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# Material Equations of Motion foı Nonlinear Theory of Thin Shells

by

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**Summary.** The material description in the general nonlinear theory of thin shells is considered. The only assumption made is the linearization of distribution of deformation over the shell thickness. Equations of motion of the shell are derived by integrating the material form of the continuum equations of motion over the shell thickness in the reference configuration.

## 1. Introduction

Equations of motion of a nonlinear shell theory in spatial description have been derived in [1] by integrating the spatial form of the continuum equations of motion over the shell thickness in a deformed configuration. Although the equations in [1] are exact and simple in form, in fact it is not possible to calculate and to interpret properly the introduced stress and couple resultants, because of the unknown varying geometry of the middle surface of the shell in the actual deformed configuration. In small strain (but large displacement!) theories it is possible to express approximately the shell middle surface geometry in deformed configuration in terms of the geometry in the reference configuration [2]. In this way one can obtain the equations of motion which, though more elaborate in form, are useful for applications.

In this note equations of motion of a nonlinear shell theory are derived by integrating the material form of the continuum equations of motion over the shell thickness in the reference configuration. Thus, the deformed shell geometry does not appear at all in the equations of motion, and stress and couple resultants are defined for the reference configuration of the shell, which is fixed and known in advance.

The obtained equations of motion are very general and hence rather elaborate in form, as the only assumption made is the linearization of the distribution of deformation over the shell thickness. This approximation is usually accepted in theories of thin shells.

The equations of motion can be used to investigate the behaviour in time of the elastic and anelastic shells with large nonlinear strains and displacements. They may, under some additional assumptions, form a basis for deriving different simplified versions of the equations of motion of non-linear theories of shells in material descriptions, which may appear to be more useful for applications.

## 2. Notations and basic relations

Absolute tensor analysis [3—6] is used as the basis of the paper.

Let $\kappa$ and $\gamma$ be the reference and actual configurations [3] of a shell (body) $\mathcal{S}$, respectively, $P=\kappa(\mathcal{X})$, $p=\gamma(\mathcal{X}, t)$, where $\mathcal{X}\in\mathcal{S}$, $P\in\mathcal{P}_\kappa\subset\mathcal{E}_3$, $p\in\mathcal{P}_\gamma\subset\mathcal{E}_3$. Let $\chi$ be a deformation with reference to $\kappa$, $\chi(P, t)=\gamma[\kappa^{-1}(P, t)]$. Equations of motion of continua in material description with reference to $\kappa$ have the form [3,8].

$$
\text{(1)}\qquad \begin{aligned} &\operatorname{Div}\mathbf{T}_\kappa+\rho_\kappa\mathbf{b}=\rho_\kappa\mathbf{a},\\ &\tilde{\mathbf{T}}=\tilde{\mathbf{T}}^T, \end{aligned}
$$

where [3]: $\mathbf{T}_\kappa$, $\tilde{\mathbf{T}}$ — the first and second Piola—Kirchhoff stress tensor, $\mathbf{T}_\kappa=\mathbf{F}\tilde{\mathbf{T}}$; $\mathbf{F}$ — the deformation gradient, $\mathbf{F}=\nabla\chi(P, t)$; $\mathbf{a}$ — the acceleration vector, $\mathbf{a}=\frac{d}{dt^2}\chi(P, t)$; $\rho_\kappa$ — the mass density in $\kappa$; $\mathbf{b}$ — the external body force vector.

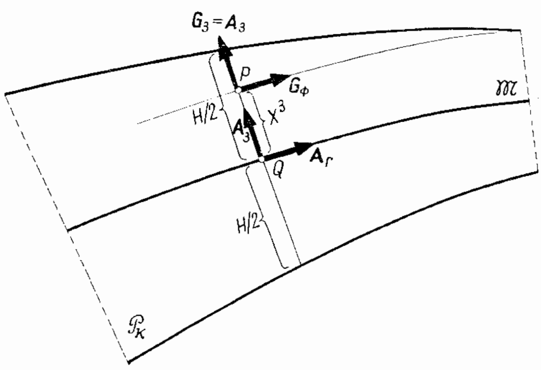
The state of strain with reference to $\kappa$ can be described in terms of the displacement vector $\mathbf{u}$ and the Green—St. Venant strain tensor $\mathbf{E}$ [3],

$$
\text{(2)}\qquad \begin{aligned} &\mathbf{u}=\chi(P, t)-P,\qquad \mathbf{F}=\mathbf{1}+\operatorname{Grad}\mathbf{u},\\ &\mathbf{E}=\tfrac{1}{2}[\operatorname{Grad}\mathbf{u}+(\operatorname{Grad}\mathbf{u})^T+(\operatorname{Grad}\mathbf{u})^T\operatorname{Grad}\mathbf{u}]. \end{aligned}
$$

