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On the Lagrangean Nonlinear Theory of Moving Shells*

The virtual work principle is used to derive two-dimensionally exact equations of the nonlinear theory of shells. All the relations are presented in terms of symmetrical stress resultant and stress couples defined with respect to some reference (undeformed) shell configuration. The theory is extended to small perturbation problems of deformed shells as well as to dynamical shell problems in a non-inertial frame of reference.

1. Introduction

The basic problems of two-dimensionally exact nonlinear shell theory were reviewed by Koiter [1]. He used the Eulerian approach in which all quantities are defined or referred to deformed shell configuration. The deformed shell middle surface geometry is not known in advance and the simple Eulerian equilibrium equations in general cannot be solved without any further simplifications. However, for small strains (but large displacements) all Eulerian relations become referred entirely to the known reference configuration and in principle can be solved [17].

In general nonlinear shell theory it is desirable to distinguish at the beginning between Eulerian and Lagrangean formulation of the theory, as it is done in three-dimensional continuum mechanics. In the Lagrangean approach all quantities are referred at the beginning to some known reference (usually undeformed) shell configuration.

Lagrangean shell theory can be constructed directly by integration of the appropriate three-dimensional continuum equations over the shell thickness. Using the second Piola-Kirchhoff stress tensor, Habip and Ebcioglu [9], Habip [10] and Pietraszkiewicz [5] discussed various dynamical nonlinear shell theories, in which three displacement and three rotation components of the shell middle surface were taken as independent deformation parameters. Shrivastava and Glockner [6] and Iwao Oshima [11] used the Piola-Kirchhoff stress tensors to obtain shell equations, in which shell deformation was defined by three displacement components of the middle surface.

The appropriate equations of shell equilibrium and natural boundary conditions for the nonlinear shell theory can also be derived using two-dimensional virtual work principle. Thin approach was used by Koiter [1], Sanders [3], Budiansky [4] and Simmonds and Danielsen [7].

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In this report we use the virtual work principle to obtain the Lagrangean nonlinear shell equations. We assume here that deformation of the shell space can be represented entirely by deformation of its middle surface. We do not restrict any strains, displacements or deflections of the shell middle surface, thus obtaining two-dimensionally exact equations of equilibrium and natural boundary conditions in terms defined with respect to some reference (undeformed) shell configuration and along its natural basis. The theory presented here is expressed in terms of symmetric stress resultant and stress couple tensors and is such that after linearization it reduces to the "best" variant of the linear shell theory discussed by Budiansky and Sanders [2]. This Lagrangean shell theory is being extended also to small perturbation problems as well as to shells vibrating in a moving non-inertial frame of reference [16].

2. Notations and basic relations

Let $\boldsymbol{r}(\vartheta^\alpha)$ and $\bar{\boldsymbol{r}}(\vartheta^\alpha)$ be the position vectors of the middle surface of a shell in the reference and deformed configurations, respectively. We will use here as far as possible the system of notations used by Koiter [1] and the author [17]. Thus, for the reference surface we will use the following geometrical quantities: the base vectors $\boldsymbol{a}_\alpha$, the unit vector normal to the surface $\boldsymbol{n}$, the metric tensor $a_{\alpha\beta}$, the curvature tensor $b_{\alpha\beta}$, the permutation tensors $\in_{\alpha\beta}$ and $\in^{\alpha\beta}$, the Gaussian curvature K, the Christoffel symbols $\Gamma_{\lambda,\alpha\beta}$, $\Gamma^\lambda_{\alpha\beta}$ and by a single vertical stroke $(\ )_{|\alpha}$ we will denote the covariant differentiation with respect to the reference surface metric. The analogous geometrical quantities for the deformed surface will be distinguished by a dash: $\bar{\boldsymbol{a}}_\alpha$, $\bar{\boldsymbol{n}}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{\in}_{\alpha\beta}$, $\bar{K}$, $\bar{\Gamma}_{\lambda.\alpha\beta}$, $\bar{\Gamma}^\lambda_{\alpha\beta}$ and by a semi-colon $(\ )_{;\alpha}$ we will denote the covariant differentiation with respect to deformed surface metric.

During deformation we have the following relations [1, 17]

$$
\begin{aligned}
&\bar{\boldsymbol{r}}=\boldsymbol{r}+\boldsymbol{u}, &&\bar{\boldsymbol{a}}_\alpha=\boldsymbol{a}_\alpha+\boldsymbol{u}_{,\alpha},\\
&\bar{\boldsymbol{a}}_\alpha=l^\kappa_{.\alpha}\boldsymbol{a}_\kappa+\varphi_\alpha\boldsymbol{n}, &&\boldsymbol{a}^\kappa=l^\kappa_{.\alpha}\bar{\boldsymbol{a}}^\alpha+n^\kappa\bar{\boldsymbol{n}},\\
&\bar{\boldsymbol{n}}=n^\kappa\boldsymbol{a}_\kappa+n\boldsymbol{n}, &&\boldsymbol{n}=\varphi_\alpha\bar{\boldsymbol{a}}^\alpha+n\bar{\boldsymbol{n}},
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
&\boldsymbol{u}=u_\alpha\boldsymbol{a}^\alpha+w\boldsymbol{n}=u^\alpha\boldsymbol{a}_\alpha+w\boldsymbol{n},\\
&l^\kappa_{.\alpha}=\delta^\kappa_\alpha+u^\kappa{}_{|\alpha}-b^\kappa_\alpha w,\\
&\varphi_\alpha=w_{,\alpha}+b^\kappa_\alpha u_\kappa,
\end{aligned}
\tag{2.2}
$$

$$
\begin{aligned}
&n_\kappa=\bar{\in}^{\alpha\beta}\in_{\lambda\kappa}\varphi_\alpha l^\lambda_{.\beta},\\
&n=\tfrac{1}{2}\bar{\in}^{\alpha\beta}\in_{\lambda\kappa}l^\lambda_{.\alpha}l^\kappa_{.\beta}.
\end{aligned}
\tag{2.3}
$$

The metric and curvature tensors for deformed surface can be found from

$$
\bar{a}_{\alpha\beta}=\bar{\boldsymbol{a}}_\alpha\cdot\bar{\boldsymbol{a}}_\beta,\quad \bar{b}_{\alpha\beta}=\bar{\boldsymbol{a}}_{\alpha;\beta}\cdot\bar{\boldsymbol{n}}=\bar{\boldsymbol{a}}_{\alpha|\beta}\cdot\bar{\boldsymbol{n}},
\tag{2.4}
$$

where in the reference metric

$$\bar{a}_{\alpha|\beta}=d^{\kappa}_{.\alpha\beta}\,a_{\kappa}+d_{\alpha\beta}\,n=\bar{a}_{\beta|\alpha},$$
$$d^{\kappa}_{.\alpha\beta}=l^{\kappa}_{.\alpha|\beta}-b^{\kappa}_{\beta}\,\varphi_{\alpha}=d^{\kappa}_{.\beta\alpha}, \tag{2.5}$$
$$d_{\alpha\beta}=\varphi_{\alpha|\beta}+b^{\kappa}_{\beta}\,l_{\kappa\alpha}=d_{\beta\alpha}.$$

Let us define the surface strain tensor $\gamma_{\alpha\beta}$ and the tensor of change of surface curvature $\kappa_{\alpha\beta}$ by the following relations

$$\gamma_{\alpha\beta}=\tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta}),\qquad \kappa_{\alpha\beta}=-(\bar{b}_{\alpha\beta}-b_{\alpha\beta}), \tag{2.6}$$

where from (2.4), (2.5) and (2.1) it follows

$$\gamma_{\alpha\beta}=\tfrac{1}{2}(l^{\kappa}_{.\alpha}\,l_{\kappa\beta}+\varphi_{\alpha}\,\varphi_{\beta}-a_{\alpha\beta}),$$
$$\kappa_{\alpha\beta}=-(n_{\kappa}\,l^{\kappa}_{.\alpha\beta}+nd_{\alpha\beta}-b_{\alpha\beta}). \tag{2.7}$$

It is important to note here that the sign in our definition for $\kappa_{\alpha\beta}$ is opposite to that used by Koiter [1] for his analogous tensor of change of curvature $\bar{\rho}_{\alpha\beta}$. Our sign convention when linearized agrees with that used by Green and Zerna [13], Naghdi [14] and Chernykh [15] for the linear theory of shells and will correspond to the usual sign convention for the stress couples. Sanders [3] and Budiansky and Sanders [2] overcame the sign convention difficulties by using the opposite sign in their definitions of curvature tensor $b_{\alpha\beta}$.

These strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ satisfy the following compatibility conditions [1]

$$\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu}+\bar{a}^{\kappa\nu}(b_{\kappa\lambda}-\kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}]=0,$$
$$K\gamma^{\kappa}_{\kappa}+\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda}-b_{\alpha\mu}\,\kappa_{\beta\lambda}+\tfrac{1}{2}\kappa_{\alpha\mu}\,\kappa_{\beta\lambda}+\tfrac{1}{2}\bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\,\gamma_{\nu\beta\lambda}]=0, \tag{2.8}$$

where we denote

$$\gamma_{\alpha\beta\lambda}\equiv\gamma_{\alpha\beta|\lambda}+\gamma_{\alpha\lambda|\beta}-\gamma_{\beta\lambda|\alpha},$$
$$\bar{a}^{\kappa\nu}=\bar{\epsilon}^{\kappa\alpha}\bar{\epsilon}^{\nu\beta}\bar{a}_{\alpha\beta}=\frac{a}{\bar{a}}\epsilon^{\kappa\alpha}\epsilon^{\nu\beta}(a_{\alpha\beta}+2\gamma_{\alpha\beta}), \tag{2.9}$$
$$\frac{\bar{a}}{a}=\tfrac{1}{2}\epsilon^{\alpha\lambda}\epsilon^{\beta\mu}(a_{\alpha\beta}+2\gamma_{\alpha\beta})(a_{\lambda\mu}+2\gamma_{\lambda\mu}).$$

In what follows instead of $\kappa_{\alpha\beta}$ we will use mainly the tensor of change of curvature $\rho_{\alpha\beta}$ defined by

$$\rho_{\alpha\beta}=\kappa_{\alpha\beta}+\tfrac{1}{2}(b^{\kappa}_{\alpha}\,\gamma_{\kappa\beta}+b^{\kappa}_{\beta}\,\gamma_{\alpha\kappa}). \tag{2.10}$$

This tensor $\rho_{\alpha\beta}$ differs only by sign from that used by Koiter [1]. This choice of the measure has one advantage. When linearized, it gives us the tensor of change of curvature discussed by Budiansky and Sanders [2] as the "best" choice of the strain measure for the linear theory of shells.

3. Eulerian theory

Koiter [1] discussed the basic set of equations for the nonlinear theory of shells using Eulerian approach, what can be summarized as follows.

Let us assume that the shell is in equilibrium under the surface load $\bar{\boldsymbol{p}}$, per unit area of deformed middle surface $\bar{S}$, and boundary force $\bar{\boldsymbol{F}}$ and couple $\bar{\boldsymbol{K}}$, per unit length of deformed surface boundary contour $\bar{C}$. Then for any further virtual (infinitesimal) displacement field

$$\delta \boldsymbol{u}=\delta \bar{u}_\alpha \bar{\boldsymbol{a}}^\alpha+\delta \bar{w} \bar{\boldsymbol{n}} \tag{3.1}$$

subject to geometrical constraints only, the principle of virtual work have the form

$$\iint_{\bar{S}}(n^{\alpha\beta}\delta\gamma_{\alpha\beta}+m^{\alpha\beta}\delta\rho_{\alpha\beta})\,d\bar{a}=\iint_{\bar{S}}\bar{\boldsymbol{p}}\cdot\delta\boldsymbol{u}\,d\bar{a}+\int_{\bar{C}}(\bar{\boldsymbol{F}}\cdot\delta\boldsymbol{u}+\bar{\boldsymbol{K}}\cdot\delta\boldsymbol{\Omega})\,d\bar{s}, \tag{3.2}$$

where

$$\begin{aligned}
\delta\gamma_{\alpha\beta}&=\tfrac{1}{2}(\delta\bar{u}_{\alpha;\beta}+\delta\bar{u}_{\beta;\alpha})-\bar{b}_{\alpha\beta}\delta\bar{w},\\
\delta\rho_{\alpha\beta}&=\delta\kappa_{\alpha\beta}+\tfrac{1}{2}(\bar{b}^\kappa_\alpha\,\delta\gamma_{\kappa\beta}+\bar{b}^\kappa_\beta\,\delta\gamma_{\kappa\alpha}),\\
\delta\kappa_{\alpha\beta}&=-\delta\bar{w}_{;\alpha\beta}-\bar{b}^\kappa_\alpha\,\delta\bar{u}_{\kappa;\beta}-\bar{b}^\kappa_\beta\,\delta\bar{u}_{\kappa;\alpha}-\bar{b}^\kappa_{\alpha;\beta}\,\delta\bar{u}_\kappa+\bar{b}^\kappa_\alpha\,\bar{b}_{\kappa\beta}\,\delta\bar{w},\\
\delta\boldsymbol{\Omega}&=\bar{\epsilon}^{\beta\alpha}(\delta\bar{\varphi}_\alpha\,\bar{\boldsymbol{a}}_\beta+\tfrac{1}{2}\,\delta\bar{\omega}_{\beta\alpha}\,\bar{\boldsymbol{n}}),
\end{aligned} \tag{3.3}$$

$$\bar{\boldsymbol{F}}=\bar{F}^\alpha\bar{\boldsymbol{a}}_\alpha+\bar{F}\bar{\boldsymbol{n}},\qquad \bar{\boldsymbol{K}}=\bar{\epsilon}_{\alpha\beta}\bar{K}^\alpha\bar{\boldsymbol{a}}^\beta,\qquad \bar{\boldsymbol{p}}=\bar{p}^\alpha\bar{\boldsymbol{a}}_\alpha+\bar{p}\bar{\boldsymbol{n}}. \tag{3.4}$$

The symmetric tensors $n^{\alpha\beta}$ and $m^{\alpha\beta}$, defined in deformed configuration $\bar{S}$ as the coefficients in the virtual work principle (3.2), are called Eulerian stress resultant and stress couple tensors, respectively.

Applying variational calculus to (3.2) we obtain the following Eulerian equations of equilibrium (1)

$$\begin{aligned}
(n^{\alpha\beta}+\tfrac{1}{2}\bar{b}^\alpha_\kappa m^{\beta\kappa}-\tfrac{1}{2}\bar{b}^\beta_\kappa m^{\alpha\kappa})_{;\alpha}-\bar{b}^\beta_\kappa m^{\alpha\kappa}{}_{;\alpha}+\bar{p}^\beta&=0,\\
m^{\alpha\beta}{}_{;\alpha\beta}+\bar{b}_{\alpha\beta}n^{\alpha\beta}+\bar{p}&=0
\end{aligned} \tag{3.5}$$

and four natural boundary conditions, which can be put in several forms. Let $\bar{\boldsymbol{v}}$ and $\boldsymbol{t}$ are unit vectors, outward normal and tangent to the deformed boundary $\bar{C}$, respectively, such that

$$\bar{\boldsymbol{v}}\times\bar{\boldsymbol{t}}=\bar{\boldsymbol{n}},\qquad \bar{\boldsymbol{v}}=\bar{v}_\alpha\bar{\boldsymbol{a}}^\alpha,\qquad \bar{\boldsymbol{t}}=\bar{t}_\alpha\bar{\boldsymbol{a}}^\alpha \tag{3.6}$$

