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Stress in isotropic elastic solid under superposed deformations

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THE EXPLICIT formulae for Cauchy stress tensor in isotropic elastic solid under successive superposition of two deformations have been given. Any deformation can be finite, small or infinitesimal one. The exact formulae for elasticity tensors of zeroth, first and second order in arbitrarily deformed reference configuration have been obtained from the Lagrangian constitutive equation. For infinitesimally and smally deformed reference configurations the elasticity tensors have been expressed in terms of elastic constants of the first or also the second order in the natural state. Derivatives of some useful functions of the second-order tensor are given in the Appendix.

Podano zamknięte wzory na tensor naprężenia Cauchy'ego w izotropowym ciele sprężystym poddanym dwu kolejnym odkształceniom. Odkształcenia te mogą być skończone, małe lub infinitezymalne. Z równania konstytutywnego Lagrange'a wyprowadzono ścisłe wzory dla tensorów sprężystości zerowego, pierwszego i drugiego rzędu w dowolnie odkształconej konfiguracji odniesienia. Jeśli konfiguracja odniesienia powstaje w wyniku odkształceń infinitezymalnych lub małych, to tensory sprężystości wyraża się za pomocą stałych sprężystych stanu naturalnego pierwszego względnie również i drugiego rzędu. W "Dodatku" podano użyteczne wzory na pochodne funkcji od tensora drugiego rzędu.

Даются замкнутые формулы для тензора напряжений Коши в изотропном упругом теле, подвергнутом двум последовательным деформациям. Эти деформации могут быть конечными, малыми или инфинитезимальными. Из определяющего уравнения Лагранжа выведены точные формулы для тензоров упругости нулевого, первого и второго порядков в произвольно деформированной конфигурации отсчета. Если конфигурации отсчета подвергнуты инфинитезимальным или малым деформациям, то тензоры упругости выражаются при помощи упругих постоянных естественного состояния первого и второго порядков. В „Дополнении" приведены полезные формулы для производных функции от тензора второго порядка.

1. Introduction

WITHIN the general theory of continuum mechanics presented by NOLL [1] and TRUESDELL and NOLL [2], the form of strain energy function for an elastic solid depends on the choice of reference configuration. The response of the elastic solid to deformation from an arbitrarily deformed reference configuration is different from the response from the unstressed natural state. This fact was referred to as deformational anisotropy by BERG [3] and URBANOWSKI [4] and can be taken into account directly by specifying the elasticity tensors in the reference configuration. These elasticity tensors are in general different from those specified in the unstressed natural state.

In this paper we discuss the stress-strain relations for an isotropic elastic solid under successive superposition of two deformations. The first deformation is assumed to connect the unstressed natural state with the deformed reference configuration, the second one connects the reference and actual configurations. Three kinds of deformations, depending on the norm of displacement gradients, have been taken into account: the finite, the small and the infinitesimal.

First of all we present the exact formulae for the elasticity tensors of the zeroth, first and second order, derived in an arbitrarily deformed reference configuration [5]. These formulae have been obtained from the Lagrangean constitutive equation with respect to the reference configuration. For infinitesimally and smally deformed reference configuration the elasticity tensors have been expressed in terms of elastic constants of the first and also the second order in the natural state, respectively.

Using the elasticity tensors obtained, six different cases of superposed deformations can be discussed. In addition to two trivial cases of successive superposition of two infinitesimal or two finite deformations, we obtain here the closed explicit formulae for the Cauchy stress tensor in four other cases of infinitesimal or small deformation superposed on a small or finite deformation.

The exact and approximate relations obtained in this paper may be useful in problems of wave propagation, vibration and stability of initially deformed isotropic elastic solid, as well as when studying the second-order effects and other similar problems of non-linear elasticity.

Most of the explicit formulae presented here are new and have been obtained using absolute tensor analysis. It is interesting to note here that TRUESDELL and NOLL [2] used extensively the notion of tensor functions, although little was said in that Ref. about their differentiation. The general rules of differentiation of tensor functions have been discussed by RYCHLEWSKI [6]. To obtain the results presented in the present paper, it was necessary to calculate effectively many derivatives of certain simple functions of the second-order tensor, which were not otherwise available. We considered it worthwhile to present some derivation formulae in the Appendix at the end of this paper. We believe that some of them may be useful also in other problems of mechanics.

2. Notations and basic relations

The absolute tensor analysis in three-dimensional Euclidean vector space [2, 6, 7] is used here. The system of notations for the continuum mechanics quantities is adopted mainly from [2].

Let the Euclidean tensors of the second order be denoted by $\mathbf{A}, \mathbf{B}, \ldots, \mathbf{H}, \mathbf{S}, \mathbf{T} \in \mathscr{C}_2$, and the metric tensor of this tensor space by $\mathbf{1} \in \mathscr{C}_2$. The Euclidean tensors of the fourth and sixth-order are denoted by $\mathbf{K}, \mathbf{L} \in \mathscr{C}_4$ and $\mathbf{M}, \mathbf{N} \in \mathscr{C}_6$. If by $\otimes$ we denote the tensor product operation, then it is evident that tensors of the types $\mathbf{1} \otimes \mathbf{1}, \mathbf{1} \otimes \mathbf{E}, \mathbf{B} \otimes \mathbf{B} \in \mathscr{C}_4$, while $\mathbf{1} \otimes \mathbf{1} \otimes \mathbf{1}, \mathbf{B} \otimes \mathbf{B} \otimes \mathbf{B}, \mathbf{G} \otimes \mathbf{K} \in \mathscr{C}_6$. Let $\mathbf{g}_i, \mathbf{g}_\alpha, \ldots$ $(i, \alpha = 1, 2, 3)$ be the triples of basic vectors of the Euclidean three-dimensional vector space $\mathscr{V} \equiv \mathscr{C}_1$. Any set of nine tensors of the types $\mathbf{g}_i \otimes \mathbf{g}_\alpha$, $\mathbf{g}_i \otimes \mathbf{g}^j$, $\mathbf{g}^\alpha \otimes \mathbf{g}^i, \ldots \in \mathscr{C}_2$, where $\mathbf{g}_i \cdot \mathbf{g}^j = \delta_i^j$, form a basis for $\mathscr{C}_2$.

Let $\mathbf{P} \in \mathscr{C}_p$, $\mathbf{Q} \in \mathscr{C}_q$, $p \geqslant q$. By $\overset{\mu,\nu}{\mathbf{P}^T} \in \mathscr{C}_p$ we denote [6, 8] the operation of transposition (μ, ν); by $\underset{\mu,\nu}{\mathrm{tr}}\,\mathbf{P} \in \mathscr{C}_{p-2}$ we denote the operation of contraction (μ, ν), $1 \leqslant \mu < \nu \leqslant p$, and the simple dot operation and the full dot operation are defined, respectively, by

$$
\begin{aligned}
\mathbf{PQ} &\equiv \underset{p,p+1}{\mathrm{tr}} (\mathbf{P} \otimes \mathbf{Q}) \in \mathscr{C}_{p+q-2}, \\
\mathbf{P} \cdot \mathbf{Q} &\equiv \underset{p-q+1,p+1}{\mathrm{tr}} \dots \underset{p,p+q}{\mathrm{tr}} (\mathbf{P} \otimes \mathbf{Q}) \in \mathscr{C}_{p-q}.
\end{aligned}
\tag{2.1}
$$

The derivatives of a tensor function $f\colon \mathscr{C}_2 \to \mathscr{C}_r$, $\mathbf{R} = f(\mathbf{A})$, are denoted here by $f_{,\mathbf{A}}, f_{,\mathbf{AA}}, \dots$ and their values at $\mathbf{A}_0 \in \mathscr{C}_2$ by $f_{,\mathbf{A}}(\mathbf{A}_0) \in \mathscr{C}_{r+2}$, $f_{,\mathbf{AA}}(\mathbf{A}_0) \in \mathscr{C}_{r+4}, \dots$. When $\mathbf{A} = g(\mathbf{B})$, $g\colon \mathscr{C}_2 \to \mathscr{C}_2$, the derivative of the tensor function $h = f \circ g$, $h(\mathbf{B}) = f[g(\mathbf{B})]$ can be found according to the following chain rule:

$$
h_{,\mathbf{B}} = f_{,\mathbf{A}} \circ g_{,\mathbf{B}} = \underset{r+1,r+3}{\mathrm{tr}}\ \underset{r+2,r+4}{\mathrm{tr}} (f_{,\mathbf{A}} \otimes g_{,\mathbf{B}}). \tag{2.2}
$$

The derivatives of some tensor functions used here are included in the Appendix at the end of this paper.