Let (Figure): $\mathcal{M}$ denote the middle surface of the shell in $\kappa$, $Q\in\mathcal{M}$; $\mathbf{A}_3(Q)$ — a unit vector field on $\mathcal{M}$, $|\mathbf{A}_3|=\mathbf{1}$, $\mathbf{A}_3\perp\mathcal{M}$; $\{X^K\}$ — normal coordinate system in $\kappa$ such that [5]

$$
\text{(3)}\qquad P=Q+X^3\,\mathbf{A}_3(Q).
$$

Greek indices will have the range (1, 2), while Latin indices will assume the range (1, 2, 3). Indices $K, L, M$ and $\Phi, \Psi, \Theta$ refer to the quantities given at $P\in\mathcal{P}_\kappa$, while $A, B, C$ and $\Gamma, \Delta, \Lambda$ refer to the quantities given at $Q\in\mathcal{M}\subset\mathcal{P}_\kappa$. This distinction




between the indices at $P$ and $Q$, which has not been used up to now in the theory of shells, comes directly from the application of the absolute tensor analysis. It enables an easy and consistent description of tensor fields on $\kappa$.

Let a set of three vector fields $\mathbf{G}_K(P)=\{\mathbf{G}_\Phi, \mathbf{G}_3\}$ and $\mathbf{A}_A(Q)=\{\mathbf{A}_\Gamma, \mathbf{A}_3\}$ form a natural basis [3] of the normal coordinate system $\{X^K\}$ at points $P$ and $Q$, respectively, connected by (3) (see the Figure). Then we get the following relations [7, 1]

$$\mathbf{1}=\mathbf{1}^T=G_{KL}\,\mathbf{G}^K\otimes\mathbf{G}^L=A_{AB}\,\mathbf{A}^A\otimes\mathbf{A}^B=\mu_K^A\,\mathbf{A}_A\otimes\mathbf{G}^K=\mu_B^L\,\mathbf{A}^B\otimes\mathbf{G}_L; \tag{4}$$

$$\begin{gathered}\mu_3^3=1, \quad \mu_3^\Gamma=\mu_\Phi^3=0,\\ \mu_\Phi^\Gamma=\mathbf{A}^\Gamma\cdot\mathbf{G}_\Phi=\delta_\Phi^\Gamma-X^3\,\delta_\Phi^\Delta\,B_\Delta^\Gamma,\\ \mu_\Delta^\Psi=\mathbf{A}_\Delta\cdot\mathbf{G}^\Psi=A_{\Gamma\Delta}\,\mu_\Phi^\Gamma\,G^{\Phi\Psi},\\ G=|G_{KL}|,\quad A=|A_{AB}|,\quad \mu=|\mu_K^A|=\sqrt{\frac{G}{A}}.\end{gathered} \tag{5}$$

Gradient and divergence operations on vector fields $\mathbf{G}_K(P)$ and $\mathbf{u}(P)$ and tensor field $\mathbf{T}(P)$, written in basis $\mathbf{G}_K(P)$ or $\mathbf{A}_A(Q)$, give us the following relations [7]:

