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Simplified Equations for the Geometrically Non-Linear Thin Elastic Shells*

The Lagrangean and canonical forms of geometrically non-linear shell equations have been simplified under various restrictive assumptions. The simplified equations contain an error introduced by the approximate constitutive equations only.

1. Introduction

In the geometrically non-linear thin elastic shell theory it is important to choose the appropriate quantities as independent variables well suited to the problem solved. The obvious choice — three displacements or three stress functions — leads to extremely complicated sets of equations which are hardly readable and difficult to use in a general discussion. The fully Lagrangean shell theory as proposed recently by Pietraszkiewicz [1] contains explicitly some deflection and displacement components, while the set of equations proposed by Simmonds and Danielson [2] is expressed in terms of finite rotation and stress function vectors. All these quantities are not so easy to handle in general shell discussions.

More promising for many shell problems and equally general seems to be the so called intrinsic approach suggested in the pioneering work of Synge and Chien [3], in which the strain or the stress measures were suggested to be taken as independent variables. For appropriate boundary conditions the solution of a non-linear shell problem can be divided into two steps. The problem for stresses and strains is solved first and displacements, if necessary, are obtained by a direct integration of the strain-displacement relations.

Using stress resultants and changes of shell curvatures Danielson [4] derived a new set of six intrinsic shell equations. For the first approximation small strain theory these equations were simplified with the help of Koiter's [5] estimates of the accuracy of two-dimensional shell constitutive equations. Recently Koiter and Simmonds [6] introduced, with the help of additional estimates for the stress derivatives in shells as obtained

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by John [7], some modifications to Danielson's equations and obtained what they called a „canonical form of intrinsic geometrically non-linear shell equations".

In this note we discuss some of the possible simplifications of the Lagrangean and the canonical shell equations for some important classes of the shell problems. Following essentially the arguments used by Koiter [8] we restict here independently the following parameters: the ratio between the bending and membrane strains, the ratio between the length of deformation pattern and Gaussian curvature of the undeformed shell middle surface, as well as deflections of the shell middle surface. Using the order-of-magnitude estimates for all terms we assume it to be possible to omit all terms of the same order as those omitted because of approximate character of the shell constitutive equations. This simplifying procedure is somewhat different from that used by Koiter [8], where the terms omitted were supposed to be small with respect to other terms in a particular simplified equation. Our simplifying procedure does not introduce any other error to the basic equations beyond that introduced already by approximate constitutive equations. Thus the solution obtained from the properly simplified equations has the same accuracy as that obtained by solving the full unsimplified set of equations.

2. Basic relations

In this note we will follow the system of notation used by Koiter [8] and Pietraszkiewicz [1, 9, 10]. With the undeformed shell middle surface, defined by the position vector $\mathbf{r}(\vartheta^\alpha)$, $\alpha=1,2$, we associate the standard surface base vectors $\mathbf{a}_\alpha=\mathbf{r}_{,\alpha}$, the metric tensor $\mathbf{a}_{\alpha\beta}=\mathbf{a}_\alpha\cdot\mathbf{a}_\beta$, the unit normal to the surface $\mathbf{n}=\frac{1}{2}\epsilon^{\alpha\beta}\mathbf{a}_\alpha\times\mathbf{a}_\beta$ and the curvature tensor $b_{\alpha\beta}= =\mathbf{a}_{\alpha,\beta}\mathbf{n}$. Similar geometrical quantities associated with deformed shell middle surface will be distinguished by a dash, for example $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\epsilon}^{\alpha\beta}$ etc.

The Lagrangean surface strain tensor $\gamma_{\alpha\beta}$ and the Lagrangean tensor of change of surface curvature $\kappa_{\alpha\beta}$ are defined by

$$\gamma_{\alpha\beta}=\tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta}),\qquad \kappa_{\alpha\beta}=-(\bar{b}_{\alpha\beta}-b_{\alpha\beta}) \tag{2.1}$$

and they satisfy the following compatibility conditions [8, 9]