Let us consider three different mappings of the body $\mathscr{B}$ into three-dimensional Euclidean point space $\mathscr{E}$ [2]: $\boldsymbol{\varkappa}_0\colon \mathscr{B} \to \mathscr{P}_0 \subset \mathscr{E}$, the natural state, unstressed; $\boldsymbol{\varkappa}\colon \mathscr{B} \to \mathscr{P}_\varkappa \subset \mathscr{E}$, a reference configuration, arbitrarily deformed; $\boldsymbol{\gamma}\colon \mathscr{B} \to \mathscr{P}_\gamma \subset \mathscr{E}$, the actual configuration. These three configurations define three deformations

$$
\chi_0 = \boldsymbol{\varkappa} \circ \boldsymbol{\varkappa}_0^{-1}, \qquad \chi = \boldsymbol{\gamma} \circ \boldsymbol{\varkappa}^{-1}, \qquad \chi^* = \boldsymbol{\gamma} \circ \boldsymbol{\varkappa}_0^{-1}, \tag{2.3}
$$

with three deformation gradients

$$
\mathbf{F}_0 = \nabla\chi_0, \qquad \mathbf{F} = \nabla\chi, \qquad \mathbf{F}^* = \nabla\chi^*. \tag{2.4}
$$

For deformation χ with deformation gradient $\mathbf{F}$, the following relations hold [2]:

$$
\begin{aligned}
&\mathbf{F} = \mathbf{1} + \mathbf{H}, \qquad \mathbf{C} = \mathbf{F}^T\mathbf{F}, \qquad \mathbf{B} = \mathbf{F}\mathbf{F}^T, \\
&\mathbf{E} = \frac{1}{2}(\mathbf{C} - \mathbf{1}), \qquad \tilde{\mathbf{E}} = \frac{1}{2}(\mathbf{H} + \mathbf{H}^T).
\end{aligned}
\tag{2.5}
$$

Similar relations hold for deformations χ_0 and χ^*. We can obtain them using deformation gradients $\mathbf{F}_0$ and $\mathbf{F}^*$ and defining in a similar way tensors $\mathbf{H}_0$, $\mathbf{C}_0$, $\mathbf{B}_0$, $\mathbf{E}_0$, $\tilde{\mathbf{E}}_0$, and $\mathbf{H}^*$, $\mathbf{C}^*$, $\mathbf{B}^*$, $\tilde{\mathbf{E}}^*$, $\mathbf{E}^*$, respectively. Between the analogous tensors defined for these three deformations we find the following relations:

$$
\begin{aligned}
\mathbf{F}^* &= \mathbf{F}\mathbf{F}_0, & \mathbf{B}^* &= \mathbf{F}\mathbf{B}_0\mathbf{F}^T, \\
\mathbf{C}^* &= \mathbf{F}_0^T\mathbf{C}\mathbf{F}_0, & \mathbf{E}^* &= \mathbf{F}_0^T\mathbf{E}\mathbf{F}_0 + \mathbf{E}_0, \\
\mathbf{H}^* &= \mathbf{H}_0 + \mathbf{H} + \mathbf{H}\mathbf{H}_0, & \tilde{\mathbf{E}}^* &= \tilde{\mathbf{E}}_0 + \tilde{\mathbf{E}} + \frac{1}{2}(\mathbf{H}\mathbf{H}_0 + \mathbf{H}_0^T\mathbf{H}^T).
\end{aligned}
\tag{2.6}
$$

The Lagrangian constitutive equation for the elastic solid has the form [2, 8]:

$$
\mathbf{S}_\varkappa = 2\varrho_\varkappa \sigma_{\varkappa,\mathbf{C}}(\mathbf{C}) = \varrho_\varkappa \tau_{\varkappa,\mathbf{E}}(\mathbf{E}), \tag{2.7}
$$

where: $\sigma_\varkappa$ and $\tau_\varkappa$ — the strain energy functions defined with respect to $\varkappa$; $\varrho_\varkappa$ — the material density in $\varkappa$; $\mathbf{S}_\varkappa$ — the second Piola-Kirchhoff stress tensor in γ, defined with respect to $\varkappa$, related to $\mathbf{T}$, the Cauchy stress tensor in γ, by the relation:

$$\mathbf{S}_\varkappa = \frac{\varrho_\varkappa}{\varrho} \mathbf{F}^{-1}\mathbf{T}(\mathbf{F}^{-1})^T. \tag{2.8}$$

For an elastic solid with a specified symmetry group, representation theorems are formulated for the strain energy functions defined with respect to an undistorted state, mainly the natural state $\varkappa_0$. In terms of σ_0 and τ_0, the strain energy functions defined with respect to $\varkappa_0$, the Lagrangian constitutive equation has the form:

$$\mathbf{S}_0 = 2\varrho_0\sigma_{0,\mathbf{C}^*}(\mathbf{C}^*) = \varrho_0\tau_{0,\mathbf{E}^*}(\mathbf{E}^*), \tag{2.9}$$

where: ϱ_0 — the material density in $\varkappa_0$; $\mathbf{S}_0$ — the second Piola-Kirchhoff stress tensor in γ, defined with respect to $\varkappa_0$, and related to $\mathbf{T}$ by:

$$\mathbf{S}_0 = \frac{\varrho_0}{\varrho} \mathbf{F}^{*-1}\mathbf{T}(\mathbf{F}^{*-1})^T. \tag{2.10}$$

For an isotropic elastic solid σ_0 and τ_0 are orthogonal invariants

$$\begin{aligned} \sigma_0(\mathbf{C}^*) &= \bar{\sigma}_0(I_{\mathbf{C}^*}, II_{\mathbf{C}^*}, III_{\mathbf{C}^*}), \\ \tau_0(\mathbf{E}^*) &= \bar{\tau}_0(I_{\mathbf{E}^*}, II_{\mathbf{E}^*}, III_{\mathbf{E}^*}), \end{aligned} \tag{2.11}$$

where the principal invariants of $\mathbf{E}^* \in \mathscr{C}_2$, for example, are defined by:

$$\begin{aligned} I_{\mathbf{E}^*} &= \mathrm{tr}\,\mathbf{E}^*, \\ II_{\mathbf{E}^*} &= \frac{1}{2}[(\mathrm{tr}\,\mathbf{E}^*)^2 - \mathrm{tr}\,\mathbf{E}^{*2}], \\ III_{\mathbf{E}^*} &= \det\mathbf{E}^* = \frac{1}{6}[(\mathrm{tr}\,\mathbf{E}^*)^3 - 3\,\mathrm{tr}\,\mathbf{E}^{*2}\mathrm{tr}\,\mathbf{E}^* + 2\,\mathrm{tr}\,\mathbf{E}^{*3}]. \end{aligned} \tag{2.12}$$

3. Elasticity tensors

Expanding the constitutive equation (2.7) into Taylor series in the neighbourhood of $\varkappa$, we obtain:

$$\mathbf{S}_\varkappa = \mathbf{T}_\varkappa + \mathbf{L}_\varkappa \cdot \mathbf{E} + \frac{1}{2!}\mathbf{M}_\varkappa \cdot (\mathbf{E} \otimes \mathbf{E}) + \ldots, \tag{3.1}$$

where

$$\begin{aligned} \mathbf{T}_\varkappa &= 2\varrho_\varkappa\sigma_{\varkappa,\mathbf{C}}(\mathbf{1}) = \varrho_\varkappa\tau_{\varkappa,\mathbf{E}}(\mathbf{0}), \\ \mathbf{L}_\varkappa &= 4\varrho_\varkappa\sigma_{\varkappa,\mathbf{CC}}(\mathbf{1}) = \varrho_\varkappa\tau_{\varkappa,\mathbf{EE}}(\mathbf{0}), \\ \mathbf{M}_\varkappa &= 8\varrho_\varkappa\sigma_{\varkappa,\mathbf{CCC}}(\mathbf{1}) = \varrho_\varkappa\tau_{\varkappa,\mathbf{EEE}}(\mathbf{0}) \end{aligned} \tag{3.2}$$

are the elasticity tensors (elasticities according to [2]) of the zeroth, first and second order, in the reference configuration $\varkappa$.