$$\begin{gathered}\operatorname{Grad}\mathbf{G}_K=\Gamma_{KL}^M\,\mathbf{G}_M\otimes\mathbf{G}^L=\mathbf{G}_M\otimes\mu_C^M\,\{(\mu_{K,\Psi}^C+\mu_K^A\,\delta_\Psi^\Delta\,\Gamma_{A\Delta}^C)\,\mathbf{G}^\Psi+\mu_{K,3}^C\,\mathbf{G}^3\},\\ \operatorname{Grad}\mathbf{u}=u^K{}_{;L}\,\mathbf{G}_K\otimes\mathbf{G}^L=\mathbf{G}_K\otimes\mu_C^K\,\{\delta_\Psi^\Delta\,u^C{}_{|\Delta}\,\mathbf{G}^\Psi+u^C{}_{,3}\,\mathbf{G}^3\},\\ \operatorname{Grad}\mathbf{T}=T^{KL}{}_{;M}\,\mathbf{G}_K\otimes\mathbf{G}_L\otimes\mathbf{G}^M=\\ =\mathbf{G}_K\otimes\mathbf{G}_L\otimes\mu_A^K\,\mu_B^L\{\delta_\Theta^\Delta\,T^{AB}{}_{|\Delta}\,\mathbf{G}^\Theta+T^{AB}{}_{,3}\,\mathbf{G}^3\},\\ \operatorname{Div}\mathbf{T}=\underset{(2,3)}{\operatorname{tr}}\ \operatorname{Grad}\mathbf{T}=T^{KL}{}_{;L}\,\mathbf{G}_K=\mu_A^K\{\delta_\Psi^\Delta\,\mu_\Delta^\Psi\,T^{A\Delta}{}_{|\Delta}+T^{A3}{}_{,3}\}\,\mathbf{G}_K,\end{gathered} \tag{6}$$

where

$$\Gamma_{KL}^M=\mathbf{G}^M\cdot\mathbf{G}_{K,L},\qquad \Gamma_{A\Delta}^C=\mathbf{A}^C\cdot\mathbf{A}_{A,\Delta} \tag{7}$$

and $(\ )_{;L}$ and $(\ )_{|\Delta}$ are the covariant derivatives of the components of tensor field, calculated with the use of the $\Gamma_{KL}^M$ and $\Gamma_{A\Delta}^C$, respectively.

From (6) it is easy to derive many relations for tensor field components in the basis $\mathbf{G}_K(P)$ in terms of their components in the basis $\mathbf{A}_A(Q)$ [7]. For instance, from $(6)_4$ we obtain [1]:

$$\begin{gathered}\mu\mu_\Phi^\Gamma\,T^{\Phi\Psi}{}_{;\Psi}=(\mu\mu_\Phi^\Gamma\,T^{\Phi\Psi}\,\delta_\Psi^\Delta)|_\Delta-\mu B_\Delta^\Gamma\,\delta_\Psi^\Delta\,T^{3\Psi}+\mu_{,3}\,\mu_\Phi^\Gamma\,T^{\Phi3},\\ \mu T^{3\Psi}{}_{;\Psi}=(\mu T^{3\Psi}\,\delta_\Psi^\Delta)|_\Delta+B_{\Gamma\Delta}\,\mu\mu_\Phi^\Gamma\,\delta_\Psi^\Delta\,T^{\Phi\Psi}+\mu_{,3}\,T^{33},\\ \mu_\Phi^\Gamma\,T^{\Phi3}{}_{;3}=(\mu_\Phi^\Gamma\,T^{\Phi3})_{,3}.\end{gathered} \tag{8}$$

### 3. Shell deformation

The deformation $\chi(P,t)$ can be expanded into Taylor series in the neighbourhood of $Q\in\mathcal{M}$ [5],

$$\chi(P,t)=\chi(Q+X^3\,\mathbf{A}_3,t)=\chi(Q,t)+X^3\,\nabla\chi(Q,t)\cdot\mathbf{A}_3+ \\ +\frac{1}{2!}(X^3)^2\,\nabla^2\,\chi(Q,t)\cdot(\mathbf{A}_3\otimes\mathbf{A}_3)+\dots \tag{9}$$

For thin shells only the linear approximation of (9) is usually taken into consideration, and from $(2)_1$ and (3) we have the expansion:

$$\mathbf{u}=\mathbf{v}+X^3\,\boldsymbol{\beta}+\ldots, \tag{10}$$

where

$$\begin{aligned} \mathbf{v}&=\chi(Q,t)-Q, \\ \boldsymbol{\beta}&=\nabla\chi(Q,t)-\mathbf{1}=(\mathrm{Grad}\,\mathbf{v})\cdot\mathbf{A}_3. \end{aligned} \tag{11}$$

From (9), (10) and (11) we have an expansion for the acceleration

$$\mathbf{a}=\ddot{\mathbf{u}}=\ddot{\mathbf{v}}+X^3\,\ddot{\boldsymbol{\beta}}+\ldots, \tag{12}$$

where

$$\begin{aligned} \ddot{\mathbf{v}}&=\frac{d^2}{dt^2}\,\chi(Q,t), \\ \ddot{\boldsymbol{\beta}}&=\frac{d^2}{dt^2}\,\nabla\chi(Q,t)=(\mathrm{Grad}\,\ddot{\mathbf{v}})\,\mathbf{A}_3. \end{aligned} \tag{13}$$

