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Variational Principles in the Geometrically Non-linear Theory of Shells Undergoing Moderate Rotations *

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Summary: A general approach to the derivation of variational principles is given for the geometrically non-linear theory of thin elastic shells undergoing moderate rotations. Starting from the principle of virtual displacements, a set of sixteen basic free functionals without subsidiary conditions is constructed. From these free functionals a number of related functionals with or without subsidiary conditions may be generated. As examples, the functionals of the total potential energy and the total complementary energy are derived.

Übersicht: Die vorliegende Arbeit enthält eine systematische Herleitung von Variationsprinzipen für die geometrisch nichtlineare Theorie dünner elastischer Schalen, in der die Quadrate der Rotationen von gleicher Größenordnung wie die Dehnungen sein können. Ausgehend vom Prinzip der virtuellen Verschiebungen wird eine Familie von sechzehn freien Funktionalen hergeleitet, die keinen Nebenbedingungen unterliegen. Von diesen freien Funktionalen kann eine Vielzahl verwandter Funktionale mit oder ohne Nebenbedingungen abgeleitet werden. Als Beispiele werden die Prinzipe vom stationären Wert des Gesamtpotentials und der komplementären Energie angegeben.

1 Introduction

The rapid development of computerized solution techniques makes it possible to calculate thin shell structures with a desired degree of accuracy within the linear as well as non-linear range of deformation. Some of the numerical methods used most frequently (e.g. finite element and finite difference energy methods) are based on appropriate variational principles.

In the linear theory of thin elastic shells various variational principles have been derived by Trefftz [1], Reissner [2—4], Naghdi [5—7], Rüdiger [8], Chernykh [9], Washizu [10] and others, which are analogous to the corresponding principles of the linear three-dimensional theory of elasticity. An extensive treatment of this subject was given recently by Abovsky et al. [11], who also discussed the extremum properties of some of the functionals they considered.

In the classical geometrically non-linear theory of shallow shells [12—14] several variational functionals have been constructed. The functionals given by Alumäe [15], Wang [16], Mushtari and Galimov [17], Grundmann [18], Huang [19], Gass [20], Stumpf [21], Harnach and Krätzig [22] and Abovsky et al. [23] are given in terms of various combinations of the displacements, strain measures, and stress measures or stress functions. In the functionals given by Aynola [24], Stumpf [21, 25] and Washizu [26] linearized strain measures and rotations as well as associated stress measures are used as independent variables.

For simplified variants of the non-linear theory of thin elastic shells undergoing moderate rotations dual extremum principles and complementary variational theorems have been pre-

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sented by Stumpf [27, 28], while the principle of stationary total potential energy was also used by Stein [29].

For the general non-linear theory of thin shells with unrestricted strains and rotations four variational principles have been formulated by Galimov [30]. In these principles the independent variables are referred to the configuration of the deformed shell surface, not usually known in advance.

In this paper a general approach to the derivation of variational principles for the geometrically non-linear theory of thin elastic shells undergoing moderate rotations is given. This theory, proposed in [31—33], is based on a consistent first approximation of the shell strain energy function. All shell quantities are referred to the known reference configuration of the undeformed shell. The theory contains as special cases the simplified versions of the non-linear shell equations derived by Mushtari and Galimov [17] (medium bending), Sanders [34] (moderately small rotations), Koiter [35] (small finite deflections), Donnell [12], Marguerre [13] and Vlasov [14] (shallow shells) as well as the non-linear theory of plates of von Kármán [36].

Starting from the principle of virtual displacements and applying the transformation procedures of Courant and Hilbert [37], a set of sixteen basic free functionals (without subsidiary conditions) is constructed. The functionals are formulated in terms of various combinations of displacements, strain measures, stress measures and reactive boundary loads as independent free field variables which are subject to variation. Among them are the Hu-Washizu principle, the Hellinger-Reissner principle and the principle of generalized total complementary energy.

The free functionals are given in two different but equivalent forms, which exhibit certain symmetry properties with respect to the geometric and static variables. The functionals denoted by $I_1 \div I_8$ are particularly convenient to apply, when the independent fields are additionally subject to certain geometric constraints. Similarly, the functionals $J_1 \div J_8$ are useful in applications when the independent fields are additionally subject to static constraints. From each of the basic functionals a variety of other, related functionals with or without subsidiary conditions may be generated. As an example, the functionals of the total potential energy and the total complementary energy are derived and a mixed functional, in which the membrane part is treated differently from the bending part, is constructed. For each of the functionals the appropriate independent fields subject to variation are clearly indicated and the full set of subsidiary and stationarity conditions is given.

Most of the functionals derived here are new even for the case of the classical non-linear theory of shallow shells. Those which have already appeared in the literature for simple versions of the non-linear shell theory are generalized here for the theory of thin elastic shells undergoing moderate rotations. Furthermore, the general case of mixed boundary conditions, with mutually complementary geometric and static quantities prescribed on the same part of the boundary, is taken into account. The shell boundary may also have corner points, where in general additional concentrated forces should be applied as a result of elimination of twisting moments acting on the boundary. Mixed boundary conditions and corner effects are frequently omitted in the works on variational principles for shell theories, but are important from the practical point of view, unless some special shell problems are considered.

2 Basic Shell Relations

Let $\mathbf{r}(\theta^\alpha) = x^k(\theta^\alpha)\, \mathbf{i}_k$ and $\bar{\mathbf{r}}(\theta^\alpha) = \bar{x}^k(\theta^\alpha)\, \mathbf{i}_k$, $k = 1, 2, 3$, be position vectors of the shell middle surface in the reference (undeformed) and deformed configurations, respectively. Here θ^α, $\alpha = 1, 2$, denotes a pair of convected surface coordinates, while x^k and $\bar{x}^k$ are the spatial components of $\mathbf{r}$ and $\bar{\mathbf{r}}$ with respect to a fixed Cartesian frame $\mathbf{i}_k$ in three-dimensional Euclidean space. With the reference shell middle surface $\mathcal{M}$ we associate standard covariant base vectors $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$, a unit normal vector $\mathbf{n} = \frac{1}{2}\, \varepsilon^{\alpha\beta}\, \mathbf{a}_\alpha \times \mathbf{a}_\beta$ and covariant components of the surface metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and of the surface curvature tensor $b_{\alpha\beta} = \mathbf{a}_{\alpha,\beta} \cdot \mathbf{n}$. Here $(\ )_{,\alpha} = \partial(\ )/\partial\theta^\alpha$, and $\varepsilon^{\alpha\beta}$ are the contravariant components of the skew-symmetric surface permutation tensor. Contravariant components $a^{\alpha\beta}$ of the metric tensor, satisfying the relations $a^{\alpha\lambda} a_{\beta\lambda} = \delta^\alpha_\beta$, where δ^α_β is the Kronecker symbol, are used to raise the indices of surface tensors defined on $\mathcal{M}$. By a vertical bar $(\ )|_\alpha$ we shall denote covariant differentiation on $\mathcal{M}$ with respect to θ^α.

The reference surface $\mathcal{M}$ is mapped uniquely into the middle surface $\bar{\mathcal{M}}$ of the deformed shell configuration by a displacement field $\mathbf{u} = \bar{\mathbf{r}} - \mathbf{r} = u^\alpha \mathbf{a}_\alpha + w\mathbf{n}$. The covariant base vectors on $\bar{\mathcal{M}}$ are $\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha} = \mathbf{a}_\alpha + (\theta_{\beta\alpha} - \omega_{\beta\alpha})\, \mathbf{a}^\beta + \varphi_\alpha \mathbf{n}$, where for subsequent use the linearized strains and rotations [31–33, 35, 38]

$$\left.\begin{aligned} \theta_{\alpha\beta} &= \tfrac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w\,, \qquad & \varphi_\alpha &= w_{,\alpha} + b_\alpha^\lambda u_\lambda\,, \\ \omega_{\alpha\beta} &= \tfrac{1}{2}\,(u_{\beta|\alpha} - u_{\alpha|\beta}) = \varepsilon_{\alpha\beta}\varphi\,, \qquad & \varphi &= \tfrac{1}{2}\,\varepsilon^{\alpha\beta} u_{\beta|\alpha} \end{aligned}\right\} \tag{2.1}$$

have been introduced.

The components of the symmetric mid-surface strain tensor and of the tensor of change of surface curvature are defined by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\,(\bar{a}_{\alpha\beta} - a_{\alpha\beta})\,, \qquad \varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})\,. \tag{2.2}$$

Here $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta$ and $\bar{b}_{\alpha\beta} = \bar{\mathbf{a}}_{\alpha,\beta} \cdot \bar{\mathbf{n}}$ are the covariant components of the metric and curvature tensors of $\bar{\mathcal{M}}$, respectively, while $\bar{\mathbf{n}} = \frac{1}{2}\,\bar{\varepsilon}^{\alpha\beta} \bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta$ is the unit normal of $\bar{\mathcal{M}}$. For a detailed discussion of the geometric relations on $\mathcal{M}$ and $\bar{\mathcal{M}}$ we refer the reader to [32, 35, 39].

In this paper we are dealing with thin shells of constant thickness $h \ll R$, where R is the smallest principal radius of curvature of $\mathcal{M}$. The shell deformation is assumed to be such that $h^2 \ll L^2$, where L is the smallest wavelength of deformation patterns on $\mathcal{M}$, [35, 40]. In general, the deformation of the neighbourhood of a point $M \in \mathcal{M}$ may be decomposed exactly into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation [31, 38]. Therefore, it is possible to construct various approximate non-linear shell theories by restricting the magnitudes of the strains and rotations of the shell material elements independently.

For the geometrically non-linear theory of shells to be considered here, the strains are assumed to be small everywhere. That is $\eta \ll 1$, where η is the largest principal strain in the shell space. As a measure of smallness of various quantities we use the common parameter $\theta = \max\left(h/L,\, h/d,\, \sqrt{h/R},\, \sqrt{\eta}\right)$, where d is the distance of the point under consideration from the shell boundary [40].

For an elastic shell a strain energy function Σ, per unit area of $\mathcal{M}$, exists. In the case of small strains everywhere and isotropic material behaviour it can be consistently simplified [31, 41, 42]. For a consistent first-approximation theory of shells the strain energy function is given, to within a relative error of $O(\theta^2)$, by the following quadratic expression

$$\Sigma = \frac{h}{2}\, H^{\alpha\beta\lambda\mu} \left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\, \varkappa_{\alpha\beta}\varkappa_{\lambda\mu}\right), \tag{2.3}$$

where

$$H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)} \left(a^{\alpha\lambda} a^{\beta\mu} + a^{\alpha\mu} a^{\beta\lambda} + \frac{2\nu}{1-\nu}\, a^{\alpha\beta} a^{\lambda\mu}\right) \tag{2.4}$$

and E and ν are Young's modulus and Poisson's ratio, respectively.

With the strain energy function (2.3) we obtain the linear constitutive equations (CE)

$$\text{CE:} \qquad N^{\alpha\beta} = \frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}} = h H^{\alpha\beta\lambda\mu} \gamma_{\lambda\mu}\,, \qquad M^{\alpha\beta} = \frac{\partial \Sigma}{\partial \varkappa_{\alpha\beta}} = \frac{h^3}{12}\, H^{\alpha\beta\lambda\mu} \varkappa_{\lambda\mu}\,, \tag{2.5}$$

where $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are symmetric stress and moment resultants consistent with the chosen surface strain measures.

Equations (2.5) can be put in matrix form

$$N^i = h H^{ij} \gamma_j\,, \qquad M^i = \frac{h^3}{12}\, H^{ij} \varkappa_j\,, \qquad i, j = 1, 2, 3\,, \tag{2.6}$$

where

$$\left.\begin{aligned} N^i &= (N^{11}, N^{22}, N^{12})^T\,, \qquad & M^i &= (M^{11}, M^{22}, M^{12})^T\,, \\ \gamma_j &= (\gamma_{11}, \gamma_{22}, \gamma_{12} + \gamma_{21})^T\,, \qquad & \varkappa_j &= (\varkappa_{11}, \varkappa_{22}, \varkappa_{12} + \varkappa_{21})^T\,, \end{aligned}\right\} \tag{2.7}$$

$$H^{ij} = \begin{bmatrix} H^{1111} & H^{1122} & H^{1112} \\ H^{2211} & H^{2222} & H^{2212} \\ H^{1211} & H^{1222} & H^{1212} \end{bmatrix}. \tag{2.8}$$

With the components of $H^{\alpha\beta\lambda\mu}$ given by (2.4), the matrix H^{ij} can be shown to be non-singular, i.e. det $(H^{ij}) \neq 0$. This assures the existence of a unique inverse of (2.6) yielding the inverse constitutive equations (IC)

$$\text{IC:} \qquad \gamma_{\alpha\beta} = \frac{1}{h} E_{\alpha\beta\lambda\mu} N^{\lambda\mu} , \qquad \varkappa_{\alpha\beta} = \frac{12}{h^3} E_{\alpha\beta\lambda\mu} M^{\lambda\mu} , \tag{2.9}$$

where

$$E_{\alpha\beta\lambda\mu} = \frac{1+\nu}{2E} \left(a_{\alpha\lambda} a_{\beta\mu} + a_{\alpha\mu} a_{\beta\lambda} - \frac{2\nu}{1+\nu} a_{\alpha\beta} a_{\lambda\mu} \right) . \tag{2.10}$$

The relation (2.9) may also be written as

$$\gamma_{\alpha\beta} = \frac{\partial \Sigma_c}{\partial N^{\alpha\beta}} , \qquad \varkappa_{\alpha\beta} = \frac{\partial \Sigma_c}{\partial M^{\alpha\beta}} , \tag{2.11}$$

where

$$\Sigma_c = \frac{1}{2h} E_{\alpha\beta\lambda\mu} \left(N^{\alpha\beta} N^{\lambda\mu} + \frac{12}{h^2} M^{\alpha\beta} M^{\lambda\mu} \right) \tag{2.12}$$

is called the complementary energy function of the shell. Equivalently, Σ_c may be constructed by means of the Legendre transformation

$$\Sigma_c(N^{\alpha\beta}, M^{\alpha\beta}) = N^{\alpha\beta} \gamma_{\alpha\beta} + M^{\alpha\beta} \varkappa_{\alpha\beta} - \Sigma(\gamma_{\alpha\beta}, \varkappa_{\alpha\beta}) . \tag{2.13}$$

The existence and uniqueness of this transformation is assured by the fact that the unique inverse of (2.5) exists. When (2.3) and (2.9) are introduced, (2.13) transforms exactly to (2.12).

An important simplification of the non-linear shell relations can be achieved by further restricting also the magnitude of the rotations of the shell material elements. The rigid-body rotation of the neighbourhood of a material point can be described by a finite rotation vector $\boldsymbol{\Omega}$, [38]. In [31, 32] the following classification of rotations has been proposed: $|\boldsymbol{\Omega}| \geq O(1)$ — finite rotations, $|\boldsymbol{\Omega}| = O(\sqrt{\theta})$ — large rotations, $|\boldsymbol{\Omega}| = O(\theta)$ — moderate rotations, $|\boldsymbol{\Omega}| = O(\theta^2)$ — small rotations. For each of these cases the strain-displacement relations (2.2) may be simplified consistently by successively neglecting those terms of relative smallness whose contribution to the strain energy function Σ lies within the error margin already implicit in assuming Σ to be of the form (2.3). This leads to consistently simplified shell relations [33] for each of the above cases.

Within the geometrically non-linear first-approximation theory of thin shells undergoing moderate rotations we then have [31—33, 43] the following complete set of equations which consists of the simplified strain-displacement relations (SD) in $\mathscr{M}$, geometric boundary conditions (GB) on $\mathscr{C}_u$, geometric corner conditions (GC) at each corner point $M_i \in \mathscr{C}_u$, equilibrium equations (EQ) in $\mathscr{M}$, static boundary conditions (SB) on $\mathscr{C}_f$ and static corner conditions (SC) at each corner point $M_j \in \mathscr{C}_f$:

$$\text{SD:} \quad \left. \begin{aligned} \gamma_{\alpha\beta} &= \theta_{\alpha\beta} + \tfrac{1}{2} \varphi_\alpha \varphi_\beta + \tfrac{1}{2} a_{\alpha\beta} \varphi^2 - \tfrac{1}{2} (\theta^\lambda_\alpha \omega_{\lambda\beta} + \theta^\lambda_\beta \omega_{\lambda\alpha}) , \\ \varkappa_{\alpha\beta} &= - \tfrac{1}{2} [\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] , \end{aligned} \right\} \tag{2.14}$$

$$\text{GB:} \quad u_\nu = u^*_\nu , \quad u_t = u^*_t , \quad w = w^* , \quad \beta = \beta^* , \tag{2.15}$$

$$\text{GC:} \quad w(s_{ui}) = w^*(s_{ui}) , \tag{2.16}$$

$$\text{EQ:} \quad \left. \begin{aligned} & [N^{\alpha\beta} - \tfrac{1}{2} (b^\alpha_\lambda M^{\lambda\beta} + b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2} (b^\alpha_\lambda M^{\lambda\beta} - b^\beta_\lambda M^{\lambda\alpha}) - \tfrac{1}{2} \omega^{\alpha\beta} N^\lambda_\lambda - \tfrac{1}{2} (\omega^{\alpha\lambda} N^\beta_\lambda + \omega^{\beta\lambda} N^\alpha_\lambda) + \\ & \quad + \tfrac{1}{2} (\theta^{\alpha\lambda} N^\beta_\lambda - \theta^{\beta\lambda} N^\alpha_\lambda)]|_\beta - b^\alpha_\beta (M^{\lambda\beta}|_\lambda + \varphi_\lambda N^{\lambda\beta}) + p^\alpha = 0 , \\ & (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})|_\beta + b_{\alpha\beta} [N^{\alpha\beta} - b^\alpha_\lambda M^{\lambda\beta} - \omega^{\alpha\lambda} N^\beta_\lambda] + p = 0 , \end{aligned} \right\} \tag{2.17}$$

$$\text{SB:}\quad \left.\begin{aligned} & [N^{\alpha\beta} - b^{\alpha}_{\lambda}M^{\lambda\beta} - \omega^{\alpha\lambda}N^{\beta}_{\lambda}]\, \nu_\alpha\nu_\beta + \tau_t M_{t\nu} = T^*_{\nu\nu} + \tau_t M^*_{t\nu}\,, \\ & [N^{\alpha\beta} - \tfrac{1}{2}(b^{\alpha}_{\lambda}M^{\lambda\beta} + b^{\beta}_{\lambda}M^{\lambda\alpha}) - \tfrac{1}{2}(b^{\alpha}_{\lambda}M^{\lambda\beta} - b^{\beta}_{\lambda}M^{\lambda\alpha}) - \tfrac{1}{2}\,\omega^{\alpha\beta}N^{\lambda}_{\lambda} - \tfrac{1}{2}(\omega^{\alpha\lambda}N^{\beta}_{\lambda} + \omega^{\beta\lambda}N^{\alpha}_{\lambda}) + \\ & \qquad + \tfrac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} - \theta^{\beta\lambda}N^{\alpha}_{\lambda})]\, t_\alpha\nu_\beta - \sigma_t M_{t\nu} = T^*_{t\nu} - \sigma_t M^*_{t\nu}\,, \\ & (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})\,\nu_\beta + \frac{d}{ds}M_{t\nu} = T^*_{n\nu} + \frac{d}{ds}M^*_{t\nu}\,, \\ & M_{\nu\nu} = M^*_{\nu\nu}\,, \end{aligned}\right\} \tag{2.18}$$

$$\text{SC:}\quad M_{t\nu}(s_{fj}+0) - M_{t\nu}(s_{fj}-0) = M^*_{t\nu}(s_{fj}+0) - M^*_{t\nu}(s_{fj}-0) \tag{2.19}$$

where $\theta_{\alpha\beta}$, $\omega_{\alpha\beta}$, φ_α and φ are given by (2.1).

In the above relations $\mathbf{p} = p^\alpha\mathbf{a}_\alpha + p\mathbf{n}$ denotes the external distributed surface load, per unit area of $\mathcal{M}$. The star added to any symbol indicates a prescribed value of this quantity at the boundary $\mathcal{C}$ of $\mathcal{M}$. As usual, we define the unit tangent $\mathbf{t}$ and the unit normal $\boldsymbol{\nu}$ of the boundary $\mathcal{C}$, given by $\theta^\alpha = \theta^\alpha(s)$ in terms of its length parameter s, by $\mathbf{t} = d\mathbf{r}/ds = t^\alpha\mathbf{a}_\alpha$ and $\boldsymbol{\nu} = \mathbf{t}\times\mathbf{n} = \nu_\beta\mathbf{a}^\beta = \varepsilon_{\beta\alpha}t^\alpha\mathbf{a}^\beta$. In (2.18) $\sigma_t = b_{\alpha\beta}t^\alpha t^\beta$ is the normal curvature of $\mathcal{C}$, while $\tau_t = -b_{\alpha\beta}\nu^\alpha t^\beta$ is its geodesic torsion. By $u_\nu = u^\alpha\nu_\alpha$, $u_t = u^\alpha t_\alpha$ and w we denote the physical components of the displacement vector $\mathbf{u} = u_\nu\boldsymbol{\nu} + u_t\mathbf{t} + w\mathbf{n}$ at $\mathcal{C}$, while $\beta = -\varphi_\alpha\nu^\alpha$ is a fourth independent geometric parameter which describes the rotation of the boundary element about the tangent to $\mathcal{C}$, [32]. The quantities $T_{\nu\nu}$, $T_{t\nu}$ and $T_{n\nu}$ defined by the corresponding expressions on the left-hand sides of (2.18), and $M_{t\nu} = M^{\alpha\beta}t_\alpha\nu_\beta$, $M_{\nu\nu} = M^{\alpha\beta}\nu_\alpha\nu_\beta$ are physical components of the resultant stress and moment vectors, per unit length of $\mathcal{C}$, which, in turn, are given by

$$\begin{aligned} \mathbf{T}^\beta = [N^{\alpha\beta} & - \tfrac{1}{2}(b^{\alpha}_{\lambda}M^{\lambda\beta} + b^{\beta}_{\lambda}M^{\lambda\alpha}) - \tfrac{1}{2}(b^{\alpha}_{\lambda}M^{\lambda\beta} - b^{\beta}_{\lambda}M^{\lambda\alpha}) - \tfrac{1}{2}\,\omega^{\alpha\beta}N^{\lambda}_{\lambda} - \\ & - \tfrac{1}{2}(\omega^{\alpha\lambda}N^{\beta}_{\lambda} + \omega^{\beta\lambda}N^{\alpha}_{\lambda}) + \tfrac{1}{2}(\theta^{\alpha\lambda}N^{\beta}_{\lambda} - \theta^{\beta\lambda}N^{\alpha}_{\lambda})]\,\mathbf{a}_\alpha + (M^{\alpha\beta}|_\alpha + \varphi_\alpha N^{\alpha\beta})\,\mathbf{n}\,, \end{aligned} \tag{2.20}$$

$$\left.\begin{aligned} & \mathbf{T}_\nu = \mathbf{T}^\beta\nu_\beta = T_{\nu\nu}\boldsymbol{\nu} + T_{t\nu}\mathbf{t} + T_{n\nu}\mathbf{n}\,, \\ & \mathbf{M}^\beta = \varepsilon_{\alpha\lambda}M^{\alpha\beta}\mathbf{a}^\lambda\,, \qquad \mathbf{M}_\nu = \mathbf{M}^\beta\nu_\beta = -M_{t\nu}\boldsymbol{\nu} + M_{\nu\nu}\mathbf{t}\,. \end{aligned}\right\} \tag{2.21}$$

By $\mathcal{C}_u$ we denote the part of $\mathcal{C}$ on which at least one of the four geometric variables of (2.15) is prescribed. By $\mathcal{C}_f$ we denote the part of $\mathcal{C}$ on which at least one component of the external force $\mathbf{T}^*_\nu = T^*_{\nu\nu}\boldsymbol{\nu} + T^*_{t\nu}\mathbf{t} + T^*_{n\nu}\mathbf{n}$ or of the external moment $\mathbf{M}^*_\nu = -M^*_{t\nu}\boldsymbol{\nu} + M^*_{\nu\nu}\mathbf{t}$ is prescribed. Thus, any of the four combinations indicated in (2.18) are possible. In general, we may prescribe on $\mathcal{C}$ either $(2.15)_1$ or $(2.18)_1$, either $(2.15)_2$ or $(2.18)_2$ etc., respectively. In the case of mixed (mutually complementary) boundary conditions [25] the boundary parts $\mathcal{C}_u$ and $\mathcal{C}_f$ coincide. They may become separated if all four GB are assumed given on $\mathcal{C}_u$ and all four SB are assumed given on $\mathcal{C}_f$. In general, $\mathcal{C}_u$ may have m corners located at $s = s_{ui}$, $i = 1, \dots, m$. Likewise, $\mathcal{C}_f$ may have n corners and points, where $M^*_{t\nu}$ is discontinuous, located at $s = s_{fj}$, $j = 1, \dots, n$. In general, such a point may belong simultaneously to both $\mathcal{C}_u$ and $\mathcal{C}_f$. Let $r \le m + n$ be the total number of these corners and discontinuity points $M_k \in \mathcal{C}$ located at $s = s_k$ $(k = 1, \dots, r)$. Then, at each M_k either (2.16) or (2.19) may be prescribed as corner condition.

The set of shell relations (2.14)—(2.19) contains, as special cases, the equations of various simplified variants of the geometrically non-linear theory of shells and plates, which have been proposed in the literature. We have indicated some of these by underlining in (2.14) those terms which do not appear in the particular variant. As a result, terms underlined in the same way do not appear in EQ and SB as well as in the variational functionals given in the following parts of this paper.

If terms marked by a double solid line are omitted, one obtains reduced shell equations equivalent to those undergoing "moderately small rotations" proposed by Sanders [34] and those undergoing "small finite deflections" given by Koiter [35]. In [34] terms, here underlined by dots, were neglected as well, while in [35] a modified tensor of change of curvature $\varrho_{\alpha\beta} \equiv \equiv \varkappa_{\alpha\beta} + \frac{1}{2}(b^\lambda_\alpha \theta_{\lambda\beta} + b^\lambda_\beta \theta_{\lambda\alpha})$ was used. The differences in the definition of the changes of curvature are negligible to within the relative error of $O(\theta^2)$ of the strain energy function (2.3). But by neglecting terms underlined by a double solid line a larger relative error of $O(\theta)$ may be introduced into (2.3). Within this larger error margin terms marked by a dot and dash line may also be omitted. This leads to the shell equations derived in [31, 33].

If the shell deformation is assumed to be such that the shell material elements undergo moderate rotations about the tangents of $\mathcal{M}$ only, while the rotations about the normal of $\mathcal{M}$ are assumed small, the terms underlined once or twice by a solid line, by dots and by a dot and dash line can be omitted. This is consistent with the error margin of the strain energy function (2.3), [31]. Such a simplified variant of the moderate rotation theory of shells was discussed in [17, 31, 33—35]. Under additional simplifying assumptions [31, 35], leading to $\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2} w_{,\alpha} w_{,\beta}$, $\varkappa_{\alpha\beta} = -w|_{\alpha\beta}$ and allowing the interchange of the sequence of covariant differentiation, this version reduces to the classical non-linear theory of shallow shells [12—14, 17, 30]. If in addition $b_{\alpha\beta} \equiv 0$ we obtain von Kármán's [36] non-linear plate equations.

Finally, if all terms underlined by broken, single solid and double solid lines are omitted, the above set of shell relations reduces to the classical linear theory of shells [6, 9]. If furthermore terms underlined by dots are omitted (or, equivalently, if $\varrho_{\alpha\beta}$ is used as the tensor of change of curvature) we obtain the so called "best" linear shell theory [42, 44].

For the following derivations it is convenient to use some abbreviated notation. We shall denote the set of displacement variables by $u \equiv (\mathbf{u}, \beta, w_k)$, the shell strain measures by $\varepsilon \equiv \equiv (\gamma_{\alpha\beta}, \varkappa_{\alpha\beta})$ and the shell stress measures by $\sigma \equiv (N^{\alpha\beta}, M^{\alpha\beta})$. We also introduce $\mathbf{Q} \equiv \mathbf{T}^\beta|_\beta$ and define the effective internal boundary force and moment (BF) on $\mathcal{C}_f$ and the internal concentrated force (CF) at each $M_k \in \mathcal{C}$ by the relations

$$\left.\begin{aligned} &\text{BF:} \quad \mathbf{P} = \mathbf{T}_\nu + \frac{d}{ds}(M_{t\nu}\mathbf{n}), \qquad M = M_{\nu\nu}, \\ &\text{CF:} \quad F_k = M_{t\nu}(s_k + 0) - M_{t\nu}(s_k - 0), \qquad k = 1, \dots, r \le m + n. \end{aligned}\right\} \tag{2.22}$$

With these the relations (2.14—18) can be written in the following abbreviated form:

$$\left.\begin{array}{llll} \text{SD:} & \gamma_{\alpha\beta} - \gamma_{\alpha\beta}(u) = 0, & \varkappa_{\alpha\beta} - \varkappa_{\alpha\beta}(u) = 0 & \text{in } \mathcal{M}, \\ \text{GB:} & \mathbf{u} - \mathbf{u}^* = \mathbf{0}, & \beta - \beta^* = 0 & \text{on } \mathcal{C}_u, \\ \text{GC:} & w_i - w_i^* = 0 & & \text{at each } M_i \in \mathcal{C}_u, \\ \text{EQ:} & \mathbf{Q}(u, \sigma) + \mathbf{p} = \mathbf{0} & & \text{in } \mathcal{M}, \\ \text{SB:} & \mathbf{P}(u, \sigma) - \mathbf{P}^* = \mathbf{0}, & M(\sigma) - M^* = 0 & \text{on } \mathcal{C}_f, \\ \text{SC:} & F_j(\sigma) - F_j^* = 0 & & \text{at each } M_j \in \mathcal{C}_f. \end{array}\right\} \tag{2.23}$$

Using the identities $\mathbf{a}_{\alpha|\beta} = b_{\alpha\beta}\mathbf{n}$, $\mathbf{n}_{|\beta} = -b^\lambda_\beta \mathbf{a}_\lambda$ and $d\mathbf{n}/ds = \tau_t \boldsymbol{\nu} - \sigma_t \mathbf{t}$ together with (2.20—22), it is easy to verify that EQ and SB in (2.23) are indeed identical with (2.17) and (2.18), respectively.

In (2.23) the dependence of certain quantities upon displacements and shell stress measures has been explicitly indicated. This symbolic notation will be used, if necessary, also for EQ, SB, SC, BF and CF, e.g. EQ (u, σ). Similarly, if the shell strain measures are introduced explicitly by CE they will be denoted in the same way, i.e. by writing e.g. EQ$[u, \sigma(\varepsilon)] \equiv$ EQ(u, ε). If the strain measures are expressed in terms of displacements with the aid of SD, we shall indicate this by writing e.g. EQ$[u, \sigma(u)] \equiv$ EQ(u). The same convenient notation will be used, if necessary, for $\mathbf{Q}$, $\mathbf{P}$, M and F_k. Likewise, if in SD the strain measures are expressed by means of the stress measures with the aid of IC, it will be indicated by writing SD$[u, \varepsilon(\sigma)] \equiv$ SD(u, σ).

3 The Principle of Virtual Displacements

Following [10], we shall derive all subsequent variational principles by starting from the principle of virtual displacements. All static relations (2.17–19) of the non-linear boundary value problem follow directly from the statement

$$\iint_{\mathcal{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\varkappa_{\alpha\beta} - \mathbf{p}\cdot\delta\mathbf{u})\,dA - \int_{\mathscr{C}_f} (\mathbf{P}^*\cdot\delta\mathbf{u} + M^*\delta\beta)\,ds - \sum_j F_j^*\delta w_j = 0\,, \qquad (3.1)$$

which is the appropriate form of the Lagrangean principle of virtual displacements for all shell theories discussed in chapter 2 [32, 33, 43]. It holds for all (additional infinitesimal) virtual displacement and strain fields satisfying SD, GB and GC.

For elastic shells the first two terms in (3.1), representing the internal virtual work, can be expressed as a variation of the shell strain energy function (2.3): $\delta\Sigma(\varepsilon) = N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\varkappa_{\alpha\beta}$. If, in addition, we assume that the external loads $\mathbf{p}$, $\mathbf{T}_\nu^*$ and $\mathbf{M}_\nu^*$ are of dead-load type, then there exist potential functions $\Phi(u) = -\mathbf{p}\cdot\mathbf{u}$, $\chi(u) = -(\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)$ and $\psi(u) = -F_j^*w_j$, the variations of which constitute the external virtual work of the various loads: $\delta\Phi(u) = -\mathbf{p}\cdot\delta\mathbf{u}$, $\delta\chi(u) = -(\mathbf{P}^*\cdot\delta\mathbf{u} + M^*\delta\beta)$ and $\delta\psi(u) = -F_j^*\delta w_j$. In this case the principle of virtual displacements (3.1) can be transformed into a variational principle of the form $\delta I = 0$, where the functional I is given by

$$I(u, \varepsilon) = \iint_{\mathcal{M}} [\Sigma(\varepsilon) - \mathbf{p}\cdot\mathbf{u}]\,dA - \int_{\mathscr{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)\,ds - \sum_j F_j^*w_j \qquad (3.2)$$

and where SD, GB and GC have to be imposed as subsidiary conditions. This principle states that among all geometrically admissible displacements and strain measures (i.e. among all those satisfying SD, GB and GC) the actual solution (u_0, ε_0) renders the functional I stationary.

The functional $I(u, \varepsilon)$ is defined here for all u and ε satisfying SD, GB and GC. Other functionals (and associated variational principles) will be defined in terms of various sets of independent field variables. Among the variety of functionals which may be constructed from $I(u, \varepsilon)$ the so-called free functionals, defined for certain sets of free field variables which are not subject to subsidiary conditions, are particularly useful. In what follows, a set of sixteen basic free functionals will be derived. From each of them a variety of other free functionals and functionals with subsidiary conditions may easily be obtained.

In order to express the various functionals in terms of different sets of independent field variables the transformation procedures suggested by Courant and Hilbert [37] will be used. The Lagrangean multiplier method is applied to eliminate subsidiary conditions and to introduce them into the functional itself. On the other hand, general solutions satisfying subsidiary conditions and/or stationarity conditions (i.e. Euler-Lagrange equations and natural boundary conditions) are used to eliminate certain independent field variables from the functionals. In both cases the transformed variational problem is equivalent to the original one as far as stationarity properties are concerned.

4 Free Functionals and Related Variational Principles

4.1 Four Independent Fields

First all subsidiary conditions of $I(u, \varepsilon)$ are introduced into the functional itself by use of the Lagrangean multiplier method. Then we obtain the free functional

$$\begin{aligned} I_1(u, \varepsilon, \sigma, f) = &\iint_{\mathcal{M}} \{\Sigma(\varepsilon) - \mathbf{p}\cdot\mathbf{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(u)] - M^{\alpha\beta}[\varkappa_{\alpha\beta} - \varkappa_{\alpha\beta}(u)]\}\,dA - \\ &- \int_{\mathscr{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)\,ds - \sum_j F_j^*w_j - \\ &- \int_{\mathscr{C}_u} [\mathbf{P}\cdot(\mathbf{u} - \mathbf{u}^*) + M(\beta - \beta^*)]\,ds - \sum_i F_i(w_i - w_i^*)\,, \end{aligned} \qquad (4.1)$$

where the right-hand sides of (2.3) and (2.14) together with (2.1) have to be introduced. Here, irrespective of previous notations, the quantities $\sigma = (N^{\alpha\beta}, M^{\alpha\beta})$ and $f = (\mathbf{P}, M, F_i)$ are momentarily considered to be two sets of Lagrange multipliers, by means of which the equations SD, GB and GC are included into the functional.

The functional I_1 is defined in terms of four fields of independent variables $(u, \varepsilon, \sigma, f)$ subject to variation: three displacement components $\mathbf{u}$ in $\mathcal{M}$, four displacement parameters $\mathbf{u}$ and β on $\mathscr{C}$, one normal displacement w_k at each corner $M_k \in \mathscr{C}$, six strain components $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ in $\mathcal{M}$, six Lagrange multipliers $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in $\mathcal{M}$, four Lagrange multipliers $\mathbf{P}$ and M on $\mathscr{C}_u$ and one Lagrange multiplier F_i at each corner $M_i \in \mathscr{C}_u$.

The associated variational principle $\delta I_1 = 0$ states that among all $(u, \varepsilon, \sigma, f)$, not restricted by any subsidiary condition, the actual solution $(u_0, \varepsilon_0, \sigma_0, f_0)$ renders the functional I_1 stationary.

Taking the first variation of I_1 involves rather lengthly operations which are given in detail in [43]. The result is

$$\begin{aligned}
\delta I_1 = &- \iint_{\mathcal{M}} [\mathbf{Q}(u, \sigma) + \mathbf{p}] \cdot \delta\mathbf{u}\, dA + \int_{\mathscr{C}_f} \{[\mathbf{P}(u, \sigma) - \mathbf{P}^*] \cdot \delta\mathbf{u} + \\
&+ [M(\sigma) - M^*]\, \delta\beta\}\, ds + \int_{\mathscr{C}_u} \{[\mathbf{P}(u, \sigma) - \mathbf{P}] \cdot \delta\mathbf{u} + [M(\sigma) - M]\, \delta\beta\}\, ds + \\
&+ \sum_j [F_j(\sigma) - F_j^*]\, \delta w_j + \sum_i [F_i(\sigma) - F_i]\, \delta w_i + \\
&+ \iint_{\mathcal{M}} \left[(hH^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu} - N^{\alpha\beta})\, \delta\gamma_{\alpha\beta} + \left(\frac{h^3}{12} H^{\alpha\beta\lambda\mu}\varkappa_{\lambda\mu} - M^{\alpha\beta}\right) \delta\varkappa_{\alpha\beta}\right] dA - \\
&- \iint_{\mathcal{M}} \{[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(u)]\, \delta N^{\alpha\beta} + [\varkappa_{\alpha\beta} - \varkappa_{\alpha\beta}(u)]\, \delta M^{\alpha\beta}\}\, dA - \\
&- \int_{\mathscr{C}_u} [(\mathbf{u} - \mathbf{u}^*) \cdot \delta\mathbf{P} + (\beta - \beta^*)\, \delta M]\, ds - \sum_i (w_i - w_i^*)\, \delta F_i\,. \qquad (4.2)
\end{aligned}$$

It can be seen from (4.2) that the stationarity conditions of the functional I_1 are EQ, SB, SC, SD, GB, GC together with additional relations, which show the Lagrange multipliers $N^{\alpha\beta}$, $M^{\alpha\beta}$ to be indeed the shell stress measures in $\mathcal{M}$, $\mathbf{P}$ and M to be effective reactive boundary force and moment on $\mathscr{C}_u$ and F_i to be the reactive concentrated force at each corner $M_i \in \mathscr{C}_u$, respectively, as has already been anticipated by using the appropriate symbols in (4.1). These latter relations are CE, BF on $\mathscr{C}_u$ and CF at all $M_i \in \mathscr{C}_u$.

Therefore, the variational principle $\delta I_1 = 0$ is equivalent to the complete set of relations of the geometrically non-linear theory of shells undergoing moderate rotations. The functional I_1 has not so far been published in the literature for any variant of the non-linear shell equations discussed in chapter 2. It may be related to the functional Π_1 given in [30], if appropriate simplifications are made and the quantities are referred to the undeformed shell geometry. Within the context of the non-linear theory of shallow shells a corresponding functional, in terms of linearized strain measures and rotations and for smooth boundaries, has been given in [24]. In accordance with the terminology of the three-dimensional theory of elasticity [10] we may call $\delta I_1 = 0$ the Hu-Washizu principle for the geometrically non-linear theory of shells undergoing moderate rotations.

By applying integration by parts together with Stokes' theorem to terms included in $\gamma_{\alpha\beta}(u)$ and $\varkappa_{\alpha\beta}(u)$ in (4.1), after some involved transformations (see again [43] for details) one obtains from I_1 an equivalent free functional

$$\begin{aligned}
J_1(u, \varepsilon, \sigma, f) = &- \iint_{\mathcal{M}} \{N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\varkappa_{\alpha\beta} - \Sigma(\varepsilon) + N^{\alpha\beta}\Phi_{\alpha\beta}(u) + [\mathbf{Q}(u, \sigma) + \mathbf{p}] \cdot \mathbf{u}\}\, dA + \\
&+ \int_{\mathscr{C}_f} \{[\mathbf{P}(u, \sigma) - \mathbf{P}^*] \cdot \mathbf{u} + [M(\sigma) - M^*]\, \beta\}\, ds + \sum_j [F_j(\sigma) - F_j^*]\, w_j + \\
&+ \int_{\mathscr{C}_u} \{[\mathbf{P}(u, \sigma) - \mathbf{P}] \cdot \mathbf{u} + [M(\sigma) - M]\, \beta + \mathbf{P} \cdot \mathbf{u}^* + M\beta^*\}\, ds + \\
&+ \sum_i [F_i(\sigma) - F_i]\, w_i + \sum_i F_i w_i^*\,. \qquad (4.3)
\end{aligned}$$

Here, taking into account (2.1), we have introduced as abbreviation for the non-linear part of SD the following quantity:

$$\Phi_{\alpha\beta}(u) = \tfrac{1}{2}\,\varphi_\alpha\varphi_\beta + \tfrac{1}{2}\,a_{\alpha\beta}\varphi^2 - \tfrac{1}{2}\,(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha})\,. \tag{4.4}$$

The functional J_1 contains the same set of independent free fields subject to variation as the functional I_1. A detailed calculation [43] shows that indeed $\delta J_1 = \delta I_1$ and, therefore, the stationarity conditions of J_1 are the same as for I_1. The functional J_1 did not appear in the literature as well for any variant of non-linear shell equations discussed in chapter 2. Using the terminology which is applied in the linear three-dimensional theory of elasticity to corresponding functionals with a similar structure [45], we may call $\delta J_1 = 0$ a principle of generalized total complementary energy for the geometrically non-linear theory of shells undergoing moderate rotations.

It is worthwhile to note certain interesting aspects of J_1. The first three terms in the surface integral of (4.3) represent the shell complementary energy function in the form of the Legendre transformation (2.13). In the next term the internal force resultants are multiplied, according to (4.4), by the non-linear part of $(2.14)_1$. These terms are completely analogous to those appearing in similar functionals of the geometrically non-linear theory of elasticity [10], in which the Kirchhoff stress tensor is multiplied by the non-linear part of the Green strain tensor. Moreover, the displacement variables u appear in (4.3) as multipliers of certain terms, which represent the static relations (i.e. EQ, SB, SC, BF and CF) of the boundary value problem. This is analogous to the case of I_1 in (4.1), where the static variables σ and f appear as multipliers of certain terms, which represent the geometric relations (SD, SB and SC) of the shell problem under consideration.

4.2 *Three Independent Fields*

In this section we present six free functionals and related variational principles, in which only three independent fields are subject to variation. The functionals can be derived from I_1 and J_1 by eliminating one of the independent fields ε, σ or f with the help of general solutions satisfying IC, CE, or SB and BF, respectively. For details and proofs of the stationarity properties we refer to [43].

First, eliminate the strain measures ε from I_1 by using the inverse constitutive equations (2.9). As a result we obtain the free functional

$$\begin{aligned} I_2(u,\sigma,f) = \iint_{\mathcal{M}} [-\Sigma_c(\sigma) + N^{\alpha\beta}\gamma_{\alpha\beta}(u) + M^{\alpha\beta}\varkappa_{\alpha\beta}(u) - \mathbf{p}\cdot\mathbf{u}]\,dA - \int_{\mathcal{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)\,ds\, - \\ - \sum_j F^*_j w_j - \int_{\mathcal{C}_u} [\mathbf{P}\cdot(\mathbf{u}-\mathbf{u}^*) + M(\beta-\beta^*)]\,ds - \sum_i F_i(w_i - w^*_i)\,. \end{aligned} \tag{4.5}$$

The variational statement $\delta I_2 = 0$ is the Hellinger-Reissner principle for the geometrically non-linear theory of shells undergoing moderate rotations.

By applying integration by parts together with Stokes' theorem to terms included in $\gamma_{\alpha\beta}(u)$ and $\varkappa_{\alpha\beta}(u)$, the functional I_2 may be transformed into an equivalent functional J_2, which also follows directly from J_1, when the strain measures ε are eliminated

$$\begin{aligned} J_2(u,\sigma,f) = &-\iint_{\mathcal{M}} \{\Sigma_c(\sigma) + N^{\alpha\beta}\Phi_{\alpha\beta}(u) + [\mathbf{Q}(u,\sigma) + \mathbf{p}]\cdot\mathbf{u}\}\,dA\, + \\ &+ \int_{\mathcal{C}_f} \{[\mathbf{P}(u,\sigma) - \mathbf{P}^*]\cdot\mathbf{u} + [M(\sigma) - M^*]\,\beta\}\,ds + \sum_j [F_j(\sigma) - F^*_j]\,w_j\, + \\ &+ \int_{\mathcal{C}_u} \{[\mathbf{P}(u,\sigma) - \mathbf{P}]\cdot\mathbf{u} + [M(\sigma) - M]\,\beta + \mathbf{P}\cdot\mathbf{u}^* + M\beta^*\}\,ds\, + \\ &+ \sum_i [F_i(\sigma) - F_i]\,w_i + \sum_i F_i w^*_i\,. \end{aligned} \tag{4.6}$$

In both functionals I_2 and J_2 the same independent free fields (u, σ, f) are subject to variation. The variations $\delta I_2 = \delta J_2$ are given by (4.2), where now the fourth line vanishes identically and SD appear in the fifth line in the transformed form $\varepsilon(\sigma) - \varepsilon(u) = 0$.

Accordingly, the stationarity conditions of I_2 and J_2 are EQ, SB, SC, BF on $\mathscr{C}_u$, CF at each $M_\iota \in \mathscr{C}_u$, SD$(u, \sigma)$, GB and GC. When the actual solution (u_0, σ_0, f_0) is known, for which I_2 and J_2 assume their stationary value, the strain measures ε may be found, if necessary, with the help of (2.9) outside the variational problem.

Next, eliminate the stress measures σ from I_1 and J_1 by using the constitutive equations (2.5) to obtain the following free functionals I_3 and J_3, defined for the same set of independent fields:

$$\begin{aligned} I_3(u, \varepsilon, f) = &- \iint_{\mathscr{M}} \{\Sigma(\varepsilon) + \mathbf{p} \cdot \mathbf{u} - hH^{\alpha\beta\lambda\mu} [\gamma_{\alpha\beta}\gamma_{\lambda\mu}(u) + (h^2/12)\, \varkappa_{\alpha\beta}\varkappa_{\lambda\mu}(u)]\}\, dA - \\ &- \int_{\mathscr{C}_f} (\mathbf{P}^* \cdot \mathbf{u} + M^*\beta)\, ds - \sum_j F_j^* w_j - \\ &- \int_{\mathscr{C}_u} [\mathbf{P} \cdot (\mathbf{u} - \mathbf{u}^*) + M(\beta - \beta^*)]\, ds - \sum_i F_i(w_i - w_i^*)\,, \end{aligned} \tag{4.7}$$

$$\begin{aligned} J_3(u, \varepsilon, f) = &- \iint_{\mathscr{M}} \{\Sigma(\varepsilon) + hH^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\Phi_{\lambda\mu}(u) + [\mathbf{Q}(u, \varepsilon) + \mathbf{p}] \cdot \mathbf{u}\}\, dA + \\ &+ \int_{\mathscr{C}_f} \{[\mathbf{P}(u, \varepsilon) - \mathbf{P}^*] \cdot \mathbf{u} + [M(\varepsilon) - M^*]\, \beta\}\, ds + \sum_j [F_j(\varepsilon) - F_j^*]\, w_j + \\ &+ \int_{\mathscr{C}_u} \{[\mathbf{P}(u, \varepsilon) - \mathbf{P}] \cdot \mathbf{u} + [M(\varepsilon) - M]\, \beta + \mathbf{P} \cdot \mathbf{u}^* + M\beta^*\}\, ds + \\ &+ \sum_i [F_i(\varepsilon) - F_i]\, w_i + \sum_i F_i w_i^*\,. \end{aligned} \tag{4.8}$$

The expressions for the variations $\delta I_3 = \delta J_3$ are given again by (4.2), where now the fourth line vanishes identically and σ is expressed everywhere in terms of ε. The stationarity conditions of I_3 and J_3 are EQ(u, ε), SB(u, ε), SC(ε), BF(u, ε) on $\mathscr{C}_u$, CF(ε) at each $M_i \in \mathscr{C}_u$, SD, GB and GC. For the actual solution $(u_0, \varepsilon_0, f_0)$, for which I_3 and J_3 assume their stationary value, the stress measures σ may be obtained with the help of (2.5).

Finally, the field f may be eliminated from I_1 and J_1 by using (2.22). One obtains the following free functionals, defined for the same set of independent fields:

$$\begin{aligned} I_4(u, \varepsilon, \sigma) = &\iint_{\mathscr{M}} \{\Sigma(\varepsilon) - \mathbf{p} \cdot \mathbf{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(u)] - M^{\alpha\beta}[\varkappa_{\alpha\beta} - \varkappa_{\alpha\beta}(u)]\}\, dA - \\ &- \int_{\mathscr{C}_f} (\mathbf{P}^* \cdot \mathbf{u} + M^*\beta)\, ds - \sum_j F_j^* w_j - \\ &- \int_{\mathscr{C}_u} [\mathbf{P}(u, \sigma) \cdot (\mathbf{u} - \mathbf{u}^*) + M(\sigma)\, (\beta - \beta^*)]\, ds - \sum_i F_i(\sigma)\, (w_i - w_i^*)\,, \end{aligned} \tag{4.9}$$

$$\begin{aligned} J_4(u, \varepsilon, \sigma) = &- \iint_{\mathscr{M}} \{N^{\alpha\beta}\gamma_{\alpha\beta} + M^{\alpha\beta}\varkappa_{\alpha\beta} - \Sigma(\varepsilon) + \\ &+ N^{\alpha\beta}\Phi_{\alpha\beta}(u) + [\mathbf{Q}(u, \sigma) + \mathbf{p}] \cdot \mathbf{u}\}\, dA + \\ &+ \int_{\mathscr{C}_f} \{[\mathbf{P}(u, \sigma) - \mathbf{P}^*] \cdot \mathbf{u} + [M(\sigma) - M^*]\, \beta\}\, ds + \sum_j [F_j(\sigma) - F_j^*]\, w_j + \\ &+ \int_{\mathscr{C}_u} [\mathbf{P}(u, \sigma) \cdot \mathbf{u}^* + M(\sigma)\, \beta^*]\, ds + \sum_i F_i(\sigma)\, w_i^*\,. \end{aligned} \tag{4.10}$$

The variations $\delta I_4 = \delta J_4$ are again given by (4.2), where the line integral over $\mathscr{C}_u$ in the second line and the sum over i in the third line vanish identically and $\delta\mathbf{P}(u, \sigma)$, $\delta M(\sigma)$ and $\delta F_j(\sigma)$ should be substituted for $\delta\mathbf{P}$, δM and δF_j in the last line, respectively. Then, the stationarity conditions of I_4 and J_4 are EQ, SB, SC, CE, SD, GB and GC.

Just as the variational principles associated with the functionals (4.1) and (4.3) the variational theorems $\delta I_4 = 0$ and $\delta J_4 = 0$ may also be termed the Hu-Washizu principle [10] and the principle of generalized total complementary energy [45], respectively, for the geometrically non-linear theory of shells undergoing moderate rotations.

The functional I_4 may be related to the functional Π_2 of [30]. Within the context of the non-linear theory of shallow shells special cases of I_4 are given in [20, 23] and, in terms of linearized strains and rotations, in [24]. It also contains a functional presented in [22] for a Donnell-Marguerre type membrane theory of shells as a special case.

4.3 *Two Independent Fields*

In this section we present six free functionals, in which only two independent fields are subject to variation. The functionals are derived in the now familiar way, by eliminating either any two of the three fields ε, σ, f from I_1 and J_1 or one of them from the corresponding three-field functional given in section 4.2. For details and proofs of the stationarity conditions we refer again to [43].

First, eliminate ε from I_3 and J_3 by using the strain-displacement relations (2.14). This gives the following free functionals:

$$\begin{aligned} I_5(u,f) = {} & \iint_{\mathcal{M}} [\Sigma(u) - \mathbf{p}\cdot\mathbf{u}]\, dA - \int_{\mathcal{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)\, ds - \sum_j F_j^* w_j - \\ & - \int_{\mathcal{C}_u} [\mathbf{P}\cdot(\mathbf{u}-\mathbf{u}^*) + M(\beta-\beta^*)]\, ds - \sum_i F_i(w_i - w_i^*)\,, \end{aligned} \tag{4.12}$$

$$\begin{aligned} J_5(u,f) = {} & - \iint_{\mathcal{M}} \{\Sigma(u) + hH^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\,\Phi_{\lambda\mu}(u) + [\mathbf{Q}(u)+\mathbf{p}]\cdot\mathbf{u}\}\, dA + \\ & + \int_{\mathcal{C}_f} \{[\mathbf{P}(u) - \mathbf{P}^*]\cdot\mathbf{u} + [M(u) - M^*]\,\beta\}\, ds + \sum_j [F_j(u) - F_j^*]\, w_j + \\ & + \int_{\mathcal{C}_u} \{[\mathbf{P}(u) - \mathbf{P}]\cdot\mathbf{u} + [M(u) - M]\,\beta + \mathbf{P}\cdot\mathbf{u}^* + M\beta^*\}\, ds + \\ & + \sum_i [F_i(u) - F_i]\, w_i + \sum_i F_i w_i^*\,. \end{aligned} \tag{4.13}$$

The stationarity conditions associated with I_5 and J_5 follow again from further reduction of (4.2) to be EQ(u), SB(u), SC(u), BF(u) on $\mathcal{C}_u$, CF(u) at each $M_i \in \mathcal{C}_u$, GB and GC. From the actual solution (u_0, f_0), for which the functionals I_5 and J_5 assume their stationary value, the shell strain and stress measures ε and σ may be obtained, if necessary, by using (2.14) and (2.5).

Next, eliminate σ from I_4 and J_4 with the aid of the constitutive equations (2.5). This leads to

$$\begin{aligned} I_6(u,\varepsilon) = {} & - \iint_{\mathcal{M}} \{\Sigma(\varepsilon) + \mathbf{p}\cdot\mathbf{u} - hH^{\alpha\beta\lambda\mu}[\gamma_{\alpha\beta}\gamma_{\lambda\mu}(u) + (h^2/12)\,\varkappa_{\alpha\beta}\varkappa_{\lambda\mu}(u)]\}\, dA - \\ & - \int_{\mathcal{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta) - \sum_j F_j^* w_j - \\ & - \int_{\mathcal{C}_u} [\mathbf{P}(u,\varepsilon)\cdot(\mathbf{u}-\mathbf{u}^*) + M(\varepsilon)\,(\beta-\beta^*)]\, ds - \sum_i F_i(\varepsilon)\,(w_i - w_i^*)\,, \end{aligned} \tag{4.14}$$

$$\begin{aligned} J_6(u,\varepsilon) = {} & - \iint_{\mathcal{M}} \{\Sigma(\varepsilon) + hH^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}\Phi_{\lambda\mu}(u) + [\mathbf{Q}(u,\varepsilon) + \mathbf{p}]\cdot\mathbf{u}\}\, dA + \\ & + \int_{\mathcal{C}_f} \{[\mathbf{P}(u,\varepsilon) - \mathbf{P}^*]\cdot\mathbf{u} + [M(\varepsilon) - M^*]\,\beta\}\, ds + \sum_j [F_j(\varepsilon) - F_j^*]\, w_j + \\ & + \int_{\mathcal{C}_u} [\mathbf{P}(u,\varepsilon)\cdot\mathbf{u}^* + M(\varepsilon)\,\beta^*]\, ds + \sum_i F_i(\varepsilon)\, w_i^*\,. \end{aligned} \tag{4.15}$$

The stationarity conditions of I_6 and J_6 are EQ(u, ε), SB(u, ε), SC(ε), SD, GB and GC. From the actual solution (u_0, ε_0) which renders the functionals I_6 and J_6 stationary, the shell stress measures σ and the reactive boundary loads f may be calculated by using (2.5) and (2.22).

Finally, the field f will be eliminated from I_2 and J_2 with the help of (2.22) to give

$$\begin{aligned} I_7(u,\sigma) = {} & \iint_{\mathcal{M}} [-\Sigma_c(\sigma) + N^{\alpha\beta}\gamma_{\alpha\beta}(u) + M^{\alpha\beta}\varkappa_{\alpha\beta}(u) - \mathbf{p}\cdot\mathbf{u}]\, dA - \\ & - \int_{\mathcal{C}_f} (\mathbf{P}^*\cdot\mathbf{u} + M^*\beta)\, ds - \sum_j F_j^* w_j - \\ & - \int_{\mathcal{C}_u} [\mathbf{P}(u,\sigma)\cdot(\mathbf{u}-\mathbf{u}^*) + M(\sigma)\,(\beta-\beta^*)]\, ds - \sum_i F_i(\sigma)\,(w_i - w_i^*)\,, \end{aligned} \tag{4.16}$$

$$\begin{aligned} J_7(u,\sigma) = {} & - \iint_{\mathcal{M}} \{\Sigma_c(\sigma) + N^{\alpha\beta}\Phi_{\alpha\beta}(u) + [\mathbf{Q}(u,\sigma) + \mathbf{p}]\cdot\mathbf{u}\}\, dA + \\ & + \int_{\mathcal{C}_f} \{[\mathbf{P}(u,\sigma) - \mathbf{P}^*]\cdot\mathbf{u} + [M(\sigma) - M^*]\,\beta\}\, ds + \sum_j [F_j(\sigma) - F_j^*]\, w_j + \\ & + \int_{\mathcal{C}_u} [\mathbf{P}(u,\sigma)\cdot\mathbf{u}^* + M(\sigma)\,\beta^*]\, ds + \sum_i F_i(\sigma)\, w_i^*\,. \end{aligned} \tag{4.17}$$

Just as the variational principle associated with the functional (4.5), the variational statement $\delta I_7 = 0$ may also be called the Hellinger-Reissner principle for the geometrically non-linear theory of shells undergoing moderate rotations. The stationarity conditions of I_7 and J_7 are EQ, SB, SC, SD(u, σ), GB and GC. When (u_0, σ_0) is known, for which I_7 and J_7 assume their stationary value, the strain measures ε and the reactive boundary loads f may be calculated from (2.9) and (2.22).

The functional I_7 corresponds to the functional $-\Pi_3$ of [30]. For the non-linear theory of shallow shells with smooth boundaries a similar functional was given in [24] in terms of linearized strains and rotations.

4.4 *Independent Displacement Field*

In certain numerical applications it is convenient to use a free functional where only one independent field is subject to variation. For the non-linear shell theory discussed here it is possible to construct two such functionals in terms of displacements as independent fields. They follow from any pair of the six two-field functionals discussed in section 4.3 by elimination of one of the independent fields. In this way we obtain

$$
\begin{aligned}
I_8(u) = & \iint_{\mathcal{M}} [\Sigma(u) - \mathbf{p} \cdot \mathbf{u}] \, dA - \int_{\mathcal{C}_f} (\mathbf{P}^* \cdot \mathbf{u} + M^*\beta) \, ds - \sum_j F_j^* w_j - \\
& - \int_{\mathcal{C}_u} [\mathbf{P}(u) \cdot (\mathbf{u} - \mathbf{u}^*) + M(u)(\beta - \beta^*)] \, ds - \sum_i F_i(u)(w_i - w_i^*) \,,
\end{aligned} \tag{4.18}
$$

$$
\begin{aligned}
J_8(u) = & - \iint_{\mathcal{M}} \{\Sigma(u) + hH^{\alpha\beta\lambda\mu}\gamma_{\alpha\beta}(u)\,\Phi_{\lambda\mu}(u) + [\mathbf{Q}(u) + \mathbf{p}] \cdot \mathbf{u}\} \, dA + \\
& + \int_{\mathcal{C}_f} \{[\mathbf{P}(u) - \mathbf{P}^*] \cdot \mathbf{u} + [M(u) - M^*]\,\beta\} \, ds + \sum_j [F_j(u) - F_j^*]\, w_j + \\
& + \int_{\mathcal{C}_u} [\mathbf{P}(u) \cdot \mathbf{u}^* + M(u)\,\beta^*] \, ds + \sum_i F_i(u)\, w_i^* \,.
\end{aligned} \tag{4.19}
$$

The first variation $\delta I_8 = \delta J_8$ takes the form

$$
\begin{aligned}
\delta I_8 = & - \iint_{\mathcal{M}} [\mathbf{Q}(u) + \mathbf{p}] \cdot \delta\mathbf{u} \, dA + \int_{\mathcal{C}_f} \{[\mathbf{P}(u) - \mathbf{P}^*] \cdot \delta\mathbf{u} + [M(u) - M^*]\,\delta\beta\} \, ds + \\
& + \sum_j [F_j(u) - F_j^*]\,\delta w_j - \int_{\mathcal{C}_u} [(\mathbf{u} - \mathbf{u}^*) \cdot \delta\mathbf{P}(u) + (\beta - \beta^*)\,\delta M(u)] \, ds - \\
& - \sum_i (w_i - w_i^*)\,\delta F_i(u) \,.
\end{aligned} \tag{4.20}
$$

From (4.20) it is seen that the stationarity conditions of I_8 and J_8 are EQ(u), SB(u), SC(u), GB and GC. For known functions u_0, for which I_8 and J_8 assume their stationary value, any other variable ε, σ or f may be calculated from (2.14), (2.5) and (2.22).

5 Other Variational Principles

In chapter 4 a family of sixteen basic free functionals has been constructed in terms of various groups of independent free fields. Special cases of these functionals can be derived easily, if only certain individual components of these fields are eliminated from $I_1 \div I_7$ or $J_1 \div J_7$.

From each free functional it is possible to generate a variety of related functionals with subsidiary conditions, for which equivalent variational statements may be given. The number of such different functionals results from the number of possible different combinations of stationarity conditions of the free functional which may be treated as subsidiary conditions for a modified variational principle. The stationarity conditions to be treated in this way may be divided into groups according to either their physical meaning (geometric, static and constitutive relations) or the domain of the shell where they apply (interior equations, boundary and corner conditions). However, each particular component of the stationarity conditions may also be treated separately. Note, that the functionals $I_1 \div I_8$ are particularly suitable for transformations, when the geometric relations of the shell boundary value problem are treated as subsidiary conditions. On the other hand, when the static relations are treated as subsidiary conditions, it is more convenient to use the functionals $J_1 \div J_8$. Here three examples of this procedure are given.

First the stationarity conditions GB and GC of the functional $I_8(u)$ will be treated as subsidiary conditions. Then the variational principle $\delta I_8 = 0$ transforms into the equivalent principle $\delta \Pi = 0$, where the functional Π is given by

$$\Pi(u) = \iint_{\mathcal{M}} [\Sigma(u) - \mathbf{p} \cdot \mathbf{u}] \, dA - \int_{\mathcal{C}_f} (\mathbf{P}^* \cdot \mathbf{u} + M^*\beta) \, ds - \sum_j F_j^* w_j \tag{5.1}$$

with GB and GC as subsidiary conditions. As stationarity conditions of $\Pi(u)$ we have EQ(u), SB(u) and SC(u).

The functional $\Pi(u)$ is the total potential energy of the geometrically non-linear theory of shells undergoing moderate rotations. The variational principle $\delta \Pi = 0$ states, that among all geometrically admissible displacements the actual solution u_0 renders the total potential energy stationary.

For the shell theory under consideration the functional Π corresponds to that given in [17], and it contains as special cases the functionals discussed in [28, 27, 29, 21, 25, 20, 26, 16] for simpler versions of the geometrically non-linear theory of shells.

Another example is provided by the functional J_7, when its stationarity conditions EQ, SB and SC are treated as subsidiary conditions. Then $\delta J_7 = 0$ transforms into an equivalent variational principle $\delta J = 0$, where

$$J(u, \sigma) = -\iint_{\mathcal{M}} [\Sigma_c(\sigma) + N^{\alpha\beta}\Phi_{\alpha\beta}(u)] \, dA + \int_{\mathcal{C}_u} [\mathbf{P}(u, \sigma) \cdot \mathbf{u}^* + M(\sigma) \, \beta^*] \, ds + \sum_i F_i(\sigma) \, w_i^* , \tag{5.2}$$

with EQ, SB and SC as subsidiary conditions. The stationarity conditions of J are SD(u, σ), GB and GC.

The functional J is the mixed total complementary energy for the non-linear theory of shells undergoing moderate rotations. With appropriate simplifications it reduces to the corresponding functionals discussed in [28, 27, 21, 25, 16] for simpler variants of the geometrically non-linear theory of shells.

It is worthwhile to note that the subsidiary conditions of the functional I given in (3.2) are the stationarity conditions of the functional J and vice versa. The variational principle $\delta J = 0$ states, that among all statically admissible displacements and stress measures (i.e. among all those satisfying EQ, SB and SC) the actual solution (u_0, σ_0) renders the functional J stationary.

As a last example, only the bending part of the geometric stationarity conditions (i.e. $(2.14)_2$, $(2.15)_{3,4}$ and (2.16)) of the functional I_4 will be treated as subsidiary conditions while simultaneously the components $\gamma_{\alpha\beta}$ and $M^{\alpha\beta}$ will be eliminated from I_4 with the aid of $(2.9)_1$ and $(2.5)_2$. Then $\delta I_4 = 0$ may be transformed into an equivalent variational principle $\delta I_m = 0$ where the functional I_m is given by

$$\begin{aligned} I_m(u, \varkappa_{\alpha\beta}, N^{\alpha\beta}) = \iint_{\mathcal{M}} \Big[-\frac{1}{2h} E_{\alpha\beta\lambda\mu} N^{\alpha\beta} N^{\lambda\mu} + \frac{h^3}{24} H^{\alpha\beta\lambda\mu} \varkappa_{\alpha\beta} \varkappa_{\lambda\mu} - \mathbf{p} \cdot \mathbf{u} + N^{\alpha\beta} \gamma_{\alpha\beta}(u) \Big] \, dA - \\ - \int_{\mathcal{C}_f} (\mathbf{P}^* \cdot \mathbf{u} + M^*\beta) \, ds - \sum_j F_j^* w_j - \\ - \int_{\mathcal{C}_u} [P_\nu(u, \varkappa_{\alpha\beta}, N^{\alpha\beta}) \, (u_\nu - u_\nu^*) + P_t(u, \varkappa_{\alpha\beta}, N^{\alpha\beta}) \, (u_t - u_t^*)] \, ds \, . \end{aligned} \tag{5.3}$$

Here $(2.14)_2$, $(2.15)_{3,4}$ and (2. 6) are the subsidiary conditions. The stationarity conditions of I_m are EQ$(u, N^{\alpha\beta})$, SB$(u, N^{\alpha\beta})$, SC(u), $\frac{1}{h} E_{\alpha\beta\lambda\mu} N^{\lambda\mu} - \gamma_{\alpha\beta}(u) = 0$ in $\mathcal{M}$ and $u_\nu - u_\nu^* = 0$, $u_t - u_t^* = 0$ on $\mathcal{C}_u$.

The functional I_m contains as special cases those given in [20] within the context of the non-linear theory of shallow shells. Note that for this simplest non-linear shell theory one has $\varkappa_{\alpha\beta} = -w|_{\alpha\beta}$ and $N^{\alpha\beta} = \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}F_{|\lambda\mu} + P^{\alpha\beta}$, where $P^{\alpha\beta}$ is a particular solution of $P^{\alpha\beta}|_\beta + p^\alpha = 0$ and F is Airy's stress function. When these relations are used to eliminate $\varkappa_{\alpha\beta}$ and $N^{\alpha\beta}$ from (5.3), the functional I_m may be transformed easily into an equivalent functional $I_s(u, F)$, which then corresponds to those given in [18—20, 23].

Proceeding in a similar way many other related functionals may be derived from the basic free functionals presented in chapter 4. Some of these modifications may be particularly convenient in applications to specific shell problems.

6 Concluding Remarks

For all geometrically non-linear shell and plate theories which have been discussed in chapter 2, the variational theorems associated with the functionals given in chapters 3–5 and those which can be generated from them are, in general, stationary principles without extremum properties. Thus, the solution which renders these functionals stationary may not always be unique. For certain simplified non-linear shell and plate theories conditions have been derived in [28, 27, 21, 25] for the stationary principle of the total potential energy to be a minimum principle and for the existence of an associated dual maximum principle of the total complementary energy. Further research concerning this question in the context of the geometrically non-linear theory of shells undergoing moderate rotations should yield corresponding results.

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