To find these elasticity tensors, the explicit formulas for $\sigma_\varkappa$ or $\tau_\varkappa$ should be known. It may be found from (2.7) and (2.8), using (2.5) and (2.6), that

$$\begin{aligned} \sigma_{\varkappa,\mathbf{C}}(\mathbf{C}) &= \mathbf{F}_0\,\sigma_{0,\mathbf{C}^*}(\mathbf{C}^*)\,\mathbf{F}_0^T, \\ \tau_{\varkappa,\mathbf{E}}(\mathbf{E}) &= \mathbf{F}_0\,\tau_{0,\mathbf{E}^*}(\mathbf{E}^*)\,\mathbf{F}_0^T, \end{aligned} \tag{3.3}$$

and assuming σ_0 or τ_0 to be given explicitly, these relations are, in fact, sufficient for our purpose.

The transformations below will be effected using the function τ_0 first. Using the chain rule (2.2) and the formulae for derivatives of principal invariants of $\mathbf{E}^*$ [2], we obtain:

$$\tau_{0,\mathbf{E}^*}(\mathbf{E}^*) = \frac{\partial \overline{\tau}_0}{\partial I_{\mathbf{E}^*}}\mathbf{1} + \frac{\partial \overline{\tau}_0}{\partial II_{\mathbf{E}^*}}(I_{\mathbf{E}^*}\mathbf{1}-\mathbf{E}^*) + \frac{\partial \overline{\tau}_0}{\partial III_{\mathbf{E}^*}}(II_{\mathbf{E}^*}\mathbf{1} - I_{\mathbf{E}^*}\mathbf{E}^* + \mathbf{E}^{*2}). \tag{3.4}$$

For a fixed deformation χ_0, tensor $\mathbf{E}^*$ depends only on $\mathbf{E}$ and any function of $\mathbf{E}^*$ is in fact the function of $\mathbf{E}$.

The relations (3.4) and (3.3) suggest the introduction of the following useful functions:

$$\begin{aligned} g_1(\mathbf{E}) &= \mathbf{B}_0, \\ g_2(\mathbf{E}) &= \mathbf{B}_0 I_{\mathbf{E}^*} - \left[\mathbf{B}_0\mathbf{E}\mathbf{B}_0 + \frac{1}{2}(\mathbf{B}_0^2 - \mathbf{B}_0)\right], \\ g_3(\mathbf{E}) &= \mathbf{B}_0 II_{\mathbf{E}^*} - \left[\mathbf{B}_0\mathbf{E}\mathbf{B}_0 + \frac{1}{2}(\mathbf{B}_0^2 - \mathbf{B}_0)\right] I_{\mathbf{E}^*} \\ &\quad + \mathbf{B}_0\mathbf{E}\mathbf{B}_0\mathbf{E}\mathbf{B}_0 + \frac{1}{2}(\mathbf{B}_0^2\mathbf{E}\mathbf{B}_0 + \mathbf{B}_0\mathbf{E}\mathbf{B}_0^2) - \mathbf{B}_0\mathbf{E}\mathbf{B}_0 + \frac{1}{4}(\mathbf{B}_0^3 - 2\mathbf{B}_0^2 + \mathbf{B}_0). \end{aligned} \tag{3.5}$$

From $(3.2)_1$, using $(3.3)_2$, (3.4) and (3.5), we obtain the elasticity tensor of the zeroth order — which is at the same time the Cauchy stress tensor in $\varkappa$ — to be

$$\mathbf{T}_\varkappa = \sum_{r=1}^{3} \tau_r \mathbf{G}_r, \tag{3.6}$$

where

$$\tau_1 = \varrho_\varkappa \left.\frac{\partial \overline{\tau}_0}{\partial I_{\mathbf{E}^*}}\right|_{\mathbf{E}=\mathbf{0}}, \quad \tau_2 = \varrho_\varkappa \left.\frac{\partial \overline{\tau}_0}{\partial II_{\mathbf{E}^*}}\right|_{\mathbf{E}=\mathbf{0}}, \quad \tau_3 = \varrho_\varkappa \left.\frac{\partial \overline{\tau}_0}{\partial III_{\mathbf{E}^*}}\right|_{\mathbf{E}=\mathbf{0}}; \tag{3.7}$$

$$\begin{aligned} \mathbf{G}_1 &= g_1(\mathbf{0}) = \mathbf{B}_0, \\ \mathbf{G}_2 &= g_2(\mathbf{0}) = \mathbf{B}_0 I_{\mathbf{E}_0} - \frac{1}{2}(\mathbf{B}_0^2 - \mathbf{B}_0), \\ \mathbf{G}_3 &= g_3(\mathbf{0}) = \mathbf{B}_0 II_{\mathbf{E}_0} - \frac{1}{2}(\mathbf{B}_0^2 - \mathbf{B}_0) I_{\mathbf{E}_0} + \frac{1}{4}(\mathbf{B}_0^3 - 2\mathbf{B}_0^2 + \mathbf{B}_0). \end{aligned} \tag{3.8}$$

To find the higher-order elasticity tensors, we note first that according to $(2.6)_2$ and the appropriate differentiation formulae

$$\mathbf{E}^*_{,\mathbf{E}} = (\mathbf{F}_0^T\mathbf{E}\mathbf{F}_0)_{,\mathbf{E}} = \frac{1}{2}\left[(\mathbf{F}_0 \otimes \mathbf{F}_0)^{\overset{1,4}{T}} + (\mathbf{F}_0 \otimes \mathbf{F}_0^T)^{\overset{1,3}{T}}\right], \tag{3.9}$$

and for any invariant $\alpha(\mathbf{E}^*)$, $\alpha\colon {}^s\mathscr{C}_2 \to \mathscr{R}$, we have:

$$\alpha_{,\mathbf{E}} = \alpha_{,\mathbf{E}^*} \circ \mathbf{E}^*_{,\mathbf{E}} = \mathbf{F}_0 \alpha_{,\mathbf{E}^*} \mathbf{F}_0^T. \tag{3.10}$$

In particular, for the principle invariants of $\mathbf{E}^*$ it follows that

$$\begin{aligned} I_{\mathbf{E}^*,\mathbf{E}} &= \mathbf{F}_0 I_{\mathbf{E}^*,\mathbf{E}^*} \mathbf{F}_0^T = g_1(\mathbf{E}),\\ II_{\mathbf{E}^*,\mathbf{E}} &= \mathbf{F}_0 II_{\mathbf{E}^*,\mathbf{E}^*} \mathbf{F}_0^T = g_2(\mathbf{E}),\\ III_{\mathbf{E}^*,\mathbf{E}} &= \mathbf{F}_0 III_{\mathbf{E}^*,\mathbf{E}^*} \mathbf{F}_0^T = g_3(\mathbf{E}). \end{aligned} \tag{3.11}$$