Using the relations following from $(6)_2$ as well as the linear approximation (10) [7], the strain tensor $\mathbf{E}$ can be put in the form:

$$\begin{aligned} \mathbf{E}(P,t)&=E_{KL}(P,t)\,\mathbf{G}^K(P)\otimes\mathbf{G}^L(P)=\delta^{\Gamma}_{\Phi}\,\delta^{\Delta}_{\Psi}\,\{{}_0E_{\Gamma\Delta}+X^3\,{}_1E_{\Gamma\Delta}+ \\ &+(X^3)^2\,{}_2E_{\Gamma\Delta}\}\,\mathbf{G}^{\Phi}\otimes\mathbf{G}^{\Psi}+\delta^{\Gamma}_{\Phi}\,\{{}_0E_{\Gamma 3}+X^3\,{}_1E_{\Gamma 3}\}\,\mathbf{G}^{\Phi}\otimes\mathbf{G}^3+{}_0E_{33}\,\mathbf{G}^3\otimes\mathbf{G}^3 \end{aligned} \tag{14}$$

where:

$$\begin{aligned} 2\,{}_0E_{\Gamma\Delta}(Q,t)&=(v_{\Gamma|\Delta}-B_{\Gamma\Delta}\,v_3)+(v_{\Delta|\Gamma}-B_{\Delta\Gamma}\,v_3)+ \\ &\quad+A^{\Lambda\Sigma}\,(v_{\Lambda|\Gamma}-B_{\Lambda\Gamma}\,v_3)\,(v_{\Sigma|\Delta}-B_{\Sigma\Delta}\,v_3)+(v_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,v_{\Lambda})\,(v_{3,\Delta}+B^{\Sigma}_{\Delta}\,v_{\Sigma}) \\ 2\,{}_1E_{\Gamma\Delta}(Q,t)&=(\beta_{\Gamma|\Delta}-B_{\Gamma\Delta}\,\beta_3)+(\beta_{\Delta|\Gamma}-B_{\Delta\Gamma}\,\beta_3)-B^{\Lambda}_{\Gamma}(v_{\Lambda|\Delta}-B_{\Lambda\Delta}v_3)-B^{\Lambda}_{\Delta}(v_{\Lambda|\Gamma}-B_{\Lambda\Gamma}v_3), \\ &\quad+A^{\Lambda\Sigma}(\beta_{\Lambda|\Gamma}-B_{\Lambda\Gamma}\,\beta_3)\,(v_{\Sigma|\Delta}-B_{\Sigma\Delta}\,v_3)+A^{\Lambda\Sigma}(v_{\Lambda|\Gamma}-B_{\Lambda\Gamma}\,v_3)\,(\beta_{\Sigma|\Delta}-B_{\Sigma\Delta}\,\beta_3)+ \\ &\quad+(\beta_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,\beta_{\Lambda})\,(v_{3,\Delta}+B^{\Sigma}_{\Delta}\,v_{\Sigma})+(v_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,v_{\Lambda})\,(\beta_{3,\Delta}+B^{\Sigma}_{\Delta}\,\beta_{\Sigma}), \\ 2\,{}_2E_{\Gamma\Delta}(Q,t)&=-B^{\Lambda}_{\Gamma}(\beta_{\Lambda|\Delta}-B_{\Lambda\Delta}\,\beta_3)-B^{\Sigma}_{\Delta}(\beta_{\Sigma|\Gamma}-B_{\Sigma\Gamma}\,\beta_3)+ \\ &\quad+A^{\Lambda\Sigma}(\beta_{\Lambda|\Gamma}-B_{\Lambda\Gamma}\,\beta_3)(\beta_{\Sigma|\Delta}-B_{\Sigma\Delta}\,\beta_3)+(\beta_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,\beta_{\Lambda})\,(\beta_{3,\Delta}+B^{\Sigma}_{\Delta}\,\beta_{\Sigma}), \\ 2\,{}_0E_{\Gamma 3}(Q,t)&=\beta_{\Gamma}+(v_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,v_{\Lambda})+A^{\Lambda\Sigma}(v_{\Lambda|\Gamma}-B_{\Lambda\Gamma}\,v_3)\,\beta_{\Sigma}+(v_{3,\Gamma}+B^{\Lambda}_{\Gamma}\,v_{\Lambda})\,\beta_3, \\ 2\,{}_1E_{\Gamma 3}(Q,t)&=\beta_{3,\Gamma}+A^{\Lambda\Sigma}\,\beta_{\Lambda|\Gamma}\,\beta_{\Sigma}+\beta_{3,\Gamma}\,\beta_3, \\ 2\,{}_0E_{33}(Q,t)&=2\beta_3+A^{\Lambda\Sigma}\,\beta_{\Lambda}\,\beta_{\Sigma}+(\beta_3)^2. \end{aligned} \tag{15}$$