Then for

$$\delta\boldsymbol{u}=\delta\bar{u}_v\bar{\boldsymbol{v}}+\delta\bar{u}_t\bar{\boldsymbol{t}}+\delta\bar{w}\bar{\boldsymbol{n}} \tag{3.7}$$

the following four conditions have to be satisfied on free $\bar{C}$

$$\begin{aligned}
(n^{\alpha\beta}-\bar{b}^\beta_\kappa m^{\alpha\kappa})\bar{v}_\alpha\bar{v}_\beta&=(\bar{F}^\beta-\bar{b}^\beta_\kappa\bar{K}^\kappa)\bar{v}_\beta,\\
(n^{\alpha\beta}-\bar{b}^\beta_\kappa m^{\alpha\kappa})\bar{v}_\alpha\bar{t}_\beta&=\bar{F}^\beta-\bar{b}^\beta_\kappa\bar{K}^\kappa)\bar{t}_\beta,\\
m^{\alpha\beta}{}_{;\beta}\bar{v}_\alpha+\frac{d}{d\bar{s}}(m^{\alpha\beta}\bar{v}_\alpha\bar{t}_\beta)&=\bar{F}+\frac{d}{d\bar{s}}(\bar{K}^\alpha\bar{t}_\alpha),\\
m^{\alpha\beta}\bar{v}_\alpha\bar{v}_\beta&=\bar{K}^\alpha\bar{v}_\alpha
\end{aligned} \tag{3.8}$$

It is important to note the simplicity and certain symmetry of our first two boundary conditions $(3.8)_{1,2}$(cf. [1]). If we introduce the normal curvature $\bar{\sigma}$ and geodesic torsion $\bar{\tau}$ of $\bar{C}$ as

$$\bar{\sigma}=\bar{b}_{\alpha\beta}\bar{t}^{\alpha}\bar{t}^{\beta}, \quad \bar{\tau}=-\bar{b}_{\alpha\beta}\bar{t}^{\alpha}\bar{v}^{\beta} \tag{3.9}$$

these two conditions can be expressed also entirely in terms of physical components

$$\begin{aligned}(n^{\alpha\beta}\bar{v}_{\alpha}\bar{v}_{\beta})+\bar{\tau}(m^{\alpha\beta}\bar{v}_{\alpha}\bar{t}_{\beta})&=\bar{F}^{\beta}\bar{v}_{\beta}+\bar{\tau}\bar{K}^{\beta}\bar{t}_{\beta},\\(n^{\alpha\beta}\bar{v}_{\alpha}\bar{t}_{\beta})-\bar{\sigma}(m^{\alpha\beta}\bar{v}_{\alpha}\bar{t}_{\beta})&=\bar{F}^{\beta}\bar{t}_{\beta}-\bar{\sigma}\bar{K}^{\beta}\bar{t}_{\beta}.\end{aligned} \tag{3.10}$$

Introducing the stress resultant and stress couple vectors by the relations

$$\begin{aligned}\boldsymbol{n}^{\alpha}&=(n^{\alpha\beta}+\tfrac{1}{2}\bar{b}^{\alpha}_{\kappa}m^{\beta\kappa}-\tfrac{1}{2}\bar{b}^{\beta}_{\kappa}m^{\alpha\kappa})\bar{\boldsymbol{a}}_{\beta}+m^{\alpha\kappa}{}_{;\kappa}\bar{\boldsymbol{n}},\\\boldsymbol{m}^{\alpha}&=\bar{\epsilon}_{\beta\lambda}m^{\alpha\beta}\bar{\boldsymbol{a}}^{\lambda}\end{aligned} \tag{3.11}$$

the equations of equilibrium (3.5) and the necessary symmetries imposed on $n^{\alpha\beta}$ and $m^{\alpha\beta}$ can be found from the following two vector equations

$$\begin{aligned}\boldsymbol{n}^{\alpha}{}_{;\alpha}+\bar{\boldsymbol{p}}&=\boldsymbol{0},\\\boldsymbol{m}^{\alpha}{}_{;\alpha}+\bar{\boldsymbol{a}}_{\alpha}\times\boldsymbol{n}^{\alpha}&=\boldsymbol{0}\end{aligned} \tag{3.12}$$

which are equivallent to the global equilibrium conditions about the origin

$$\begin{aligned}\int_{\bar{C}}\boldsymbol{n}^{\alpha}\bar{v}_{\alpha}d\bar{s}+\iint_{\bar{S}}\bar{\boldsymbol{p}}d\bar{a}&=\boldsymbol{0},\\\int_{\bar{C}}(\boldsymbol{m}^{\alpha}+\bar{\boldsymbol{r}}\times\boldsymbol{n}^{\alpha})\bar{v}_{\alpha}d\bar{s}+\iint_{\bar{S}}(\bar{\boldsymbol{r}}\times\bar{\boldsymbol{p}})d\bar{a}&=\boldsymbol{0}.\end{aligned} \tag{3.13}$$

It is important to note here, that the equations (3.5), (3.12) and (3.13) are two-dimensionally exact, what can be verified by an independent derivation of the Eulerian shell equations integrating three-dimensional continuum equations over the shell thickness [14].

4. Lagrangean theory

It is easy to note, that the Eulerian stress resultant and stress couple tensors $n^{\alpha\beta}$ and $m^{\alpha\beta}$, defined in (3.2), are referred to deformed shell configuration. Thus the vectors $\boldsymbol{n}^{\alpha}$ and $\boldsymbol{m}^{\alpha}$ are measured per unit length of the deformed coordinate curves θ^{α}. Usually the geometry of the deformed shell is not known in advance and it is not possible to solve the equilibrium equations (3.5) without any further simplifications.

In the exact nonlinear theory of shells it is desirable to deal with the quantities associated entirely with the known reference (undeformed) shell configuration.

Let us assume the shell to be in equilibrium, under the surface load $\boldsymbol{p}$, per unit area of the reference shell middle surfaces, and the boundary force $\boldsymbol{F}$ and couple $\boldsymbol{K}$, per unit length of the reference surface boundary contour C. If the virtual displacement field (3.1) is resolved in the reference surface basis

$$\delta \boldsymbol{u} = \delta u_\alpha \boldsymbol{a}^\alpha + \delta w \boldsymbol{n} \tag{4.1}$$

then the principle of virtual work (3.2) can be put in the following Lagrangean form

$$\iint_S (N^{\alpha\beta} \delta\gamma_{\alpha\beta} + M^{\alpha\beta} \delta\rho_{\alpha\beta})\, da = \iint_S \boldsymbol{p} \cdot \delta \boldsymbol{u} da + \int_C (\boldsymbol{F} \cdot \delta \boldsymbol{u} + \boldsymbol{K} \cdot \delta \boldsymbol{\Omega})\, ds, \tag{4.2}$$

where here we have $(3.3)_2$ and

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \tfrac{1}{2}(l^{\kappa}_{.\alpha}\, \delta l_{\kappa\beta} + l^{\kappa}_{.\beta}\, \delta l_{\kappa\alpha} + \varphi_\alpha\, \delta\varphi_\beta + \varphi_\beta\, \delta\varphi_\alpha), \\ \delta\kappa_{\alpha\beta} &= -\boldsymbol{n} \cdot \delta \boldsymbol{d}_{\alpha\beta} - n^\kappa \delta d_{\kappa\alpha\beta} - d_{\alpha\beta}\, \delta n - d_{\kappa\alpha\beta}\, \delta n^\kappa, \\ \boldsymbol{F} = F^\alpha \boldsymbol{a}_\alpha + F\boldsymbol{n}, &\quad \boldsymbol{K} = \in_{\alpha\beta} K^\alpha \boldsymbol{a}^\beta + K\boldsymbol{n}, \quad \boldsymbol{p} = p^\alpha \boldsymbol{a}_\alpha + p\boldsymbol{n}. \end{aligned} \tag{4.3}$$

The symmetric tensors $N^{\alpha\beta}$ and $M^{\alpha\beta}$, defined with respect to the reference (underformed) shell configuration in (4.2), will be called the Lagrangean stress resultant and couple tensors, respectively.