Now, from (3.5), (3.11) and the appropriate differentiation formulae, we obtain:

$$\begin{aligned} &g_{1,\mathbf{E}}(\mathbf{E}) = \mathbf{0},\\ &g_{2,\mathbf{E}}(\mathbf{E}) = \mathbf{B}_0 \otimes \mathbf{B}_0 - \frac{1}{2}\,[(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}],\\ &g_{3,\mathbf{E}}(\mathbf{E}) = \mathbf{B}_0 \otimes \mathbf{B}_0 I_{\mathbf{E}^*} - \frac{1}{2}\,[(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}] I_{\mathbf{E}^*}\\ &\quad - \frac{1}{2}\,[\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0] - (\mathbf{B}_0\mathbf{E}\mathbf{B}_0 \otimes \mathbf{B}_0 + \mathbf{B}_0 \otimes \mathbf{B}_0\mathbf{E}\mathbf{B}_0)\\ &\quad + \frac{1}{4}\,[\{\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0\}^{\overset{1,4}{T}} + \{\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0\}^{\overset{1,3}{T}}]\\ &\quad + \frac{1}{4}\,[(\mathbf{B}_0\mathbf{E}\mathbf{B}_0 \otimes \mathbf{B}_0 + \mathbf{B}_0 \otimes \mathbf{B}_0\mathbf{E}\mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0\mathbf{E}\mathbf{B}_0 \otimes \mathbf{B}_0 + \mathbf{B}_0 \otimes \mathbf{B}_0\mathbf{E}\mathbf{B}_0)^{\overset{1,3}{T}}]. \end{aligned} \tag{3.12}$$

Let us define the tensors

$$\begin{aligned} \mathbf{K}_1 &= g_{1,\mathbf{E}}(\mathbf{0}) = \mathbf{0},\\ \mathbf{K}_2 &= g_{2,\mathbf{E}}(\mathbf{0}) = \mathbf{B}_0 \otimes \mathbf{B}_0 - \frac{1}{2}\,[(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}],\\ \mathbf{K}_3 &= g_{3,\mathbf{E}}(\mathbf{0}) = \mathbf{B}_0 \otimes \mathbf{B}_0 I_{\mathbf{E}_0} - \frac{1}{2}\,[(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}] I_{\mathbf{E}_0}\\ &\quad - \frac{1}{2}\,[\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0]\\ &\quad + \frac{1}{4}\,[\{\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0\}^{\overset{1,4}{T}} + \{\mathbf{B}_0 \otimes (\mathbf{B}_0^2 - \mathbf{B}_0) + (\mathbf{B}_0^2 - \mathbf{B}_0) \otimes \mathbf{B}_0\}^{\overset{1,3}{T}}]; \end{aligned} \tag{3.13}$$

$$\begin{aligned} &\mathbf{N}_1 = g_{1,\mathbf{EE}}(\mathbf{0}) = \mathbf{0}, \qquad \mathbf{N}_2 = g_{2,\mathbf{EE}}(\mathbf{0}) = \mathbf{0},\\ &\mathbf{N}_3 = g_{3,\mathbf{EE}}(\mathbf{0}) = \mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0 - \frac{1}{2}\,\{\mathbf{B}_0 \otimes [(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}]\\ &\quad + [(\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,6}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,5}{T}}] + [(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}] \otimes \mathbf{B}_0\}\\ &\quad + \frac{1}{8}\,\{(\mathbf{B}_0 \otimes [(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}] + [(\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,6}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,5}{T}}])^{\overset{1,4}{T}}\\ &\quad + (\mathbf{B}_0 \otimes [(\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,4}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,3}{T}}] + [(\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,6}{T}} + (\mathbf{B}_0 \otimes \mathbf{B}_0 \otimes \mathbf{B}_0)^{\overset{1,5}{T}}])^{\overset{1,3}{T}}\}. \end{aligned} \tag{3.14}$$

Differentiating $(3.3)_2$ according to (3.2), and using (2.2), (3.7), (3.8), (3.13) and (3.14), for the elasticity tensors of the first and second order in $\varkappa$ we obtain:

$$
\begin{aligned}
\mathbf{L}_\varkappa &= \sum_{r=1}^{3}\Big(\tau_r\mathbf{K}_r+\sum_{s=1}^{3}\tau_{rs}\mathbf{G}_r\otimes\mathbf{G}_s\Big),\\
\mathbf{M}_\varkappa &= \sum_{r=1}^{3}\Big\{\tau_r\mathbf{N}_r+\sum_{s=1}^{3}\Big[\tau_{rs}(\mathbf{K}_r\otimes\mathbf{G}_s+\{\mathbf{K}_r\otimes\mathbf{G}_s\}^{\overset{3,5}{T}\,\overset{4,6}{T}}\\
&\qquad\qquad +\mathbf{G}_r\otimes\mathbf{K}_s)+\sum_{t=1}^{3}\tau_{rst}\mathbf{G}_r\otimes\mathbf{G}_s\otimes\mathbf{G}_t\Big]\Big\},
\end{aligned}
\tag{3.15}
$$

where

$$
\tau_{12}=\varrho_\varkappa\frac{\partial\bar{\tau}_0}{\partial I_{\mathbf{E}^*}\partial II_{\mathbf{E}^*}}\bigg|_{\mathbf{E}=\mathbf{0}},\ \ldots,\qquad \tau_{312}=\varrho_\varkappa\frac{\partial\bar{\tau}_0}{\partial III_{\mathbf{E}^*}\partial I_{\mathbf{E}^*}\partial II_{\mathbf{E}^*}}\bigg|_{\mathbf{E}=\mathbf{0}},\ \ldots
\tag{3.16}
$$

It is worthwhile once more to point out here that the relations (3.6) and (3.15) obtained for the elasticity tensors of the isotropic elastic solid are exact and valid for arbitrarily deformed reference configuration $\varkappa$. The relations analogous to (3.5) can be found in [2]. The explicit exact relations for $\mathbf{L}_\varkappa$ and $\mathbf{M}_\varkappa$ given here in (3.15) have not been discussed in the literature and are new, [5].

In a special case when $\varkappa$ is a natural state, $\varkappa=\varkappa_0$, then $\mathbf{E}_0=\mathbf{0}$, $\mathbf{B}_0=\mathbf{1}$, $\mathbf{T}_\varkappa=\mathbf{T}_0=\mathbf{0}$ and the elasticity tensors of the first and second order in $\varkappa_0$ have the form:

$$
\begin{aligned}
\mathbf{L}_0 &= \lambda\mathbf{1}\otimes\mathbf{1}+\mu[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}],\\
\mathbf{M}_0 &= \nu_1\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}+\nu_2\{\mathbf{1}\otimes[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]\\
&\quad +[(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})^{\overset{1,6}{T}}+(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})^{\overset{1,5}{T}}]+[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]\otimes\mathbf{1}\}\\
&\quad +\nu_3\{(\mathbf{1}\otimes[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]+[(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})^{\overset{1,6}{T}}+(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})^{\overset{1,5}{T}}])^{\overset{1,4}{T}}\\
&\quad +(\mathbf{1}\otimes[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]+[(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1}^{\overset{1,6}{T}}+(\mathbf{1}\otimes\mathbf{1}\otimes\mathbf{1})^{\overset{1,5}{T}}])^{\overset{1,3}{T}}\},
\end{aligned}
\tag{3.17}
$$

where

$$
\begin{aligned}
\lambda &= \vartheta_{11}+\vartheta_2,\qquad \mu=-\frac{1}{2}\vartheta_2,\\
\nu_1 &= \vartheta_{111}+3\vartheta_{12}+\vartheta_3,\qquad \nu_2=-\frac{1}{2}(\vartheta_{12}+\vartheta_3),\qquad \nu_3=\frac{1}{4}\vartheta_3,
\end{aligned}
\tag{3.18}
$$

$$
\vartheta_r=\tau_r|_{\varkappa=\varkappa_0},\qquad \vartheta_{rs}=\tau_{rs}|_{\varkappa=\varkappa_0},\qquad \vartheta_{rst}=\tau_{rst}|_{\varkappa=\varkappa_0}.
\tag{3.19}
$$

The second-order elastic constants ν_1, ν_2, ν_3 and the relation $(3.17)_2$ written in components with respect to the natural basis of a coordinate system given in $\varkappa_0$, have been introduced in [9].

