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Some Exact Reduction of the Non-Linear Shell Compatibility Conditions

Under KIRCHHOFF-LOVE type constraints the deformation of a thin shell is described entirely by deformation of its middle surface. The components of the surface LAGRANGEan strain tensor $\gamma_{\alpha\beta} = \frac{1}{2} (\bar{a}_{\alpha\beta} - a_{\alpha\beta})$ and the LAGRANGEan tensor of change of surface curvature $\varkappa_{\alpha\beta} = \dot{-} (\bar{b}_{\alpha\beta} - b_{\alpha\beta})$ have to satisfy the following compatibility conditions [1]

$$\begin{aligned} &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu} [\varkappa_{\beta\lambda|\mu} + \bar{a}^{\varkappa\nu}(b_{\varkappa\lambda} - \varkappa_{\varkappa\lambda})\, \gamma_{\nu\beta\mu}] = 0 \,, \\ &K\gamma^{\varkappa}_{\varkappa} + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu} [\gamma_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\varkappa_{\beta\lambda} + \tfrac{1}{2} (\varkappa_{\alpha\mu}\varkappa_{\beta\lambda} + \bar{a}^{\varkappa\nu}\gamma_{\varkappa\alpha\mu}\gamma_{\nu\beta\lambda})] = 0 \end{aligned} \qquad (1)$$

where

$$\gamma_{\varkappa\alpha\mu} = \gamma_{\varkappa\alpha|\mu} + \gamma_{\varkappa\mu|\alpha} - \gamma_{\alpha\mu|\varkappa} .$$

The exact relations (1) form the extremely complicated set of three non-linear differential equations for $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ to be satisfied during the surface deformation. In order to simplify them some additional assumptions about the magnitude of strains, deflections, displacements, length of deformation patterns etc. are usually introduced. Then the order-of-magnitude estimates provide reasonable arguments for omission of some terms in (1) shown to be small within the class of shell problems to be solved [1, 2]. As a consequence, the displacement field satisfying exactly the simplified compatibility conditions do not carry exactly the reference surface into another (deformed) surface in 3-dimensional EUCLIDean space. One would expect the greatest error of (1) to be in the limiting case of the linear theory, when all the non-linear terms have been omitted as small. However we show here, that (1) can be reduced exactly to the linear form known from the linear theory of shells.

During the deformation of the shell as a 3-dimensional body, the components of GREEN's strain tensor $\eta_{ij} = \frac{1}{2}(\bar{g}_{ij} - g_{ij})$ have to satisfy the following compatibility conditions

$$\varepsilon^{rij}\varepsilon^{skl}(2\eta_{il;jk} + \bar{g}^{mn}\eta_{mil}\eta_{njk}) = 0 \tag{2}$$

where

$$\eta_{mil} = \eta_{mi;l} + \eta_{ml;i} - \eta_{il;m} .$$

In normal convected coordinate system $\vartheta^i = (\vartheta^\alpha, \zeta)$ the relations (1) follow from (2) by specifying $r = \alpha$, $s = 3$ and $r = 3$, $s = 3$ together with $\zeta = 0$.

In terms of components of spatial displacement gradient we obtain

$$\begin{aligned} &\eta_{il} = e_{il} + \tfrac{1}{2} g^{mn} v_{m;i} v_{n;l}, \qquad \eta_{mil} = F^p_m v_{p;il} , \\ &g^{ij} = F^i_m F^j_n \bar{g}^{mn} , \qquad e_{il} = \tfrac{1}{2}(v_{i;l} + v_{l;i}) , \qquad F^p_m = \delta^p_m + v^p_{;m} . \end{aligned} \tag{3}$$

That allows us to transform (2) exactly [3] to the linear form, which for $r = \alpha$, $s = 3$ and $r = 3$, $s = 3$ reads

$$\varepsilon^{3\alpha\beta}\varepsilon^{3\lambda\mu}(e_{\beta\mu;3\lambda} - e_{3\mu;\beta\lambda}) = 0 , \qquad \varepsilon^{3\alpha\beta}\varepsilon^{3\lambda\mu} e_{\alpha\mu;\beta\lambda} = 0 . \tag{4}$$

In normal convected coordinates we obtain

$$\begin{aligned} e_{\alpha\beta} = {} & \vartheta_{\alpha\beta} + \zeta\,[\tfrac{1}{2}(n_{\alpha|\beta} + n_{\beta|\alpha}) - b_{\alpha\beta}(n-1) - \tfrac{1}{2} b^\varkappa_\alpha(\vartheta_{\varkappa\beta} - \omega_{\varkappa\beta}) - \tfrac{1}{2} b^\varkappa_\beta(\vartheta_{\varkappa\alpha} - \omega_{\varkappa\alpha})] \\ & - \zeta^2\,[\tfrac{1}{2}(b^\varkappa_\alpha n_{\varkappa|\beta} + b^\varkappa_\beta n_{\varkappa|\alpha}) - b^\varkappa_\alpha b_{\varkappa\beta}(n-1)] , \\ e_{\alpha 3} = {} & \tfrac{1}{2}(\varphi_\alpha + n_\alpha) + \zeta\,\tfrac{1}{2} n_{,\alpha} , \qquad e_{33} = n - 1 \end{aligned} \tag{5}$$

where

$$\vartheta_{\alpha\beta} = \tfrac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w$$

and the relations for φ_α, $\omega_{\alpha\beta}$, n_α and n are given in [1, 4].

Let us expand in (4) the double spatial covariant derivative, next express spatial CHRISTOFFEL symbols and permutation tensors in terms of the surface quantities and ζ, and then put $\zeta = 0$. During these purely algebraic transformations some non-linear terms cancel each other, others happen to be symmetric in α, β or λ, μ indices and can be contracted as well. Finally [5], we arrive at the following linear relations

$$\begin{aligned} &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\,[(\varrho_{\beta\lambda} - \tfrac{1}{2} b^\varkappa_\beta \vartheta_{\varkappa\lambda} - \tfrac{1}{2} b^\varkappa_\lambda \vartheta_{\varkappa\beta})_{|\mu} + b^\varkappa_\lambda(\vartheta_{\varkappa\beta|\mu} + \vartheta_{\varkappa\mu|\beta} - \vartheta_{\beta\mu|\varkappa})] = 0 , \\ &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}\,[\vartheta_{\alpha\mu|\beta\lambda} - b_{\alpha\mu}\varrho_{\beta\lambda}] = 0 \end{aligned} \tag{6}$$

where

$$\varrho_{\alpha\beta} = -\tfrac{1}{2}(\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\varkappa_\alpha \omega_{\beta\varkappa} + b^\varkappa_\beta \omega_{\alpha\varkappa}) .$$

The relations (6) are recognized to be the compatibility conditions of the linear theory of shells expressed in terms of the "best" linear strain measures $\vartheta_{\alpha\beta}$ and $\varrho_{\alpha\beta}$.

The reduction presented above is exact within the KIRCHHOFF-LOVE type constraints. Thus the conditions (6) are satisfied not only within the assumptions of the linear theory of shells, but remain exactly satisfied for unrestricted deformation of the shell middle surface [6].

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