Using the relations

$$d\bar{a} = \sqrt{\frac{\bar{a}}{a}}\, da, \quad \bar{\nu}_\alpha d\bar{s} = \sqrt{\frac{\bar{a}}{a}}\, \nu_\alpha ds \tag{4.4}$$

it is easy to establish the relations between the Lagrangean and Eulerian quantities to be

$$N^{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\, n^{\alpha\beta}, \quad M^{\alpha\beta} = \sqrt{\frac{\bar{a}}{a}}\, m^{\alpha\beta}. \tag{4.5}$$

To transform (4.2) let us note that in vector form

$$\begin{aligned} \delta\gamma_{\alpha\beta} &= \tfrac{1}{2}\delta(\bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_\beta) = \tfrac{1}{2}(\delta \boldsymbol{u}_{,\alpha} \cdot \bar{\boldsymbol{a}}_\beta + \bar{\boldsymbol{a}}_\alpha \cdot \delta \boldsymbol{u}_{,\beta}) \\ \delta\kappa_{\alpha\beta} &= -\delta(\bar{\boldsymbol{a}}_{\alpha;\beta} \cdot \bar{\boldsymbol{n}}) = -[(\delta \boldsymbol{u}_{,\alpha})_{|\beta} - \bar{a}^{\kappa\lambda} \gamma_{\lambda\alpha\beta} \delta \boldsymbol{u}_{,\kappa}] \cdot \bar{\boldsymbol{n}} \end{aligned} \tag{4.6}$$

and the left-hand side of (4.2), representing the internal virtual work (IVW) can be transformed to the form

$$\mathrm{IVW} = \int_C [(Q^{\alpha\beta} \bar{\boldsymbol{a}}_\beta + Q^\alpha \bar{\boldsymbol{n}}) \cdot \delta \boldsymbol{u} - M^{\alpha\beta} \bar{\boldsymbol{n}} \cdot \delta \bar{\boldsymbol{u}}_{,\beta}]\, \nu_\alpha ds - \iint_S (Q^{\alpha\beta} \bar{\boldsymbol{a}}_\beta + Q^\alpha \bar{\boldsymbol{n}})_{|\alpha} \cdot \delta \boldsymbol{u}\, da, \tag{4.7}$$

where

$$\begin{aligned} Q^{\alpha\beta} &= N^{\alpha\beta} + \tfrac{1}{2} \bar{b}^\alpha_\kappa M^{\kappa\beta} - \tfrac{1}{2} \bar{b}^\beta_\kappa M^{\alpha\kappa}, \\ Q^\alpha &= M^{\alpha\beta}{}_{|\alpha} + \bar{a}^{\alpha\nu} \gamma_{\nu\lambda\mu} M^{\lambda\mu}, \\ \bar{b}^\alpha_\kappa &= \bar{a}^{\alpha\nu}(b_{\nu\kappa} - \kappa_{\nu\kappa}). \end{aligned} \tag{4.8}$$

The second part of the line integral (4.7) can be transformed further as follows

$$\begin{aligned} -\int_C M^{\alpha\beta} \bar{\boldsymbol{n}} \cdot \delta \boldsymbol{u}_{,\beta}\, \nu_\alpha ds &= +\int_C [M^{\alpha\beta} \bar{\boldsymbol{n}}_{,\beta} \cdot \delta \boldsymbol{u} - M^{\alpha\beta} (\bar{\boldsymbol{n}} \cdot \delta \boldsymbol{u})_{,\beta}]\, \nu_\alpha ds = \\ &= -\int_C \bar{b}^\beta_\kappa M^{\alpha\kappa} \bar{\boldsymbol{a}}_\beta \cdot \delta \boldsymbol{u} \nu_\alpha ds + \int_C [(M^{\alpha\beta} \nu_\alpha t_\beta)_{,t} \bar{\boldsymbol{n}} \cdot \delta \boldsymbol{u} - \\ &\quad - (M^{\alpha\beta} \nu_\alpha t_\beta \bar{\boldsymbol{n}} \cdot \delta \boldsymbol{u})_{,t} - M^{\alpha\beta} \nu_\alpha \nu_\beta (\bar{\boldsymbol{n}} \cdot \delta \boldsymbol{u})_{,\nu}]\, ds, \end{aligned} \tag{4.9}$$

where by $(\ )_{,t}$ and $(\ )_{,\nu}$ we denote directional derivatives at the reference shell boundary C, in directions defined by the unit vectors $\boldsymbol{\nu}$ and $\boldsymbol{t}$, outward normal and tangent to the reference boundary C, respectively.

To transform the right-hand side of (4.2), representing the external virtual work (EVW) it is necessary to transform only the last line integral as follows

$$\int_C \boldsymbol{K}\cdot\delta\boldsymbol{\Omega}\,ds=\tfrac{1}{2}\int_C \boldsymbol{K}\cdot(\bar{\boldsymbol{a}}^\alpha\times\delta\bar{\boldsymbol{a}}_\alpha+\bar{\boldsymbol{n}}\times\delta\bar{\boldsymbol{n}})\,ds=-\int_C R^\beta\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}_{,\beta}\,ds=$$
$$=\int_C \bar{b}^\beta_\kappa R^\kappa\bar{\boldsymbol{a}}_\beta\cdot\delta\boldsymbol{u}\,ds+\int_C[(R^\beta t_\beta)_{,t}\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}-(R^\beta t_\beta\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u})_{,t}-$$
$$-R^\beta\nu_\beta(\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u})_{,\nu}]\,ds, \tag{4.10}$$

where

$$R^\beta=\bar{\epsilon}^{\beta\alpha}\boldsymbol{K}\cdot\bar{\boldsymbol{a}}_\alpha=\sqrt{\frac{a}{\bar{a}}}\,\epsilon^{\beta\alpha}(\epsilon_{\lambda\mu}K^\lambda l^\mu_{.\alpha}+K\varphi_\alpha). \tag{4.11}$$

Using (4.7), (4.9) and (4.10) it is easy to show that the Lagrangean form of the virtual work principle (4.2) gives us the following Lagrangean equilibrium equations

$$\begin{aligned}&(Q^{\alpha\lambda}l^\beta_{.\lambda}+Q^\alpha n^\beta)_{|\alpha}-b^\beta_\alpha(Q^{\alpha\lambda}\varphi_\lambda+Q^\alpha n)+p^\beta=0,\\&(Q^{\alpha\lambda}\varphi_\lambda+Q^\alpha n)_{|\alpha}+b_{\alpha\beta}(Q^{\alpha\lambda}l^\beta_{.\lambda}+Q^\alpha n^\beta)+p=0\end{aligned} \tag{4.12}$$

and four natural boundary conditions, which we present here in the form similar to that of (3.8) for the Eulerian theory. Thus if at the boundary C

$$\delta\boldsymbol{u}=\delta u_\nu\boldsymbol{\nu}+\delta u_t\boldsymbol{t}+\delta w\,\boldsymbol{n} \tag{4.13}$$

then the following conditions have to be satisfied on free C

$$\begin{aligned}&\{(Q^{\alpha\beta}-\bar{b}^\beta_\kappa M^{\alpha\kappa})\nu_\alpha l^\lambda_{.\beta}+[Q^\alpha\nu_\alpha+(M^{\alpha\beta}\nu_\alpha t_\beta)_{,t}]n^\lambda\}\nu_\lambda=\{F^\lambda-\bar{b}^\beta_\kappa R^\kappa l^\lambda_{.\beta}+(R^\beta t_\beta)_{,t}n^\lambda\}\nu_\lambda,\\&\{(Q^{\alpha\beta}-\bar{b}^\beta_\kappa M^{\alpha\kappa})\nu_\alpha l^\lambda_{.\beta}+[Q^\alpha\nu_\alpha+(M^{\alpha\beta}\nu_\alpha t_\beta)_{,t}]n^\lambda\}t_\lambda=\{F^\lambda-\bar{b}^\beta_\kappa R^\kappa l^\lambda_{.\beta}+(R^\beta t_\beta)_{,t}n^\lambda\}t_\lambda,\\&(Q^{\alpha\beta}-\bar{b}^\beta_\kappa M^{\alpha\kappa})\nu_\alpha\varphi_\beta+[Q^\alpha\nu_\alpha+(M^{\alpha\beta}\nu_\alpha t_\beta)_{,t}]n=F-\bar{b}^\beta_\kappa R^\kappa\varphi_\beta+(R^\beta t_\beta)_{,t}n,\\&M^{\alpha\beta}\nu_\alpha\nu_\beta=R^\beta\nu_\beta.\end{aligned} \tag{4.14}$$