Similar relations can also be obtained in terms of the strain energy function $\sigma_\varkappa(\mathbf{C})$. The equivalent formulae are only more complicated in the present case since for $\mathbf{C}=\mathbf{1}$ some terms do not dissapear, as was the case for $\mathbf{E}=\mathbf{0}$ when we used $\tau_\varkappa(\mathbf{E})$. In the present case, we also introduce the functions:

$$
\begin{aligned}
\bar{g}_1(\mathbf{C}) &= I_{\mathbf{C}^*,\mathbf{C}}=\mathbf{B}_0,\\
\bar{g}_2(\mathbf{C}) &= II_{\mathbf{C}^*,\mathbf{C}}=\mathbf{B}_0 I_{\mathbf{C}^*}-\mathbf{B}_0\mathbf{C}\mathbf{B}_0,\\
\bar{g}_3(\mathbf{C}) &= III_{\mathbf{C}^*,\mathbf{C}}=\mathbf{B}_0 II_{\mathbf{C}^*}-\mathbf{B}_0\mathbf{C}\mathbf{B}_0 I_{\mathbf{C}^*}+\mathbf{B}_0\mathbf{C}\mathbf{B}_0\mathbf{C}\mathbf{B}_0,
\end{aligned}
\tag{3.20}
$$

and from (3.2) by analogous transformations we obtain the exact relations for the elasticity tensors in $\boldsymbol{\varkappa}$:

$$
(3.21)\qquad
\begin{aligned}
\mathbf{T}_\varkappa &= 2\sum_{r=1}^{3} \sigma_r \overline{\mathbf{G}}_r,\\
\mathbf{L}_\varkappa &= 4\sum_{r=1}^{3}\Big(\sigma_r \overline{\mathbf{K}}_r + \sum_{r=1}^{3}\sigma_{rs}\overline{\mathbf{G}}_r\otimes\overline{\mathbf{G}}_s\Big),\\
\mathbf{M}_\varkappa &= 8\sum_{r=1}^{3}\Big\{\sigma_r\overline{\mathbf{N}}_r + \sum_{s=1}^{3}\Big[\sigma_{rs}(\overline{\mathbf{K}}_r\otimes\overline{\mathbf{G}}_s + \{\overline{\mathbf{K}}_r\otimes\overline{\mathbf{G}}_s\}^{\overset{3,5}{T}\ \overset{4,6}{T}} + \overline{\mathbf{G}}_r\otimes\overline{\mathbf{K}}_s)\\
&\qquad\qquad + \sum_{t=1}^{3}\sigma_{rst}\overline{\mathbf{G}}_r\otimes\overline{\mathbf{G}}_s\otimes\overline{\mathbf{G}}_t\Big]\Big\},
\end{aligned}
$$

where

$$
(3.22)\qquad
\begin{gathered}
\overline{\mathbf{G}}_r = \overline{g}_r(\mathbf{1}),\quad \overline{\mathbf{K}}_r = g_{r,\mathbf{C}}(\mathbf{1}),\quad \overline{\mathbf{N}}_r = g_{r,\mathbf{CC}}(\mathbf{1}),\\
\sigma_{12} = \varrho_\varkappa \frac{\partial\overline{\sigma}_0}{\partial I_{\mathbf{C}^*}\,\partial II_{\mathbf{C}^*}}\Big|_{\mathbf{C}=\mathbf{1}},\ \ldots,\qquad \sigma_{312} = \varrho_\varkappa\frac{\partial\overline{\sigma}_0}{\partial III_{\mathbf{C}^*}\,\partial I_{\mathbf{C}^*}\,\partial II_{\mathbf{C}^*}}\Big|_{\mathbf{C}=\mathbf{1}},\ \ldots
\end{gathered}
$$

The explicit formulas for $\overline{\mathbf{G}}_r$, $\overline{\mathbf{K}}_r$ and $\overline{\mathbf{N}}_r$ can easily be obtained from $\overline{g}_r(\mathbf{C})$ in exactly the same way as those found before for $\mathbf{G}_r$, $\mathbf{K}_r$ and $\mathbf{N}_r$ from $g_r(\mathbf{E})$, and we do not present them here. When $\boldsymbol{\varkappa} = \boldsymbol{\varkappa}_0$, we obtain the relations (3.17), where the elastic constants λ, μ and ν_1, ν_2, ν_3 are defined in terms of σ_0 by some more complicated relations, for example:

$$
(3.23)\qquad
\begin{aligned}
\lambda &= 4(\xi_2+\xi_3)+4[\xi_{11}+2\xi_{12}+\xi_{13}+2(\xi_{21}+2\xi_{22}+\xi_{23})+\xi_{31}+\xi_{32}+\xi_{33}],\\
\mu &= -2(\xi_2+\xi_3),
\end{aligned}
$$

where

$$
(3.24)\qquad \xi_r = \sigma_r|_{\varkappa=\varkappa_0},\qquad \xi_{rs} = \sigma_{rs}|_{\varkappa=\varkappa_0}.
$$

4. Smally strained reference configuration

The exact results for $\mathbf{T}_\varkappa$, $\mathbf{L}_\varkappa$ and $\mathbf{M}_\varkappa$ have been derived analytically assuming the explicit form of the strain energy functions τ_0 or σ_0 to be given for the isotropic elastic solid.

For a wide class of elastic solids, the explicit analytic formulae for the strain energy functions have not as yet been established. The material properties of the solid are usually described in terms of some elastic constants of the first and second order defined in the natural state $\boldsymbol{\varkappa}_0$. These constants, which are determined experimentally, enable us to describe satisfactorily the behaviour of the solid for the most important class of relatively small deformations from the natural state.

The magnitude of deformation χ_0 is usually described by a norm of displacement gradient $\mathbf{H}_0$, defined by $|\mathbf{H}_0| = (\mathbf{H}_0\cdot\mathbf{H}_0)^{1/2} \sim \varepsilon$. In this paper, we consider deformation χ_0 to be small if $1+\varepsilon^3 \approx 1$ and infinitesimal if $1+\varepsilon^2 \approx 1$.

For small deformation χ_0, the elasticity tensors $\mathbf{T}_\varkappa$, $\mathbf{L}_\varkappa$ and $\mathbf{M}_\varkappa$, describing the solid material properties in a reference configuration $\boldsymbol{\varkappa}$, can be found approximately, by expanding them in the neighbourhood of $\boldsymbol{\varkappa}_0$ into Taylor series with respect to $\mathbf{H}_0$.