$$\begin{gathered}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu}+\bar{a}^{\kappa\nu}(b_{\kappa\lambda}-\kappa_{\kappa\lambda})(\gamma_{\nu\beta|\mu}+\gamma_{\nu\mu|\beta}-\gamma_{\beta\mu|\nu})]=0,\\ K\gamma^\kappa_\kappa+\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda}-b_{\alpha\mu}\kappa_{\beta\lambda}+\tfrac{1}{2}\{\kappa_{\alpha\mu}\kappa_{\beta\lambda}+\\ +\bar{a}^{\kappa\nu}(\gamma_{\kappa\alpha|\mu}+\gamma_{\kappa\mu|\alpha}-\gamma_{\alpha\mu|\kappa})(\gamma_{\nu\beta|\lambda}+\gamma_{\nu\lambda|\beta}-\gamma_{\beta\lambda|\nu})\}]=0.\end{gathered} \tag{2.2}$$

In what follows we shall use, instead of $\kappa_{\alpha\beta}$, the modified tensor $\rho_{\alpha\beta}$ defined by

$$\rho_{\alpha\beta}=\kappa_{\alpha\beta}+\tfrac{1}{2}(b^\kappa_\alpha\gamma_{\kappa\beta}+b^\kappa_\beta\gamma_{\kappa\alpha}) \tag{2.3}$$

the linear part of which is exactly the measure supposed by Budiansky and Sanders [11] to be the „best" for the linear shell theory.

The fully Lagrangean equilibrium equations, expressed in terms of the Lagrangean symmetric stress resultants $N^{\alpha\beta}$ and stress couples $M^{\alpha\beta}$, have been derived by Pietraszkiewicz [1] in the form

$$(Q^{\alpha\lambda}l^{\beta}_{.\lambda}+Q^{\alpha}n^{\beta})_{|\alpha}-b^{\beta}_{\alpha}(Q^{\alpha\lambda}\varphi_{\lambda}+Q^{\alpha}n)+p^{\beta}=0,$$
$$(Q^{\alpha\lambda}\varphi_{\lambda}+Q^{\alpha}n)_{|\alpha}+b_{\alpha\beta}(Q^{\alpha\lambda}l^{\beta}_{.\lambda}+Q^{\alpha}n^{\beta})+p=0, \tag{2.4}$$

where

$$Q^{\alpha\beta}=N^{\alpha\beta}+\tfrac{1}{2}\bar{b}^{\alpha}_{\kappa}M^{\kappa\beta}-\tfrac{1}{2}\bar{b}^{\beta}_{\kappa}M^{\alpha\kappa},$$
$$Q^{\alpha}=M^{\alpha\beta}\big|_{\beta}+\bar{a}^{\alpha\nu}(\gamma_{\nu\lambda|\mu}+\gamma_{\nu\mu|\lambda}-\gamma_{\lambda\mu|\nu})M^{\lambda\mu} \tag{2.5}$$

and the other quantities are defined by the relations

$$\bar{\mathbf{a}}_{\alpha}=l^{\kappa}_{.\alpha}\mathbf{a}_{\kappa}+\varphi_{\alpha}\mathbf{n},\qquad \bar{\mathbf{n}}=n^{\kappa}\mathbf{a}_{\kappa}+n\mathbf{n}. \tag{2.6}$$

Danielson [4] showed that it is possible to express the Eulerian equilibrium equations derived by Koiter [5] in terms of Lagrangean quantities only, to obtain

$$N^{\alpha\beta}\big|_{\alpha}+\bar{a}^{\beta\mu}(2\gamma_{\mu\lambda|\alpha}-\gamma_{\lambda\alpha|\mu})N^{\lambda\alpha}-\tfrac{1}{2}(\bar{b}^{\beta}_{\kappa}M^{\kappa\alpha}-\bar{b}^{\alpha}_{\kappa}M^{\kappa\beta})_{|\alpha}-\bar{b}^{\beta}_{\kappa}M^{\kappa\alpha}\big|_{\alpha}+$$
$$-\bar{b}^{\beta\mu}(2\gamma_{\mu\lambda|\alpha}-\gamma_{\lambda\alpha|\mu})M^{\lambda\alpha}+\sqrt{\frac{\bar{a}}{a}}\,\bar{p}^{\beta}=0, \tag{2.7}$$

$$M^{\alpha\beta}\big|_{\alpha\beta}+[\bar{a}^{\alpha\mu}(2\gamma_{\mu\lambda|\beta}-\gamma_{\lambda\beta|\mu})M^{\lambda\beta}]_{|\alpha}+\bar{b}_{\alpha\beta}N^{\alpha\beta}+\sqrt{\frac{\bar{a}}{a}}\,\bar{p}=0.$$

This form of equilibrium equations is particularly suitable in the intrinsic approach.