If we introduce the Lagrangean stress resultant and stress couple vectors by the relations

$$\boldsymbol{N}^\alpha=Q^{\alpha\beta}\bar{\boldsymbol{a}}_\beta+Q^\alpha\bar{\boldsymbol{n}},\qquad \boldsymbol{M}^\alpha=\bar{\epsilon}_{\beta\lambda}M^{\alpha\beta}\bar{\boldsymbol{a}}^\lambda, \tag{4.15}$$

where (4.8), (2.1) and (2.9) should be used, it is easy to show that the Lagrangean equations of equilibrium (4.12) and all the symmetries imposed on $N^{\alpha\beta}$ and $M^{\alpha\beta}$ can be found from the following Lagrangean vector equilibrium equations

$$\begin{aligned}&\boldsymbol{N}^\alpha{}_{|\alpha}+\boldsymbol{p}=0,\\&\boldsymbol{M}^\alpha{}_{|\alpha}+\bar{\boldsymbol{a}}_\alpha\times\boldsymbol{N}^\alpha=0\end{aligned} \tag{4.16}$$

which are equivallent to the Lagrangean global equilibrium conditions about the origin

$$\int_C \boldsymbol{N}^\alpha \nu_\alpha ds + \iint_S \boldsymbol{p}\, da = \boldsymbol{0}, \tag{4.17}$$
$$\int_C (\boldsymbol{M}^\alpha + \bar{\boldsymbol{r}} \times \boldsymbol{N}^\alpha) \nu_\alpha ds + \iint_S (\bar{\boldsymbol{r}} \times \boldsymbol{p})\, da = \boldsymbol{0}.$$

It is easy to show that under (4.4) and (4.5) we have

$$\boldsymbol{N}^\alpha = \sqrt{\frac{\bar{a}}{a}}\,\boldsymbol{n}^\alpha, \quad \boldsymbol{M}^\alpha = \sqrt{\frac{\bar{a}}{a}}\,\boldsymbol{m}^\alpha, \quad \boldsymbol{p} = \sqrt{\frac{\bar{a}}{a}}\,\bar{\boldsymbol{p}} \tag{4.18}$$

and (4.17) can be obtained directly from (3.13).

The equations (4.12), (4.15), (4.16) and (4.17) are also two-dimensionally exact, valid for arbitrarily large strains, displacements, deflections and rotations of the shell middle surface.

The vector equations (4.16) enable us to discuss the obtained results from purely geometrical point of view. Indead (4.12) can be looked at as the component form of (4.16) along the reference basis $\boldsymbol{a}_\alpha$, $\boldsymbol{n}$.

It is easy to note that our equations (4.12), (4.15) are different from those proposed before for the Lagrangean shell theory. Our theory is formulated entirely in terms of symmetric stress resultant and stress couple tensors. This results here directly from defining them as the coefficients in the virtual work principle (4.2), and not by integration over the shell thickness as it is done in [5, 6, 9 - 12]. Our equations are two-dimensionally exact. It means that, as yet, we have not restricted any strains, displacements or deflections of the shell middle surface. The equations are written entirely in known reference shell geometry by means of quantities defined with respect to the reference configuration. For any particular shell problem our equations in principle can be solved without any further simplification.

On the other hand our equations are much more complex than those of the Eulerian theory (3.5) and (3.8). Although the full discussion of various consistent simplifications of our equations will be published separately, it is interesting to note here the following two extreme special cases. If the flexural strains are supposed to be much smaller than the extensional strains, all stress couple terms can be ignored, and the resulting membrane equations are those derived by Budiansky [4] for the Lagrangean membrane nonlinear shell theory. Complete linearisation of our equations for infinitesimal strains, displacements and deflections leads to the „best" variant of the linear theory of shells [1, 2].

5. Small perturbation problems

Large class of problems, such like dynamical stability or small vibrations of deformed shell, can be solved using the exact theory of small deformations superposed on the large finite deformation [13]. Let us assume that the deformed shell $\bar{S}$ undergoes some small perturbation and becomes S'.

Thus

$$\boldsymbol{r}'=\bar{\boldsymbol{r}}+\varepsilon\overset{*}{\boldsymbol{n}},\quad a'_\alpha=\bar{\boldsymbol{a}}_\alpha+\varepsilon\overset{*}{\boldsymbol{a}}_\alpha,\quad \bar{\boldsymbol{n}}'=\boldsymbol{n}+\varepsilon\overset{*}{\boldsymbol{n}},$$
$$a'_{\alpha\beta}=\bar{a}_{\alpha\beta}+\varepsilon\overset{*}{a}_{\alpha\beta},\qquad b'_{\alpha\beta}=\bar{b}_{\alpha\beta}+\varepsilon\overset{*}{b}_{\alpha\beta},\tag{5.1}$$
$$\gamma'_{\alpha\beta}=\gamma_{\alpha\beta}+\varepsilon\overset{*}{\gamma}_{\alpha\beta},\quad \kappa'_{\alpha\beta}=\kappa_{\alpha\beta}+\varepsilon\overset{*}{\kappa}_{\alpha\beta},\quad \text{etc.},$$

where all geometrical quantities with star can be found easily in terms of undeformed geometry and perturbed displacement field. Thus, for example, from (5.1) and (2.1) to (2.3) we find

$$\overset{*}{\boldsymbol{a}}_\alpha=\overset{*}{l}{}^\kappa_{.\alpha}\,\boldsymbol{a}_\kappa+\overset{*}{\varphi}_\kappa\,\boldsymbol{n},\qquad \overset{*}{\boldsymbol{n}}=\overset{*}{n}{}^\kappa\boldsymbol{a}_\kappa+\overset{*}{\varphi}_\kappa\,\boldsymbol{n},\tag{5.2}$$
$$\overset{*}{l}{}^\kappa_{.\alpha}=\overset{*}{u}{}^\kappa{}_{|\alpha}-b^\kappa_\alpha\overset{*}{w},\qquad \overset{*}{\varphi}_\alpha=\overset{*}{w}_{,\alpha}+b^\kappa_\alpha\overset{*}{u}_\kappa,$$

$$\overset{*}{\gamma}_{\alpha\beta}=\tfrac{1}{2}(\overset{*}{l}{}^\kappa_{.\alpha}\overset{*}{l}_{\kappa\beta}+\overset{*}{l}{}^\kappa_{.\alpha}l_{\kappa\beta}+\varphi_\alpha\overset{*}{\varphi}_\beta+\overset{*}{\varphi}_\alpha\varphi_\beta),\tag{5.3}$$
$$\overset{*}{\kappa}_{\alpha\beta}=-d_{\alpha\beta}\overset{*}{n}-\overset{*}{d}_{\alpha\beta}n-d_{\kappa\alpha\beta}\overset{*}{n}{}^\kappa-\overset{*}{d}_{\kappa\alpha\beta}n^\kappa,$$