The expressions (15) furnish an adequate material description of shell deformation when only a linear approximation of (9) is taken into consideration.

## 4. Equations of motion

Let us write Eq. $(1)_1$ in basis $\mathbf{A}_{\Delta}(Q)$ in order to form three component equations. After using (8) we have

$$\begin{aligned} &(\mu\mu^{\Gamma}_{\Phi}\,T^{\Phi\Psi}_{\kappa}\,\delta^{\Delta}_{\Psi})|_{\Delta}-B^{\Gamma}_{\Delta}\,\mu T^{3\Psi}_{\kappa}\,\delta^{\Delta}_{\Psi}+\mu\mu^{\Gamma}_{\Phi}\,T^{\Phi 3}_{\kappa}+\rho_{\kappa}\,\mu\mu^{\Gamma}_{\Phi}\,b^{\Phi}=\rho_{\kappa}\,\mu\mu^{\Gamma}_{\Phi}\,a^{\Phi}, \\ &(\mu T^{3\Psi}_{\kappa}\,\delta^{\Delta}_{\Psi})|_{\Delta}+B_{\Gamma\Delta}\,\mu\mu^{\Gamma}_{\Phi}\,T^{\Phi\Psi}_{\kappa}\,\delta^{\Delta}_{\Psi}+(\mu T^{33}_{\kappa})_{,3}+\rho_{\kappa}\,\mu b^3=\rho_{\kappa}\,\mu a^3, \end{aligned} \tag{16}$$

Multiplying (16) by $X^3$ we arrive in turn at:

$$
\begin{aligned}
&(\mu\mu^{\Gamma}_{\Phi} T^{\Phi\Psi}_{\kappa}\, \delta^{\Delta}_{\Psi} X^3)|_{\Delta} - B^{\Gamma}_{\Delta}\, \mu T^{3\Psi}_{\kappa}\, \delta^{\Delta}_{\Psi} X^3 - \mu\mu^{\Gamma}_{\Phi} T^{\Phi 3}_{\kappa} + \\
&\qquad + (\mu\mu^{\Gamma}_{\Phi} T^{\Phi 3}_{\kappa} X^3)_{,3} + \rho_{\kappa}\, \mu\mu^{\Gamma}_{\Phi}\, b^{\Phi} X^3 = \rho_{\kappa}\, \mu\mu^{\Gamma}_{\Phi}\, a^{\Phi} X^3 , \\
&(\mu T^{3\Psi}_{\kappa}\, \delta^{\Delta}_{\Psi} X^3)|_{\Delta} + B_{\Gamma\Delta}\, \mu\mu^{\Gamma}_{\Phi} T^{\Phi\Psi}_{\kappa}\, \delta^{\Delta}_{\Psi} X^3 - \mu T^{33}_{\kappa} + \\
&\qquad + (\mu T^{33}_{\kappa} X^3)_{,3} + \rho_{\kappa}\, \mu b^3 X^3 = \rho_{\kappa}\, \mu a^3 X^3 .
\end{aligned} \tag{17}
$$

With the help of $(2)_2$ and $(6)_2$ we have for the components of $\mathbf{T}_{\kappa}$ in the basis $\mathbf{G}_K \otimes \mathbf{G}_L$

$$
\begin{aligned}
&T^{\Phi L}_{\kappa} = [\delta^{\Phi}_{\Theta} + \mu^{\Phi}_{\Gamma} (u^{\Gamma}|_{\Delta} - B^{\Gamma}_{\Delta} u^3)\, \delta^{\Delta}_{\Theta}]\, \tilde{T}^{\Theta L} + \mu^{\Phi}_{\Gamma} u^{\Gamma}_{,3}\, \tilde{T}^{3L} , \\
&T^{3L}_{\kappa} = [(u^3_{,\Delta} + B_{\Delta\Gamma} u^{\Gamma})\, \delta^{\Delta}_{\Theta}]\, \tilde{T}^{\Theta L} + (1 + u^3_{,3})\, \tilde{T}^{3L} .
\end{aligned} \tag{18}
$$

