\} ds'. \tag{3.4}$$

After integrating and making use of the static-geometric analogy (2.6) as well as the relations (3.1) and (3.2), from the relation (3.4) we obtain the following expressions for the vector of stress function $\mathbf{v}^*$ and that of "rotation of the stress function" $\mathbf{\Omega}_t^*$ at any point M of the curve Γ [5, 4, 2, 12]:

$$\begin{aligned} \mathbf{\Omega}_t^* &= \mathbf{\Omega}_t^{*\circ} - \frac{1}{kL}[\mathbf{F} - \overset{s}{\mathbf{F}}], \\ \mathbf{v}^* &= \mathbf{v}^{*\circ} + \mathbf{\Omega}_t^{*\circ}\times(\mathbf{r} - \mathbf{r}^\circ) - \frac{1}{kL^2}\{[\mathbf{B}_{(j)} - \overset{s}{\mathbf{B}}_{(j)}] + [\mathbf{F} - \overset{s}{\mathbf{F}}]\times L(\mathbf{r} - \mathbf{R}^j)\}, \end{aligned} \tag{3.5}$$

where $\mathbf{\Omega}_t^\circ$, $\mathbf{v}^{*\circ}$ are the values of $\mathbf{\Omega}_t^*$ and $\mathbf{v}^*$ at the point M_0; $\overset{s}{\mathbf{F}}$, $\overset{s}{\mathbf{B}}_{(j)}$ are the total vectors of the force and the couple (3.4) of the section forces obtained from the assumed particular integral of the equilibrium equations (the first terms in the relations (3.3)).

4. The Multivaluedness of the Stress Functions

Let us consider the case in which the curve Γ considered in Sec. 3 constitutes a closed contour Γ. Denoting the increment of any quantity W on the closed contour Γ by $[W]_\Gamma$, we find from (3.5), [4],

$$
(4.1)\qquad
\begin{aligned}
[\boldsymbol{\Omega}^*_t]_\Gamma &= -\frac{1}{kL}\{[\mathbf{F}]_\Gamma - [\overset{s}{\mathbf{F}}]_\Gamma\},\\
[\mathbf{v}^*]_\Gamma &= -\frac{1}{kL^2}\{([\mathbf{B}_{(j)}]_\Gamma - [\overset{s}{\mathbf{B}}_{(j)}]_\Gamma) + ([\mathbf{F}]_\Gamma - [\overset{s}{\mathbf{F}}]_\Gamma)\times L(\mathbf{r}-\mathbf{R}^j)\}.
\end{aligned}
$$

Let us consider the shell with $(m+1)$ times connected region with closed inner edges Γ_i $(i = 1, \ldots, m)$, (Fig. 2). Let us assume that the edge load, which is known for each contour Γ_i, has been reduced to the total vectors of the force $\mathbf{F}^i$ and the couple $\mathbf{B}^i_{(i)}$ with respect to the point

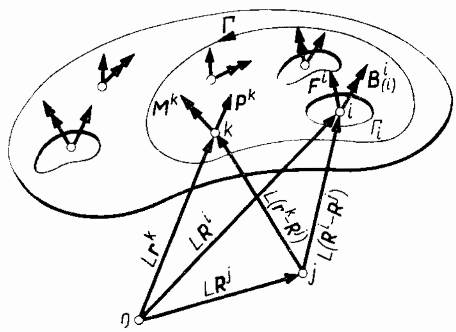

FIG. 2.

i associated with the contour Γ_i. Let the shell be loaded by a distributed load $\frac{1}{L}\mathbf{p}$ and, in addition, by concentrated loads with the force vectors $\mathbf{P}^k$ and couple vectors $\mathbf{M}^k$ acting on the points k of the shell $(k = 1, \ldots, n)$. Let us consider a closed contour Γ (Fig. 2) containing in its interior p inner edges Γ_i $(i = 1, \ldots, p \leqslant m)$ and q points of application of concentrated loads k $(k = 1, \ldots, q \leqslant n)$. The equilibrium conditions of the fragment of the shell contained inside the contour Γ, which must be satisfied by the searched full solution, have the form