$$\overset{*}{n}{}^\kappa=-2n^\kappa\bar{a}^{\alpha\beta}\overset{*}{\gamma}_{\alpha\beta}+\sqrt{\frac{a}{\bar{a}}}\in^{\alpha\beta}\in^{\lambda\kappa}(\varphi_\alpha\overset{*}{l}_{\lambda\beta}+\overset{*}{\varphi}_\alpha l_{\lambda\beta}),\tag{5.4}$$
$$\overset{*}{n}=-2n\bar{a}^{\alpha\beta}\overset{*}{\gamma}_{\alpha\beta}+\sqrt{\frac{a}{\bar{a}}}\in^{\alpha\beta}\in^{\lambda\mu}l_{\alpha\lambda}\overset{*}{l}_{\beta\mu},$$

$$\overset{*}{a}_{\alpha\beta}=2\overset{*}{\gamma}_{\alpha\beta},\qquad \overset{*}{b}_{\alpha\beta}=-\overset{*}{\kappa}_{\alpha\beta},\tag{5.5}$$
$$\overset{*}{a}{}^{\kappa\nu}=-2\bar{a}^{\kappa\nu}\bar{a}^{\alpha\beta}\overset{*}{\gamma}_{\alpha\beta}+2\frac{a}{\bar{a}}\in^{\kappa\alpha}\in^{\nu\beta}\overset{*}{\gamma}_{\alpha\beta},\qquad \text{etc.}$$

Let us assume also that

$$\boldsymbol{p}'=\boldsymbol{p}+\varepsilon\overset{*}{\boldsymbol{p}},\qquad \boldsymbol{F}'=\boldsymbol{F}+\varepsilon\overset{*}{\boldsymbol{F}},\qquad \boldsymbol{K}'=\boldsymbol{K}+\varepsilon\overset{*}{\boldsymbol{K}}',\tag{5.6}$$
$$N'^{\alpha\beta}=N^{\alpha\beta}+\varepsilon\overset{*}{N}{}^{\alpha\beta},\qquad M'^{\alpha\beta}=M^{\alpha\beta}+\varepsilon\overset{*}{M}{}^{\alpha\beta}.$$

If we now write down the equilibrium equations (4.12) for deformed perturbed shell S', and substract (4.12) valid for $\bar{S}$, we obtain the following Lagrangean equations to be satisfied for the perturbed quantities

$$(\overset{*}{Q}{}^{\alpha\lambda}l^\beta_{.\lambda}+\overset{*}{Q}{}^\alpha n^\beta)_{|\alpha}-b^\beta_\alpha(\overset{*}{Q}{}^{\alpha\lambda}\varphi_\lambda+\overset{*}{Q}{}^\alpha n)+(Q^{\alpha\lambda}\overset{*}{l}{}^\beta_{.\lambda}+Q^\alpha\overset{*}{n}{}^\beta)_{|\alpha}-b^\beta_\alpha(Q^{\alpha\lambda}\overset{*}{\varphi}_\lambda+Q^\alpha\overset{*}{n})+$$
$$+[\tfrac{1}{2}(\overset{*}{b}{}^\alpha_\kappa M^{\lambda\kappa}-\overset{*}{b}{}^\lambda_\kappa M^{\alpha\kappa})l^\beta_{.\lambda}+(\overset{*}{a}{}^{\alpha\lambda}\gamma_{\lambda\mu\kappa}+\bar{a}^{\alpha\lambda}\overset{*}{\gamma}_{\lambda\mu\kappa})M^{\mu\kappa}n^\beta]_{|\alpha}-$$

$$-b^{\beta}_{\alpha}[\tfrac{1}{2}(\overset{*}{b}{}^{\alpha}_{\kappa} M^{\lambda\kappa}-\overset{*}{b}{}^{\lambda}_{\kappa} M^{\alpha\kappa})\,\varphi_{\lambda}+(\overset{*}{a}{}^{\alpha\lambda}\gamma_{\lambda\mu\kappa}+\bar{a}^{\alpha\lambda}\overset{*}{\gamma}_{\lambda\mu\kappa})\,M^{\mu\kappa}n]+\overset{*}{p}{}^{\beta}=0, \tag{5.7}$$

$$(\overset{*}{Q}{}^{\alpha\lambda}\varphi_{\lambda}+\overset{*}{Q}{}^{\alpha}n)_{|\alpha}+b_{\alpha\beta}(\overset{*}{Q}{}^{\alpha\lambda}l^{\beta}_{.\lambda}+\overset{*}{Q}{}^{\alpha}n^{\beta})+(Q^{\alpha\lambda}\overset{*}{\varphi}_{\lambda}+Q^{\alpha}\overset{*}{n})_{|\alpha}+b_{\alpha\beta}(Q^{\alpha\lambda}\overset{*}{l}{}^{\beta}_{.\lambda}+Q^{\alpha}\overset{*}{n}{}^{\beta})+$$

$$+[\tfrac{1}{2}(\overset{*}{b}{}^{\alpha}_{\kappa} M^{\lambda\kappa}-\overset{*}{b}{}^{\lambda}_{\kappa} M^{\alpha\kappa})\,\varphi_{\lambda}+(\overset{*}{a}{}^{\alpha\lambda}\gamma_{\lambda\mu\kappa}+\bar{a}^{\alpha\lambda}\overset{*}{\gamma}_{\lambda\mu\kappa})\,M^{\mu\kappa}n]_{|\alpha}+$$

$$+b_{\alpha\beta}[\tfrac{1}{2}(\overset{*}{b}{}^{\alpha}_{\kappa} M^{\lambda\kappa}-\overset{*}{b}{}^{\lambda}_{\kappa} M^{\alpha\kappa})\,l^{\beta}_{.\lambda}+(\overset{*}{a}{}^{\alpha\lambda}\gamma_{\lambda\mu\kappa}+\bar{a}^{\alpha\lambda}\overset{*}{\gamma}_{\lambda\mu\kappa})\,M^{\mu\kappa}n^{\beta}]+\overset{*}{p}=0,$$

where we denoted

$$\begin{aligned}
\overset{*}{Q}{}^{\alpha\lambda}&=\overset{*}{N}{}^{\alpha\lambda}+\tfrac{1}{2}\bar{b}^{\alpha}_{\kappa}\overset{*}{M}{}^{\lambda\kappa}-\tfrac{1}{2}\bar{b}^{\lambda}_{\kappa}\overset{*}{M}{}^{\alpha\kappa},\\
\overset{*}{Q}{}^{\alpha}&=\overset{*}{M}{}^{\lambda\alpha}{}_{|\lambda}+\bar{a}^{\alpha\lambda}\gamma_{\lambda\mu\kappa}\overset{*}{M}{}^{\mu\kappa},\\
\overset{*}{b}{}^{\alpha}_{\kappa}&=\overset{*}{a}{}^{\alpha\beta}\bar{b}_{\kappa\beta}+\bar{a}^{\alpha\beta}\overset{*}{b}_{\kappa\beta},\\
\overset{*}{\gamma}_{\lambda\mu\kappa}&=\overset{*}{\gamma}_{\lambda\mu|\kappa}+\overset{*}{\gamma}_{\lambda\kappa|\mu}-\overset{*}{\gamma}_{\mu\kappa|\lambda}.
\end{aligned} \tag{5.8}$$