The relations (16), (17) and (18) are exact. The linear approximation (10) in the basis $\mathbf{A}_{\Delta}(Q)$ takes the form:

$$
u^{\Delta}(P, t) = v^{\Delta}(Q, t) + X^3\, \beta^{\Delta}(Q, t) . \tag{19}
$$

Let us integrate Eqs. (16) and (17) with respect to $X^3$ across the thickness $H = H(Q)$ of the shell in the reference configuration $\kappa$. Using (18) and (19), we obtain the following equations of motion:

$$
\begin{aligned}
&\{(\delta^{\Delta}_{\Gamma} + v^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} v^3)\, N^{\Gamma\Delta} + [\beta^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} (1 + \beta^3)]\, M^{\Gamma\Delta} + \beta^{\Delta} N^{\Delta 3}\}|_{\Delta} - \\
&\qquad - B^{\Delta}_{\Delta} \{(v^3_{,\Gamma} + B_{\Gamma\Sigma} v^{\Sigma})\, N^{\Gamma\Delta} + (\beta^3_{,\Gamma} + B_{\Gamma\Sigma} \beta^{\Sigma})\, M^{\Gamma\Delta} + (1 + \beta^3)\, N^{\Delta 3}\} + \\
&\qquad + P^{\Delta} = {}_0\rho \ddot{v}^{\Delta} + {}_1\rho \ddot{\beta}^{\Delta} , \\
&\{(v^3_{,\Gamma} + B_{\Gamma\Sigma} v^{\Sigma})\, N^{\Gamma\Delta} + (\beta^3_{,\Gamma} + B_{\Gamma\Sigma} \beta^{\Sigma})\, M^{\Gamma\Delta} + (1 + \beta^3)\, N^{\Delta 3}\}|_{\Delta} + \\
&\qquad + B_{\Delta\Delta} \{(\delta^{\Delta}_{\Gamma} + v^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} v^3)\, N^{\Gamma\Delta} + [\beta^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} (1 + \beta^3)]\, M^{\Gamma\Delta} + \beta^{\Delta} N^{\Delta 3}\} + \\
&\qquad + P^3 = {}_0\rho \ddot{v}^3 + {}_1\rho \ddot{\beta}^3 , \\
&\{(\delta^{\Delta}_{\Gamma} + v^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} v^3)\, M^{\Gamma\Delta} + [\beta^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} (1 + \beta^3)]\, K^{\Gamma\Delta} + \beta^{\Delta} M^{\Delta 3}\}|_{\Delta} - \\
&\qquad - B^{\Delta}_{\Delta} \{(v^3_{,\Gamma} + B_{\Gamma\Sigma} v^{\Sigma})\, M^{\Gamma\Delta} + (\beta^3_{,\Gamma} + B_{\Gamma\Sigma} \beta^{\Sigma})\, K^{\Gamma\Delta} + (1 + \beta^3)\, M^{\Delta 3}\} - \\
&\qquad - \{(\delta^{\Delta}_{\Gamma} + v^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} v^3)\, N^{\Gamma 3} + [\beta^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} (1 + \beta^3)]\, M^{\Gamma 3} + \beta^{\Delta} N^{33}\} + \\
&\qquad + M^{\Delta} = {}_1\rho \ddot{v}^{\Delta} + {}_2\rho \ddot{\beta}^{\Delta} , \\
&\{(v^3_{,\Gamma} + B_{\Gamma\Sigma} v^{\Sigma})\, M^{\Gamma\Delta} + (\beta^3_{,\Gamma} + B_{\Gamma\Sigma} \beta^{\Sigma})\, K^{\Gamma\Delta} + (1 + \beta^3)\, M^{\Delta 3}\}|_{\Delta} + \\
&\qquad + B_{\Delta\Delta} \{(\delta^{\Delta}_{\Gamma} + v^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} v^3)\, M^{\Gamma\Delta} + [\beta^{\Delta}|_{\Gamma} - B^{\Delta}_{\Gamma} (1 + \beta^3)]\, K^{\Gamma\Delta} + \beta^{\Delta} M^{\Delta 3}\} - \\
&\qquad - \{(v^3_{,\Gamma} + B_{\Gamma\Sigma} v^{\Sigma})\, N^{\Gamma 3} + (\beta^3_{,\Gamma} + B_{\Gamma\Sigma} \beta^{\Sigma})\, M^{\Gamma 3} + (1 + \beta^3)\, N^{33}\} + \\
&\qquad + M^3 = {}_1\rho \ddot{v}^3 + {}_2\rho \ddot{\beta}^3 ,
\end{aligned} \tag{20}
$$

