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W. Pietraszkiewicz

Consistent Second Approximation to the Elastic Strain Energy of a Shell

For a homogeneous linearly-elastic solid the strain energy function σ, per unit volume of an undeformed reference configuration, and the Lagrangean constitutive equations, expressed in terms of the second Piola-Kirchhoff stress tensor S^{ij}, have the forms

$$\sigma = \tfrac{1}{2} L^{ijkl} E_{ij} E_{kl} , \qquad S^{ij} = L^{ijkl} E_{kl} , \qquad (i, j = 1, 2, 3) . \tag{1}$$

Here E_{ij} is the Green strain tensor and L^{ijkl} is the linear elasticity tensor.

In the undeformed reference configuration of an elastic shell of small thickness h we introduce a normal coordinate system ϑ^α, ζ such that ζ is the distance from the reference shell middle surface $\mathcal{M}$. The shell strain energy function $\boldsymbol{\Sigma}$, per unit area of $\mathcal{M}$, is defined by

$$\boldsymbol{\Sigma} = \int_{-h/2}^{h/2} \sigma \mu \, \mathrm{d}\zeta , \qquad \mu = 1 - \zeta 2H + \zeta^2 K , \tag{2}$$

where H and K are the mean and Gaussian curvatures of $\mathcal{M}$, respectively.

For a material having elastic symmetry with respect to the surface $\mathcal{M}$ we can eliminate E_{33} from $(1)_1$ by using $(1)_2$, which together with $E_{\beta 3} = E_{3\beta}$ gives

$$\sigma = \frac{(S^{33})^2}{2L^{3333}} + \frac{1}{2} H^{\alpha\beta\lambda\mu} E_{\alpha\beta} E_{\lambda\mu} + 2L^{3\beta 3\mu} E_{3\beta} E_{3\mu} , \tag{3}$$

$$H^{\alpha\beta\lambda\mu} = L^{\alpha\beta\lambda\mu} - \frac{L^{\alpha\beta 33} L^{33\lambda\mu}}{L^{3333}} , \qquad (\alpha, \beta = 1, 2) . \tag{4}$$

Let us expand the elasticities and strains in (3) into series with respect to ζ

$$H^{\alpha\beta\lambda\mu} = \sum_{n=0}^{\infty} \zeta^n H_n^{\alpha\beta\lambda\mu} , \qquad L^{3\beta 3\mu} = \sum_{n=0}^{\infty} \zeta^n L_n^{3\beta 3\mu} , \tag{5}$$

$$E_{\alpha\beta} = \gamma_{\alpha\beta} + \zeta \tfrac{1}{2} (\pi_{\alpha\beta} + \pi_{\beta\alpha}) + \zeta^2 \delta_{\alpha\beta} + \dots , \qquad E_{3\beta} = \gamma_{3\beta} + \zeta \tfrac{1}{2} \pi_{3\beta} + \dots \tag{6}$$

Let $\boldsymbol{a}_i$ and $\bar{\boldsymbol{a}}_i$ be bases of the convected coordinate system ϑ^α, ζ calculated at the shell middle surface in the reference and deformed configurations, respectively. Note that the unit base vector $\boldsymbol{a}_3 \equiv \boldsymbol{n}$, orthogonal to $\mathcal{M}$, deforms into the base vector $\bar{\boldsymbol{a}}_3$ which, in general, is neither unit nor orthogonal to the deformed shell middle surface $\bar{\mathcal{M}}$, $\bar{\boldsymbol{a}}_3 \neq \bar{\boldsymbol{n}}$. In the general case of deformation for $\bar{\boldsymbol{a}}_3$ we obtain

$$\bar{\boldsymbol{a}}_3 = \sqrt{(1 + 2\gamma_{33})(1 - 2\bar{a}^{3\beta}\gamma_{3\beta})}\,\bar{\boldsymbol{n}} + 2\gamma_{3\beta}\bar{\boldsymbol{a}}^\beta , \qquad \gamma_{33} = E_{33}|_{\zeta=0} . \tag{7}$$

When a displacement field is assumed to be linear across the shell thickness, the shell strain measures γ_{ij}, $\pi_{i\beta}$ and $\delta_{\alpha\beta}$ are exactly quadratic functions of two independent displacemental variables: $\boldsymbol{u}$, the displacement vector of $\mathcal{M}$, and the vector $\boldsymbol{\beta} = \bar{\boldsymbol{a}}_3 - \boldsymbol{n}$, [1, 2].

