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FINITE ROTATIONS IN SHELLS

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A general theory of finite rotations in the non-linear theory of shells is presented. The finite rotation tensor and an equivalent finite rotation vector of the principal directions of strain are expressed at the shell middle surface in terms of two independent vector displacement parameters. Some relations in terms of the finite rotations are given.

The total rotation of the shell boundary element is shown to consist of two subsequent rotations: due to a pure stretch along principal directions of strain and due to a rigid-body rotation of the principal directions. Exact formulae are derived for a total finite rotation tensor and for an equivalent total finite rotation vector of the shell boundary element. With the help of purely geometric considerations three general forms of geometrical boundary conditions are formulated.

By using additional simplifying assumptions some formulae concerning finite rotations are derived for the Kirchhoff–Love shell theory, for the geometrically non-linear theory of shells and for the first-approximation theory of thin elastic shells. Within K–L theory three variants of statical boundary conditions are formulated. Each of them is energetically compatible with the respective variant of geometrical boundary conditions.

1. Introduction

The deformation of a neighbourhood of any continuum particle can be exactly decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation of the principal directions. The rotations are conventionally described by a proper orthogonal tensor **R** [25, 26]. An alternative description of the rotations is possible either by means of three angles (usually Euler angles) or a finite rotation vector **Ω**, [9, 21]. The rotation parameters appear explicitly in various relations of the continuum mechanics and in the analytical mechanics of a rigid-body motion.

When strains are assumed to be small, only small rotations of material elements may appear in a truly three-dimensional elastic body with boundary conditions preventing a rigid-body motion, [10]. However, within such geometrically non-linear theory of elasticity large rotations may appear in thin bodies such as beams, thin-walled beams, plates and shells. The rolling of a sheet of paper into a cylinder is a trivial example of this phenomena. Therefore, there is a substantial qualitative difference in the behaviour of thin bodies and of those of truly three-dimensional nature. This suggests, that the rotational part of deformation should play a more important role in the non-linear theories of thin bodies than in general problems of continuum mechanics.

The shell literature is not free from confusions about analytical representation of the finite rotations. Usually the linearized rotations or the middle surface displacement gradients are used, apparently on the intuitive grounds, to describe rotations also within the non-linear range of the shell deformation.

Within the Kirchhoff–Love non-linear theory of shells Simmonds and Danielson [23, 24] used the finite rotation vector $\mathbf{\Omega}$ as an independent kinematical variable of the shell theory. The rotation of a shell boundary element was described by Novozhilov and Shamina [11] in terms of a total finite rotation vector $\mathbf{\Omega}_t$ of the boundary. In these works the finite rotation vectors were introduced in a descriptive manner, without relating them to basic kinematical parameters of shell deformation, such as displacements of the shell middle surface or components of a deformation gradient tensor. The theory of finite rotations in shells subject to K–L constraints was developed by the author [15–17], where also various relations between the finite rotation parameters and midsurface displacements were presented.

The consistent classification of approximate variants of shell equations in terms of restricted rotations was given in [16] for the first-approximation theory of thin isotropic elastic shells. The rotations of the shell material element were defined to be small, moderate, large or finite depending on the order of magnitude of the rotation angle ω as compared with a small parameter θ introduced in [5, 7] and for each of the cases the consistent set of shell equations was given, [17]. Some results on a shell deformation with small strains but unrestricted rotations were obtained by Wempner [27] and Galimov [4].

In this report a general theory of finite rotations in shells is developed, [15]. At the shell middle surface we define a shell deformation gradient tensor $\mathbf{G}$, which provides a complete and exact information about deformation in a neighbourhood of the shell middle surface particles. The displacement field is described by means of two independent parameters: a displacement vector $\mathbf{u}$ of the shell middle surface and a vector $\boldsymbol{\beta}$ describing the change of tangents to the material fibres initially orthogonal to the reference shell middle surface. A polar decomposition theorem applied to $\mathbf{G}$ allows to derive an exact formula for the finite rotation tensor $\mathbf{R}$ in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$. The rotations are also described by means of a finite rotation vector $\mathbf{\Omega}$, for which three equivalent exact formulae in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$ are given.

A thorough analysis of deformation of a shell boundary element is presented. The total rotation of the element consists of two subsequent rotations. The first rotation appears as a result of a pure stretch of the boundary element along principal directions of strain. The second rotation is associated with a rigid-body rotation of the principal directions. Exact formulae are derived for a total finite rotation tensor $\mathbf{R}_t$ and for a total finite rotation vector $\mathbf{\Omega}_t$ of the shell boundary element. Then purely geometrical considerations analogous to those of [11] lead to three general forms of geometrical boundary conditions expressed in terms of either $\mathbf{u}$ and $\boldsymbol{\beta}$, or $\mathbf{\Omega}_t$ and the physical components γ_{tt}, γ_{3t}, γ_{33} of the Green strain tensor or the vector $\mathbf{k}_t$ of change of the boundary curvature and the strains γ_{tt}, γ_{3t}, γ_{33}, respectively. The vector $\mathbf{k}_t$ is shown to depend only upon the shell strain measures.

The relations given in Sections 2–5 are exact at the shell middle surface and are valid for unrestricted strains and unrestricted rotations of the shell material elements. The main results are simplified in Section 6 by imposing additional assumptions commonly used in constructing some variants of shell equations.

Within the Kirchhoff–Love constraints the appropriate formulae reduce to those given in [3, 11, 15–17]. Additionally, three variants of statical boundary conditions are constructed. Each of them is energetically compatible with the respective variant of geometrical boundary conditions. Within the small strains the reduced formulae for rotation parameters are derived with accuracy up to the second-order terms. Two limiting cases of truly finite rotations and of small rotations are given, the later one is shown to agree with [20]. Finally, some relations for the first-approximation theory of isotropic elastic shells are presented. In this case the change of the shell thickness is also taken into account, [17].

2. Notation and preliminary relations

Let $\mathbf{p}(\theta^i)$ and $\bar{\mathbf{p}}(\theta^i)$, $i = 1, 2, 3$ are position vectors of a shell particle in the reference and deformed configurations, respectively, connected by a deformation $\bar{\mathbf{p}} = \chi(\mathbf{p})$. Here θ^i are curvilinear convected coordinates with the reference and deformed bases $\mathbf{g}_i = \mathbf{p}_{,i}$ and $\bar{\mathbf{g}}_i = \bar{\mathbf{p}}_{,i}$, respectively.

In the Lagrangean description the displacement vector of the particle is given by

$$\mathbf{v} = \chi(\mathbf{p}) - \mathbf{p} = v^i \mathbf{g}_i. \tag{1}$$

Since the $d\mathbf{p} = \mathbf{g}_i\, d\theta^i$ for differentials we obtain

$$d\bar{\mathbf{p}} = \bar{\mathbf{g}}_i\, d\theta^i = (\mathbf{g}_i + \mathbf{v}_{,i})\, d\theta^i = \mathbf{F}\, d\mathbf{p}, \tag{2}$$

$$\mathbf{F} = \mathbf{1} + \text{grad}\, \mathbf{v} = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i \tag{3}$$

where $\mathbf{F}$ is the spatial deformation gradient tensor and $\mathbf{1} = g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j$ is the metric tensor of the three-dimensional Euclidean space.

In the reference shell configuration we assume θ^i to form a normal system such that $\mathbf{p}(\theta^i) = \mathbf{r}(\theta^\alpha) + \theta^3\mathbf{a}_3(\theta^\alpha)$, $\alpha = 1, 2$, where $\mathbf{r}$ is a position vector of the reference shell middle surface $\mathcal{M}$, $\mathbf{a}_3 \equiv \mathbf{n}$ is a unit normal to $\mathcal{M}$, $\theta^3 \equiv \zeta$ is a distance from $\mathcal{M}$ and $-\frac{1}{2}h \leqslant \zeta \leqslant \frac{1}{2}h$, where h is a small shell thickness. With the surface $\mathcal{M}$ we associate standard covariant base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, covariant components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ of the surface metric tensor $\mathbf{a}$ and covariant components $b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$ of the surface curvature tensor $\mathbf{b}$. Similar geometric quantities associated with deformed middle surface $\bar{\mathcal{M}} = \chi(\mathcal{M})$ are marked by a dash: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, etc. Other notation concerning the surface geometry follow that of [6, 16].

In this report we shall use mainly spatial bases $\mathbf{a}_a$ and $\bar{\mathbf{a}}_a$, $a = 1, 2, 3$, calculated at the shell middle surface, which give covariant components of the spatial metric tensor $a_{ab} = \mathbf{a}_a \cdot \mathbf{a}_b$ and $\bar{a}_{ab} = \bar{\mathbf{a}}_a \cdot \bar{\mathbf{a}}_b$ with determinants $a = |a_{ab}|$ and $\bar{a} = |\bar{a}_{ab}|$, respectively. Note that the base vector $\mathbf{a}_3 \equiv \mathbf{n}$ deforms into the base vector $\bar{\mathbf{a}}_3$ which, in general, is neither unit nor normal to the surface $\bar{\mathcal{M}}$, $\bar{\mathbf{a}}_3 \neq \bar{\mathbf{n}}$.

By expanding $\mathbf{v}$ and $\mathbf{F}$ into series with respect to ζ we obtain

$$\mathbf{v} = \mathbf{u} + \zeta\boldsymbol{\beta} + \cdots, \qquad \mathbf{F} = (\mathbf{G} - \zeta\bar{\boldsymbol{\lambda}}\mathbf{G} + \cdots)\mathbf{g}^{-1} \tag{4}$$

where the shell deformation gradient tensor $\mathbf{G}$ is defined by

$$\mathbf{G} = \mathbf{F}|_{\zeta=0} = \bar{\mathbf{a}}_a \otimes \mathbf{a}^a, \qquad \mathbf{G}^{-1} = \mathbf{a}_a \otimes \bar{\mathbf{a}}^a \tag{5}$$

and

$$\mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} = u_\alpha\mathbf{a}^\alpha + w\mathbf{n}, \qquad \boldsymbol{\beta} = \bar{\mathbf{a}}_3 - \mathbf{n} = \beta_\alpha\mathbf{a}^\alpha + \beta\mathbf{n}, \tag{6}$$

$$\bar{\boldsymbol{\lambda}} = -\bar{\mathbf{a}}_{3,\beta} \otimes \bar{\mathbf{a}}^\beta, \qquad \mathbf{g}^{-1} = \delta^b_j\mathbf{a}_b \otimes \mathbf{g}^j. \tag{7}$$

The Green strain tensor $\mathbf{E} = \frac{1}{2}(\mathbf{F}^{\mathrm{T}}\mathbf{F} - \mathbf{1})$ with (4) takes the form

$$\mathbf{E} = \mathbf{g}^{-1}\left[\boldsymbol{\gamma} + \zeta\tfrac{1}{2}(\boldsymbol{\pi} + \boldsymbol{\pi}^{\mathrm{T}}) + \zeta^2\boldsymbol{\mu} + \cdots\right]\mathbf{g}^{-1} \tag{8}$$

where the shell strain measures are defined by

$$\boldsymbol{\gamma} = \tfrac{1}{2}(\mathbf{G}^{\mathrm{T}}\mathbf{G} - \mathbf{1}), \qquad \boldsymbol{\pi} = -(\mathbf{G}^{\mathrm{T}}\bar{\boldsymbol{\lambda}}\mathbf{G} - \mathbf{b}), \qquad \boldsymbol{\mu} = \tfrac{1}{2}(\mathbf{G}^{\mathrm{T}}\bar{\boldsymbol{\lambda}}^{\mathrm{T}}\bar{\boldsymbol{\lambda}}\mathbf{G} - \mathbf{b}^2). \tag{9}$$

Within the linear approximation (4) the shell strain measures become exactly quadratic functions of **u** and $\boldsymbol{\beta}$ and their derivatives [12 ,15]

$$\begin{aligned}
2\gamma_{\alpha\beta} &= \varphi_{\alpha\beta} + \varphi_{\beta\alpha} + a^{\lambda\mu}\varphi_{\lambda\alpha}\varphi_{\mu\beta} + \varphi_{\alpha}\varphi_{\beta},\\
2\pi_{(\alpha\beta)} &= \psi_{\alpha\beta} + \psi_{\beta\alpha} - b^{\lambda}_{\alpha}\varphi_{\lambda\beta} - b^{\lambda}_{\beta}\varphi_{\lambda\alpha} + a^{\lambda\mu}(\varphi_{\lambda\alpha}\psi_{\mu\beta} + \varphi_{\lambda\beta}\psi_{\mu\alpha})\\
&\quad + \varphi_{\alpha}\psi_{\beta} + \varphi_{\beta}\psi_{\alpha},\\
2\mu_{\alpha\beta} &= a^{\lambda\mu}\psi_{\lambda\alpha}\psi_{\mu\beta} - b^{\lambda}_{\alpha}\psi_{\lambda\beta} - b^{\lambda}_{\beta}\psi_{\lambda\alpha} + \psi_{\alpha}\psi_{\beta},\\
2\gamma_{3\alpha} &= \varphi_{\alpha} + \beta_{\alpha} + a^{\lambda\mu}\varphi_{\lambda\alpha}\beta_{\mu} + \varphi_{\alpha}\beta,\\
\pi_{3\alpha} &= \psi_{\alpha} - b^{\lambda}_{\alpha}\beta_{\lambda} + a^{\lambda\mu}\psi_{\lambda\alpha}\beta_{\mu} + \psi_{\alpha}\beta,\\
2\gamma_{33} &= 2\beta + a^{\lambda\mu}\beta_{\lambda}\beta_{\mu} + \beta^{2}
\end{aligned} \tag{10}$$

where

$$\begin{aligned}
\mathbf{u}_{,\beta} &= \varphi_{\alpha\beta}\mathbf{a}^{\alpha} + \varphi_{\beta}\mathbf{n}, &\quad \boldsymbol{\beta}_{,\beta} &= \psi_{\alpha\beta}\mathbf{a}^{\alpha} + \psi_{\beta}\mathbf{n},\\
\varphi_{\alpha\beta} &= u_{\alpha|\beta} - b_{\alpha\beta}w, & \varphi_{\beta} &= w_{,\beta} + b^{\lambda}_{\beta}u_{\lambda},\\
\psi_{\alpha\beta} &= \beta_{\alpha|\beta} - b_{\alpha\beta}\beta, & \psi_{\beta} &= \beta_{,\beta} + b^{\lambda}_{\beta}\beta_{\lambda}
\end{aligned} \tag{11}$$

and $(\ )_{|\beta}$ denotes the surface covariant derivative in the reference surface metric $a_{\alpha\beta}$.

Within the linear approximation (4) we have $\pi_{3\alpha} = \gamma_{33,\alpha}$ and $\mu_{\alpha\beta}$ are expressible in terms of γ_{ab} and $\pi_{(\alpha\beta)}$ according to

$$\boldsymbol{\mu} = \tfrac{1}{2}\left[(\mathbf{b} - \boldsymbol{\pi}^{\mathrm{T}})(\mathbf{1} + 2\boldsymbol{\gamma})^{-1}(\mathbf{b} - \boldsymbol{\pi}) - \mathbf{b}^{2}\right]. \tag{12}$$

From geometrical considerations we obtain

$$\begin{aligned}
\bar{a}^{ad} &= \frac{1}{2}\frac{a}{\bar{a}}\varepsilon^{abc}\varepsilon^{def}(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}),\\
\frac{\bar{a}}{a} &= \frac{1}{6}\varepsilon^{abc}\varepsilon^{def}(a_{ad} + 2\gamma_{ad})(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}),\\
\bar{\varepsilon}_{abc} &= \sqrt{\frac{\bar{a}}{a}}\,\varepsilon_{abc}, \qquad \bar{\varepsilon}^{abc} = \sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{abc}
\end{aligned} \tag{13}$$

where ε_{abc} and ε^{abc} are components of the spatial permutation tensor calculated at $\mathfrak{M}$.

The vector $\bar{\mathbf{a}}_3$ can be related to $\bar{\mathbf{n}}$ by [18]

$$\bar{\mathbf{a}}_3 = 2\gamma_{3\beta}\bar{\mathbf{a}}^{\beta} + \sqrt{(1 + 2\gamma_{33})(1 - 2\bar{a}^{3\beta}\gamma_{3\beta})}\,\bar{\mathbf{n}}. \tag{14}$$

3. Finite rotation tensor and vector

According to (4) the shell deformation gradient tensor **G** defined in (5) provides a complete and exact information about the deformation of a neighbourhood of the shell middle surface particles. By applying the polar decomposition theorem [25] the tensor is represented by

$$\mathbf{G} = \mathbf{R}\mathbf{U} = \mathbf{V}\mathbf{R}, \qquad \mathbf{G}^{-1} = \mathbf{U}^{-1}\mathbf{R}^{\mathrm{T}} = \mathbf{R}^{\mathrm{T}}\mathbf{V}^{-1} \tag{15}$$

where **U** and **V** are the right and left stretch tensors, respectively, and **R** is a finite rotation tensor. Decomposition of **G** in terms of **U** is compatible with the Lagrangian description preferred in this work, in terms of **V** it is compatible with the Eulerian description.

By the formulae $(4)_1$ and $(15)_1$ the deformation of a neighbourhood about a particle of the shell middle surface is decomposed analytically into a rigid-body translation, a pure stretch along principal directions of **U** (or **V**) and a rigid body rotation of the principal directions.

From (5) and (15) we obtain

$$\begin{aligned} \bar{\mathbf{a}}_a &= \mathbf{G}\mathbf{a}_a = \mathbf{R}\check{\mathbf{a}}_a = \mathbf{V}\overset{*}{\mathbf{a}}_a, \\ \bar{\mathbf{a}}^a &= (\mathbf{G}^{-1})^{\mathrm{T}}\mathbf{a}^a = \mathbf{R}\check{\mathbf{a}}^a = \mathbf{V}^{-1}\overset{*}{\mathbf{a}}{}^a \end{aligned} \tag{16}$$

where two intermediate bases are defined by

$$\check{\mathbf{a}}_a = \mathbf{U}\mathbf{a}_a = \mathbf{R}^{\mathrm{T}}\bar{\mathbf{a}}_a, \qquad \overset{*}{\mathbf{a}}_a = \mathbf{R}\mathbf{a}_a = \mathbf{V}^{-1}\bar{\mathbf{a}}_a \tag{17}$$

Using (5) and (16) the following formulae are obtained

$$\begin{aligned} &\mathbf{U} = \check{\mathbf{a}}_a \otimes \mathbf{a}^a, \qquad \mathbf{V} = \bar{\mathbf{a}}_a \otimes \overset{*}{\mathbf{a}}{}^a, \\ &\mathbf{R} = \bar{\mathbf{a}}_a \otimes \check{\mathbf{a}}^a = \overset{*}{\mathbf{a}}_a \otimes \mathbf{a}^a \end{aligned} \tag{18}$$

In what follows it is convenient to introduce an extension tensor $\check{\boldsymbol{\gamma}}$ defined by

$$\check{\boldsymbol{\gamma}} = \mathbf{U} - \mathbf{1} = \sqrt{\mathbf{1} + 2\boldsymbol{\gamma}} - \mathbf{1} = \check{\gamma}_{ab}\mathbf{a}^a \otimes \mathbf{a}^b \tag{19}$$

in terms of which we have

$$\begin{aligned} &\check{\mathbf{a}}_a = (\delta_a^b + \check{\gamma}_a^b)\mathbf{a}_b, \qquad \gamma_{ab} = \left(\delta_a^c + \tfrac{1}{2}\check{\gamma}_a^c\right)\check{\gamma}_{cb}, \\ &\sqrt{\frac{\bar{a}}{a}} = \frac{1}{6}\varepsilon^{abc}\varepsilon_{def}(\delta_a^d + \check{\gamma}_a^d)(\delta_b^e + \check{\gamma}_b^e)(\delta_c^f + \check{\gamma}_c^f). \end{aligned} \tag{20}$$

Note that γ is exactly quadratic in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$, but many shell relations contain $\sqrt{(\bar{a}/a)}$ and therefore are non-rational in terms of γ_{ab}. On the other hand, $\check{\gamma}$ is defined to depend upon $\mathbf{u}$ and $\boldsymbol{\beta}$ through the non-rational relation (19), but $\sqrt{(\bar{a}/a)}$ in (20) and in the relations containing it become polynomials in $\check{\gamma}_{ab}$.

The symmetric tensor γ has three real eigenvalues γ_r in three orthogonal principal directions defined by a triad of unit vectors $\mathbf{h}_r$, which satisfy the set of equations $\gamma\mathbf{h}_r = \gamma_r\mathbf{h}_r$, $\not\Sigma r$. In the basis $\mathbf{h}_r$ the tensor γ takes the form

$$\gamma = \gamma_1\mathbf{h}_1 \otimes \mathbf{h}_1 + \gamma_2\mathbf{h}_2 \otimes \mathbf{h}_2 + \gamma_3\mathbf{h}_3 \otimes \mathbf{h}_3. \tag{21}$$

It follows from (10) that γ_{ab}, and therefore γ_r and $\mathbf{h}_r$, depend only upon $\mathbf{u}$ and $\boldsymbol{\beta}$ and the geometry of $\mathcal{M}$.

The symmetric tensors $\mathbf{U}$ and $\check{\gamma}$ are coaxial with γ and have therefore the same eigenvectors $\mathbf{h}_r$. Their eigenvalues are related by

$$\check{\gamma}_r = U_r - 1 = \sqrt{1 + 2\gamma_r} - 1. \tag{22}$$

The exact formula for $\mathbf{R}$ follows from (18), (6) and (20) to be

$$\mathbf{R} = \left[(\mathbf{a}_\alpha + \mathbf{u}_{,\alpha})\bar{a}^{\alpha b} + (\mathbf{n} + \boldsymbol{\beta})\bar{a}^{3b}\right] \otimes (\delta_b^c + \check{\gamma}_b^c)\mathbf{a}_c. \tag{23}$$

Since (23) involves $\check{\gamma}_b^c$, for unrestricted strains it is a non-rational expression in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$.

The proper orthogonal tensor $\mathbf{R}$ has one real eigenvalue equal to $+1$ and two complex conjugate eigenvalues $\cos\omega \pm i\sin\omega$. Let $\mathbf{e}$ be a unit vector satisfying $\mathbf{Re} = +\mathbf{e}$. If $\mathbf{e}_1 \perp \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e} \times \mathbf{e}_1$ are unit vectors of the remaining principal directions then [1, 15]

$$\begin{aligned} \mathbf{R} &= \cos\omega(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - \sin\omega(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) + \mathbf{e} \otimes \mathbf{e} \\ &= \cos\omega\mathbf{1} + \sin\omega\mathbf{S} + (1 - \cos\omega)\mathbf{e} \otimes \mathbf{e} \end{aligned} \tag{24}$$

where $\mathbf{S} = \mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2 = -\mathbf{S}^{\mathrm{T}}$ is a skew tensor whose axial vector is $\mathbf{e}$ and ω is the angle of rotation about the rotation axis defined by $\mathbf{e}$.

The parameters $\mathbf{e}$ and ω are defined uniquely by the tensor $\mathbf{R}$. If $R_{kl} = \mathbf{i}_k \cdot \mathbf{R}\mathbf{i}_l$ are components of $\mathbf{R}$ with respect to the Cartesian frame then [8]

$$\mathbf{e} = -\frac{e_{klm}R_{kl}}{2\sin\omega}\mathbf{i}_m, \qquad \cos\omega = \frac{1}{2}(R_{kk} - 1) \tag{25}$$

where e_{klm} is a permutation symbol. Since

$$\mathbf{r} = x^k\mathbf{i}_k, \qquad \mathbf{a}_\alpha = x^k_{,\alpha}\mathbf{i}_k, \qquad \mathbf{n} = \tfrac{1}{2}\varepsilon^{\alpha\beta}x^k_{,\alpha}x^l_{,\beta}e_{klm}\mathbf{i}_m \tag{26}$$

it follows from (23) that R_{kl} depend only upon the geometry of $\mathfrak{M}$ and the displacement parameters $\mathbf{u}$ and $\boldsymbol{\beta}$.

In what follows it is convenient to describe the rotational part of shell deformation by means of an equivalent finite rotation vector $\boldsymbol{\Omega}$. The direction of $\boldsymbol{\Omega}$ is defined by $\mathbf{e}$ and the magnitude is taken here to be $|\sin\omega|$. For $|\omega| < \pi$ we have

$$\boldsymbol{\Omega} = \sin\omega\mathbf{e}. \tag{27}$$

Note that $\boldsymbol{\Omega}$ defined in this way is not a vector in the usual sense. In particular, the rules of superposition of finite rotation vectors [9] are different from the usual addition rules of a linear vector space.

By using $\boldsymbol{\Omega}$ we obtain

$$\begin{aligned} \bar{\mathbf{a}}_a \equiv \mathbf{R}\check{\mathbf{a}}_a &= \check{\mathbf{a}}_a + \boldsymbol{\Omega} \times \check{\mathbf{a}}_a + \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \check{\mathbf{a}}_a) \\ &= \cos\omega\check{\mathbf{a}}_a + \boldsymbol{\Omega} \times \check{\mathbf{a}}_a + \frac{1}{2\cos^2\omega/2}(\boldsymbol{\Omega} \cdot \check{\mathbf{a}}_a)\boldsymbol{\Omega} \end{aligned} \tag{28}$$

which also shows how the finite rotation is accomplished by means of $\boldsymbol{\Omega}$.

The formulae (28) and $(25)_1$, expressing $\boldsymbol{\Omega}$ in terms of R_{kl} and therefore in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$, give its components with respect to the Cartesian basis $\mathbf{i}_k$. In theoretical considerations it is convenient to have $\boldsymbol{\Omega}$ expressed directly in the reference basis $\mathbf{a}_a$. Multiplying $(28)_2$ by $\bar{\varepsilon}^{abc}\check{\mathbf{a}}_b$ and making use of the symmetry we obtain [15, 21]

$$2\boldsymbol{\Omega} = \bar{\varepsilon}^{abc}(\bar{\mathbf{a}}_a \cdot \check{\mathbf{a}}_b)\check{\mathbf{a}}_c = \sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{abc}(\delta_b^e + \check{\gamma}_b^e)(\delta_c^f + \check{\gamma}_c^f)G_{ea}\mathbf{a}_f \tag{29}$$

where the components of $\mathbf{G}$ are defined by

$$G_{ab} = \mathbf{a}_a \cdot \mathbf{G}\mathbf{a}_b = \mathbf{a}_a \cdot \bar{\mathbf{a}}_b = \begin{bmatrix} l_{\alpha\beta}, & \beta_\alpha \\ \varphi_\beta, & 1+\beta \end{bmatrix}, \qquad l_{\alpha\beta} = a_{\alpha\beta} + \varphi_{\alpha\beta}. \tag{30}$$

By using (20) and some other polynomial identities involving $\check{\gamma}_{ab}$ the formula (29) is transformed [15] into an alternative simpler form

$$2\boldsymbol{\Omega} = \check{\mathbf{a}}_a \times \bar{\mathbf{a}}^a = \varepsilon_{def}(\delta_a^d + \check{\gamma}_a^d)\bar{a}^{ab}G^e_{.b}\mathbf{a}^f. \tag{31}$$

When written partly in terms of the surface quantities (31) leads to the following formula

$$2\boldsymbol{\Omega} = \varepsilon_{\lambda\mu}\left\{\left(\delta_a^3 + \check{\gamma}_a^3\right)\left[\bar{a}^{a\beta} l_{\cdot\beta}^{\lambda} + \bar{a}^{a3}\beta^{\lambda}\right] - \left(\delta_a^{\lambda} + \check{\gamma}_a^{\lambda}\right)\left[\bar{a}^{a\beta}\varphi_{\beta} + \bar{a}^{a3}(1+\beta)\right]\right\}\mathbf{a}^{\mu} + \varepsilon_{\lambda\mu}\left(\delta_a^{\lambda} + \check{\gamma}_a^{\lambda}\right)\left[\bar{a}^{a\beta} l_{\cdot\beta}^{\mu} + \bar{a}^{a3}\beta^{\mu}\right]\mathbf{n}. \tag{32}$$

According to (10)–(12) and (19) the exact formula (32) give us quite complicated non-rational expression for $\boldsymbol{\Omega}$ in terms of $\mathbf{u}$ and $\boldsymbol{\beta}$.

The rotation of the shell material fibres coinciding with principal directions of strains are described completely by $\mathbf{R}$ or $\boldsymbol{\Omega}$. Other shell fibres may suffer a rotation also during the pure stretch along principal directions of strain. Sometimes it is convenient to replace these two rotations by one equivalent total rotation. This approach is used in Section 5 to describe the total rotation of the shell boundary element.

4. Relations in terms of finite rotations

By using the finite rotation tensor $\mathbf{R}$ or the finite rotation vector $\boldsymbol{\Omega}$ it is possible to obtain many identities and geometrical relations [15] which are very useful in the non-linear shell theory.

From (28) it follows that

$$\begin{aligned} \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta^{\beta}} &= (\mathbf{R} - \mathbf{1})\mathbf{a}_{\beta} + \check{\gamma}_{\beta}^{a}\mathbf{R}\mathbf{a}_a \\ &= \check{\gamma}_{\beta}^{a}\mathbf{a}_a + \boldsymbol{\Omega}\times\check{\mathbf{a}}_{\beta} + \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\check{\mathbf{a}}_{\beta}), \\ \boldsymbol{\beta} &= (\mathbf{R} - \mathbf{1})\mathbf{n} + \check{\gamma}_3^{a}\mathbf{R}\mathbf{a}_a \\ &= \check{\gamma}_3^{a}\mathbf{a}_a + \boldsymbol{\Omega}\times\check{\mathbf{a}}_3 + \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\check{\mathbf{a}}_3). \end{aligned} \tag{33}$$

When $(6)_2$ and $(11)_1$ are used this leads to the formulae for $\varphi_{\alpha\beta}$, φ_{β}, β_{α} and β in terms of $\boldsymbol{\Omega}$ and $\check{\gamma}_{ab}$.

Differentiation of $\mathbf{R}$ and $\boldsymbol{\Omega}$ along convected coordinate lines of a three-dimensional continua was discussed by Shield [22] and Shamina [21], respectively. Describing rotations by $\boldsymbol{\Omega}$, at the reference shell middle surface we have

$$\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}\theta^{\beta}} = \cos\omega\,\mathbf{k}_{\beta} + \frac{1}{2}\boldsymbol{\Omega}\times\mathbf{k}_{\beta} - \frac{1}{4\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{k}_{\beta}). \tag{34}$$

The vector $\mathbf{k}_\beta$ is expressible entirely by means of the shell strain measures

$$\mathbf{k}_\beta = \bar{\varepsilon}^{aef}(\gamma_{e\beta;\,a} - A_{ea\beta})\check{\mathbf{a}}_f, \qquad A_{ea\beta} = \tfrac{1}{2}a^{gh}\check{\gamma}_{eg}\check{\gamma}_{ah;\,\beta} \tag{35}$$

where $(\ )_{;\,a}$ is the spatial covariant derivative calculated at $\mathfrak{M}$ by means of a_{ab}.

The integrability condition of the equations (34) can be expressed in terms of $\mathbf{k}_\beta$ by

$$\varepsilon^{\alpha\beta}\left(\mathbf{k}_{\beta|\alpha} + \tfrac{1}{2}\mathbf{k}_\alpha \times \mathbf{k}_\beta\right) = \mathbf{0}. \tag{36}$$

When (35) and (20) are introduced into (36) we obtain three compatibility conditions in terms of γ_{ab} and $\pi_{(\alpha\beta)}$. They assure the existence of displacement parameters $\mathbf{u}$ and $\boldsymbol{\beta}$ compatible with the two independent shell strain measures.

Since at $\mathfrak{M}$

$$\begin{aligned}
&\gamma_{\alpha\beta;\,3} = \pi_{(\alpha\beta)} + b^\kappa_\alpha\gamma_{\kappa\beta} + b^\kappa_\beta\gamma_{\kappa\alpha}, \qquad \gamma_{3\alpha;\,\beta} = \gamma_{3\alpha|\beta} + b^\kappa_\beta\gamma_{\kappa\alpha} - b_{\alpha\beta}\gamma_{33},\\
&\gamma_{\alpha\mu;\beta} = \gamma_{\alpha\mu|\beta} - b_{\alpha\beta}\gamma_{3\mu} - b_{\mu\beta}\gamma_{3\alpha}, \qquad \gamma_{33;\,\beta} = \gamma_{33,\,\beta} + 2b^\kappa_\beta\gamma_{\kappa 3}
\end{aligned} \tag{37}$$

we are able to solve (35) with respect to $\pi_{(\alpha\beta)}$ and obtain

$$\begin{aligned}
\pi_{(\alpha\beta)} = {}&\tfrac{1}{2}\left(\bar{\varepsilon}_{3\alpha\lambda}\mathbf{k}_\beta + \bar{\varepsilon}_{3\beta\lambda}\mathbf{k}_\alpha\right)\cdot\check{\mathbf{a}}^\lambda - \tfrac{1}{2}\left(b^\kappa_\alpha\gamma_{\kappa\beta} + b^\kappa_\beta\gamma_{\kappa\alpha}\right) - b_{\alpha\beta}\gamma_{33}\\
&+ \tfrac{1}{2}\left(\gamma_{3\alpha|\beta} + \gamma_{3\beta|\alpha}\right) + \tfrac{1}{2}\left(A_{\alpha3\beta} + A_{\beta3\alpha} - A_{3\alpha\beta} - A_{3\beta\alpha}\right).
\end{aligned} \tag{38}$$

By inverting (34) we can express $\mathbf{k}_\beta$ entirely in terms of the finite rotation vector

$$\mathbf{k}_\beta = \frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}\theta^\beta} + \frac{1}{2\cos^2\omega/2}\,\frac{\mathrm{d}\boldsymbol{\Omega}}{\mathrm{d}\theta^\beta}\times\boldsymbol{\Omega} + \frac{\mathrm{d}\omega}{\mathrm{d}\theta^\beta}\,tg\ \omega/2\ \boldsymbol{\Omega} \tag{39}$$

which together with (38) leads to an exact formula for $\pi_{(\alpha\beta)}$ in terms of finite rotations and strains.

5. Deformation of a shell boundary

Let $\mathcal{C}$ be a boundary curve at $\mathfrak{M}$ defined by $\theta^\alpha = \theta^\alpha(s)$ where s is the length parameter of $\mathcal{C}$. We assume, that in the reference configuration the lateral shell boundary surface $\partial\mathcal{P}$ is rectilinear and orthogonal to $\mathfrak{M}$

along $\mathcal{C}$. The position vector of any $P \in \partial\mathcal{P}$ is given by

$$\mathbf{p} = \mathbf{p}(s, \zeta) = \mathbf{r}(s) + \zeta\mathbf{n}(s). \tag{40}$$

With each $M \in \mathcal{C}$ we associate vectors: $\mathbf{t} = \mathrm{d}\mathbf{r}/\mathrm{d}s$, the unit tangent to $\mathcal{C}$, and $\boldsymbol{\nu} = \mathbf{t} \times \mathbf{n}$, the outward unit normal.

After the shell deformation $\partial\mathcal{P}$ is transformed into a surface $\partial\bar{\mathcal{P}}$ which, in general, is neither rectilinear nor orthogonal to $\bar{\mathcal{M}}$ along $\bar{\mathcal{C}}$. In the neighbourhood of $\bar{\mathcal{C}}$ we have the following expansion for the position vector of $\bar{P} \in \partial\bar{\mathcal{P}}$

$$\bar{\mathbf{p}} = \bar{\mathbf{p}}(s, \zeta) = \bar{\mathbf{r}}(s) + \zeta\bar{\mathbf{a}}_3(s) + \cdots \tag{41}$$

which allows us to use the rectilinear approximation to $\partial\bar{\mathcal{P}}$ in the neighbourhood of $\bar{\mathcal{C}}$.

During the shell deformation the orthonormal triad $\mathbf{t}, \mathbf{n}, \boldsymbol{\nu}$ transforms into a skew triad

$$\begin{aligned} \bar{\mathbf{a}}_t &= \frac{\mathrm{d}\bar{\mathbf{r}}}{\mathrm{d}s} = \bar{\mathbf{a}}_\alpha t^\alpha, \qquad \bar{\mathbf{a}}_3 = \mathbf{n} + \boldsymbol{\beta}, \\ \bar{\mathbf{a}}_\nu &= \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 = \sqrt{\frac{\bar{a}}{a}}\ \bar{\mathbf{a}}^\alpha \nu_\alpha. \end{aligned} \tag{42}$$

The lengths of the vectors are

$$\begin{aligned} \bar{a}_t &= |\bar{\mathbf{a}}_t| = \sqrt{1 + 2\gamma_{tt}}\ , \qquad \bar{a}_3 = |\bar{\mathbf{a}}_3| = \sqrt{1 + 2\gamma_{33}}\ , \\ \bar{a}_\nu &= |\bar{\mathbf{a}}_\nu| = \sqrt{(1 + 2\gamma_{tt})(1 + 2\gamma_{33}) - 4\gamma_{3t}^2} \end{aligned} \tag{43}$$

where $\gamma_{tt} = \gamma_{\alpha\beta}t^\alpha t^\beta$, $\gamma_{3t} = \gamma_{3\alpha}t^\alpha$ and γ_{33} are physical components of strain at the boundary.

In what follows it is convenient to introduce a vector

$$\bar{\mathbf{a}}_m = \bar{\mathbf{a}}_\nu \times \bar{\mathbf{a}}_t = \bar{a}_t^2\bar{\mathbf{a}}_3 - 2\gamma_{3t}\bar{\mathbf{a}}_t, \qquad \bar{a}_m = |\bar{\mathbf{a}}_m| = \bar{a}_\nu\bar{a}_t \tag{44}$$

which together with $\bar{\mathbf{a}}_\nu$ and $\bar{\mathbf{a}}_t$ forms an orthogonal triad along $\bar{\mathcal{C}}$.

From the polar decomposition of $\mathbf{G}$ it follows that there are intermediate vectors $\check{\mathbf{a}}_\nu$, $\check{\mathbf{a}}_t$, $\check{\mathbf{a}}_3$ and $\check{\mathbf{a}}_m$ such that formulae (28) hold, for example

$$\bar{\mathbf{a}}_t = \mathbf{R}\check{\mathbf{a}}_t = \check{\mathbf{a}}_t + \boldsymbol{\Omega} \times \check{\mathbf{a}}_t + \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \check{\mathbf{a}}_t). \tag{45}$$

The vectors follow from $(20)_1$ to be

$$\begin{aligned} \check{\mathbf{a}}_t &= \check{\gamma}_{\nu t}\boldsymbol{\nu} + (1 + \check{\gamma}_{tt})\mathbf{t} + \check{\gamma}_{3t}\mathbf{n}, \\ \check{\mathbf{a}}_3 &= \check{\gamma}_{3\nu}\boldsymbol{\nu} + \check{\gamma}_{3t}\mathbf{t} + (1 + \check{\gamma}_{33})\mathbf{n}, \\ \check{\mathbf{a}}_\nu &= \check{\mathbf{a}}_t \times \check{\mathbf{a}}_3, \qquad \check{\mathbf{a}}_m = \check{\mathbf{a}}_\nu \times \check{\mathbf{a}}_t. \end{aligned} \tag{46}$$

Since $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ do not coincide, in general, with principal directions of strain, the transformation of $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ into $\check{\mathbf{a}}_\nu$, $\check{\mathbf{a}}_t$, $\check{\mathbf{a}}_m$ consists of extensions by appropriate factors (43) and $(44)_2$ and of a finite rotation performed by an orthogonal tensor $\check{\mathbf{R}}_t$ or a finite rotation vector $\check{\boldsymbol{\Omega}}_t = \check{\mathbf{e}}_t \sin \check{\omega}_t$. They are computed with the help of (46) from the formulae analogous to (28) and (31):

$$\begin{aligned} \check{\mathbf{R}}_t &= \check{\boldsymbol{\nu}} \otimes \boldsymbol{\nu} + \check{\mathbf{t}} \otimes \mathbf{t} + \check{\mathbf{m}} \otimes \mathbf{n}, \\ 2\check{\boldsymbol{\Omega}}_t &= \boldsymbol{\nu} \times \check{\boldsymbol{\nu}} + \mathbf{t} \times \check{\mathbf{t}} + \mathbf{n} \times \check{\mathbf{m}} \end{aligned} \tag{47}$$

where

$$\check{\boldsymbol{\nu}} = \frac{\check{\mathbf{a}}_\nu}{\bar{a}_\nu}, \qquad \check{\mathbf{t}} = \frac{\check{\mathbf{a}}_t}{\bar{a}_t}, \qquad \check{\mathbf{m}} = \frac{\check{\mathbf{a}}_m}{\bar{a}_m}. \tag{48}$$

Therefore, for a typical vector $\check{\mathbf{a}}_t$ we obtain, for example,

$$\check{\mathbf{a}}_t = \bar{a}_t \check{\mathbf{R}}_t \mathbf{t} = \bar{a}_t \left[\mathbf{t} + \check{\boldsymbol{\Omega}}_t \times \mathbf{t} + \frac{1}{2\cos^2 \check{\omega}_t/2} \check{\boldsymbol{\Omega}}_t \times (\check{\boldsymbol{\Omega}}_t \times \mathbf{t}) \right]. \tag{49}$$

It is convenient to replace the two successive rotations performed by $\check{\mathbf{R}}_t$ or $\check{\boldsymbol{\Omega}}_t$ and $\mathbf{R}$ or $\boldsymbol{\Omega}$ by a single equivalent rotation performed by a total finite rotation tensor $\mathbf{R}_t = \mathbf{R}\check{\mathbf{R}}_t$ or a total finite rotation vector $\boldsymbol{\Omega}_t = \sin \omega_t \mathbf{e}_t$. Applying the superposition rule of finite rotation vectors [9] for $\boldsymbol{\Omega}_t$ we obtain

$$\boldsymbol{\Omega}_t = \left(1 - \frac{\check{\boldsymbol{\Omega}}_t \cdot \boldsymbol{\Omega}}{4\cos^2 \frac{1}{2}\check{\omega}_t \cos^2 \frac{1}{2}\omega} \right) \left[\cos^2 \tfrac{1}{2}\omega \check{\boldsymbol{\Omega}}_t + \cos^2 \tfrac{1}{2}\check{\omega}_t \boldsymbol{\Omega} + \tfrac{1}{2}\boldsymbol{\Omega} \times \check{\boldsymbol{\Omega}}_t \right]. \tag{50}$$

Therefore, the transformation $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ into $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_t$, $\bar{\mathbf{a}}_m$ takes the final form

$$\begin{bmatrix} \bar{\mathbf{a}}_\nu \\ \bar{\mathbf{a}}_t \\ \bar{\mathbf{a}}_m \end{bmatrix} = \mathbf{R}_t \begin{bmatrix} \bar{a}_\nu \boldsymbol{\nu} \\ \bar{a}_t \mathbf{t} \\ \bar{a}_m \mathbf{n} \end{bmatrix} = \begin{bmatrix} \bar{a}_\nu \boldsymbol{\nu} \\ \bar{a}_t \mathbf{t} \\ \bar{a}_m \mathbf{n} \end{bmatrix} + \boldsymbol{\Omega}_t \times \begin{bmatrix} \bar{a}_\nu \boldsymbol{\nu} \\ \bar{a}_t \mathbf{t} \\ \bar{a}_m \mathbf{n} \end{bmatrix}$$

$$+ \frac{1}{2\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times \left(\boldsymbol{\Omega}_t \times \begin{bmatrix} \bar{a}_\nu \boldsymbol{\nu} \\ \bar{a}_t \mathbf{t} \\ \bar{a}_m \mathbf{n} \end{bmatrix} \right). \tag{51}$$

From $(44)_1$ it follows that

$$\bar{\mathbf{a}}_3 = \frac{1}{\bar{a}_t^2} (2\gamma_{3t} \bar{\mathbf{a}}_t + \bar{\mathbf{a}}_m) \tag{52}$$

which together with (51) gives an exact relation for $\bar{\mathbf{a}}_3$ as well.

Let us remind differentiation rules of $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ along $\mathcal{C}$ and along a curve $\mathcal{C}_\nu$ orthogonal to $\mathcal{C}$, whose length parameter is s_ν, [2, 16]

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{bmatrix} \boldsymbol{\nu} \\ \mathbf{t} \\ \mathbf{n} \end{bmatrix} = \boldsymbol{\omega}_t \times \begin{bmatrix} \boldsymbol{\nu} \\ \mathbf{t} \\ \mathbf{n} \end{bmatrix}, \qquad \frac{\mathrm{d}}{\mathrm{d}s_\nu} \begin{bmatrix} \boldsymbol{\nu} \\ \mathbf{t} \\ \mathbf{n} \end{bmatrix} = -\boldsymbol{\omega}_\nu \times \begin{bmatrix} \boldsymbol{\nu} \\ \mathbf{t} \\ \mathbf{n} \end{bmatrix}, \tag{53}$$

$$\boldsymbol{\omega}_t = \sigma_t \boldsymbol{\nu} + \tau_t \mathbf{t} + \kappa_t \mathbf{n}, \qquad \boldsymbol{\omega}_\nu = \tau_\nu \boldsymbol{\nu} + \sigma_\nu \mathbf{t} + \kappa_\nu \mathbf{n}$$

where

$$\sigma_t = b_{\alpha\beta} t^\alpha t^\beta, \qquad \tau_t = -b_{\alpha\beta} \nu^\alpha t^\beta, \qquad \kappa_t = t_\alpha \nu^\alpha|_\beta t^\beta,$$

$$\sigma_\nu = b_{\alpha\beta} \nu^\alpha \nu^\beta, \qquad \tau_\nu = \tau_t, \qquad \kappa_\nu = \nu_\alpha t^\alpha|_\beta \nu^\beta. \tag{54}$$

Here σ_t is the normal curvature, τ_t is the geodesic torsion and κ_t is the geodesic curvature of $\mathcal{C}$, respectively, and σ_ν, τ_ν, κ_ν are similar parameters of $\mathcal{C}_\nu$.

Differentiation rules of $\bar{\boldsymbol{\nu}} = \bar{\mathbf{a}}_\nu / \bar{a}_\nu$, $\bar{\mathbf{t}} = \bar{\mathbf{a}}_t / \bar{a}_t$, $\bar{\mathbf{m}} = \bar{\mathbf{a}}_m / \bar{a}_m$ along $\bar{\mathcal{C}}$ it is convenient to construct in a form similar to that of (53). Since $\mathrm{d}\bar{s} = \bar{a}_t \, \mathrm{d}s$, where $\bar{s}$ is a length parameter along $\bar{\mathcal{C}}$, let us assume that

$$\frac{\mathrm{d}}{\mathrm{d}s} \begin{bmatrix} \bar{\boldsymbol{\nu}} \\ \bar{\mathbf{t}} \\ \bar{\mathbf{m}} \end{bmatrix} = \bar{\boldsymbol{\omega}}_t \times \begin{bmatrix} \bar{\boldsymbol{\nu}} \\ \bar{\mathbf{t}} \\ \bar{\mathbf{m}} \end{bmatrix}, \qquad \bar{\boldsymbol{\omega}}_t = \bar{a}_t (\bar{\sigma}_t \bar{\boldsymbol{\nu}} + \bar{\tau}_t \bar{\mathbf{t}} + \bar{\kappa}_t \bar{\mathbf{m}}). \tag{55}$$

Note that $\bar{\mathbf{m}}$ as defined here is not orthogonal to $\bar{\mathcal{M}}$ and $\bar{\boldsymbol{\nu}}$ does not rest on a plane tangent to $\bar{\mathcal{M}}$. Therefore, the parameters $\bar{\sigma}_t$, $\bar{\tau}_t$, $\bar{\kappa}_t$ cannot be related directly to the geometry of $\overline{\mathfrak{M}}$, although they do describe the curvature properties of the boundary curve $\bar{\mathcal{C}}$. This makes a significant difference when compared with analogous relations at the shell boundary deformed under K–L constraints, which were discussed in [11, 16].

Differentiating the identity $\bar{\mathbf{m}} \cdot \boldsymbol{\nu} - \bar{\boldsymbol{\nu}} \cdot \mathbf{n} = 2\boldsymbol{\Omega}_t \cdot \mathbf{t}$ and introducing the vector $\check{\boldsymbol{\omega}}_t$ such that

$$\begin{aligned} \check{\boldsymbol{\omega}}_t &= \bar{a}_t(\bar{\sigma}_t \boldsymbol{\nu} + \bar{\tau}_t \mathbf{t} + \bar{\kappa}_t \mathbf{n}), \\ \bar{\boldsymbol{\omega}}_t &= \check{\boldsymbol{\omega}}_t + \boldsymbol{\Omega}_t \times \check{\boldsymbol{\omega}}_t + \frac{1}{2\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \check{\boldsymbol{\omega}}_t) \end{aligned} \tag{56}$$

after involved transformations (see [15]) we obtain the formula analogous to (34) for differentiation of the total finite rotation vector

$$\frac{\mathrm{d}\boldsymbol{\Omega}_t}{\mathrm{d}s} = \cos \omega_t \mathbf{k}_t + \frac{1}{2}\boldsymbol{\Omega}_t \times \mathbf{k}_t - \frac{1}{4\cos^2 \omega_t/2} \boldsymbol{\Omega}_t \times (\boldsymbol{\Omega}_t \times \mathbf{k}_t). \tag{57}$$

Here $\mathbf{k}_t$ defined by

$$\begin{aligned} \mathbf{k}_t &= \check{\boldsymbol{\omega}}_t - \boldsymbol{\omega}_t = -k_{tt}\boldsymbol{\nu} + k_{\nu t}\mathbf{t} - k_{nt}\mathbf{n}, \\ -k_{tt} &= \bar{a}_t\bar{\sigma}_t - \sigma_t, \qquad k_{\nu t} = \bar{a}_t\bar{\tau}_t - \tau_t, \qquad -k_{nt} = \bar{a}_t\bar{\kappa}_t - \kappa_t \end{aligned} \tag{58}$$

is the vector of change of curvature of the shell boundary contour.

In order to calculate (58), let us differentiate $\bar{\mathbf{a}}_t$, $\bar{\mathbf{a}}_3$ and $\bar{\mathbf{a}}_\nu$ to obtain

$$\begin{aligned} \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s} &= \bar{\mathbf{a}}_{\alpha,\beta} t^\alpha t^\beta + \bar{\mathbf{a}}_\alpha t^\alpha_{,\beta} t^\beta = t^\alpha|_\beta t^\beta \bar{\mathbf{a}}_\alpha + \sigma_t \bar{\mathbf{a}}_3 + \bar{a}^{dc}\gamma_{c\alpha\beta} t^\alpha t^\beta \bar{\mathbf{a}}_d, \\ \frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s} &= \bar{\mathbf{a}}_{3,\beta} t^\beta = -b^\lambda_\beta t^\beta \bar{\mathbf{a}}_\lambda + \gamma_{c3\beta} t^\beta \bar{\mathbf{a}}^c, \\ \gamma_{cab} &= \gamma_{ca;b} + \gamma_{cb;a} - \gamma_{ab;c}, \\ \frac{\mathrm{d}\bar{\mathbf{a}}_\nu}{\mathrm{d}s} &= \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s} \times \bar{\mathbf{a}}_3 + \bar{\mathbf{a}}_t \times \frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s} \\ &= -\sqrt{\frac{\bar{a}}{a}}\, \epsilon_{\lambda\mu}(t^\lambda|_\beta + \bar{a}^{\lambda c}\gamma_{c\alpha\beta} t^\alpha) t^\beta \bar{\mathbf{a}}^\mu - \sqrt{\frac{\bar{a}}{a}}\, \tau_t \bar{\mathbf{a}}^3 \\ &\quad + \sqrt{\frac{\bar{a}}{a}}\, \nu_\lambda(\bar{a}^{3c}\bar{\mathbf{a}}^\lambda - \bar{a}^{\lambda c}\bar{\mathbf{a}}^3)\gamma_{c3\beta} t^\beta \end{aligned} \tag{59}$$

and from $(44)_1$ we also get

$$\frac{\mathrm{d}\bar{\mathbf{a}}_m}{\mathrm{d}s} = 2\frac{\mathrm{d}\gamma_{tt}}{\mathrm{d}s}\bar{\mathbf{a}}_3 + \bar{a}_t^2 \frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s} - 2\frac{\mathrm{d}\gamma_{3t}}{\mathrm{d}s}\bar{\mathbf{a}}_t - 2\gamma_{3t}\frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s}. \tag{60}$$

It follows from (55), that the curvature parameters of $\bar{\mathcal{C}}$ are defined by

$$
\begin{aligned}
\bar{a}_m \bar{a}_t^2 \bar{\sigma}_t &= \bar{\mathbf{a}}_m \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s} = -\bar{\mathbf{a}}_t \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_m}{\mathrm{d}s},\\
\bar{a}_\nu \bar{a}_t \bar{a}_m \bar{\tau}_t &= \bar{\mathbf{a}}_\nu \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_m}{\mathrm{d}s} = -\bar{\mathbf{a}}_m \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_\nu}{\mathrm{d}s},\\
\bar{a}_\nu \bar{a}_t^2 \bar{\kappa}_t &= \bar{\mathbf{a}}_t \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_\nu}{\mathrm{d}s} = -\bar{\mathbf{a}}_\nu \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s}.
\end{aligned}
\tag{61}
$$

Combining (58)–(61) the following formulae for components of $\mathbf{k}_t$ are obtained

$$
\begin{aligned}
-k_{tt} &= \frac{1}{\bar{a}_t \bar{a}_m}\left[\bar{a}_t^2\left(2\frac{\mathrm{d}\gamma_{3t}}{\mathrm{d}s} + \sigma_t - \pi_{tt}\right) - 2\gamma_{3t}\frac{\mathrm{d}\gamma_{tt}}{\mathrm{d}s}\right] - \sigma_t,\\
k_{\nu t} &= \frac{1}{\bar{a}_\nu \bar{a}_m}\sqrt{\frac{\bar{a}}{a}}\ \Big[\bar{a}_t^2\left(\tau_t + \nu_\lambda \bar{a}^{\lambda c}\gamma_{c3\beta} t^\beta\right)\\
&\quad + 2\gamma_{3t}\left(\kappa_t - \nu_\lambda \bar{a}^{\lambda c}\gamma_{c\alpha\beta} t^\alpha t^\beta\right)\Big] - \tau_t,\\
-k_{nt} &= \frac{1}{\bar{a}_\nu \bar{a}_t}\sqrt{\frac{\bar{a}}{a}}\ \left(\kappa_t - \nu_\lambda \bar{a}^{\lambda c}\gamma_{c\alpha\beta} t^\alpha t^\beta\right) - \kappa_t.
\end{aligned}
\tag{62}
$$

In order to have expressed $(62)_{2,3}$ also by means of physical components of strain measures the following extended formulae [15] may be used in (61) and (58):

$$
\begin{aligned}
\bar{\mathbf{a}}_\nu \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s} &= \sqrt{\frac{\bar{a}}{a}}\ \tau_t + \sqrt{\frac{a}{\bar{a}}}\ \Bigg\{\Bigg[\frac{\mathrm{d}\gamma_{3\nu}}{\mathrm{d}s} - \frac{\mathrm{d}\gamma_{3t}}{\mathrm{d}s_\nu} + \pi_{(\nu t)} + \kappa_\nu \gamma_{3\nu}\\
&\quad + 2\sigma_t \gamma_{\nu t} - 2\tau_t \gamma_{\nu\nu} - \kappa_t \gamma_{3t}\Bigg]\left[(1 + 2\gamma_{tt})(1 + 2\gamma_{33}) - 4\gamma_{3t}^2\right]\\
&\quad - \left(\frac{\mathrm{d}\gamma_{33}}{\mathrm{d}s} + 2\sigma_t \gamma_{3t} - 2\tau_t \gamma_{3\nu}\right)\left[2\gamma_{3\nu}(1 + 2\gamma_{tt}) - 4\gamma_{\nu t}\gamma_{3t}\right]\\
&\quad - \left(\pi_{tt} + 2\sigma_t \gamma_{tt} - 2\tau_t \gamma_{\nu t}\right)\left[2\gamma_{\nu t}(1 + 2\gamma_{33}) - 4\gamma_{3\nu}\gamma_{3t}\right]\Bigg\},
\end{aligned}
\tag{63}
$$

$$\bar{\mathbf{a}}_\nu \cdot \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s} = -\sqrt{\frac{\bar{a}}{a}}\,\kappa_t - \sqrt{\frac{a}{\bar{a}}}\left\{\left[\frac{\mathrm{d}\gamma_{tt}}{\mathrm{d}s_\nu} - 2\frac{\mathrm{d}\gamma_{\nu t}}{\mathrm{d}s} - 2\kappa_\nu\gamma_{\nu t}\right.\right.$$

$$\left. + 2\sigma_t\gamma_{3\nu} + 2\kappa_t(\gamma_{tt} - \gamma_{\nu\nu})\right]\left[(1 + 2\gamma_{tt})(1 + 2\gamma_{33}) - 4\gamma_{3t}^2\right]$$

$$+ \left(2\frac{\mathrm{d}\gamma_{3t}}{\mathrm{d}s} - \pi_{tt} - 2\sigma_t\gamma_{33} + 2\kappa_t\gamma_{3\nu}\right)$$

$$\times\left[2\gamma_{3\nu}(1 + 2\gamma_{tt}) - 4\gamma_{\nu t}\gamma_{3t}\right]$$

$$+ \left(\frac{\mathrm{d}\gamma_{tt}}{\mathrm{d}s} - 2\sigma_t\gamma_{3t} + 2\kappa_t\gamma_{\nu t}\right)$$

$$\left.\times\left[2\gamma_{\nu t}(1 + 2\gamma_{33}) - 4\gamma_{3\nu}\gamma_{3t}\right]\right\}$$

where

$$\begin{aligned}\frac{\bar{a}}{a} = {} & 1 + 2(\gamma_{\nu\nu} + \gamma_{tt} + \gamma_{33}) \\ & + 4(\gamma_{\nu\nu}\gamma_{tt} + \gamma_{\nu\nu}\gamma_{33} + \gamma_{tt}\gamma_{33} - \gamma_{\nu t}^2 - \gamma_{3\nu}^2 - \gamma_{3t}^2) \\ & + 8(\gamma_{\nu\nu}\gamma_{tt}\gamma_{33} + 2\gamma_{\nu t}\gamma_{3\nu}\gamma_{3t} - \gamma_{\nu\nu}\gamma_{3t}^2 - \gamma_{tt}\gamma_{3\nu}^2 - \gamma_{33}\gamma_{\nu t}^2).\end{aligned} \tag{64}$$

The relations (62) show that the vector $\mathbf{k}_t$ is expressible entirely by means of the shell strain measures at the boundary.

According to (41) and (6) a rectilinear surface $\bar{\mathbf{p}} = \bar{\mathbf{p}}(s, \zeta)$ tangent to $\partial\bar{\mathcal{P}}$ at $\bar{\mathcal{C}}$ is uniquely defined if we assume two vector functions

$$\mathbf{u} = \mathbf{u}^*(s), \qquad \boldsymbol{\beta} = \boldsymbol{\beta}^*(s) \quad \text{at } \mathcal{C}. \tag{65}$$

This gives a general form of the displacemental boundary conditions in the non-linear theory of shells.

The same rectilinear surface may be defined implicitly, with an accuracy up to a constant translation in space, by differential equations

$$\frac{\partial\bar{\mathbf{p}}}{\mathrm{d}s} = \bar{\mathbf{a}}_t + \zeta\frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s}, \qquad \frac{\partial\bar{\mathbf{p}}}{\partial\zeta} = \bar{\mathbf{a}}_3, \qquad \frac{\mathrm{d}\bar{\mathbf{r}}}{\mathrm{d}s} = \bar{\mathbf{a}}_t. \tag{66}$$

According to (51) and (52) the equations are uniquely established if $\boldsymbol{\Omega}_t$ and

$\gamma_{tt}, \gamma_{3t}, \gamma_{33}$ are known along $\mathcal{C}$. Therefore, within the general non-linear theory of shells the kinematical boundary conditions have the following form

$$\begin{aligned} &\mathbf{\Omega}_t = \mathbf{\Omega}_t^*(s), \\ &\gamma_{tt} = \gamma_{tt}^*(s), \qquad \gamma_{3t} = \gamma_{3t}^*(s), \qquad \gamma_{33} = \gamma_{33}^*(s) \quad \text{at } \mathcal{C}. \end{aligned} \tag{67}$$

Let us differentiate again the equations (66) to obtain

$$\frac{\partial^2 \bar{\mathbf{p}}}{\partial s^2} = \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s} + \zeta \frac{\mathrm{d}^2 \bar{\mathbf{a}}_3}{\mathrm{d}s^2}, \qquad \frac{\partial^2 \bar{\mathbf{p}}}{\partial s \partial \zeta} = \frac{\mathrm{d}\bar{\mathbf{a}}_3}{\mathrm{d}s}, \qquad \frac{\mathrm{d}^2 \bar{\mathbf{r}}}{\mathrm{d}s^2} = \frac{\mathrm{d}\bar{\mathbf{a}}_t}{\mathrm{d}s}. \tag{68}$$

These differential equations also define implicitly the same rectilinear surface $\bar{\mathbf{p}} = \bar{\mathbf{p}}(s, \zeta)$ tangent to $\partial\bar{\mathcal{P}}$ at $\bar{\mathcal{C}}$, with accuracy up to a translation in space linearly varying with s (that is, up to a rigid-body motion). The equations (68) are uniquely established if the values of $\mathrm{d}\bar{\mathbf{a}}_t/\mathrm{d}s$ and $\mathrm{d}\bar{\mathbf{a}}_3/\mathrm{d}s$ are known along $\mathcal{C}$. According to (55), (58) and (62) these vectors are expressible in terms of $\mathbf{k}_t$ and $\gamma_{tt}, \gamma_{3t}, \gamma_{33}$ which are sufficient for establishing (68). Therefore, within the general non-linear theory of shells the deformational boundary conditions take the following form

$$\begin{aligned} &\mathbf{k}_t = \mathbf{k}_t^*(s), \\ &\gamma_{tt} = \gamma_{tt}^*(s), \qquad \gamma_{3t} = \gamma_{3t}^*(s), \qquad \gamma_{33} = \gamma_{33}^*(s) \quad \text{at } \mathcal{C}. \end{aligned} \tag{69}$$

When values of $\mathbf{u}$ and $\boldsymbol{\beta}$ are known along $\mathcal{C}$, the values of $\mathbf{\Omega}_t, \gamma_{tt}, \gamma_{3t}, \gamma_{33}$ and $\mathbf{k}_t$ can easily be calculated by differentiation procedures (see [15]). However, if only values of $\mathbf{k}_t$ and $\gamma_{tt}, \gamma_{3t}, \gamma_{33}$ along $\mathcal{C}$ are known in advance, in order to obtain values of $\mathbf{\Omega}_t$ the differential equation (57) should be solved. The structure of (57) is analogous to that describing the motion of a rigid body about a fixed point [9] and the methods of solutions developed in analytical mechanics may be of assistance in calculating $\mathbf{\Omega}_t$ from the known $\mathbf{k}_t$.

It follows from (42) and (52) that

$$\mathbf{u}(s) = \mathbf{u}_0 + \int_{M_0}^{M} (\bar{\mathbf{a}}_t - \mathbf{t})\, \mathrm{d}s, \qquad \boldsymbol{\beta}(s) = \frac{1}{\bar{a}_t^2}(2\gamma_{3t}\bar{\mathbf{a}}_t + \bar{\mathbf{a}}_m) - \mathbf{n}. \tag{70}$$

Therefore, if only values of $\mathbf{\Omega}_t$ and $\gamma_{tt}, \gamma_{3t}, \gamma_{33}$ are known along $\mathcal{C}$, (51) and (70) give the values for $\mathbf{u}$ and $\boldsymbol{\beta}$ as well.

6. Some simplified results

All the formulae presented in Section 2–5 have been obtained by taking into account the linear terms in the expansions (4). The theory of finite rotations in shells developed there is three-dimensionally exact at the shell middle surface, since higher-order terms of the expansions (4) cannot affect the change in slope of tangents to material fibres calculated at the shell middle surface.

Here the main results of Section 2–5 are simplified by using additional assumptions. Some of the simplified formulae are related to those already known in the literature.

6.1. Kirchhoff–Love theory

According to K–L constraints, the material fibres that were orthogonal to the reference shell middle surface $\mathcal{M}$, after the shell deformation remain orthogonal to the deformed surface $\bar{\mathcal{M}}$ and do not change their lengths. Therefore we assume $\bar{\mathbf{a}}_3 \equiv \bar{\mathbf{n}}$.

Although the results obtained under K–L constraints are meaningless within a large-strain shell theory (since the first-order effect due to change in the shell thickness is ignored), they are quite useful as a starting point for further simplifications towards the first-approximation geometrically non-linear theory of thin isotropic and elastic shells.

Under K–L constraints we have [13, 14, 16]

$$\bar{\mathbf{a}}_\alpha = l^{\lambda}_{\cdot\alpha}\mathbf{a}_\lambda + \varphi_\alpha \mathbf{n}, \qquad \bar{\mathbf{n}} = n_\alpha \mathbf{a}^\alpha + n\mathbf{n},$$

$$n_\mu = \sqrt{\frac{a}{\bar{a}}}\ \varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\varphi_\alpha l^{\lambda}_{\cdot\beta}, \qquad n = \frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\ \varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu} l^{\lambda}_{\cdot\alpha} l^{\mu}_{\cdot\beta}, \tag{71}$$

$$\mathbf{G} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \qquad \bar{\boldsymbol{\lambda}} = \bar{\mathbf{b}}, \qquad \boldsymbol{\beta} = \bar{\mathbf{n}} - \mathbf{n},$$

$$\gamma_{\alpha\beta} = \frac{1}{2}\left(\bar{a}_{\alpha\beta} - a_{\alpha\beta}\right) = \frac{1}{2}\left(l^{\lambda}_{\cdot\alpha} l_{\lambda\beta} + \varphi_\alpha\varphi_\beta - a_{\alpha\beta}\right),$$

$$\begin{aligned}\pi_{(\alpha\beta)} \equiv \kappa_{\alpha\beta} &= -\left(\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right)\\ &= -\left[n\left(\varphi_{\alpha|\beta} + b^{\lambda}_{\beta} l_{\lambda\alpha}\right) + n_\lambda\left(l^{\lambda}_{\cdot\alpha|\beta} - b^{\lambda}_{\beta}\varphi_\alpha\right) - b_{\alpha\beta}\right],\end{aligned} \tag{72}$$

$$\mu_{\alpha\beta} \equiv \nu_{\alpha\beta} = \frac{1}{2}\left(\bar{b}^{\lambda}_{\alpha}\bar{b}_{\lambda\beta} - b^{\lambda}_{\alpha} b_{\lambda\beta}\right), \qquad \gamma_{3\alpha} = \gamma_{33} = \pi_{3\alpha} = 0,$$

$$\frac{\bar{a}}{a} = 1 + 2\gamma^{\alpha}_{\alpha} + 2\left(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}\right).$$

The formulae (23) and (32) reduce to

$$\begin{aligned} \mathbf{R} &= \bar{a}^{\alpha\beta}\left(\delta^{\lambda}_{\beta} + \check{\gamma}^{\lambda}_{\beta}\right)\left(\mathbf{a}_{\alpha} + \mathbf{u}_{,\alpha}\right) \otimes \mathbf{a}_{\lambda} + \left(n_{\alpha}\mathbf{a}^{\alpha} + n\mathbf{n}\right) \otimes \mathbf{n}, \\ 2\boldsymbol{\Omega} &= \varepsilon_{\lambda\mu}\left[n^{\lambda} - \bar{a}^{\alpha\beta}\left(\delta^{\lambda}_{\alpha} + \check{\gamma}^{\lambda}_{\alpha}\right)\varphi_{\beta}\right]\mathbf{a}^{\mu} + \varepsilon_{\lambda\mu}\bar{a}^{\alpha\beta}\left(\delta^{\lambda}_{\alpha} + \check{\gamma}^{\lambda}_{\alpha}\right)l^{\mu}_{\cdot\beta}\mathbf{n} \end{aligned} \tag{73}$$

which are exactly the same as derived in [16].

By simplifying (35) and (38) we obtain

$$\begin{aligned} \mathbf{k}_{\beta} &= \sqrt{\frac{a}{\bar{a}}}\ \varepsilon^{\lambda\mu}\left[\left(\kappa_{\beta\lambda} + b^{\alpha}_{\beta}\check{\gamma}_{\alpha\lambda}\right)\check{\mathbf{a}}_{\mu} + \left(\gamma_{\beta\mu|\lambda} - \frac{1}{2}\check{\gamma}^{\kappa}_{\mu}\check{\gamma}_{\kappa\lambda|\beta}\right)\mathbf{n}\right], \\ \kappa_{\alpha\beta} &= \frac{1}{2}\left(\bar{\varepsilon}_{\alpha\lambda}\mathbf{k}_{\beta} + \bar{\varepsilon}_{\beta\lambda}\mathbf{k}_{\alpha}\right)\cdot\check{\mathbf{a}}^{\lambda} - \frac{1}{2}\left(b^{\lambda}_{\alpha}\check{\gamma}_{\lambda\beta} + b^{\lambda}_{\beta}\check{\gamma}_{\lambda\alpha}\right). \end{aligned} \tag{74}$$

The formulae have been given in [15, 16] and independently by Chernykh and Shamina [3].

Under K–L constraints $\check{\mathbf{m}} = \mathbf{n}$, $\bar{\mathbf{m}} = \bar{\mathbf{n}}$. If $\boldsymbol{\beta} = \beta_{\nu}\bar{\mathbf{a}}_{\nu} + \beta_{t}\bar{\mathbf{a}}_{t} + \beta\bar{\mathbf{n}}$, then

$$\begin{aligned} \beta_{t} &= -\frac{1}{1 + 2\gamma_{tt}}\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s}\cdot\mathbf{n}, \\ \beta &= 1 - \sqrt{1 - (1 + 2\gamma_{tt})\left(\beta_{\nu}^{2} + \beta_{t}^{2}\right)} \end{aligned} \tag{75}$$

and at $\mathcal{C}$ only $\mathbf{u}$ and β_{ν} are independent. Therefore, in the three variants of geometrical boundary conditions we can assume values either for $\mathbf{u}$ and β_{ν}, or for $\boldsymbol{\Omega}_{t}$ and γ_{tt}, or for $\mathbf{k}_{t}$ and γ_{tt} at $\mathcal{C}$. The simplified formula for the vector $\boldsymbol{\Omega}_{t}$ follows from (73) and (50), where now

$$\begin{aligned} \check{\boldsymbol{\Omega}}_{t} &= -\frac{\check{\gamma}_{\nu t}}{\sqrt{1 + 2\gamma_{tt}}}\mathbf{n}, \\ 2\cos^{2}\check{\omega}_{t}/2 &= \frac{\sqrt{1 + 2\gamma_{tt}} + 1 + \check{\gamma}_{tt}}{\sqrt{1 + 2\gamma_{tt}}}. \end{aligned} \tag{76}$$

The components of $\mathbf{k}_{t}$ are simplified to

$$\begin{aligned} -k_{tt} &= \frac{1}{\sqrt{1 + 2\gamma_{tt}}}\left(\sigma_{t} - \kappa_{tt}\right) - \sigma_{t}, \\ k_{\nu t} &= \frac{1}{\sqrt{1 + 2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\ \left[\tau_{t} + \nu_{\lambda}\bar{a}^{\lambda\mu}\left(\kappa_{\mu\beta} + 2b^{\kappa}_{\beta}\gamma_{\mu\kappa}\right)t^{\beta}\right] - \tau_{t}, \\ -k_{nt} &= \frac{1}{1 + 2\gamma_{tt}} \\ &\quad\times\sqrt{\frac{\bar{a}}{a}}\ \left[\kappa_{t} - \nu_{\lambda}\bar{a}^{\lambda\mu}\left(\gamma_{\mu\alpha|\beta} + \gamma_{\mu\beta|\alpha} - \gamma_{\alpha\beta|\mu}\right)t^{\alpha}t^{\beta}\right] - \kappa_{t}. \end{aligned} \tag{77}$$

The relations (75)–$(77)_{1,3}$ agree with those derived by Novozhilov and Shamina [11] (see also [16]). The formula $(77)_2$ differs in form from that obtained in [11], since in definition of $\bar{\tau}_t$ we have used the second scalar product in $(61)_2$. Using the first scalar product in $(61)_2$ we would obtain here an equivalent relation

$$k_{\nu t} = -\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\ \nu_\lambda \bar{a}^{\lambda\alpha}(b_{\alpha\beta} - \kappa_{\alpha\beta})t^\beta - \tau_t \tag{78}$$

which agrees with that of [11].