The approximate constitutive equations for thin isotropic elastic shells undergoing small strains have the form [5, 6]

$$N^{\alpha\beta}=\frac{1}{A(1-\nu^2)}[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\kappa}_{\kappa}]+O(Eh\eta\varepsilon^2),$$
$$M^{\alpha\beta}=D[(1-\nu)\rho^{\alpha\beta}+\nu a^{\alpha\beta}\rho^{\kappa}_{\kappa}]+O(Eh^2\eta\varepsilon^2), \tag{2.8}$$

or

$$\gamma_{\alpha\beta}=A[(1+\nu)N_{\alpha\beta}-\nu a_{\alpha\beta}N^{\kappa}_{\kappa}]+O(\eta\varepsilon^2),$$
$$\rho_{\alpha\beta}=\frac{1}{D(1-\nu^2)}[(1+\nu)M_{\alpha\beta}-\nu a_{\alpha\beta}M^{\kappa}_{\kappa}]+O\left(\frac{\eta\varepsilon^2}{h}\right), \tag{2.9}$$

where

$$A=\frac{1}{Eh},\qquad D=\frac{Eh^3}{12(1-\nu^2)},$$
$$\varepsilon\equiv\max\left(\frac{h}{L},\frac{h}{d},\sqrt{\frac{h}{R}},\sqrt{\eta}\right), \tag{2.10}$$

and h is the constant shell thickness, R – the smallest principal radius of the undeformed middle surface curvature, L – the minimum length of deformation pattern, d – a distance from the shell boundary, η – the maximum value of strain and the symbol $O(\ )$ means „the order of".

In the order-of-magnitude estimates we will use the following estimates for the various parameters

$$a_{\alpha\beta}\sim 1, \quad b_{\alpha\beta}\sim\frac{1}{R}, \quad (\;)_{|\alpha}\sim\frac{(\;)}{\lambda},$$
$$\left(\frac{h}{L}\right)^2\ll 1, \quad \frac{h}{R}\ll 1, \quad \eta\ll 1, \tag{2.11}$$

where

$$\lambda=\frac{h}{\varepsilon}=\min\left(L, d, \sqrt{hR}, \frac{h}{\sqrt{\eta}}\right). \tag{2.12}$$

Following the line suggested by the works of Danielson [4] and Koiter and Simmonds [6] it is possible to considerably simplify the compatibility (2.2) and the equilibrium (2.7) equations and obtain the following canonical shell equations