All such expansions have the form:

$$f(\mathbf{H}_0) = f(0)+f_{,\mathbf{H}_0}(0)\cdot\mathbf{H}_0+\frac{1}{2!}f_{,\mathbf{H}_0\mathbf{H}_0}(0)\cdot(\mathbf{H}_0\otimes\mathbf{H}_0)+\ldots, \tag{4.1}$$

where $f_{,\mathbf{H}_0}, f_{,\mathbf{H}_0\mathbf{H}_0}$, ... are obtained using the chain rule (2.2) and derivatives of the appropriate tensor functions given in the Appendix. For various terms of (3.6) and (3.15), we obtain the following expansions [8, 10]:

$$\begin{aligned}\mathbf{B}_0 &= \mathbf{1}+2\tilde{\mathbf{E}}_0+\mathbf{H}_0\mathbf{H}_0^T,\\ \mathbf{B}_0^2-\mathbf{B}_0 &= 2\tilde{\mathbf{E}}_0+(4\tilde{\mathbf{E}}_0^2+\mathbf{H}_0\mathbf{H}_0^T)+\ldots,\\ \mathbf{B}_0^3-2\mathbf{B}_0^2+\mathbf{B}_0 &= 4\tilde{\mathbf{E}}_0^2+\ldots,\end{aligned} \tag{4.2}$$

$$\begin{aligned}&\mathbf{E}_0 = \tilde{\mathbf{E}}_0+\frac{1}{2}\mathbf{H}_0^T\mathbf{H}_0, \quad \mathbf{E}_0^2 = \tilde{\mathbf{E}}_0^2+\ldots, \quad \mathbf{E}_0^3 = +\ldots\\ &I_{\mathbf{E}_0} = I_{\tilde{\mathbf{E}}_0}+\frac{1}{2}I_{\mathbf{H}_0^T\mathbf{H}_0}, \quad II_{\mathbf{E}_0} = II_{\tilde{\mathbf{E}}_0}+\ldots, \quad III_{\mathbf{E}_0} = +\ldots\end{aligned} \tag{4.3}$$

$$\begin{aligned}&\frac{\varrho_0}{\varrho_\varkappa} = (\det\mathbf{C}_0)^{1/2} = 1+I_{\tilde{\mathbf{E}}_0}+\left(-\frac{1}{2}I_{\tilde{\mathbf{E}}_0}^2+2II_{\tilde{\mathbf{E}}_0}+\frac{1}{2}I_{\mathbf{H}_0^T\mathbf{H}_0}\right)+\ldots\\ &\frac{\varrho_\varkappa}{\varrho_0} = 1-I_{\tilde{\mathbf{E}}_0}+\left(\frac{3}{2}I_{\tilde{\mathbf{E}}_0}^2-2II_{\tilde{\mathbf{E}}_0}-I_{\mathbf{H}_0^T\mathbf{H}_0}\right)+\ldots\end{aligned} \tag{4.4}$$

$$\begin{aligned}\tau_r &= \vartheta_r+(\vartheta_{r1}-\vartheta_r)I_{\tilde{\mathbf{E}}_0}+\left[\frac{1}{2}(\vartheta_{r1}-\vartheta_r)I_{\mathbf{H}_0^T\mathbf{H}_0}\right.\\ &\qquad\left.+\left(\frac{1}{2}\vartheta_{r11}-\vartheta_{r1}+\frac{3}{2}\vartheta_r\right)I_{\tilde{\mathbf{E}}_0}^2+(\vartheta_{r2}-2\vartheta_r)II_{\tilde{\mathbf{E}}_0}\right]+\ldots\\ \tau_{rs} &= \vartheta_{rs}+(\vartheta_{rs1}-\vartheta_{rs})I_{\tilde{\mathbf{E}}_0}+\ldots\\ \tau_{rst} &= \vartheta_{rst}+\ldots.\end{aligned} \tag{4.5}$$

From (3.8), (3.13) and (3.14), and using (4.2) and (4.3), we can find the expansions of $\mathbf{G}_r$, $\mathbf{K}_r$ and $\mathbf{N}_r$, then put them together with (4.5) into (3.6) and (3.15), and make use of (3.18) to introduce the elastic constants λ, μ and ν_1, ν_2, ν_3. After lengthy but elementary algebraic transformations, we obtain the following expansions for the elasticity tensors:

$$\begin{aligned}\mathbf{T}_\varkappa &= \left[\lambda+\left(\frac{\nu_1}{2}+\nu_2-\lambda\right)I_{\tilde{\mathbf{E}}_0}\right]I_{\tilde{\mathbf{E}}_0}\mathbf{1}+2[\mu+(\nu_2+\lambda-\mu)I_{\tilde{\mathbf{E}}_0}]\tilde{\mathbf{E}}_0\\ &\qquad+\left(\frac{\lambda}{2}I_{\mathbf{H}_0^T\mathbf{H}_0}-2\nu_2 II_{\tilde{\mathbf{E}}_0}\right)\mathbf{1}+\mu\mathbf{H}_0\mathbf{H}_0^T+4(\nu_3+\mu)\tilde{\mathbf{E}}_0^2+\ldots\\ \mathbf{L}_\varkappa &= [\lambda+(\nu_1-\lambda)I_{\tilde{\mathbf{E}}_0}]\mathbf{1}\otimes\mathbf{1}+[\mu+(\nu_2-\mu)I_{\tilde{\mathbf{E}}_0}][(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]\\ &\qquad+2(\nu_2+\lambda)(\mathbf{1}\otimes\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}}_0\otimes\mathbf{1})\\ &\qquad+2(\nu_3+\mu)[(\mathbf{1}\otimes\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}}_0\otimes\mathbf{1})^{\overset{1,4}{T}}+(\mathbf{1}\otimes\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}}_0\otimes\mathbf{1})^{\overset{1,3}{T}}]+\ldots\\ \mathbf{M}_\varkappa &= \mathbf{M}_0+\ldots\end{aligned} \tag{4.6}$$

The relation for $\mathbf{T}_\varkappa$ analogous to $(4.6)_1$ with different second-order elastic constants has been obtained in [10, 11]. The explicit relation for $\mathbf{L}_\varkappa$ given here in $(4.6)_2$ has not been discussed in the literature and is new [5].

For infinitesimal deformation χ_0, it follows from $(4.6)_2$ that

$$(4.7) \qquad \begin{aligned} \mathbf{T}_\varkappa &= \lambda I_{\tilde{\mathbf{E}}_0}\mathbf{1}+2\mu\tilde{\mathbf{E}}_0+\ldots \\ \mathbf{L}_\varkappa &= \mathbf{L}_0+\ldots, \end{aligned}$$

and this result is well known from the classical linear theory of elasticity.

5. Stress under superposed deformations

The exact (3.6), (3.15) and the approximate (4.6) relations for elasticity tensors in arbitrarily and smally deformed reference configuration, respectively, make it possible to formulate the basic field equations of the non-linear theory of elasticity with various degrees of approximation. Here we discuss in greater detail only the stress-deformation relations under successive superposition of two various deformations.

We have introduced here three different kinds of deformations: the finite, the small and the infinitesimal. Thus there may be at most nine different cases of superposition of these deformations. We do not discuss here, however, the superposition of the finite on small or infinitesimal deformations, or the small on an infinitesimal deformation, since these cases are physically unjustified. For the six remaining cases of superposition we present the explicit relations for the Cauchy stress tensor $\mathbf{T}$.

a) Successive superposition of two finite deformations. In this, the most general, case from (2.9) and (2.10) we have:

$$(5.1) \qquad \mathbf{T} = \varrho\mathbf{F}^*\tau_{0,\mathbf{E}^*}(\mathbf{E}^*)\,\mathbf{F}^{*T},$$

and this relation is useful only within the exact non-linear elasticity [2].

b) Small deformation superposed on a finite deformation. The appropriate relation for $\mathbf{T}$ can be obtained from (2.8) using (3.1), $(2.5)_1$ and an expansion for $\varrho/\varrho_\varkappa$ similar to $(4.4)_2$. Retaining only the terms quadratic in $\mathbf{H}$, we obtain:

$$(5.2) \qquad \begin{aligned} \mathbf{T} = {} & \mathbf{T}_\varkappa - I_{\tilde{\mathbf{E}}_0}\mathbf{T}_\varkappa + \mathbf{H}\mathbf{T}_\varkappa + \mathbf{T}_\varkappa\mathbf{H}^T + \mathbf{L}_\varkappa\cdot\tilde{\mathbf{E}} + \left(\frac{3}{2}I_{\tilde{\mathbf{E}}}^2 - 2II_{\tilde{\mathbf{E}}} - I_{\mathbf{H}^T\mathbf{H}}\right)\mathbf{T}_\varkappa \\ & - I_{\tilde{\mathbf{E}}}(\mathbf{H}\mathbf{T}_\varkappa + \mathbf{T}_\varkappa\mathbf{H} + \mathbf{L}_\varkappa\cdot\tilde{\mathbf{E}}) + \mathbf{H}\mathbf{T}_\varkappa\mathbf{H} + \mathbf{H}(\mathbf{L}_\varkappa\cdot\tilde{\mathbf{E}}) + (\mathbf{L}_\varkappa\cdot\tilde{\mathbf{E}})\mathbf{H}^T \\ & + \frac{1}{2}\mathbf{L}_\varkappa\cdot(\mathbf{H}^T\mathbf{H}) + \frac{1}{2}\mathbf{M}_\varkappa\cdot(\tilde{\mathbf{E}}\otimes\tilde{\mathbf{E}}) + \ldots, \end{aligned}$$

where $\mathbf{T}_\varkappa$, $\mathbf{L}_\varkappa$ and $\mathbf{M}_\varkappa$ are given by (3.6) and (3.15).