After substitution of (5) and (6) into (3), (4) and (2) we obtain

$$\sigma = \frac{(S^{33})^2}{2L^{3333}} + \sum_{n=0}^{\infty} \zeta^n \sigma_n , \qquad S = \int_{-h/2}^{h/2} \frac{(S^{33})^2}{2L^{3333}} \mu \, d\zeta , \tag{8}$$

$$\boldsymbol{\Sigma} = S + h\sigma_0 + \sum_{n=2,4,6,\ldots}^{\infty} \frac{h^{n+1}}{2^n(n+1)} (\sigma_n - 2H\sigma_{n-1} + K\sigma_{n-2}) \tag{9}$$

where, in particular,

$$\begin{aligned}
\sigma_0 &= \tfrac{1}{2} \underset{0}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + 2\underset{0}{L}^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu} ,\\
\sigma_1 &= \underset{0}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\pi_{(\lambda\mu)} + \tfrac{1}{2} \underset{1}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + 2\underset{0}{L}^{3\beta3\mu}\gamma_{3\beta}\pi_{3\mu} + 2\underset{1}{L}^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu} ,\\
\sigma_2 &= \underset{0}{H}^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\delta_{\lambda\mu} + \tfrac{1}{2}\pi_{(\alpha\beta)}\pi_{(\lambda\mu)}) + \underset{1}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\pi_{(\lambda\mu)} + \tfrac{1}{2} \underset{2}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \tfrac{1}{2} \underset{0}{L}^{3\beta3\mu}\pi_{3\beta}\pi_{3\mu} +\\
&\qquad + 2\underset{1}{L}^{3\beta3\mu}\gamma_{3\beta}\pi_{3\mu} + 2\underset{2}{L}^{3\beta3\mu}\gamma_{3\beta}\gamma_{3\mu}
\end{aligned} \tag{10}$$

and $\pi_{(\alpha\beta)}$ is the symmetric part of $\pi_{\alpha\beta}$. The formula (9) gives an exact representation for the shell strain energy function in the form of an infinite series with respect to the shell thickness h.

Let us assume now the strains to be small everywhere in the shell: $\eta \ll 1$, $\eta = \max |E_r|$, where E_r are three eigenvalues of E_{ij}. Let us choose at $\mathcal{M}$ a coordinate system ϑ^α such that the metric and curvature tensors of $\mathcal{M}$ are estimated by $a_{\alpha\beta} = O(1)$ and $b_{\alpha\beta} = O(1/R)$, respectively, where R is the smallest radius of curvature of $\mathcal{M}$. In the case of an isotropic elastic shell, loaded only at its lateral boundaries, the stresses S^{ij} in the interior domain of the shell can be exactly estimated as follows [3]:

$$S^{\alpha\beta} = O(\eta E) , \qquad S^{3\beta} = O(E\eta\vartheta) , \qquad S^{33} = O(E\eta\vartheta^2) . \tag{11}$$

Here E is the Young's modulus and the small parameter ϑ, redefined in [4] by using qualitative arguments, has the form $\vartheta = \max\left(\frac{h}{L}, \frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\right)$, where L is the smallest wavelength of deformation patterns at $\mathcal{M}$ and d is the distance from the lateral shell boundary. According to $(1)_2$ the estimates of stresses (11) imply also the appropriate estimates for strains: $E_{\alpha\beta} = O(\eta)$, $E_{3\beta} = O(\eta\vartheta)$ and $E_{33} = O(\nu\eta)$, where ν is the Poisson's ratio. For the bending state of strain $\gamma_{\alpha\beta} \sim h\pi_{(\alpha\beta)}$. Using (6) we are able to estimate the shell strain measures as follows:

$$\gamma_{\alpha\beta} = O(\eta) , \qquad h\pi_{(\alpha\beta)} = O(\eta) , \qquad \gamma_{3\beta} = O(\eta\vartheta) , \qquad \gamma_{33} = O(\eta\boldsymbol{\nu}) \tag{12}$$

and estimates $h^2\delta_{\alpha\beta} = O(\eta\vartheta^2)$ and $h\pi_{3\beta} = O(\nu\eta\vartheta)$ follow directly from (12) in the case of approximately linear distribution of displacement field across the shell thickness.