$$
(4.2)\qquad
\begin{aligned}
[\mathbf{F}]_\Gamma &= L\iint_{S_\Gamma}\mathbf{p}'dS' + \sum_{k=1}^{q}\mathbf{P}^k + \sum_{i=1}^{p}\mathbf{F}^i,\\
[\mathbf{B}_{(j)}]_\Gamma &= L^2\iint_{S_\Gamma}(\mathbf{r}'-\mathbf{R}^j)\times\mathbf{p}'dS' + \sum_{k=1}^{q}[\mathbf{M}^k + L(\mathbf{r}^k-\mathbf{R}^j)\times\mathbf{P}^k]\\
&\quad + \sum_{i=1}^{p}[\mathbf{B}^i_{(i)} + L(\mathbf{R}^i-\mathbf{R}^j)\times\mathbf{F}^i].
\end{aligned}
$$

The particular integral of the equations of equilibrium (2.3) can be assumed in various manners. The particular integral involves directly the distributed load only. The construction of the particular integral of Eqs. (2.3) in the multiply connected region S_Γ loaded by the surface load $\frac{1}{L}\mathbf{p}$, can be facilitated by transforming the region S_Γ into a simply connected region $S_\Gamma + \sum_{i=1}^{m} S_{\Gamma_i}$ of the shell loaded by the distributed load

$$\frac{1}{L}\,\overline{\mathbf{p}} = \begin{cases} \frac{1}{L}\mathbf{p} \text{ for the region } S_\Gamma, \\ \frac{1}{L}\mathbf{p}_i \text{ for the regions } S_{\Gamma_i},\ i=1,\dots,m, \end{cases}$$

where $\frac{1}{L}\mathbf{p}_i$ are fictitious distributed loads assumed for each prolonged region S_{Γ_i} of the shell, so that the particular integral of Eqs. (2.3) can easily be obtained.

Into the particular integral of Eqs. (2.3) we can introduce in an indirect manner, by means of the boundary conditions on the contours Γ_i, any of the loads $\mathbf{F}^i$ and $\mathbf{B}^i_{(i)}$ acting on the contours Γ_i in their complete or partial value. The points k of application of the concentrated loads may also be treated as indefinitely small closed inner edges with displacements and deformations continuous at the points k (cf. [8]). Any of the loads $\mathbf{P}^k$ and $\mathbf{M}^k$ can, therefore, also be introduced indirectly in their complete or partial value in the particular integral. Thus, the particular integral will satisfy the following equilibrium conditions, (cf. (4.2))

$$
\begin{aligned}
[\overset{s}{\mathbf{F}}]_\Gamma &= L \iint_{S_\Gamma + \sum_i S_{\Gamma_i}} \overline{\mathbf{p}}' dS' + \sum_{k=1}^{q} \overset{s}{\mathbf{P}}{}^k + \sum_{i=1}^{p} \overset{s}{\mathbf{F}}{}^i, \\
[\overset{s}{\mathbf{B}}_{(j)}]_\Gamma &= L^2 \iint_{S_\Gamma + \sum_i S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^j) \times \overline{\mathbf{p}}' dS' + \sum_{k=1}^{q} [\overset{s}{\mathbf{M}}{}^k + L(\mathbf{r}^k - \mathbf{R}^j) \times \overset{s}{\mathbf{P}}{}^k] \\
&\quad + \sum_{i=1}^{p} [\overset{s}{\mathbf{B}}{}^i_{(i)} + L(\mathbf{R}^i - \mathbf{R}^j) \times \overset{s}{\mathbf{F}}{}^i],
\end{aligned} \tag{4.3}
$$

where $\overset{s}{\mathbf{P}}{}^k$, $\overset{s}{\mathbf{M}}{}^k$, $\overset{s}{\mathbf{F}}{}^i$, $\overset{s}{\mathbf{B}}{}^i_{(i)}$ denote the vectors of the section forces corresponding to the assumed particular integral. These vectors may differ, in general, from the corresponding vectors $\mathbf{P}^k$, $\mathbf{M}^k$, $\mathbf{F}^i$, $\mathbf{B}^i_{(i)}$.