c) Infinitesimal deformation superposed on a finite deformation. Retaining in (5.2) only the terms linear in $\mathbf{H}$, we have:

$$(5.3) \qquad \mathbf{T} = \mathbf{T}_\varkappa - I_{\tilde{\mathbf{E}}}\mathbf{T}_\varkappa + \mathbf{H}\mathbf{T}_\varkappa + \mathbf{T}_\varkappa\mathbf{H}^T + \mathbf{L}_\varkappa\cdot\tilde{\mathbf{E}} + \ldots$$

The general relations of this kind are well known [2, 12, 13, 14], but they are specified usually only for homogeneous deformations. Here the explicit form of (5.3) can easily be found for the isotropic elastic solid, if we make use of (3.6) and $(3.15)_1$. In particular, the last term of (5.3) is the linear combination of the following terms:

$$
\begin{aligned}
&\mathbf{G}_1\cdot\tilde{\mathbf{E}} = \operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0),\\
(5.4)\qquad &\mathbf{G}_2\cdot\tilde{\mathbf{E}} = I_{\mathbf{E}_0}\operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0)-\operatorname{tr}(\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T\tilde{\mathbf{E}}),\\
&\mathbf{G}_3\cdot\tilde{\mathbf{E}} = II_{\mathbf{E}_0}\operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0)-I_{\mathbf{E}_0}\operatorname{tr}(\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T\tilde{\mathbf{E}})+\operatorname{tr}(\mathbf{F}_0\mathbf{E}_0^2\mathbf{F}_0^T\tilde{\mathbf{E}})
\end{aligned}
$$

$$
\begin{aligned}
&\mathbf{K}_1\cdot\tilde{\mathbf{E}} = \mathbf{0},\\
(5.5)\qquad &\mathbf{K}_2\cdot\tilde{\mathbf{E}} = \mathbf{B}_0\operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0)-\mathbf{B}_0\tilde{\mathbf{E}}\mathbf{B}_0,\\
&\mathbf{K}_3\cdot\tilde{\mathbf{E}} = \mathbf{B}_0 I_{\mathbf{E}_0}\operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0)-\mathbf{B}_0\tilde{\mathbf{E}}\mathbf{B}_0 I_{\mathbf{E}_0}-\mathbf{B}_0\operatorname{tr}(\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T\tilde{\mathbf{E}})\\
&\qquad -\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T\operatorname{tr}(\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{F}_0)+\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T\tilde{\mathbf{E}}\mathbf{B}_0+\mathbf{B}_0\tilde{\mathbf{E}}\mathbf{F}_0\mathbf{E}_0\mathbf{F}_0^T.
\end{aligned}
$$

d) Successive superposition of two small deformations. The appropriate relation for $\mathbf{T}$ follows from (5.2) if the expression of (4.6) are used for the elasticity tensors. Retaining only the terms square in $\mathbf{H}_0$ or $\mathbf{H}$ and their products, after extensive but elementary transformations we obtain:

$$
\begin{aligned}
(5.6)\qquad \mathbf{T} = &\left[\lambda+\left(\frac{\nu_1}{2}+\nu_2-\lambda\right)I_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}\right]I_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}\mathbf{1}+2[\mu+(\nu_2+\lambda-\mu)I_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}](\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})\\
&+\left[\frac{\lambda}{2}I_{(\mathbf{H}_0+\mathbf{H})^T(\mathbf{H}_0+\mathbf{H})}+\frac{\lambda}{2}I_{(\mathbf{H}\mathbf{H}_0+\mathbf{H}_0^T\mathbf{H}^T)}-2\nu_2 II_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}\right]\mathbf{1}+\mu(\mathbf{H}_0+\mathbf{H})(\mathbf{H}_0+\mathbf{H})^T\\
&+\mu(\mathbf{H}\mathbf{H}_0+\mathbf{H}_0^T\mathbf{H}^T)+4(\nu_3+\mu)(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})^2+\ldots
\end{aligned}
$$

A similar result in terms of different elastic constants and applying a certain perturbation technique was obtained in [10].

Note that in this case we need to know only the elastic constants of the first- and second-order in $\varkappa_0$, and not the explicit form of the strain energy function. For such small deformation non-linear theory of elasticity, it is possible to formulate all field equations which would be applicable beyond the limits of classical linear elasticity theory.

e) Infinitesimal deformation superposed on a small deformation. For this case, it suffices to omit in (5.6) all the terms which are quadratic in $\mathbf{H}$, to obtain:

$$
\begin{aligned}
(5.7)\qquad \mathbf{T} = &\left[\lambda+\left(\frac{\nu_1}{2}+\nu_2-\lambda\right)I_{\tilde{\mathbf{E}}_0}\right]I_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}\mathbf{1}+\left(\frac{\nu_1}{2}+\nu_2-\lambda\right)I_{\tilde{\mathbf{E}}}I_{\tilde{\mathbf{E}}_0}\mathbf{1}\\
&+2[\mu+(\nu_2+\lambda-\mu)I_{\tilde{\mathbf{E}}_0}](\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})+2(\nu_2+\lambda-\mu)I_{\tilde{\mathbf{E}}}\tilde{\mathbf{E}}_0\\
&+\frac{\lambda}{2}[I_{(\mathbf{H}_0+\mathbf{H})\mathbf{H}_0^T}+2I_{\mathbf{H}\tilde{\mathbf{E}}_0}+I_{\mathbf{H}_0^T\mathbf{H}^T}]\mathbf{1}-2\nu_2[II_{\tilde{\mathbf{E}}_0}+I_{\tilde{\mathbf{E}}_0}I_{\tilde{\mathbf{E}}}-I_{\tilde{\mathbf{E}}_0\tilde{\mathbf{E}}}]\mathbf{1}\\
&+\mu[(\mathbf{H}_0+\mathbf{H})\mathbf{H}_0^T+2\mathbf{H}\tilde{\mathbf{E}}_0+\mathbf{H}_0^T\mathbf{H}^T]+4(\nu_3+\mu)(\tilde{\mathbf{E}}_0^2+\tilde{\mathbf{E}}_0\tilde{\mathbf{E}}+\tilde{\mathbf{E}}\tilde{\mathbf{E}}_0)+\ldots
\end{aligned}
$$

The same result follows also from (5.3) if the expressions $(4.6)_{1,2}$ are used.

f) Successive superposition of two infinitesimal deformations. Retaining in (5.7) only the terms linear in $\mathbf{H}_0$ and $\mathbf{H}$, we obtain:

$$
(5.8)\qquad \mathbf{T} = \lambda I_{(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})}\mathbf{1}+2\mu(\tilde{\mathbf{E}}_0+\tilde{\mathbf{E}})+\ldots,
$$

which gives us the well known superposition principle of the classical linear theory of elasticity.