Let $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the Lagrangian stress and couple resultant tensors of the shell in equilibrium. For any additional virtual displacement field $\delta\mathbf{u}$ subject to geometrical constraints the Lagrangian internal virtual work takes the form

$$IVW = \int\int_{\mathcal{M}} \left(N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right) \mathrm{d}A = -\int\int_{\mathcal{M}} (\mathbf{G}\mathbf{N}^\beta)_{|\beta}\cdot\delta\mathbf{u}\,\mathrm{d}A + J_c,$$

$$J_c = \int_{\mathcal{C}} \left(\mathbf{P}_\nu\cdot\delta\mathbf{u} + \overline{M}_{\nu\nu}\bar{\mathbf{a}}_t\cdot\delta\boldsymbol{\Omega}_t\right)\mathrm{d}s + \overline{M}_{t\nu}\bar{\mathbf{n}}\cdot\delta\mathbf{u}|_{\mathcal{C}} \tag{79}$$

where

$$\mathbf{N}^\beta = (N^{\alpha\beta} - \bar{b}^\alpha_\lambda M^{\lambda\beta})\mathbf{a}_\alpha + \left[M^{\alpha\beta}|_\alpha + \bar{a}^{\beta\kappa}(2\gamma_{\kappa\lambda|\mu} - \gamma_{\lambda\mu|\kappa})M^{\lambda\mu}\right]\mathbf{n},$$

$$\mathbf{P}_\nu = \mathbf{G}\mathbf{N}^\beta\nu_\beta + \frac{\mathrm{d}}{\mathrm{d}s}\left(\overline{M}_{t\nu}\bar{\mathbf{n}}\right),$$

$$\overline{M}_{t\nu} = \frac{1}{1+2\gamma_{tt}}M^{\alpha\beta}(\delta^\lambda_\alpha + 2\gamma^\lambda_\alpha)t_\lambda\nu_\beta, \tag{80}$$

$$\overline{M}_{\nu\nu} = \frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}\ M^{\alpha\beta}\nu_\alpha\nu_\beta,$$

$$\overline{M}_{t\nu}\bar{\mathbf{n}}\cdot\delta\mathbf{u}|_{\mathcal{C}} = \sum_{M_n}\left[\overline{M}_{t\nu}(s_n+0) - \overline{M}_{t\nu}(s_n-0)\right]\bar{\mathbf{n}}(s_n)\cdot\delta\mathbf{u}(s_n)$$

and M_n, $n = 1, 2, \ldots, N$ are corners of $\mathcal{C}$ labelled by $s = s_n$.

Since $\delta\beta_\nu = (\delta\boldsymbol{\Omega}_t \times \bar{\mathbf{n}})\cdot\bar{\boldsymbol{\nu}} = \delta\boldsymbol{\Omega}_t\cdot\bar{\mathbf{t}}$ it follows from the structure of J_c in (79) that the effective internal force $\mathbf{P}_\nu$ and the moment $\sqrt{1+2\gamma_{tt}}\ \overline{M}_{\nu\nu}$ are statical quantities at $\mathcal{C}$, which produce work on variations of displacemental variables $\mathbf{u}$ and β_ν.

Let $\mathbf{F}_\nu$ and $\mathbf{B}_\nu(0)$ be a total force and a total couple, with respect to an origin 0 in space, of all internal stress and couple resultants acting along a part of the boundary. In the Lagrangian description these vectors are defined by

$$\mathbf{F}_\nu = \mathbf{F}_\nu^0 + \int_{M_0}^{M} \mathbf{P}_\nu \, \mathrm{d}s, \qquad \mathbf{B}_\nu(0) = \mathbf{B}_\nu^0(0) + \int_{M_0}^{M} \left(\overline{M}_{\nu\nu}\bar{\mathbf{a}}_t + \bar{\mathbf{r}} \times \mathbf{P}_\nu\right) \mathrm{d}s \tag{81}$$

where $\mathbf{F}_\nu^0$ and $\mathbf{B}_\nu^0(0)$ are initial values of $\mathbf{F}_\nu$ and $\mathbf{B}_\nu(0)$ at $M = M_0$.

Differentiating $(81)_1$ and $\mathbf{B}_\nu = \mathbf{B}_\nu(0) - \bar{\mathbf{r}} \times \mathbf{F}_\nu$, the total couple with respect to a current point $\overline{M}$ of the deformed boundary, we obtain

$$\frac{\mathrm{d}\mathbf{F}_\nu}{\mathrm{d}s} = \mathbf{P}_\nu, \qquad \frac{\mathrm{d}\mathbf{B}_\nu}{\mathrm{d}s} = \overline{M}_{\nu\nu}\bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{F}_\nu. \tag{82}$$

Let us differentiate $\delta\mathbf{u}$ and $\delta\mathbf{\Omega}_t$ and take into account that $\mathrm{d}\bar{s} = \sqrt{1 + 2\gamma_{tt}}\, \mathrm{d}s$, which gives

$$\frac{\mathrm{d}\delta\mathbf{\Omega}_t}{\mathrm{d}s} = \sqrt{1 + 2\gamma_{tt}}\, \delta\mathbf{k}_t, \qquad \frac{\mathrm{d}\delta\mathbf{u}}{\mathrm{d}s} = \delta\bar{\gamma}_{tt}\bar{\mathbf{a}}_t + \delta\mathbf{\Omega}_t \times \bar{\mathbf{a}}_t. \tag{83}$$

Here $\delta\bar{\gamma}_{tt} = \delta\gamma_{\alpha\beta}\bar{t}^\alpha\bar{t}^\beta = \delta\gamma_{tt}/\bar{a}_t^2$ where $\delta\gamma_{tt} = \delta\gamma_{\alpha\beta}t^\alpha t^\beta$.

It is possible now to transform the boundary part of (79) as follows

$$\begin{aligned} J_c &= \int_{M_0}^{M}\left[\left(\overline{M}_{\nu\nu}\bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{F}_\nu\right)\cdot \delta\mathbf{\Omega}_t - \frac{\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu}{1 + 2\gamma_{tt}}\delta\gamma_{tt}\right]\mathrm{d}s \\ &\quad + \left(\overline{M}_{t\nu}\bar{\mathbf{n}} + \mathbf{F}_\nu\right)\cdot \delta\mathbf{u}\Big|_{M_0}^{M} \\ &= -\int_{M_0}^{M}\left(\sqrt{1 + 2\gamma_{tt}}\,\mathbf{B}_\nu \cdot \delta\mathbf{k}_t + \frac{\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu}{1 + 2\gamma_{tt}}\delta\gamma_{tt}\right)\mathrm{d}s \\ &\quad + \left[\left(\overline{M}_{t\nu}\bar{\mathbf{n}} + \mathbf{F}_\nu\right)\cdot \delta\mathbf{u} + \mathbf{B}_\nu \cdot \delta\mathbf{\Omega}_t\right]\Big|_{M_0}^{M} \end{aligned} \tag{84}$$

The relations (84) show that during the virtual deformation some statical parameters produce work on variations of geometrical parameters $\mathbf{\Omega}_t$, γ_{tt} and $\mathbf{k}_t$, γ_{tt} of the shell boundary. Therefore within K–L constraints to each of the geometrical quantity there corresponds a statical quantity as follows [19]:

$$\begin{aligned} &\mathbf{u} \leftrightarrow \mathbf{P}_\nu, \qquad \beta_\nu \leftrightarrow \sqrt{1 + 2\gamma_{tt}}\,\overline{M}_{\nu\nu}, \\ &\mathbf{\Omega}_t \leftrightarrow \overline{M}_{\nu\nu}\bar{\mathbf{a}}_t - \bar{\mathbf{a}}_t \times \mathbf{F}_\nu, \qquad \gamma_{tt} \leftrightarrow -\frac{\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu}{1 + 2\gamma_{tt}}, \\ &\mathbf{k}_t \leftrightarrow -\sqrt{1 + 2\gamma_{tt}}\,\mathbf{B}_\nu, \qquad \gamma_{tt} \leftrightarrow -\frac{\bar{\mathbf{a}}_t \cdot \mathbf{F}_\nu}{1 + 2\gamma_{tt}}. \end{aligned} \tag{85}$$

Assuming the statical parameters to have prescribed values at $\mathcal{C}$, three variants of statical boundary conditions for the K–L non-linear theory of shells are obtained. They are energetically compatible with the displacemental, kinematical and deformational boundary conditions, respectively. The constant terms appearing outside the integration in (84) lead to additional discontinuity conditions to be satisfied at each corner M_n of $\mathcal{C}$.

6.2. Geometrically non-linear theory

Let us take the coordinate system at $\mathfrak{M}$ such that $a_r \sim O(1)$, where a_r are eigenvalues of the spatial shell metric tensor **1**. Suppose the strains to be small everywhere in the shell: $\eta \ll 1$, $\eta = \max_r E_r$, where E_r are three eigenvalues of **E**. All strain parameters are supposed to have here the same order of magnitude:

$$\gamma_{\alpha\beta} \sim \gamma_{3\beta} \sim \gamma_{33} \sim O(\eta),\ h\pi_{(\alpha\beta)} \sim O(\eta).$$

Under these conditions it is permissible to omit in the exact relations some terms which are small with respect to unity. In the process of simplification it is suggested to keep also the second-order terms in some intermediate formulae. Many shell relations are calculated as a difference between terms of the same order and in such a case the second-order terms become of primary importance.

Under small strains we have

$$\begin{aligned}
\bar{a}_{ab} &= a_{ab} + 2\gamma_{ab} = a_{ab} + O(\eta),\\
\bar{a}^{ab} &= a^{ab} - 2\gamma_{ab} + O(\eta^2) = a^{ab} + O(\eta),\\
\frac{\bar{a}}{a} &= 1 + 2\gamma^a_a + O(\eta^2) = 1 + O(\eta), \qquad \check{\gamma}_{ab} = \gamma_{ab} + O(\eta^2),\\
\bar{\mathbf{a}}_3 &= 2\gamma^\alpha_3 \bar{\mathbf{a}}_\alpha + (1+\gamma_{33})\bar{\mathbf{n}} + O(\eta^2).
\end{aligned} \tag{86}$$

The formulae (23) and (32) are simplified to

$$\begin{aligned}
\mathbf{R} =& \left[l_{\lambda\alpha}(\delta^\alpha_c - \gamma^\alpha_c) + \beta_\lambda(a_{3c} - \gamma_{3c}) \right]\mathbf{a}^\lambda \otimes \mathbf{a}^c\\
&+ \left[\varphi_\alpha(\delta^\alpha_c - \gamma^\alpha_c) + (1+\beta)(a_{3c} - \gamma_{3c}) \right]\mathbf{n} \otimes \mathbf{a}^c + O(\eta^2),\\
2\Omega =& \varepsilon^{\lambda\mu}\left[-l_{\lambda\alpha}\gamma^\alpha_3 + \beta_\lambda(1-\gamma_{33}) - \varphi_\alpha(\delta^\alpha_\lambda - \gamma^\alpha_\lambda) + (1+\beta)\gamma_{3\lambda} \right]\mathbf{a}_\mu\\
&+ \varepsilon^{\lambda\mu}\left[\varphi_{\mu\alpha}(\delta^\alpha_\lambda - \gamma^\alpha_\lambda) - \beta_\mu\gamma_{3\lambda} \right]\mathbf{n} + O(\eta^2).
\end{aligned} \tag{87}$$

By simplifying (47) with (46) at the shell boundary we obtain

$$\begin{aligned} \check{\mathbf{R}}_t &= \mathbf{1} + \gamma_{3\nu}(\boldsymbol{\nu} \otimes \mathbf{n} - \mathbf{n} \otimes \boldsymbol{\nu}) - \gamma_{3t}(\mathbf{t} \otimes \mathbf{n} - \mathbf{n} \otimes \mathbf{t}) \\ &\quad + \gamma_{\nu t}(\boldsymbol{\nu} \otimes \mathbf{t} - \mathbf{t} \otimes \boldsymbol{\nu}) + O(\eta^2) \\ \check{\boldsymbol{\Omega}}_t &= \gamma_{3t}\boldsymbol{\nu} + \gamma_{3\nu}\mathbf{t} - \gamma_{\nu t}\mathbf{n} + O(\eta^2), \qquad \cos^2 \check{\omega}_t/2 = 1 + O(\eta^2) \end{aligned} \tag{88}$$

and the superposition formula (50) reduces to

$$\boldsymbol{\Omega}_t = \cos^2 \omega^+/2\check{\boldsymbol{\Omega}}_t + \boldsymbol{\Omega} + \frac{1}{2}\boldsymbol{\Omega}^+ \times \check{\boldsymbol{\Omega}}_t - \frac{\check{\boldsymbol{\Omega}}_t \cdot \boldsymbol{\Omega}^+}{4\cos^2 \omega^+/2}\boldsymbol{\Omega}^+ + O(\eta^2) \tag{89}$$

where

$$\begin{aligned} \boldsymbol{\Omega}^+ &= \tfrac{1}{2}\varepsilon^{\lambda\mu}\left[(\beta_\lambda - \varphi_\lambda)\mathbf{a}_\mu + \varphi_{\lambda\mu}\mathbf{n}\right], \\ \cos^2 \omega^+/2 &= \tfrac{1}{4}(\operatorname{tr}\mathbf{G} + 1). \end{aligned} \tag{90}$$

By omitting quadratic terms in (35) and (38) we obtain

$$\begin{aligned} \mathbf{k}_\beta &= \varepsilon^{\lambda\mu}\left[(\pi_{(\beta\lambda)} + b^\kappa_\beta\gamma_{\kappa\lambda} - \gamma_{3\beta|\lambda} + b_{\beta\lambda}\gamma_{33})\mathbf{a}_\mu \right. \\ &\quad \left. + (\gamma_{\beta\mu|\lambda} - b_{\beta\lambda}\gamma_{3\mu})\mathbf{n}\right], \\ \pi_{(\alpha\beta)} &= \tfrac{1}{2}(\varepsilon_{\alpha\lambda}\mathbf{k}_\beta + \varepsilon_{\beta\lambda}\mathbf{k}_\alpha)\cdot\mathbf{a}^\lambda - \tfrac{1}{2}(b^\kappa_\alpha\gamma_{\kappa\beta} + b^\kappa_\beta\gamma_{\kappa\alpha}) \\ &\quad - b_{\alpha\beta}\gamma_{33} + \tfrac{1}{2}(\gamma_{3\alpha|\beta} + \gamma_{3\beta|\alpha}). \end{aligned} \tag{91}$$

The relations (62) and (63) are simplified by dropping terms which are small with respect to unity [15]

$$\begin{aligned} k_{tt} &= \pi_{tt} - 2\frac{d\gamma_{3t}}{ds} + \sigma_t(\gamma_{tt} + \gamma_{33}), \\ k_{\nu t} &= \pi_{(\nu t)} - \frac{d\gamma_{3t}}{ds_\nu} + \frac{d\gamma_{3\nu}}{ds} + \kappa_\nu\gamma_{3\nu} + 2(\sigma_t - \pi_{tt})\gamma_{\nu t} \\ &\quad - \tau_t(\gamma_{\nu\nu} + \gamma_{33}) + \kappa_t\gamma_{3t}, \\ k_{nt} &= 2\frac{d\gamma_{\nu t}}{ds} - \frac{d\gamma_{tt}}{ds_\nu} + 2\kappa_\nu\gamma_{\nu t} - 2(\sigma_t - \pi_{tt})\gamma_{3\nu} - \kappa_t(\gamma_{tt} - \gamma_{\nu\nu}). \end{aligned} \tag{92}$$

Let us interpret (91) and (92) by means of $\kappa_{\alpha\beta}$, the change of middle surface curvature. It follows from $(86)_3$ that under small strains [18]

$$\pi_{(\alpha\beta)} = \kappa_{\alpha\beta} + \gamma_{3\alpha|\beta} + \gamma_{3\beta|\alpha} - b_{\alpha\beta}\gamma_{33} \tag{93}$$

which gives

$$\mathbf{k}_\beta = \varepsilon^{\lambda\mu}\left[\left(\kappa_{\beta\lambda} + b^\kappa_\beta\gamma_{\kappa\lambda} - \gamma_{3\lambda|\beta}\right)\mathbf{a}_\mu + \left(\gamma_{\beta\mu|\lambda} - b_{\beta\lambda}\gamma_{3\mu}\right)\mathbf{n}\right], \tag{94}$$

$$k_{tt} = \kappa_{tt} + 2\kappa_t\gamma_{3\nu} + \sigma_t\gamma_{tt},$$

$$k_{\nu t} = \kappa_{\nu t} + 2\frac{\mathrm{d}\gamma_{3\nu}}{\mathrm{d}s} + 2(\sigma_t - \kappa_{tt})\gamma_{\nu t} - \tau_t\gamma_{\nu\nu}, \tag{95}$$

$$k_{nt} = 2\frac{\mathrm{d}\gamma_{\nu t}}{\mathrm{d}s} - \frac{d\gamma_{tt}}{\mathrm{d}s_\nu} + 2\kappa_\nu\gamma_{\nu t} - 2(\sigma_t - \kappa_{tt})\gamma_{3\nu} - \kappa_t(\gamma_{tt} - \gamma_{\nu\nu}).$$

Some further simplification of the relations given here is possible provided the order of magnitude of rotations with respect to η is known. Two limiting cases may be considered.

For truly finite rotations the rotation angle $\omega \sim O(1)$ and thus $\varphi_\alpha \sim \varphi_{\alpha\beta} \sim \beta_\alpha \sim \beta \sim O(1)$. Therefore, in $(87)_2$ and (89) we can omit strains as compared with rotations and obtain

$$\Omega = \Omega^+ + O(\eta), \qquad \Omega_t = \Omega^+ + O(\eta). \tag{96}$$

Within small rotations $\omega \sim O(\eta)$ and therefore $\varphi_\alpha \sim \varphi_{\alpha\beta} \sim \beta_\alpha \sim \beta \sim O(\eta)$. Then all the relations can be linearized and the formulae $(87)_2$, $(88)_2$ and (92) reduce to those obtained in [20] for the Reissner-type linear theory of shells.

6.3. *First-approximation theory of isotropic elastic shells*

For an isotropic elastic shell loaded at its lateral boundaries the exact error estimates for stresses given by John [5] allow to obtain also the estimates for the shell strain measures [15]:

$$\gamma_{\alpha\beta} \sim h\pi_{(\alpha\beta)} \sim O(\eta), \qquad \mu_{\alpha\beta} \sim O(\eta\theta^2),$$

$$\gamma_{3\beta} \sim h\pi_{3\beta} \sim O(\eta\theta), \qquad \gamma_{33} = -\frac{\nu}{1-\nu}\gamma^\alpha_\alpha + O(\eta\theta^2) = O(\nu\eta).$$

Here ν is the Poisson's ratio and the small parameter θ, redefined in [7] by using qualitative arguments, is given by $\theta = \max(h/L, h/d, \sqrt{(h/R)}, \sqrt{\eta})$, where L is the smallest wavelength of deformation patterns at $\mathfrak{M}$, d is the distance from the lateral shell boundary and R is the smallest radius of curvature of $\mathfrak{M}$.

According to $(86)_3$ we have

$$\beta_\alpha = (1 + \gamma_{33})n_\alpha + O(\eta\theta), \qquad 1 + \beta = (1 + \gamma_{33})n + O(\eta\theta) \tag{97}$$

which allow to simplify (87)–(90) to the forms

$$
\begin{aligned}
\mathbf{R} &= l_{\lambda\alpha}(\delta^{\alpha}_{\mu} - \gamma^{\alpha}_{\mu})\mathbf{a}^{\lambda} \otimes \mathbf{a}^{\mu} + (1 - \gamma^{\kappa}_{\kappa})(\varphi_{\alpha} l^{\alpha}_{\cdot\lambda} - \varphi_{\lambda} l^{\alpha}_{\cdot\alpha})\mathbf{a}^{\lambda} \otimes \mathbf{n} \\
&\quad + \varphi_{\alpha}(\delta^{\alpha}_{\mu} - \gamma^{\alpha}_{\mu})\mathbf{n} \otimes \mathbf{a}^{\mu} \\
&\quad + \tfrac{1}{2}(1 - \gamma^{\kappa}_{\kappa})\big(l^{\alpha}_{\cdot\alpha} l^{\beta}_{\cdot\beta} - l^{\alpha}_{\cdot\beta} l^{\beta}_{\cdot\alpha}\big)\mathbf{n} \otimes \mathbf{n} + O(\eta\theta), \\
2\boldsymbol{\Omega} &= \varepsilon^{\lambda\mu}\big\{\big[(1 - \gamma^{\kappa}_{\kappa})(\varphi_{\alpha} l^{\alpha}_{\cdot\lambda} - \varphi_{\lambda} l^{\alpha}_{\cdot\alpha}) - \varphi_{\alpha}(\delta^{\alpha}_{\lambda} - \gamma^{\alpha}_{\lambda})\big]\mathbf{a}_{\mu} \\
&\qquad + \varphi_{\mu\alpha}(\delta^{\alpha}_{\lambda} - \gamma^{\alpha}_{\lambda})\mathbf{n}\big\} + O(\eta\theta), \qquad\qquad (98) \\
\check{\mathbf{R}}_{t} &= \mathbf{1} + \gamma_{\nu t}(\boldsymbol{\nu} \otimes \mathbf{t} - \mathbf{t} \otimes \boldsymbol{\nu}) + O(\eta\theta), \qquad \check{\boldsymbol{\Omega}}_{t} = -\gamma_{\nu t}\mathbf{n} + O(\eta\theta), \\
\boldsymbol{\Omega}^{+} &= \tfrac{1}{2}\varepsilon^{\lambda\mu}\big\{\big[(2 + \varphi^{\kappa}_{\cdot\kappa})\varphi_{\lambda} - \varphi^{\kappa}\varphi_{\kappa\lambda}\big]\mathbf{a}_{\mu} + \varphi_{\mu\lambda}\mathbf{n}\big\}.
\end{aligned}
$$

By omitting terms with $\gamma_{3\nu}$ in (95) for deformational quantities we obtain [15, 16].

$$
\begin{aligned}
k_{tt} &= \kappa_{tt} + \sigma_{t}\gamma_{tt}, \\
k_{\nu t} &= \kappa_{\nu t} + 2(\sigma_{t} - \kappa_{tt})\gamma_{\nu t} - \tau_{t}\gamma_{\nu\nu}, \qquad\qquad (99) \\
k_{nt} &= 2\frac{\mathrm{d}\gamma_{\nu t}}{\mathrm{d}s} - \frac{\mathrm{d}\gamma_{tt}}{\mathrm{d}s_{\nu}} + 2\kappa_{\nu}\gamma_{\nu t} - \kappa_{t}(\gamma_{tt} - \gamma_{\nu\nu}).
\end{aligned}
$$

Simplifying the statical values in (85), within the first-approximation geometrically non-linear theory we can assume along $\mathcal{C}$ either $\mathbf{P}_{\nu}$ and $M_{\nu\nu}$, or $M_{\nu\nu}\bar{\mathbf{t}} - \bar{\mathbf{t}} \times \mathbf{F}_{\nu}$ and $-\bar{\mathbf{t}} \cdot \mathbf{F}_{\nu}$, or $-\mathbf{B}_{\nu}$ and $-\bar{\mathbf{t}} \cdot \mathbf{F}_{\nu}$, where

$$
\begin{aligned}
\mathbf{P}_{\nu} &= P_{\nu\nu}\bar{\boldsymbol{\nu}} + P_{t\nu}\bar{\mathbf{t}} + P_{n\nu}\bar{\mathbf{n}}, \qquad \mathbf{F}_{\nu} = \mathbf{F}^{0}_{\nu} + \int_{M_0}^{M}\mathbf{P}_{\nu}\,\mathrm{d}s \\
\mathbf{B}_{\nu} &= \mathbf{B}^{0}_{\nu}(0) + \int_{M_0}^{M}(M_{\nu\nu}\bar{\mathbf{t}} + \bar{\mathbf{r}} \times \mathbf{P}_{\nu})\,\mathrm{d}s - \bar{\mathbf{r}} \times \mathbf{F}_{\nu}, \qquad\qquad (100) \\
P_{\nu\nu} &= N_{\nu\nu} - (\sigma_{\nu} - \kappa_{\nu\nu})M_{\nu\nu} + 2(\tau_{t} + \kappa_{\nu t})M_{t\nu}, \\
P_{t\nu} &= N_{t\nu} + (\tau_{t} + \kappa_{\nu t})M_{\nu\nu} - 2(\sigma_{t} - \kappa_{tt})M_{t\nu}, \qquad\qquad (101) \\
P_{n\nu} &= \frac{\mathrm{d}M_{\nu\nu}}{\mathrm{d}s_{\nu}} + 2\frac{\mathrm{d}M_{t\nu}}{\mathrm{d}s} + \kappa_{t}(M_{\nu\nu} - M_{tt}) + 2\kappa_{\nu}M_{t\nu}.
\end{aligned}
$$

The formulae (101) were obtained in [16].

Linearization of relations of the first-approximation theory leads to the formulae which agree with those presented by Chernykh [2] for the classical linear theory of shells.

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