The appropriate boundary conditions for the perturbed quantities can be found similarly from (4.15) to obtain

$$\{(\overset{*}{Q}{}^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}\overset{*}{M}{}^{\alpha\kappa})\,\nu_{\alpha}\,l^{\lambda}_{.\beta}-\overset{*}{b}{}^{\beta}_{\kappa}M^{\alpha\kappa}\nu_{\alpha}\,l^{\lambda}_{.\beta}+(Q^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}M^{\alpha\kappa})\,\nu_{\alpha}\,\overset{*}{l}{}^{\lambda}_{.\beta}+$$

$$+[\overset{*}{Q}{}^{\alpha}\nu_{\alpha}+(\overset{*}{M}{}^{\alpha\beta}\nu_{\alpha}\,t_{\beta})_{,t}]\,n^{\lambda}+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}\,t_{\beta})_{,t}]\,\overset{*}{n}{}^{\lambda}\}\,\nu_{\lambda}=$$

$$=\{\overset{*}{F}{}^{\lambda}-(\bar{b}^{\beta}_{\kappa}\overset{*}{R}{}^{\kappa}+\overset{*}{b}{}^{\beta}_{\kappa}R^{\kappa})\,l^{\lambda}_{.\beta}-\bar{b}^{\beta}_{\kappa}R^{\kappa}\overset{*}{l}{}^{\lambda}_{.\beta}+(\overset{*}{R}{}^{\beta}t_{\beta})_{,t}\,n^{\lambda}+(R^{\beta}t_{\beta})_{,t}\,\overset{*}{n}{}^{\lambda}\}\,\nu_{\lambda},$$

$$\{(\overset{*}{Q}{}^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}\overset{*}{M}{}^{\alpha\kappa})\,\nu_{\alpha}\,l^{\lambda}_{.\beta}-\overset{*}{b}{}^{\beta}_{\kappa}M^{\alpha\kappa}\nu_{\alpha}\,l^{\lambda}_{.\beta}+(Q^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}M^{\alpha\kappa})\,\nu_{\alpha}\,\overset{*}{l}{}^{\lambda}_{.\beta}+$$

$$+[\overset{*}{Q}{}^{\alpha}\nu_{\alpha}+(\overset{*}{M}{}^{\alpha\beta}\nu_{\alpha}\,t_{\beta})_{,t}]\,n^{\lambda}+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}\,t_{\beta})_{,t}]\,\overset{*}{n}{}^{\lambda}\}\,t_{\lambda}=$$

$$=\{\overset{*}{F}{}^{\lambda}-(\bar{b}^{\beta}_{\kappa}\overset{*}{R}{}^{\kappa}+\overset{*}{b}{}^{\beta}_{\kappa}R^{\kappa})\,l^{\lambda}_{.\beta}-\bar{b}^{\beta}_{\kappa}R^{\kappa}\overset{*}{l}{}^{\lambda}_{.\beta}+(\overset{*}{R}{}^{\beta}t_{\beta})_{,t}\,n^{\lambda}+(R^{\beta}t_{\beta})_{,t}\,\overset{*}{n}{}^{\lambda}\}\,\nu_{\lambda}, \tag{5.9}$$

$$(\overset{*}{Q}{}^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}M^{\alpha\kappa})\,\nu_{\alpha}\,\varphi_{\beta}-\overset{*}{b}{}^{\beta}_{\kappa}M^{\alpha\kappa}\nu_{\alpha}\,\varphi_{\beta}+(Q^{\alpha\beta}-\bar{b}^{\beta}_{\kappa}M^{\alpha\kappa})\,\nu_{\alpha}\,\overset{*}{\varphi}_{\beta}+$$

$$+[\overset{*}{Q}{}^{\alpha}\nu_{\alpha}+(\overset{*}{M}{}^{\alpha\beta}\nu_{\alpha}t_{\beta})_{,t}]\,n+[Q^{\alpha}\nu_{\alpha}+(M^{\alpha\beta}\nu_{\alpha}\,t_{\beta})_{,t}]\,\overset{*}{n}=$$

$$=\overset{*}{F}-(\bar{b}^{\beta}_{\kappa}\overset{*}{R}{}^{\kappa}\varphi_{\beta}+\overset{*}{b}{}^{\beta}_{\kappa}R^{\kappa}\varphi_{\beta})-\bar{b}^{\beta}_{\kappa}R^{\kappa}\overset{*}{\varphi}_{\beta}+(\overset{*}{R}{}^{\beta}t_{\beta})_{,t}\,n+(R^{\beta}t_{\beta})_{,t}\,\overset{*}{n},$$

$$\overset{*}{M}{}^{\alpha\beta}\nu_{\alpha}\,\nu_{\beta}=\overset{*}{R}{}^{\beta}\nu_{\beta}.$$

6. Dynamics of shells in a non-inertial frame

Up to now our Eulerian or Lagrangean equations can be used only for statical shell problems. It is easy, however, to extend them also to dynamical shell problems using the vector equations (3.13) or (4.17). Thus if dynamical effects are taken into account then (4.17) should be replaced by [8]

$$\begin{aligned} &\int_C \boldsymbol{N}^\alpha \nu_\alpha \, ds + \iint_S \boldsymbol{p} \, da = \iint_S \rho \boldsymbol{a} \, da, \\ &\int_C (\boldsymbol{M}^\alpha + \bar{\boldsymbol{r}} \times \boldsymbol{N}^\alpha) \nu_\alpha \, ds + \iint_S (\boldsymbol{m} + \bar{\boldsymbol{r}} \times \boldsymbol{p}) \, da = \iint_S \rho (\bar{\boldsymbol{r}} \times \boldsymbol{a} + \boldsymbol{I} \cdot \boldsymbol{\varepsilon}) \, da, \end{aligned} \tag{6.1}$$

where ρ is the mass per unit area of the reference shell middle surface, $\boldsymbol{m}$ — external surface moment per unit area of the reference shell middle surface, $\boldsymbol{\varepsilon}$ — angular acceleration vector of the shell deformation, $\boldsymbol{I}$ — moment of inertia tensor. Solving the dynamical problem we should known ρ, $\boldsymbol{m}$ and $\boldsymbol{I}$ in advance. Here, looking only at the conventional shell theory, we assume $\boldsymbol{m}=\boldsymbol{I}=\boldsymbol{O}$.

Inertial frame of reference $\{O_0, \boldsymbol{i}_i\}$ can be defined [16] by choosing a point O_0 in three-dimensional Euchidean point space and three unit orthogonal vectors $\boldsymbol{i}_m$, $m=1,2,3$. In this frame the motion of the shell middle surface points $\bar{P}(t)$ can be described by

$$\begin{aligned} &\bar{\boldsymbol{r}}_0(t) = \overrightarrow{O_0 P}(t) = \bar{r}_{0m}(t) \boldsymbol{i}_m = (r_0^\beta + u^\beta) \boldsymbol{a}_\beta + (r_0 + w) \boldsymbol{n}, \\ &\boldsymbol{v}(t) = \frac{d}{dt} \bar{\boldsymbol{r}}_0(t) = \dot{u}^\beta \boldsymbol{a}_\beta + \dot{w} \boldsymbol{n}, \\ &\boldsymbol{a}(t) = \frac{d^2}{dt^2} \bar{\boldsymbol{r}}_0(t) = \ddot{u}^\beta \boldsymbol{a}_\beta + \ddot{w} \boldsymbol{n} \end{aligned} \tag{6.2}$$

and appropriate Lagrangean equations of motion follow easily as component representation in $\boldsymbol{a}_\beta$, $\boldsymbol{n}$ basis of

$$\boldsymbol{N}^\alpha|_\alpha + \boldsymbol{p} = \boldsymbol{a}. \tag{6.3}$$

Let us assume further the another moving non-inertial frame of reference $\{O(t), \boldsymbol{k}_i(t)\}$, defined by a moving point $O(t)$ and three moving unit orthogonal vectors $\boldsymbol{k}_i(t)$, $i=1,2,3$. In this moving frame the motion of the same $\bar{P}(t)$ is described by

$$\bar{\boldsymbol{r}}(t) = \overrightarrow{O(t)P}(t) = \bar{r}_i(t) \boldsymbol{k}_i(t) = (r^\beta + u^\beta) \boldsymbol{a}_\beta + (r + w) \boldsymbol{n}. \tag{6.4}$$