where:

$$
\begin{aligned}
&\begin{bmatrix} N^{\Gamma\Delta} \\ M^{\Gamma\Delta} \\ K^{\Gamma\Delta} \end{bmatrix} = \delta^{\Gamma}_{\Phi}\, \delta^{\Delta}_{\Psi} \int_{-H/2}^{H/2} \mu \tilde{T}^{\Phi\Psi} \begin{bmatrix} 1 \\ X^3 \\ (X^3)^2 \end{bmatrix} dX^3 = \begin{bmatrix} N^{\Delta\Gamma} \\ M^{\Delta\Gamma} \\ K^{\Delta\Gamma} \end{bmatrix} , \\
&\begin{bmatrix} N^{\Delta 3} \\ M^{\Delta 3} \end{bmatrix} = \delta^{\Delta}_{B} \int_{-H/2}^{H/2} \mu \tilde{T}^{3B} \begin{bmatrix} 1 \\ X^3 \end{bmatrix} dX^3 ,
\end{aligned} \tag{21}
$$

$$\begin{bmatrix} \mathcal{B}^A \\ \mathcal{M}^A \end{bmatrix} = \int\limits_{-H/2}^{H/2} \rho_\kappa \, \mu \mu^A_K b^K \begin{bmatrix} 1 \\ X^3 \end{bmatrix} dX^3, \qquad \begin{bmatrix} {}_0\rho \\ {}_1\rho \\ {}_2\rho \end{bmatrix} = \int\limits_{-H/2}^{H/2} \rho_\kappa \, \mu \begin{bmatrix} 1 \\ X^3 \\ (X^3)^2 \end{bmatrix} dX^3,$$

$$\begin{bmatrix} \mathcal{C}^\Gamma \\ \mathcal{L}^\Gamma \end{bmatrix} = \{[\mu^\Gamma_\Phi + \delta^\Delta_\Phi (v^\Gamma|_\Delta - B^\Gamma_\Delta v^3) + \delta^\Delta_\Phi (\beta^\Gamma|_\Delta - B^\Gamma_\Delta \beta^3) X^3] \, v\tilde{T}^{\Phi 3} + \beta^\Gamma \mu \tilde{T}^{33}\} \begin{bmatrix} 1 \\ X^3 \end{bmatrix} \Bigg|_{-H/2}^{H/2},$$

$$\begin{bmatrix} \mathcal{C}^3 \\ \mathcal{L}^3 \end{bmatrix} = \{[\delta^\Delta_\Phi (v^3_{,\Delta} + B_{\Gamma\Delta} v^\Gamma) + \delta^\Delta_\Phi (\beta^3_{,\Delta} + B_{\Gamma\Delta} \beta^\Gamma) X^3] \, \mu \tilde{T}^{\Phi 3} + (1+\beta^3) \mu \tilde{T}^{33}\} \begin{bmatrix} 1 \\ X^3 \end{bmatrix} \Bigg|_{-H/2}^{H/2},$$

$$P^A = \mathcal{C}^A + \mathcal{B}^A, \qquad M^A = \mathcal{L}^A + \mathcal{M}^A.$$

The Eqs. (20) obtained form the general set of equations of motion for the non-linear theory of thin shells in the material description.

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В. Петрашкевич, **Материальные уравнения движения нелинейной теории тонких оболочек**

**Содержание.** В работе обсуждается материальное описание общей нелинейной теории движения тонких оболочек, предполагая линейность распределения деформации на толщине оболочки. Уравнения движения оболочки получены путем интегрирования материальной формы уравнений движения непрерывной среды по толщине оболочки в начальном недеформированном состоянии.