$$\begin{gathered}
\rho^{\beta}_{\alpha|\beta}-\rho^{\beta}_{\beta|\alpha}+\tfrac{1}{2}(1+\nu)A(b^{\lambda}_{\alpha}N^{\beta}_{\lambda}-b^{\beta}_{\lambda}N^{\lambda}_{\alpha})_{|\beta}-Ab^{\beta}_{\alpha}N^{\lambda}_{\lambda|\beta}+\\
+(1+\nu)A(\rho^{\beta}_{\alpha}N^{\lambda}_{\lambda|\beta}+\rho^{\beta}_{\lambda}N^{\lambda}_{\beta|\alpha})-\nu A\rho^{\lambda}_{\lambda}N^{\beta}_{\beta|\alpha}-2A(1+\nu)(b^{\beta}_{\alpha}-\rho^{\beta}_{\alpha})\bar{p}_{\beta}=O\left(\frac{\eta\varepsilon^4}{h\lambda}\right),\\
AN^{\alpha}_{\alpha}|^{\beta}_{\beta}+b^{\alpha}_{\beta}\rho^{\beta}_{\alpha}-b^{\alpha}_{\alpha}\rho^{\beta}_{\beta}-\tfrac{1}{2}\rho^{\alpha}_{\beta}\rho^{\beta}_{\alpha}+\tfrac{1}{2}\rho^{\alpha}_{\alpha}\rho^{\beta}_{\beta}+A(1+\nu)\bar{p}^{\alpha}|_{\alpha}=O\left(\frac{\eta\varepsilon^2}{\lambda^2}\right), \qquad (2.13)\\
N^{\beta}_{\alpha|\beta}-\tfrac{1}{2}(1-\nu)D(b^{\lambda}_{\alpha}\rho^{\beta}_{\lambda}-b^{\beta}_{\lambda}\rho^{\lambda}_{\alpha})_{|\beta}-Db^{\beta}_{\alpha}\rho^{\lambda}_{\lambda|\beta}+D(\rho^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}-\tfrac{1}{2}\delta^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}\rho^{\kappa}_{\kappa})_{|\beta}+2A(N^{\lambda}_{\alpha}N^{\beta}_{\lambda})_{|\beta}-\\
-\tfrac{1}{2}A\left[(1-\nu)N^{\kappa}_{\lambda}N^{\lambda}_{\kappa}+\nu N^{\kappa}_{\kappa}N^{\lambda}_{\lambda}\right]_{|\alpha}+2A\left[(1+\nu)N^{\lambda}_{\alpha}\bar{p}_{\lambda}-\nu N^{\lambda}_{\lambda}\bar{p}_{\alpha}\right]+(1+\gamma^{\lambda}_{\lambda})\bar{p}_{\alpha}=O\left(Eh\frac{\eta\varepsilon^4}{\lambda}\right),\\
D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta}+(b^{\alpha}_{\beta}-\rho^{\alpha}_{\beta})N^{\beta}_{\alpha}+\bar{p}=O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right).
\end{gathered}$$

These equations differ from those presented in [6] by the sign convention for $\rho_{\alpha\beta}$ and by taking into account the surface forces $\bar{p}_{\alpha}$ and $\bar{p}$.

It is important to point out here that the selection of $N^{\alpha\beta}$ and $\rho_{\alpha\beta}$ as the independent shell variables introduces the smallest error into the intrinsic equations. If we decide, for example, to eliminate $N^{\alpha\beta}$ from equilibrium equations (2.7) the error introduced to $(2.7)_1$ by $(2.8)_1$ happens to be $O(Eh\eta\varepsilon^2/\lambda)$ and all terms, except the first one, should be omitted within the same accuracy.

The approximate constitutive equations (2.8) and (2.9) may also be used to simplify the Lagrangean equilibrium equations (2.4). Thus after elimination of $M^{\alpha\beta}$ and $\gamma_{\alpha\beta}$ from (2.5) we obtain

$$\begin{gathered}
Q^{\alpha\beta}=N^{\alpha\beta}+\tfrac{1}{2}D(1-\nu)(b^{\alpha}_{\kappa}\rho^{\kappa\beta}-b^{\beta}_{\kappa}\rho^{\alpha\kappa})+O(Eh\eta\varepsilon^4),\\
Q^{\alpha}=D\rho^{\kappa|\alpha}_{\kappa}+O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda}\right),
\end{gathered} \tag{2.14}$$

which simplifies considerably the equations (2.4).

The relations (2.13) or (2.4) with (2.14) contain only an error introduced by the approximate constitutive equations. However, there are many shell problems for which the type of solution expected can be predicted in advance. If for a particular shell problem we restrict at the beginning the domain of some properly chosen parameters, then the solution can be obtained with the same accuracy from a considerably simplified set of equations. In the following chapters some possibilities of this kind will be discussed.

3. Restrictions on the strain ratios

Let γ be the maximum value of extensional strains and ρ the maximum value of change of curvature at certain shell point on the middle surface. Then $\rho h/\gamma$ describes the ratio between bending and membrane deformations.

In respect to the restrictions put on the ratio $\rho h/\gamma$ it is possible to distinguish the following types of shell problems:

a) $\frac{\rho h}{\gamma} \lesssim \varepsilon^2$ membrane shell theory,

b) $\frac{1}{\varepsilon} \lesssim \frac{\rho h}{\gamma} \lesssim \varepsilon$ bending shell theory,

c) $\frac{\rho h}{\gamma} \gtrsim \frac{1}{\varepsilon^2}$ inextensional bending shell theory.

For the *membrane shell theory* order-of-magnitude estimation of all terms with $\rho_{\alpha\beta}$ allows to omit in (2.13) a lot of terms of the same order as the error of the equations, and the canonical shell equations can be simplified to the form

$$\rho^{\beta}_{\alpha|\beta} - \rho^{\beta}_{\beta|\alpha} + \tfrac{1}{2}(1+\nu)A(b^{\lambda}_{\alpha}N^{\beta}_{\lambda} - b^{\beta}_{\lambda}N^{\lambda}_{\alpha})_{|\beta} - Ab^{\beta}_{\alpha}N^{\lambda}_{\lambda|\beta} - 2A(1+\nu)b^{\beta}_{\alpha}\bar{p}_{\beta} = O\left(\frac{\eta\varepsilon^4}{h\lambda}\right),$$

$$AN^{\alpha}_{\alpha}|^{\beta}_{\beta} + A(1+\nu)\bar{p}^{\alpha}|_{\alpha} = O\left(\frac{\eta\varepsilon^2}{\lambda^2}\right), \tag{3.1}$$

$$N^{\beta}_{\alpha|\beta} + 2A(N^{\lambda}_{\alpha}N^{\beta}_{\lambda})_{|\beta} - \tfrac{1}{2}A[(1-\nu)N^{\kappa}_{\lambda}N^{\lambda}_{\kappa} + \nu N^{\kappa}_{\kappa}N^{\lambda}_{\lambda}]_{|\alpha} +$$
$$+ 2A[(1+\nu)N^{\lambda}_{\alpha}\bar{p}_{\lambda} - \nu N^{\lambda}_{\lambda}\bar{p}_{\alpha}] + (1+\gamma^{\lambda}_{\lambda})\bar{p}_{\alpha} = O\left(Eh\frac{\eta\varepsilon^4}{\lambda}\right),$$

$$b^{\alpha}_{\beta}N^{\beta}_{\alpha} + \bar{p} = O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right).$$

It is easy to see that the membrane equilibrium equations $(3.1)_{3,4}$ can be solved in terms of $N^{\alpha\beta}$ without any reference to the compatibility. Because of that the membrane shell problems are sometimes called statically determined. The additional independent equation $(3.1)_2$ to be satisfied by $N^{\alpha\beta}$ shows that within geometrically non-linear shell theory the membrane state can occur under particular circumstances only. The most interesting result to be noted here is that under the error of canonical equations it is necessary to take into account the linear as well as the quadratic terms in $N^{\alpha\beta}$.

For the *inextensional bending theory* order-of-magnitude estimation of all terms with $N^{\alpha\beta}$ allows to omit in (2.13) a lot of terms of the same order as the error of the equations, and the canonical shell equations can be simplified to the form

$$\rho^{\beta}_{\alpha|\beta}-\rho^{\beta}_{\beta|\alpha}=O\left(\frac{\eta\varepsilon^4}{h\lambda}\right),$$

$$b^{\alpha}_{\beta}\rho^{\beta}_{\alpha}-b^{\alpha}_{\alpha}\rho^{\beta}_{\beta}-\tfrac{1}{2}\rho^{\alpha}_{\beta}\rho^{\beta}_{\alpha}+\tfrac{1}{2}\rho^{\alpha}_{\alpha}\rho^{\beta}_{\beta}=O\left(\frac{\eta\varepsilon^2}{\lambda^2}\right),$$

$$N^{\beta}_{\alpha|\beta}-\tfrac{1}{2}(1-\nu)D(b^{\lambda}_{\alpha}\rho^{\beta}_{\lambda}-b^{\beta}_{\lambda}\rho^{\lambda}_{\alpha})_{|\beta}-Db^{\beta}_{\alpha}\rho^{\lambda}_{\lambda|\beta}+$$
$$+D(\rho^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}-\tfrac{1}{2}\delta^{\beta}_{\alpha}\rho^{\lambda}_{\lambda}\rho^{\kappa}_{\kappa})_{|\beta}+(1+\gamma^{\lambda}_{\lambda})\bar{p}_{\alpha}=O\left(Eh\frac{\eta\varepsilon^4}{\lambda}\right), \tag{3.2}$$

$$D\rho^{\alpha}_{\alpha}|^{\beta}_{\beta}+\bar{p}=O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right).$$

In analogy to membrane equations it is easy to see that compatibility equations $(3.2)_{1,2}$ can be solved here in terms of $\rho_{\alpha\beta}$ without any reference to the equilibrium. Because of that, the inextensional bending shell problems are sometimes called geometrically determined. The additional independent equation $(3.2)_4$ to be satisfied by $\rho_{\alpha\beta}$ shows that within geometrically non-linear shell theory the inextensional bending state can occur under particular circumstances only. It is also interesting to note that under the error of canonical equations it is necessary to take into account the linear as well as the quadratic terms in $\rho_{\alpha\beta}$.

The simplified sets of equations (3.1) and (3.2) are not supposed to be used for solving problems of shells with singular middle surface points, because in such problems they occasionally may lead to inadequate results. For example, if we apply the membrane equation (3.1) to a flat membrane, for which $b^{\beta}_{\alpha}=0$, then equation $(3.1)_4$ becomes indefinite. The similar problem appears also within the linear theory of shells, where as typical the following singular surfaces were noted: plate, infinite cylinder, cone, toroid. However, the simplified equations are not supposed to be applied even to shells with points near to singular such as very long cylinders or very shallow shells.

4. Restrictions on the length of deformation pattern

In the first approximation shell theory it is possible to solve problems in which the length of deformation pattern is at least one order of magnitude greater than the thickness of the shell (2.11). For some problems it is still desirable to put an upper bound on the length of deformation pattern by the relation [8]

$$|K|L^2\lesssim\varepsilon^2. \tag{4.1}$$

This type of deformation is typical for bending shell problems for which we do not need to use $N^{\alpha\beta}$ and $\rho_{\alpha\beta}$ as independent variables and so high accuracy in $(2.13)_{1,3}$. This allows us to reduce the canonical equations. If we admit the use of other quantities then the error introduced into the equations (2.13) from constitutive equations $(2.8)_1$ and $(2.9)_2$ is larger than indicated in (2.13), and equilibrium $(2.13)_3$ and compatibility $(2.13)_1$ equations reduce to

$$\rho^{\beta}_{\alpha|\beta}-\rho^{\beta}_{\beta|\alpha}=O\left(\frac{\eta\varepsilon^2}{h\lambda}\right),$$
$$N^{\beta}_{\alpha|\beta}+\bar{p}_{\alpha}=O\left(Eh\frac{\eta\varepsilon^2}{\lambda}\right). \tag{4.2}$$

Now it is possible to use the fact that under (4.1) the order of the surface covariant differentiation is approximately immaterial. Thus it is possible to introduce a curvature function W and a stress function F by the relations

$$\rho^{\beta}_{\alpha}=W|^{\beta}_{\alpha}+O\left(\frac{\eta\varepsilon^2}{h}\right),\qquad N^{\beta}_{\alpha}=\delta^{\beta\kappa}_{\alpha\lambda}F|^{\lambda}_{\kappa}+P^{\beta}_{\alpha}+O(Eh\eta\varepsilon^2) \tag{4.3}$$

which satisfy the equations (4.2) within the same error. The remained equations $(2.13)_{2,4}$ have, in terms of W and F, the form

$$\begin{aligned}&AF|^{\alpha\beta}_{\alpha\beta}-\delta^{\alpha\lambda}_{\beta\kappa}(b^{\beta}_{\alpha}-\tfrac{1}{2}W|^{\beta}_{\alpha})W|^{\kappa}_{\lambda}+A\,[P^{\alpha}_{\alpha}|^{\beta}_{\beta}-(1+\nu)P^{\alpha}_{\beta}|^{\beta}_{\alpha}]=O\left(\frac{\eta\varepsilon^2}{\lambda^2}\right),\\ &DW|^{\alpha\beta}_{\alpha\beta}+\delta^{\alpha\lambda}_{\beta\kappa}(b^{\beta}_{\alpha}-W|^{\beta}_{\alpha})F|^{\kappa}_{\lambda}+(b^{\alpha}_{\beta}-W|^{\alpha}_{\beta})P^{\beta}_{\alpha}+\bar{p}=O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right),\end{aligned} \tag{4.4}$$

and these are, with accuracy up to different sign conventions, the quasi-shallow shell equations as obtained by Koiter [8].