Appendix

Derivatives of some functions of the second-order tensor

The formulae presented here for derivatives of tensor functions $f: \mathscr{C}_2 \to \mathscr{C}_2$ and $f: \mathscr{C}_2 \to \mathscr{C}_4$ have been obtained bearing in mind that for any $\mathbf{B} \in \mathscr{C}_2$ and $\alpha \in \mathscr{R}$

$$f_{,\mathbf{A}}(\mathbf{A}) \cdot \mathbf{B} = \frac{d}{d\alpha} f(\mathbf{A} + \alpha \mathbf{B})|_{\alpha=0}.$$

For example, for $f(\mathbf{A}) = \mathbf{A}^2$ we have:

$$f_{,\mathbf{A}}(\mathbf{A}) \cdot \mathbf{B} = \frac{d}{d\alpha} [(\mathbf{A} + \alpha \mathbf{B})(\mathbf{A} + \alpha \mathbf{B})]|_{\alpha=0} = \mathbf{BA} + \mathbf{AB}$$

and

$$(\mathbf{A} \otimes \mathbf{1})^{\overset{1,4}{T}} \cdot \mathbf{B} = A^{ij} g^{kl} B_{mn} (\mathbf{g}_k \cdot \mathbf{g}^m)(\mathbf{g}_i \cdot \mathbf{g}^n) \mathbf{g}_l \otimes \mathbf{g}_j = \mathbf{BA},$$

$$(\mathbf{1} \otimes \mathbf{A}^T)^{\overset{1,4}{T}} \cdot \mathbf{B} = g^{ij} A^{kl} B_{mn} (\mathbf{g}_l \cdot \mathbf{g}^m)(\mathbf{g}_j \cdot \mathbf{g}^n) \mathbf{g}_k \otimes \mathbf{g}_j = \mathbf{AB}.$$

The formulae for derivatives can be presented in many other equivalent forms; for example, we have also:

$$\mathbf{AB} = (\mathbf{A} \otimes \mathbf{1})^{\overset{2,3}{T}} \cdot \mathbf{B} = [\mathbf{A}(\mathbf{1} \otimes \mathbf{1})^{\overset{1,4}{T}}] \cdot \mathbf{B} = [\mathbf{A}(\mathbf{1} \otimes \mathbf{1})^{\overset{2,3}{T}}] \cdot \mathbf{B} = \ldots$$

Many similar tensor identities have been given in [8].

Derivatives of some simple functions of the second-order tensor are given in Table 1.

Table 1

No.	$f(\mathbf{A})$	$f_{,\mathbf{A}}(\mathbf{A})$	Remarks
1	$\mathbf{A}$	$(\mathbf{1} \otimes \mathbf{1})^{\overset{1,4}{T}}$	[6, 8]
2	$\mathbf{A}^T$	$(\mathbf{1} \otimes \mathbf{1})^{\overset{1,3}{T}}$	
3	$\mathbf{A}^T\mathbf{A}$	$(\mathbf{A} \otimes \mathbf{1})^{\overset{1,3}{T}} + (\mathbf{1} \otimes \mathbf{A})^{\overset{1,4}{T}}$	[6, 8]
4	$\mathbf{A}\mathbf{A}^T$	$(\mathbf{A}^T \otimes \mathbf{1})^{\overset{1,4}{T}} + (\mathbf{1} \otimes \mathbf{A})^{\overset{1,3}{T}}$	
5	$\mathbf{PAQ}$	$(\mathbf{Q} \otimes \mathbf{P}^T)^{\overset{1,4}{T}}$	$\mathbf{P}, \mathbf{Q} \in \mathscr{C}_2$
6	$\mathbf{PA}^T\mathbf{Q}$	$(\mathbf{Q} \otimes \mathbf{P})^{\overset{1,3}{T}}$	
7	$g(\mathbf{A})\, h(\mathbf{A})$	$\{[h(\mathbf{A})]^{T}_{2}[g_{,\mathbf{A}}(\mathbf{A})]^{\overset{1,2}{T}}\}^{\overset{1,2}{T}} + g(\mathbf{A}) h_{,\mathbf{A}}(\mathbf{A})$	$g: \mathscr{C}_2 \to \mathscr{C}_2$
8	$\{g(\mathbf{A}) \otimes h(\mathbf{A})\}^{\overset{\mu,\nu}{T}}$	$\{[g_{,\mathbf{A}}(\mathbf{A}) \otimes h(\mathbf{A})]^{\overset{3,5}{T}\,\overset{4,6}{T}} + g(\mathbf{A}) \otimes h_{,\mathbf{A}}(\mathbf{A})\}^{\overset{\mu,\nu}{T}}$	$h: \mathscr{C}_2 \to \mathscr{C}_2$

In applications to continuum mechanics, the argument $\mathbf{A}$ of f is usually the symmetric tensor $\mathbf{A} \in {}^s\mathscr{C}_2$. To find $f_{,\mathbf{A}}$ at ${}^s\mathscr{C}_2$, we extend f into $\mathscr{C}_2$, defining $\bar{f}(\mathbf{A}) = f\left[\frac{1}{2}(\mathbf{A}+\mathbf{A}^T)\right]$, then find $\bar{f}_{,\mathbf{A}}$ at $\mathscr{C}_2$, and finally restrict $\mathscr{C}_2$ to ${}^s\mathscr{C}_2$ by putting $\mathbf{A} = \mathbf{A}^T$.

Derivatives of some tensor functions of symmetric argument are given in Table 2.

Table 2

No.	$f(\mathbf{A})$	$f_{,\mathbf{A}}(\mathbf{A})$	Remarks
1	$\mathbf{A}$	$\frac{1}{2}[(\mathbf{1}\otimes\mathbf{1})^{\overset{1,4}{T}} + (\mathbf{1}\otimes\mathbf{1})^{\overset{1,3}{T}}]$	[6, 8]
2	$\mathbf{A}^2$	$\frac{1}{2}[(\mathbf{A}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{A})^{\overset{1,4}{T}} + (\mathbf{A}\otimes\mathbf{1}+\mathbf{1}\otimes\mathbf{A})^{\overset{1,3}{T}}]$	[8]
3	$\mathbf{PAQ}$	$\frac{1}{2}[(\mathbf{Q}\otimes\mathbf{P}^T)^{\overset{1,4}{T}} + (\mathbf{Q}\otimes\mathbf{P})^{\overset{1,3}{T}}]$	
4	$\mathbf{PAQAR}$	$\frac{1}{2}[(\mathbf{QAR}\otimes\mathbf{P}^T+\mathbf{R}\otimes\mathbf{Q}^T\mathbf{A}\mathbf{P}^T)^{\overset{1,4}{T}} + (\mathbf{QAR}\otimes\mathbf{P}+\mathbf{R}\otimes\mathbf{PAQ})^{\overset{1,3}{T}}]$	
5	$\mathbf{PAQ}\otimes\mathbf{R}$	$\frac{1}{2}[(\mathbf{Q}\otimes\mathbf{R}\otimes\mathbf{P}^T)^{\overset{1,6}{T}} + (\mathbf{Q}\otimes\mathbf{R}\otimes\mathbf{P})^{\overset{1,5}{T}}]$	$\mathbf{P},\mathbf{Q}\in\mathscr{C}_2$ $\mathbf{R},\mathbf{S}\in\mathscr{C}_2$
6	$\mathbf{P}\otimes\mathbf{QAR}$	$\frac{1}{2}\mathbf{P}\otimes[(\mathbf{R}\otimes\mathbf{Q}^T)^{\overset{1,4}{T}} + (\mathbf{R}\otimes\mathbf{Q})^{\overset{1,3}{T}}]$	
7	$\mathbf{PAQ}\otimes\mathbf{RAS}$	$\frac{1}{2}[(\mathbf{Q}\otimes\mathbf{RAS}\otimes\mathbf{P}^T)^{\overset{1,6}{T}} + (\mathbf{Q}\otimes\mathbf{RAS}\otimes\mathbf{P})^{\overset{1,5}{T}}$	
		$+\mathbf{PAQ}\otimes(\mathbf{S}\otimes\mathbf{R}^T)^{\overset{1,4}{T}} + \mathbf{PAQ}\otimes(\mathbf{S}\otimes\mathbf{R})^{\overset{1,3}{T}}]$	

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