It is possible now to estimate the orders of magnitude of all terms in the shell strain energy function (9). In particular, within an error $O(Eh\eta^2\vartheta^4)$ we obtain

$$\begin{aligned}
S &= O(Eh\eta^2\vartheta^4) , \qquad \frac{h^3}{12} K\sigma_0 = O(Eh\eta^2\vartheta^4) ,\\
-\frac{h^3}{12} 2H\sigma_1 &= -\frac{h^3}{12} 2H\underset{0}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\pi_{(\lambda\mu)} + O(Eh\eta^2\vartheta^4) ,\\
\frac{h^3}{12}\sigma_2 &= \frac{h^3}{12}\underset{0}{H}^{\alpha\beta\lambda\mu}\left(\frac{1}{2}\pi_{(\alpha\beta)}\pi_{(\lambda\mu)} + \gamma_{\alpha\beta}\delta_{\lambda\mu}\right) + \frac{h^3}{24}\underset{0}{L}^{3\beta3\mu}\pi_{3\beta}\pi_{3\mu} + \frac{h^3}{12}\underset{1}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\pi_{(\lambda\mu)} + O(Eh\eta^2\vartheta^4) .
\end{aligned} \tag{13}$$

The orders of magnitude of all other terms under the sign of a sum in (8) are much smaller than $O(Eh\eta^2\vartheta^4)$.

Therefore, within this error the shell strain energy function may be consistently approximated by the following expression

$$\begin{aligned}
\boldsymbol{\Sigma} = \frac{h}{2}\underset{0}{H}^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\pi_{(\alpha\beta)}\pi_{(\lambda\mu)}\right) + 2h\underset{0}{L}^{3\beta3\mu}\left(k^2\gamma_{3\beta}\gamma_{3\mu} + l^2\frac{h^2}{48}\pi_{3\beta}\pi_{3\mu}\right) +\\
+ \frac{h^3}{12}\underset{0}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\left(\delta_{\lambda\mu} - 2H\pi_{(\lambda\mu)}\right) + \frac{h^3}{12}\underset{1}{H}^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\pi_{(\lambda\mu)} + O(Eh\eta^2\vartheta^4) , \quad k^2 = \frac{5}{6} , \quad l^2 = \frac{7}{10} .
\end{aligned} \tag{14}$$

The shell elasticity tensors appearing in (14), in the case of an isotropic elastic material, take the forms

$$\begin{aligned}
\underset{0}{H}^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right), \qquad \underset{0}{L}^{3\beta3\mu} = \frac{E}{2(1+\nu)}a^{\beta\mu} ,\\
\underset{1}{H}^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)}\left[2(a^{\alpha\lambda}b^{\beta\mu} + b^{\alpha\lambda}a^{\beta\mu}) + 2(a^{\alpha\mu}b^{\beta\lambda} + b^{\alpha\mu}a^{\beta\lambda}) + \frac{4\nu}{1-\nu}(a^{\alpha\beta}b^{\lambda\mu} + b^{\alpha\beta}a^{\lambda\mu})\right].
\end{aligned} \tag{15}$$

With the help of (7) we are able to interpret the shell strain measure $\pi_{(\alpha\beta)}$ in terms of the tensor of change of curvature of the shell middle surface $\varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})$ through the relation

$$\pi_{(\alpha\beta)} = \tfrac{1}{2} (\bar{\boldsymbol{a}}_\alpha \cdot \bar{\boldsymbol{a}}_{3,\beta} + \bar{\boldsymbol{a}}_\beta \cdot \bar{\boldsymbol{a}}_{3,\alpha}) - \boldsymbol{a}_\alpha \cdot \boldsymbol{n}_{,\beta}$$

$$= \varkappa_{\alpha\beta} + \gamma_{3\alpha}|_\beta + \gamma_{3\beta}|_\alpha - (b_{\alpha\beta} - \varkappa_{\alpha\beta})\, \gamma_{33} + O\left(\frac{\eta^2\vartheta^2}{h}\right) = \varkappa_{\alpha\beta} + O\left(\frac{\eta\vartheta^2}{h}\right), \tag{16}$$

where $(\ )|_\alpha$ is the covariant derivative with respect to the reference middle surface metric.