5. Restrictions on deflections

In the general case the deformation of the shell middle surface can be divided into three successive steps: rigid body translation, rigid body rotation, and pure deformation. Geometry of the surface deformation with the use of the finite rotation vector has been discussed by Simmonds and Danielson [2]*). Here we use deflections rather than finite rotation vector components, since they are already included in our Lagrangean equilibrium equations (2.4).

For the shell deformation we have [8, 9]

$$l_{\alpha\beta}=a_{\alpha\beta}+\vartheta_{\alpha\beta}-\omega_{\alpha\beta},\qquad \varphi_{\alpha}=w_{,\alpha}+b^{\kappa}_{\alpha}u_{\kappa}, \tag{5.1}$$

where

$$\begin{aligned}&\vartheta_{\alpha\beta}=\tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}w,\\ &\omega_{\alpha\beta}=\tfrac{1}{2}(u_{\beta|\alpha}-u_{\alpha|\beta})=\epsilon_{\alpha\beta}\omega,\end{aligned} \tag{5.2}$$

and u_α, w are the Lagrangean components of the displacement vector.

Let φ be the greatest deflection of the normal to the shell middle surface and ω be the greatest rotation about the normal. For the small finite deflection shell theory we use the estimates

$$\omega\sim\varepsilon,\qquad \varphi\sim\varepsilon \tag{5.3}$$

which leads to the following estimates

$$\begin{aligned}&\gamma_{\alpha\beta}=\vartheta_{\alpha\beta}+\tfrac{1}{2}(\underline{a_{\alpha\beta}\omega^2}+\varphi_{\alpha}\varphi_{\beta})+O(\eta\varepsilon),\\ &\rho_{\alpha\beta}=-\tfrac{1}{2}(\varphi_{\alpha|\beta}+\varphi_{\beta|\alpha}+\underline{b^{\kappa}_{\alpha}\omega_{\beta\kappa}+b^{\kappa}_{\beta}\omega_{\alpha\kappa}})+O\left(\frac{\eta\varepsilon}{\lambda}\right),\end{aligned} \tag{5.4}$$

$$n^{\alpha}=-\varphi^{\alpha}-\underline{\varphi^{\lambda}\omega^{\alpha}_{\lambda}}+O(\eta\varepsilon),\qquad n=1+O(\eta). \tag{5.5}$$

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*) General theory of finite rotations in shells has been developed recently by Pietraszkiewicz [12]. Consult also [13, 14].

With the help of (2.14), (5.4) and (5.5) the simplified Lagrangean equilibrium equations (2.4) can be obtained in the form

$$[N^{\alpha\lambda} l^{\beta}_{.\lambda} + \tfrac{1}{2} D(1-\nu)(b^{\alpha}_{\kappa}\rho^{\kappa\lambda} - b^{\lambda}_{\kappa}\rho^{\kappa\alpha})(\delta^{\beta}_{\lambda} - \underline{a^{\beta\mu}\omega_{\mu\lambda}}) - D\rho^{\kappa}_{\kappa}|^{\alpha}(\varphi^{\beta} + \underline{\varphi_{\lambda}\omega^{\lambda\beta}})]_{|\alpha} -$$
$$- b^{\beta}_{\alpha}(N^{\alpha\lambda}\varphi_{\lambda} + D\rho^{\kappa}_{\kappa}|^{\alpha}) + p^{\beta} = O\left(Eh\frac{\eta\varepsilon^4}{\lambda}\right), \tag{5.6}$$
$$N^{\alpha\lambda}\varphi_{\lambda|\alpha} + D\rho^{\kappa}_{\kappa}|^{\alpha}_{\alpha} + b_{\alpha\beta}N^{\alpha\lambda}(\delta^{\beta}_{\lambda} - \underline{a^{\beta\mu}\omega_{\mu\lambda}}) + p = O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right).$$

If we use a more restrictive assumption $\omega \sim \varepsilon^2$ together with $\varphi \sim \varepsilon$ then all underlined terms may also be omitted.

Equations $(5.6)_1$ can be simplified even further for problems which are supposed to be solved in displacements. In such a case the constitutive equation $(2.8)_1$ introduces an error $O\left(Eh\frac{\eta\varepsilon^2}{\lambda}\right)$ and equations $(5.6)_1$ become

$$[N^{\alpha\lambda}(\delta^{\beta}_{\lambda} - \underline{a^{\beta\mu}\omega_{\mu\lambda}})]_{|\alpha} + p^{\beta} = O\left(Eh\frac{\eta\varepsilon^2}{\lambda}\right). \tag{5.7}$$

Finally, introducing $(2.8)_1$, (5.2), (5.4) and (5.5) into (5.7) and $(5.6)_2$ we obtain the set of equations for small finite deflection shell theory to be solved in displacements.

It is worthwhile to point out at the end that the classical von Karman-type non-linear shell theory is based on simultaneous restrictions of the following quantities

$$\varphi \sim \varepsilon, \quad \omega \sim \varepsilon^2, \quad |K| L^2 \lesssim \varepsilon^2,$$
$$\frac{u}{v} \sim \varepsilon, \quad p^{\beta} \lesssim Eh\frac{\eta\varepsilon^2}{\lambda} \tag{5.8}$$

under which

$$\gamma_{\alpha\beta} = \vartheta_{\alpha\beta} + \tfrac{1}{2} w_{,\alpha} w_{,\beta} + O(\eta\varepsilon^2),$$
$$\rho_{\alpha\beta} = -w|_{\alpha\beta} + O\left(\frac{\eta\varepsilon}{\lambda}\right), \tag{5.9}$$

and the classical non-linear shell equations may be shown to have the known form

$$AF|^{\alpha\beta}_{\alpha\beta} + \delta^{\alpha\lambda}_{\beta\kappa}(b^{\beta}_{\alpha} + \tfrac{1}{2} w|^{\beta}_{\alpha})\, w|^{\kappa}_{\lambda} = O\left(\frac{\eta\varepsilon^2}{\lambda}\right),$$
$$Dw|^{\alpha\beta}_{\alpha\beta} - \delta^{\alpha\lambda}_{\beta\kappa}(b^{\beta}_{\alpha} + w|^{\beta}_{\alpha})\, F|^{\kappa}_{\lambda} = p + O\left(Eh^2\frac{\eta\varepsilon^2}{\lambda^2}\right). \tag{5.10}$$

These equations, called also the shallow shell equations, are used most frequently in the up-today literature.

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Uproszczenie równań geometrycznie nieliniowej teorii cienkich powłok sprężystych

Streszczenie

W pracy rozważono możliwości uproszczeń równań podstawowych, geometrycznie nieliniowej teorii cienkich powłok sprężystych. Uproszczeń dokonano przewidując charakter nieznanego rozwiązania, a w szczególności stosunek między odkształceniem membranowym i giętnym, długością fali deformacji i krzywizną Gaussa powłoki, a także używając różnych ograniczeń na obroty otoczenia powierzchni środkowej powłok. W każdym szczególnym przypadku, równania uproszczone uzyskano pomijając małe człony rzędu błędu zawsze wprowadzanego do równań podstawowych w przybliżonej postaci równań konstytutywnych.

Упрощение уравнений геометрически нелинейной теории тонких упругих оболочек

Резюме

В работе рассматриваются возможности упрощений основных уравнений геометрически нелинейной теории тонких упругих оболочек. Упрощения проводились предвидывая характер неизвестного решения, а в особенности отношения между мембранной и изгибной деформациями, длиной деформационной волны и кривизной Гаусса оболочки, а также пользуясь различными ограничениями на обороты окрестности серединной поверхности оболочки. В каждом отдельном случае упрощенные уравнения получались пренебрегая малыми членами порядка погрешности, всегда вводимой в основные уравнения из приближенной формы конститутивных уравнений.