If the contour Γ contains in its interior only one of the contours Γ_i (or points k) then assuming that the point j (Fig. 1) coincides with the point i (or k) the relations (4.2) and (4.3) are considerably simplified.

Direct substitution of (4.2) and (4.3) into (4.1) shows that the stress functions are, in general, multivalued ones in the region S_Γ of the shell. The multivalued part of the stress function vector may be separated from the solution by representing $\mathbf{\Omega}_t^*$ and $\mathbf{v}^*$, formally, as follows (cf. [4,5])

$$
\begin{aligned}
\mathbf{\Omega}_t^* &= \mathbf{\Omega}_t^{*0} + \sum_{i=1}^{m+n} \mathbf{\Omega}^{*i} \Phi_i(\theta^\alpha) + \overset{p}{\mathbf{\Omega}}{}_t^*, \\
\mathbf{v}^* &= \mathbf{v}^{*0} + \mathbf{\Omega}_t^{*0} \times (\mathbf{r} - \mathbf{r}^0) + \sum_{i=1}^{m+n} \{\mathbf{v}^{*i} + \mathbf{\Omega}^{*i} \times (\mathbf{r} - \mathbf{R}^i)\} \Phi_i(\theta^\alpha) + \overset{p}{\mathbf{v}}{}^*.
\end{aligned} \tag{4.4}
$$

where $\overset{p}{\mathbf{v}}{}^*$, $\overset{p}{\boldsymbol{\Omega}}{}^*_t$ denote the single-valued correctional term of the solution; $\Phi_i(\theta^\alpha)$ is a multivalued scalar function of the contour Γ_i which only on passing round the contour Γ_i have a unit increment, but its derivatives in every direction are single-valued functions; $\boldsymbol{\Omega}^{*i}$, $\mathbf{v}^{*i}$ are the stress function dislocation parameters of the contour Γ_i (a term introduced by ČERNYKH [4]).

According to (4.1), the parameters $\boldsymbol{\Omega}^{*i}$ and $\mathbf{v}^{*i}$ can be expressed by the relation

$$\boldsymbol{\Omega}^{*i} = -\frac{1}{kL}\{[\mathbf{F}]_{\Gamma_i'} - [\overset{s}{\mathbf{F}}]_{\Gamma_i'}\}, \quad \mathbf{v}^{*i} = -\frac{1}{kL^2}\{[\mathbf{B}_{(i)}]_{\Gamma_i'} - [\overset{s}{\mathbf{B}}_{(i)}]_{\Gamma_i'}\}. \tag{4.5}$$

For a contour Γ_i' containing one of the inner closed edges Γ_i, we have from (4.2) and (4.3)(¹):

$$\begin{aligned} \boldsymbol{\Omega}^{*i} &= -\frac{1}{kL}\Big[\mathbf{F}^i - \overset{s}{\mathbf{F}}{}^i - L\iint_{S_{\Gamma_i}} \mathbf{p}_t' dS'\Big], \\ \mathbf{v}^{*i} &= -\frac{1}{kL^2}\Big[\mathbf{B}^i_{(i)} - \overset{s}{\mathbf{B}}{}^i_{(i)} - L^2\iint_{S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^i)\times \mathbf{p}_t' dS'\Big]. \end{aligned} \tag{4.6}$$

For a contour Γ_k' containing one of the concentrated load application points k, we have from (4.2) and (4.3)

$$\boldsymbol{\Omega}^{*k} = -\frac{1}{kL}[\mathbf{P}^k - \overset{s}{\mathbf{P}}{}^k], \quad \mathbf{v}^{*k} = -\frac{1}{kL^2}[\mathbf{M}^k - \overset{s}{\mathbf{M}}{}^k]. \tag{4.7}$$

5. Conclusions

From the relations (4.4), (4.6) and (4.7) results the following conclusions:

1) For simply connected shells loaded by a distributed load and concentrated loads, the vector of the stress function $\mathbf{v}^*$ is, in general, a multivalued function in the region of the shell. The vector $\mathbf{v}^*$ is single-valued in the entire region of the shell in the following particular cases only:

(¹) The result, analogous to $(4.6)_2$, obtained by ČERNYKH ([5] VIII, Sec. 4, Eq. (4.4)) is not exact. The fictitious distributed load $\mathbf{p}_i$ on S_{Γ_i} is not considered there. The terms $\overset{s}{\mathbf{F}}{}^i$ and $\overset{s}{\mathbf{B}}{}^i_{(i)}$ are also rejected. It was observed that "Rejecting the quantity $\overset{s}{\mathbf{B}}$ which is not essential in the problem under consideration we obtain..." ([5], p. 156). This is not correct, in general, and is justified only in those cases in which the distributed load $\mathbf{p} = 0$ and, in addition, if we assume that $\mathbf{p}_i = 0$ on each S_{Γ_i} and the particular integral is zero in the entire region of the shell, $\overset{s}{\mathbf{F}}{}^i = \overset{s}{\mathbf{B}}{}^i_{(i)} = 0$ $(i = 1, \ldots, m)$, $\overset{s}{\mathbf{P}}{}^k = \overset{s}{\mathbf{M}}{}^k = 0$ $(k = 1, \ldots, n)$. In addition, in order to transform the couple $\mathbf{B}^j$ about a point M of the curve Γ into the couple $\mathbf{B}^j_{(j)}$ about a fixed point j (cf. Fig. 1), ČERNYKH [5] makes use of the inaccurate relation $\mathbf{B}^j_{(j)} = \mathbf{B}^j + \mathbf{r}\times\mathbf{F}^j$ ([5], VIII, §4, below Eq. (4.3)) instead of the correct relation $\mathbf{B}^j_{(j)} = \mathbf{B}^j + (\mathbf{r} - \mathbf{R}^j)\times\mathbf{F}^j$. He did not explain there whether the point j was assumed in addition to coincide with the origin 0, or not.

1a) There are no concentrated loads, that is $\mathbf{P}^k = \mathbf{M}^k = 0$ ($k = 1, \ldots, n$). The particular integral of Eqs. (2.3) for a simply connected region is continuous and the dislocation parameters vanish identically according to (4.7).

1b) The particular integral of Eqs. (2.3), constructed for a multiply connected region with infinitely small inner edges Γ_k cutting out the points k ($k = 1, \ldots, n$) has been chosen in such a manner that for each pair of values $\mathbf{P}^k$ and $\mathbf{M}^k$ ($k = 1, \ldots, n$) the relations

$$(5.1) \qquad \overset{s}{\mathbf{P}}{}^k \overset{!}{=} \mathbf{P}^k, \quad \overset{s}{\mathbf{M}}{}^k \overset{!}{=} \mathbf{M}^k$$

are satisfied.

2) For multiply connected shells the stress function vector $\mathbf{v}^*$ is, in general, a multivalued function in the entire region of the shell. It is single-valued in the entire region of the shell in the following particular cases only:

2a) We have $\mathbf{P}^k = \mathbf{M}^k = 0$ ($k = 1, \ldots, n$) and $\mathbf{F}^i = \mathbf{B}^i_{(i)} = 0$ ($i = 1, \ldots, m$). Thus the shell is loaded by a distributed load and a self-equilibrated load over the contours Γ_i. The particular integral of Eqs. (2.3) for a simply connected region $S_\Gamma + \sum_{i=1}^{m} S_{\Gamma_i}$ of the prolonged shell is continuous. From the conditions of general equilibrium (4.3) of each region S_{Γ_i} of the shell we obtain for the particular integral

$$\overset{s}{\mathbf{F}}{}^i = -L \iint_{S_{\Gamma_i}} \mathbf{p}'_i dS', \qquad \overset{s}{\mathbf{B}}{}^i_{(i)} = -L^2 \iint_{S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^i) \times \mathbf{p}'_i dS'.$$

According to (4.6), the dislocation parameters $\mathbf{\Omega}^{*i}$ and $\mathbf{v}^{*i}$ will, therefore, be zero.

2b) The particular integral of Eqs. (2.3), constructed for a multiply connected region with indefinitely small inner edges Γ_k cutting out the points k ($k = 1, \ldots, n$) and inner edges Γ_i ($i = 1, \ldots, m$), has been chosen in such a manner that for each pair of non-zero values of $\mathbf{P}^k$ and $\mathbf{M}^k$ the relations (5.1) are satisfied, and for each pair of non-zero values of $\mathbf{F}^i$ and $\mathbf{B}^i_{(i)}$, if we pass round each of the inner contours Γ_i, we have

$$(5.2) \qquad \overset{s}{\mathbf{F}}{}^i + L \iint_{S_{\Gamma_i}} \mathbf{p}'_i dS' \overset{!}{=} \mathbf{F}^i, \qquad \overset{s}{\mathbf{B}}{}^i_{(i)} + L^2 \iint_{S_{\Gamma_i}} (\mathbf{r}' - \mathbf{R}^i) \times \mathbf{p}'_i dS' \overset{!}{=} \mathbf{B}^i_{(i)}.$$

Then, in agreement with (4.6), the dislocation parameters $\mathbf{\Omega}^{*i}$ and $\mathbf{v}^{*i}$ are zero.

The particular integrals mentioned in the conclusions 1b) and 2b) theoretically can always be constructed. In practice it may happen, however, that such a particular integral cannot be constructed for certain contour Γ_i or point k. In such a case, if we solve the problem in stress functions, the vector $\mathbf{v}^*$ should be treated, after calculating the parameters $\mathbf{\Omega}^{*i}$ and $\mathbf{v}^{*i}$ according to (4.5), as a multivalued function $(4.4)_2$ with respect to passing round these contours.

6. Final Remarks

The above considerations are of a qualitative character only because they indicate the occurrence of the problem of multivaluedness of the stress functions. The problem of how to build up multivalued stress functions has not yet been analysed sufficiently in the general case.

As follows from (4.4), the first stage of the solution is the construction of a multivalued funtion $\Phi_i(\theta^\alpha)$ having required properties. For shells of revolution with concentric edges we can, in view of the axial symmetry of the shell, use the function $\Phi = \theta/2\pi$ [4,11], which was also made use of in the paper [8] for the investigation of the type of singularity in shells under concentrated loads. For shallow shells it was shown in the present author's paper [9], that the type of multivaluedness of the vector $\mathbf{v}^*$, in the direction of the vector $\mathbf{e}_3$ normal to the plane of projection (horizontal plane) of the shell (cf. [6,13]), is the same as for a plane stress state where methods of constructing multivalued solutions are known (cf. [10]). However, thc problem of constructing the multivalued part of the vector $\mathbf{v}^*$ is still unsolved, in the general case.

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Streszczenie

WIELOZNACZNOŚĆ FUNKCJI NAPRĘŻEŃ W LINIOWEJ TEORII POWŁOK

Rozważono zagadnienie wieloznaczności wektora funkcji naprężeń w liniowej teorii powłok. Rozpatrzono powłokę o obszarze wielospójnym obciążoną obciążeniem powierzchniowym, skupionymi siłami i momentami oraz niesamozrównoważonym obciążeniem na wewnętrznych brzegowych konturach zamkniętych. Sformułowano wnioski jak należy przyjmować całkę szczególną równań równowagi by funkcje naprężeń były jednoznaczne w całym wielospójnym obszarze powłoki.

Резюме

МНОГОЗНАЧНОСТЬ ФУНКЦИЙ НАПРЯЖЕНИЙ В ЛИНЕЙНОЙ ТЕОРИИ ОБОЛОЧЕК

Рассматривается задача многозначности вектора функции напряжений в линейной теории оболочек. Рассмотрена оболочка многосвязной области, нагруженная поверхностной нагрузкой, сосредоточенными силами и моментами, а также неуравновешенной нагрузкой на внутренних краевых контурах. Формулируются условия при каких следует принимать частное решение уравнений равновесия, чтобы функции напряжений были однозначными во всей многосвязной области оболочки.

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