Let the motion of the inertial frame $\{O_0, \boldsymbol{i}_m\}$ with respect to the moving frame $\{O(t), \boldsymbol{k}_i(t)\}$ is defined by a vector function $\boldsymbol{c}(t)$ and an orthogonal tensor function $\boldsymbol{Q}(t)$ such that [16]

$$\boldsymbol{c} = c_i(t) \boldsymbol{k}_i, \quad \boldsymbol{Q} = Q_{ij}(t) \boldsymbol{k}_i \otimes \boldsymbol{k}_j, \quad \bar{\boldsymbol{r}} = \boldsymbol{c} + \boldsymbol{Q} \bar{\boldsymbol{r}}_0. \tag{6.5}$$

Then for absolute acceleration in the moving frame we can prove [18, 16] the formula

$$\boldsymbol{a} = \ddot{\bar{\boldsymbol{r}}} - \ddot{\boldsymbol{c}} - 2\boldsymbol{A}(\dot{\bar{\boldsymbol{r}}} - \dot{\boldsymbol{c}}) - (\dot{\boldsymbol{A}} - \boldsymbol{A}^2)(\bar{\boldsymbol{r}} - \boldsymbol{c}), \tag{6.6}$$

where here

$$A=\dot{Q}Q^T, \quad \dot{A}=A^2-Q\ddot{Q}^T, \\ A=-A^T, \quad \dot{A}=-\dot{A}^T \tag{6.7}$$

and the dot means the time derivative with respect to the moving frame, keeping $\boldsymbol{k}_i$ constant.

The orthogonal tensor components Q_{ij} can be expressed in terms of only three functions, Euler angles $\psi(t)$, $\varphi(t)$, $\theta(t)$ for example [16]

$$Q_{ij}=\begin{bmatrix} \cos\varphi\cos\psi-\sin\varphi\sin\psi\cos\theta, & -\sin\varphi\cos\psi-\cos\varphi\sin\psi\cos\theta, & \sin\psi\sin\theta \\ \cos\varphi\sin\psi+\sin\varphi\cos\psi\cos\theta, & -\sin\varphi\sin\psi+\cos\varphi\cos\psi\cos\theta, & -\cos\psi\sin\theta \\ \sin\varphi\sin\theta, & \cos\varphi\sin\theta, & \cos\theta \end{bmatrix} \tag{6.8}$$

For any reference surface geometry, defined in the moving frame by $\boldsymbol{r}=\boldsymbol{r}(\theta^\alpha)$ we can easily express $\boldsymbol{k}_i$ in terms of $\boldsymbol{a}_\beta$, $\boldsymbol{n}$ and Lagrangean equations of motion follows as component representation of (6.3) in $\boldsymbol{a}_\beta$, $\boldsymbol{n}$ basis, using (6.6) for absolute acceleration.

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References

[1] W. T. Koiter, *On the nonlinear theory of thin elastic shells.* Proc. Kon. Ned. Ak. Wet., Ser. B, vol. 69 (1966), No. 1, p. 1-54.

[2] B. Budiansky, J. L. Sanders, *On the „best" first-order linear shell theory.* In „Progress in Applied Mechanics", Prager Anniversary volume, vol. 192, Macmillan, New York 1963, p. 129 - 140.

[3] J. L. Sanders, *Nonlinear theories for thin shells.* Quart. Appl. Math., vol. 21, 1963, p. 21-36.

[4] B. Budiansky, *Notes on nonlinear shell theory.* J. Appl. Mech., Trans. ASME, Ser. E, vol. 35, June 1968, p. 393-401.

[5] W. Pietraszkiewicz, *Material equations of motion for nonlinear theory of thin shells.* Bull. Acad. Polon. Sci., Sér. sci. tech., vol. 19 (1971) No. 6, p. 261-266.

[6] J. P. Shrivastava, P. G. Glockner, *Lagrangean formulation of statics of shells.* J. Eng. Mech. Div., Proc. ASCE, EM5, October 1970, p. 547 - 563.

[7] J. G. Simmonds, D. A. Danielsen, *Nonlinear shell theory with a finite rotation vector.* Proc. Kon. Ned. Ak. Wet., Ser. B, vol. 73 (1970) p. 460-478.

[8] J. G. Simmonds, D. A. Danielsen, *Nonlinear shell theory with finite rotation and stress — function vectors.* ASME publ., Paper No. 72 — APM-CC.

[9] L. M. Habip, I. K. Ebcioglu, *On the equations of motion of shells in the reference state.* Ing.-Arch., vol. 34 (1965), p. 28 - 32.

[10] L. M. Habip, *Theory of elastic shells in the reference state*, Ing.-Arch., vol. 34 (1965) p. 228-237.

[11] Iwao Oshima et al., *On nonlinear shell theories.* Bull. JSME, vol. 13 (1970) No. 64, p. 1155-1164.

[12] A. Sawczuk, *Lagrangean formulation of large deflection theories for plastic shells.* Bull. Acad. Polon. Sci, Série sci. tech., vol. 19 (1971), No. 6, p. 247-252.

[13] A. E. Green, W. Zerna, *Theoretical elasticity.* 2 nd Ed., Oxford 1968.

[14] P. M. Naghdi, *Foundations of elastic shell theory.* In „Progress in Solid Mechanics" vol. IV, Amsterdam 1963.

[15] К. Ф. Черных, *Линейная теория оболочек*, т. II. Ленинград 1965.

[16] W. Pietraszkiewicz, *Równania ruchu ośrodka ciągłego w nieinercjalnym układzie odniesienia.* Biuletyn IMP PAN, No. 10 (675), 1971.

[17] W. Pietraszkiewicz, *Seminars on the nonlinear theory of shells*. Biuletyn IMP PAN, No. 74 (743), 1973.
[18] C. Truesdell, W. Noll, *The nonlinear field theories of mechanics*. In „Encyclopedia of Physics" vol. III/3, Springer-Verlag, 1965.

O nieliniowej teorii powłok w ruchu

Streszczenie

Podstawowe zależności nieliniowej teorii powłok uzyskano z dwuwymiarowej zasady prac wirtualnych. Przedstawiono teorię powłok zarówno w ujęciu Eulera, gdy wszystkie zależności formułowane są w nieznanej geometrii konfiguracji odkształconej powłoki, jak i w ujęciu Lagrange'a, gdy wszystkie zależności formułowane są w znanej geometrii ustalonej konfiguracji odniesienia powłoki. Przedstawiony w pracy nowy wariant nieliniowej teorii powłok w ujęciu Lagrange'a zawiera tylko symetryczne tensory sił i momentów wewnętrznych i stanowi naturalne uogólnienie „najlepszego" wariantu liniowej teorii powłok. Uzyskane wyniki są dwuwymiarowo ścisłe dla powierzchni środkowej powłoki.

Podstawowe zależności teorii powłok w ujęciu Lagrange'a rozszerzono również na zagadnienie małych perturbacji stanu odkształconego oraz sformułowano podstawowe związki nieliniowej dynamiki powłok poruszających się w nieinercjalnym układzie odniesienia.

К нелинейной теории движущихся оболочек

Резюме

Основные зависимости нелинейной теории оболочек получены из двумерного принципа виртуальной работы. Рассмотрена теория оболочек в эйлеровом представлении, когда все зависимости формулируются в неизвестной геометрии деформированной срединной поверхности оболочки, а также в лагранжевом представлении, когда все зависимости формулируются в известной геометрии недеформированной срединной поверхности оболочки.

Предлагаемая новая нелинейная теория оболочек в лагранжевом представлении выражается только через симметрические тензоры внутренних сил и моментов и является обобщением „наилучшего" варианта линейной теории оболочек. Результаты полученные в работе являются двумерно точными для срединной поверхности оболочки.

Основные зависимости теории оболочек в лагранжевом представлении обобщены на проблемы малых пертурбаций деформированного состояния оболочки. Получены также основные зависимости нелинейной динамики оболочек движущихся в неинерциальной системе отсчета.