Admiting a greater error in (14) we obtain

$$\Sigma = \frac{h}{2} \underset{0}{H}^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \varkappa_{\alpha\beta}\varkappa_{\lambda\mu}\right) + O(Eh\eta^2\vartheta^2)\,. \tag{17}$$

This is the well known consistent first approximation to the shell strain energy function [4, 5, 2]. The formula (17) includes the main contributions to the elastic strain energy of a shell due to stretching and bending of the shell middle surface as well as due to the transverse strains. The last one is included by using in (17) the modified shell elasticity tensor.

The expression (14) is the consistent second approximation to the shell strain energy function. It refines in a consistent manner the shell strain energy (17) of the first-approximation theory by taking into account also all the secondary contributions to the elastic strain energy of a shell. These contributions come from the transverse shear, from the change of curvature due to shear and from various couplings between the stretching, bending and the secondary bending of the shell. The secondary contributions from the transverse strains are also taken into account in (14) by using there the modified shell elasticity tensors. However, the actual distribution of the transverse strains over the shell thickness can easily be recovered by using the inverse of $(1)_2$. In the case of isotropy we obtain

$$E_{33} = -\frac{\nu}{1-\nu} (\gamma_\alpha^\alpha + \zeta\varkappa_\alpha^\alpha) + O(\eta\vartheta^2)\,, \tag{18}$$

$$= \frac{(1+\nu)(1-2\nu)}{1-\nu} \frac{S^{33}}{E} - \frac{\nu}{1-\nu} \{\gamma_\alpha^\alpha + \zeta(\pi_\alpha^\alpha + 2b^{\alpha\beta}\gamma_{\alpha\beta}) +$$

$$+ \zeta^2[\delta_\alpha^\alpha + 2b^{\alpha\beta}\pi_{(\alpha\beta)} + 3b_\lambda^\alpha b^{\lambda\beta}\gamma_{33}]\} + O(\eta\vartheta^4)\,. \tag{19}$$

The formula (18) is compatible with the consistent first-approximation theory while (19) is compatible with the consistent second-approximation theory.

By differentiating the refined strain energy function (14) with respect to the proper shell strain measures we obtain the constitutive equations for the stress resultants $N^{\alpha\beta}$, the stress couples $M^{\alpha\beta}$, the shearing forces $2N^{3\beta}$, the stress couples due to shear $M^{3\beta}$ and for the secondary couples $K^{\alpha\beta}$, together with appropriate error estimates of the constitutive equations:

$$\begin{aligned}
N^{\alpha\beta} &= h\underset{0}{H}^{\alpha\beta\lambda\mu}\left[\gamma_{\lambda\mu} + \frac{h^2}{12}(\delta_{\lambda\mu} - 2H\pi_{(\lambda\mu)})\right] + \frac{h^3}{12}\underset{1}{H}^{\alpha\beta\lambda\mu}\pi_{(\lambda\mu)} + O(Eh\eta\vartheta^4)\,,\\
M^{\alpha\beta} &= \frac{h^3}{12}\underset{0}{H}^{\alpha\beta\lambda\mu}(\pi_{(\lambda\mu)} - 2H\gamma_{\lambda\mu}) + \frac{h^3}{12}\underset{1}{H}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + O(Eh^2\eta\vartheta^4)\,,\\
N^{3\beta} &= 2k^2h\underset{0}{L}^{3\beta3\mu}\gamma_{3\mu} + O(Eh\eta\vartheta^3)\,, \qquad M^{3\beta} = l^2\frac{h^3}{12}\underset{0}{L}^{3\beta3\mu}\pi_{3\mu} + O(Eh^2\eta\vartheta^3)\,,\\
K^{\alpha\beta} &= \frac{h^3}{12}\underset{0}{H}^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} + O(Eh^3\eta\vartheta^2)\,.
\end{aligned} \tag{20}$$

All these strain and stress measures should be included into the system of equations of the consistent second-approximation geometrically non-linear bending theory of thin elastic shells [1].

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Anschrift: Doc. dr. hab. inż. WOJCIECH PIETRASZKIEWICZ, Instytut Maszyn Przepływowych PAN, Ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland