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Three Forms of Geometrically Non-Linear Bending Shell Equations*

The shell deformation under Kirchhoff-Love constraints is decomposed into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation. Some formulae for the finite rotations are given. The shell equilibrium equations, geometric boundary conditions and energetically compatible with them static boundary conditions are derived in terms of Lagrangean quantities and consistently simplified under small elastic strains and bending theory. The basic shell equations are expressed in terms of either strains and changes of curvature, or strains and finite rotations, or displacements only.

1. Introduction

The basic set of equations for the first-approximation geometrically non-linear theory of thin elastic shells may be presented in terms of various quantities chosen as independent variables.

The most appealing way is to formulate solution of a shell problem in terms of the displacement vector $\boldsymbol{u}$ of its middle surface. The vector describes completely the deformation of the shell space, since within an error of the first-approximation theory it is permissible to assume the Kirchhoff-Love constraints on the shell deformation. However, the change in shell thickness during deformation should be taken into account in the constitutive equations. The resulting set of such three Lagrangean displacemental equilibrium equations become extremely complex [1 - 3].

Two other general forms of shell equations have recently been proposed. Danielson [4], Koiter and Simmonds [5] and the author [2] discussed various forms of shell equations by using as independent variables differently defined internal stress resultants $N^{\alpha\beta}$ and changes of curvatures $\kappa_{\alpha\beta}$. Simmonds and Danielson [6, 7] derived a set of shell equations in terms of finite rotation and stress function vectors $\boldsymbol{\Omega}$ and $\boldsymbol{F}$ as independent variables.

These three forms of shell equations differ in the reduction procedures used in the process of derivation, which depends upon the chosen independent variables. In [2, 3] we reduced consistently the relations between the strain measures and displacements and only

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then the displacemental shell equations were obtained from the virtual work principle. In the reduction procedure used in [4 - 7] it was assumed to be permissible to omit directly in the shell equations some terms, which were of the same order as the error already introduced into the equations by the approximate constitutive equations. Besides, in [5, 2] some error estimates derived by John [8] were taken into consideration. It is also interesting to note that in the three forms of shell equations the component relations were obtained from the appropriate vector relations by resolving them in different surface bases: in the reference basis [1 - 3], in the deformed basis [4, 5, 2] and in an intermediate basis [6, 7] which was the result of a rigid-body rotation of the reference basis by the finite rotation vector. Therefore, it seems almost impossible to compare the three forms of shell equations for the general case of geometrically non-linear theory of shells.

Recently it was shown in [9] how to decompose analytically the deformation of a neighbourhood about a point of the shell middle surface into three separate steps: a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation of the principal directions. This decomposition and the theory of finite rotations in shells as developed in [2, 3, 9] allows to discuss here, within one general scheme, three sets of shell equations: in terms of the shell strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, in terms of the surface strain tensor $\gamma_{\alpha\beta}$ and the finite rotation vector $\boldsymbol{\Omega}$ or in terms of the displacement vector $\boldsymbol{u}$. These three sets of equations are constructed here for a special case of the bending theory of shells in which the small strains in the shell space caused by the stretching and bending of its middle surface are assumed to be of comparable order in the whole internal shell region.

The reduced bending shell equations in terms of the strain measures presented here agree with those obtained by Koiter [10] in differently defined changes of curvatures. The appropriate deformational boundary conditions are derived here by consistently reducing the deformational quantities given in [11, 2]. We present here also the appropriate statical boundary conditions, which are energetically compatible with the deformational ones.

The bending shell equations in terms of $\gamma_{\alpha\beta}$ and $\boldsymbol{\Omega}$ follow directly from the equations in terms of strain measures by expressing $\kappa_{\alpha\beta}$ by means of $\boldsymbol{\Omega}$. The appropriate kinematical boundary conditions are obtained here by reducing those given in [11, 2]. We present here also the appropriate statical boundary conditions, which are energetically compatible with the kinematical ones.

Finally, we discuss here also a displacemental form of bending shell equations by expressing all quantities appearing in the previous two sets of shell equations in terms of displacements. The appropriate geometrical and statical boundary conditions are those given in [2].

In effect, the solution of any geometrically non-linear bending shell problem is divided into three steps. In the first step the strains and changes of curvatures (and therefore the stresses within the shell space) are obtained. In the second step the finite rotation vector is found from the known strain measures. The displacement field is calculated in the third step from the finite rotations and strains. However, in many shell problems it is the stress field that is sought for and after the first simple step the solution of the shell problem may be postponed.

2. Notation and preliminary relations

Let $\boldsymbol{r}(\vartheta^\alpha)=x^k(\vartheta^\alpha)\boldsymbol{i}_k$ and $\bar{\boldsymbol{r}}(\vartheta^\alpha)=\bar{x}^k(\vartheta^\alpha)\boldsymbol{i}_k$, $k=1, 2, 3$, be position vectors of a surface in the reference and deformed configurations, respectively. Here ϑ^α, $\alpha=1, 2$, is a pair of the surface convected coordinates and x^k and $\bar{x}^k$ are components of $\boldsymbol{r}$ and $\bar{\boldsymbol{r}}$ in a common Cartesian frame. With the reference surface $\mathscr{M}$ we associate standard surface covariant base vectors $\boldsymbol{a}_\alpha=\boldsymbol{r}_{,\alpha}$, covariant components of a metric tensor $a_{\alpha\beta}=\boldsymbol{a}_\alpha\cdot\boldsymbol{a}_\beta$ with determinant $a=|a_{\alpha\beta}|$, a unit vector normal to $\mathscr{M}$, $\boldsymbol{n}=\frac{1}{2}\varepsilon^{\alpha\beta}\boldsymbol{a}_\alpha\times\boldsymbol{a}_\beta$ and covariant components of the curvature tensor $b_{\alpha\beta}=\boldsymbol{a}_{\alpha,\beta}\cdot\boldsymbol{n}$, where $\varepsilon^{\alpha\beta}$ are contravariant components of a permutation tensor. Similar geometric quantities associated with deformed surface $\overline{\mathscr{M}}$ are marked by a dash: $\bar{\boldsymbol{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{\boldsymbol{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}^{\alpha\beta}$. Other details of the notation used in this paper are given in [2, 10].

The basic vectors of the deformed surface can be expressed in terms of the geometry of $\mathscr{M}$ and a displacement vector $\boldsymbol{u}=\bar{\boldsymbol{r}}-\boldsymbol{r}=u_\alpha\boldsymbol{a}^\alpha+w\boldsymbol{n}$ by

$$\bar{\boldsymbol{a}}_\alpha=l_{\lambda\alpha}\boldsymbol{a}^\lambda+\varphi_\alpha\boldsymbol{n}, \qquad \bar{\boldsymbol{n}}=n_\lambda\boldsymbol{a}^\lambda+n\boldsymbol{n}, \tag{2.1}$$

where

$$\begin{gathered} l_{\alpha\beta}=a_{\alpha\beta}+\vartheta_{\alpha\beta}-\omega_{\alpha\beta}, \\ \vartheta_{\alpha\beta}=\tfrac{1}{2}(u_{\alpha|\beta}+u_{\beta|\alpha})-b_{\alpha\beta}w, \qquad \varphi_\alpha=w_{,\alpha}+b^\lambda_\alpha u_\lambda, \\ \omega_{\alpha\beta}=\tfrac{1}{2}(u_{\beta|\alpha}-u_{\alpha|\beta})=\varepsilon_{\alpha\beta}\varphi, \qquad \varphi=\tfrac{1}{2}\varepsilon^{\alpha\beta}u_{\beta|\alpha}, \end{gathered} \tag{2.2}$$

$$n_\mu=\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}\varphi_\alpha l^\lambda_{.\beta}, \qquad n=\frac{1}{2}\sqrt{\frac{a}{\bar{a}}}\,\varepsilon^{\alpha\beta}\varepsilon_{\lambda\mu}l^\lambda_{.\alpha}l^\mu_{.\beta}. \tag{2.3}$$

The components of the Lagrangean surface strain tensor $\gamma_{\alpha\beta}$ and of the tensor of change of surface curvature $\kappa_{\alpha\beta}$ are defined by

$$\gamma_{\alpha\beta}=\tfrac{1}{2}(\bar{a}_{\alpha\beta}-a_{\alpha\beta})=\vartheta_{\alpha\beta}+\tfrac{1}{2}(\vartheta^\lambda_\alpha-\omega^\lambda_{.\alpha})(\vartheta_{\lambda\beta}-\omega_{\lambda\beta})+\tfrac{1}{2}\varphi_\alpha\varphi_\beta, \tag{2.4}$$

$$\kappa_{\alpha\beta}=-(\bar{b}_{\alpha\beta}-b_{\alpha\beta})=-[n(\varphi_{\alpha|\beta}+b^\lambda_\beta l_{\lambda\alpha})+n_\lambda(l^\lambda_{.\alpha|\beta}-b^\lambda_\beta\varphi_\alpha)-b_{\alpha\beta}]. \tag{2.5}$$

They satisfy the following compatibility conditions

$$\begin{gathered} \varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\kappa_{\beta\lambda|\mu}+\bar{a}^{\kappa\nu}(b_{\kappa\lambda}-\kappa_{\kappa\lambda})\gamma_{\nu\beta\mu}]=0, \\ K\gamma^\kappa_\kappa+\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}[\gamma_{\alpha\mu|\beta\lambda}-b_{\alpha\mu}\kappa_{\beta\lambda}+\tfrac{1}{2}(\kappa_{\alpha\mu}\kappa_{\beta\lambda}+\bar{a}^{\kappa\nu}\gamma_{\kappa\alpha\mu}\gamma_{\nu\beta\lambda})]=0, \end{gathered} \tag{2.6}$$

where

$$\gamma_{\nu\beta\mu}=\gamma_{\nu\beta|\mu}+\gamma_{\nu\mu|\beta}-\gamma_{\beta\mu|\nu}. \tag{2.7}$$

3. Decomposition of shell deformation

The geometry of deformation of a thin shell and its boundary has been discussed in detail in [2]. Here we remind some basic results.

Under Kirchhoff-Love constraints deformation of a shell is entirely described by deformation of its middle surface. Complete information about deformation of neighbourhood of any particle at the shell middle surface is contained in the shell deformation gradient tensor $\boldsymbol{G}$ which has the form

$$\boldsymbol{G}=\bar{\boldsymbol{a}}_\alpha\otimes\boldsymbol{a}^\alpha+\bar{\boldsymbol{n}}\otimes\boldsymbol{n}, \qquad \boldsymbol{G}^{-1}=\boldsymbol{a}_\alpha\otimes\bar{\boldsymbol{a}}^\alpha+\boldsymbol{n}\otimes\bar{\boldsymbol{n}}. \tag{3.1}$$

By applying the polar decomposition theorem the tensor G can be represented by

$$G=RU=VR, \quad G^{-1}=U^{-1}R^{T}=R^{T}V^{-1}, \tag{3.2}$$

where U and V are the right and left stretch tensors, respectively, and R is a finite rotation tensor. Decomposition of G in terms of U is compatible with Lagrangean description, in terms of V it is compatible with Eulerian one.

By the formulae (3.2) the deformation of a neighbourhood about a particle of the shell middle surface is decomposed analytically into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation of the principal directions. Lagrangean and Eulerian descriptions differ here only by different orders of these elementary deformations.

It follows from (3.1) and (3.2) that

$$\bar{a}_\alpha=R\breve{a}_\alpha=V\overset{*}{a}_\alpha, \quad \bar{n}=Rn, \quad \bar{a}^\alpha=R\breve{a}^\alpha=V^{-1}\overset{*}{a}^\alpha, \tag{3.3}$$

where

$$\begin{aligned} \breve{a}_\alpha&=Ua_\alpha, \quad \breve{a}^\alpha=U^{-1}a^\alpha,\\ \overset{*}{a}_\alpha&=Ra_\alpha, \quad \overset{*}{a}^\alpha=Ra^\alpha . \end{aligned} \tag{3.4}$$

The first intermediate basis $\breve{a}_\alpha$, n is obtained by stretching the reference basis a_α, n along the principal directions of U. The second intermediate basis is obtained by a rigid-body rotation of the reference basis with the help of R.

By using (3.3) and (3.4) we obtain

$$U=\breve{a}_\alpha\otimes a^\alpha+n\otimes n, \quad R=\bar{a}_\alpha\otimes\breve{a}^\alpha+\bar{n}\otimes n. \tag{3.5}$$

In what follows it is convenient to define some Lagrangean strain tensors co-axial with U. Besides the strain tensor $\gamma=\frac{1}{2}(U^2-1)$ with components (2.4) we shall also use here the modified strain tensor $\breve{\gamma}$ defined by

$$\begin{aligned} \breve{\gamma}&=U-1=\sqrt{1+2\gamma}-1=\breve{\gamma}_{\alpha\beta}\,a^\alpha\otimes a^\beta,\\ 2\gamma_{\alpha\beta}&=2\breve{\gamma}_{\alpha\beta}+\breve{\gamma}^\lambda_\alpha\breve{\gamma}_{\lambda\beta}, \quad \breve{a}_\alpha=(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)\,a_\lambda, \end{aligned} \tag{3.6}$$

where $1=a_\alpha\otimes a^\alpha+n\otimes n$ is the metric tensor of a three-dimensional Euclidean space calculated at $\mathcal{M}$. Note that $\gamma_{\alpha\beta}$ are quadratic in terms of displacements, but many geometrical relations containing $\sqrt{\bar{a}/a}$ are non-rational in terms of $\gamma_{\alpha\beta}$. On the other hand, we define $\breve{\gamma}_{\alpha\beta}$ to depend upon displacements through the complicated non-rational relation (3.6), but then the geometrical relations containing $\sqrt{\bar{a}/a}$ become polynomials in $\breve{\gamma}_{\alpha\beta}$. Within the small strains $\gamma_{\alpha\beta}$ and $\breve{\gamma}_{\alpha\beta}$ coincide.

The general formula for R in terms of displacements follows from $(3.5)_2$ to be

$$R=\bar{a}^{\alpha\beta}(a_\alpha+u_{,\alpha})\otimes(a_\beta+\breve{\gamma}_{\beta\lambda}\,a^\lambda)+(n_\alpha\,a^\alpha+nn)\otimes n. \tag{3.7}$$

The proper orthogonal tensor R describes an axis of rotation in space defined by a unit vector e and an angle of rotation ω about the axis of rotation. The rotational part of deformation may also be described by means of an equivalent finite rotation vector $\Omega=\sin\omega e$, which is defined uniquely by the tensor R. The general formula for Ω in terms of displacements takes the form

$$\Omega=\tfrac{1}{2}\varepsilon_{\lambda\mu}\{[n^\lambda-\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)\varphi_\beta]\,a^\mu+\bar{a}^{\alpha\beta}(\delta^\lambda_\alpha+\breve{\gamma}^\lambda_\alpha)\,l^\mu_{.\beta}\,n\}. \tag{3.8}$$

The action of $\boldsymbol{\Omega}$ on $\breve{\boldsymbol{a}}_\alpha$ is calculated according to

$$\bar{\boldsymbol{a}}_\alpha=\breve{\boldsymbol{a}}_\alpha+\boldsymbol{\Omega}\times\breve{\boldsymbol{a}}_\alpha+\frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\breve{\boldsymbol{a}}_\alpha) \tag{3.9}$$

and the same formula relates $\bar{\boldsymbol{n}}$ to $\boldsymbol{n}$ and $\overset{*}{\boldsymbol{a}}_\alpha$ to $\boldsymbol{a}_\alpha$.

Differentiation of $\boldsymbol{\Omega}$ along surface coordinate lines follows from the rule derived by Shamina [12] for three-dimensional case to be

$$\boldsymbol{\Omega}_{,\beta}=\cos\omega\boldsymbol{k}_\beta+\tfrac{1}{2}\boldsymbol{\Omega}\times\boldsymbol{k}_\beta-\frac{1}{4\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{k}_\beta), \tag{3.10}$$

where $\boldsymbol{k}_\beta$ is a vector of change of curvature of the coordinate lines. For the vector $\boldsymbol{k}_\beta$ we obtained the following formula

$$\boldsymbol{k}_\beta=\boldsymbol{\Omega}_{,\beta}+\frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega}_{,\beta}\times\boldsymbol{\Omega}+\omega_{,\beta}\operatorname{tg}\omega/2\,\boldsymbol{\Omega}= \tag{3.11}$$

$$=\bar{\varepsilon}^{\lambda\mu}[(\kappa_{\beta\lambda}+b^\kappa_\beta\breve{\gamma}_{\kappa\lambda})\,\breve{\boldsymbol{a}}_\mu+(\gamma_{\beta\mu|\lambda}-\tfrac{1}{2}\breve{\gamma}^\kappa_\mu\breve{\gamma}_{\kappa\lambda|\beta})\,\boldsymbol{n}]\,. \tag{3.12}$$

The integrability condition of (3.10) takes the form

$$\bar{\varepsilon}^{\alpha\beta}(\boldsymbol{k}_{\beta|\alpha}+\tfrac{1}{2}\boldsymbol{k}_\alpha\times\boldsymbol{k}_\beta)=\boldsymbol{0}\,. \tag{3.13}$$

The component form of (3.13) with (3.12) with respect to the reference basis is equivalent to the compatibility conditions (2.6).

Taking into account that

$$\bar{\varepsilon}_{\alpha\lambda}\,\breve{\boldsymbol{a}}^\lambda=(\delta^\kappa_\alpha+\breve{\gamma}^\kappa_\alpha)\,\varepsilon_{\kappa\lambda}\,\boldsymbol{a}^\lambda \tag{3.14}$$

we are able to invert (3.12) and obtain

$$\kappa_{\alpha\beta}=\tfrac{1}{2}\varepsilon_{\kappa\lambda}[(\delta^\kappa_\alpha+\breve{\gamma}^\kappa_\alpha)\boldsymbol{k}_\beta+(\delta^\kappa_\beta+\breve{\gamma}^\kappa_\beta)\boldsymbol{k}_\alpha]\cdot\boldsymbol{a}^\lambda-\tfrac{1}{2}(b^\lambda_\alpha\breve{\gamma}_{\lambda\beta}+b^\lambda_\beta\breve{\gamma}_{\lambda\alpha})\,. \tag{3.15}$$

This relation, together with (3.11), gives an exact expression for $\kappa_{\alpha\beta}$ in terms of $\boldsymbol{\Omega}$ and $\breve{\gamma}_{\alpha\beta}$.

With the reference boundary curve $\mathscr{C}$ defined by $\vartheta^\alpha=\vartheta^\alpha(s)$, where s is the length parameter, we associate the unit tangent $\boldsymbol{t}=d\boldsymbol{r}/ds$ and the outward unit normal $\boldsymbol{v}=\boldsymbol{t}\times\boldsymbol{n}$. The orthonormal triad $\boldsymbol{v}$, $\boldsymbol{t}$, $\boldsymbol{n}$ does not coincide, in general, with the principal directions of strain. The rotation of the triad from the reference to the deformed configuration is described by a total finite rotation vector $\boldsymbol{\Omega}_t$, which is the result of superposition of a finite rotation of the principal directions of strain on a finite rotation due to a pure stretching along the principal directions. The exact formula for $\boldsymbol{\Omega}_t$ in terms of displacements follows from the superposition rule for finite rotation vectors [2, 3] or directly from the relation

$$2\boldsymbol{\Omega}_t=\frac{1}{\sqrt{1+2\gamma_{tt}}}(\boldsymbol{v}\times\bar{\boldsymbol{a}}_v+\boldsymbol{t}\times\bar{\boldsymbol{a}}_t)+\boldsymbol{n}\times\bar{\boldsymbol{n}}\,, \tag{3.16}$$

where

$$\bar{\boldsymbol{a}}_t=\frac{d\bar{\boldsymbol{r}}}{ds}=\bar{\boldsymbol{a}}_\alpha t^\alpha\,,\ \ \bar{\boldsymbol{a}}_v=\bar{\boldsymbol{a}}_t\times\bar{\boldsymbol{n}}\,. \tag{3.17}$$

Deformation of the shell lateral boundary is described by deformation of the triad $\boldsymbol{v}$, $\boldsymbol{t}$, $\boldsymbol{n}$. The detailed discussion given in [11, 2] shows that in order to establish the deformed lateral boundary each of the following three groups of geometric quantities, defining three different types of geometric boundary conditions, may be assumed given at the boundary:

1) displacemental b.c.: $\boldsymbol{u}$ and $\beta_\nu = \dfrac{(\bar{\boldsymbol{n}}-\boldsymbol{n})\cdot\bar{\boldsymbol{a}}_\nu}{1+2\gamma_{tt}}$ at $\mathscr{C}$,

2) kinematical b.c.: $\boldsymbol{\Omega}_t$ and $\gamma_{tt}=\gamma_{\alpha\beta}t^\alpha t^\beta$ at $\mathscr{C}$,

3) deformational b.c.: $\boldsymbol{k}_t$ and γ_{tt} at $\mathscr{C}$.

The vector of change of boundary curvature $\boldsymbol{k}_t$ is calculated according to

$$\boldsymbol{k}_t = -k_{tt}\,\boldsymbol{\nu} + k_{\nu t}\,\boldsymbol{t} - k_{nt}\,\boldsymbol{n}, \tag{3.18}$$

where

$$\begin{aligned} -k_{tt} &= \frac{1}{\sqrt{1+2\gamma_{tt}}}(\sigma_t-\kappa_{tt})-\sigma_t, \\ k_{\nu t} &= -\frac{1}{\sqrt{1+2\gamma_{tt}}}\sqrt{\frac{\bar{a}}{a}}\,\nu_\kappa\,\bar{a}^{\kappa\alpha}(b_{\alpha\beta}-\kappa_{\alpha\beta})\,t^\beta-\tau_t, \\ -k_{nt} &= \frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}}(\kappa_t-\nu_\kappa\,\bar{a}^{\kappa\lambda}\gamma_{\lambda\alpha\beta}\,t^\alpha t^\beta)-\kappa_t, \end{aligned} \tag{3.19}$$

and σ_t, τ_t, κ_t are the normal curvature, the geodesic torsion and the geodesic curvature of $\mathscr{C}$, respectively.

4. Equilibrium equations and statical boundary conditions

The non-linear shell equations can be presented either in Eulerian or in Lagrangean descriptions. We prefer here to use the Lagrangean approach, since in this case all quantities are related to the known geometry of the reference shell middle surface $\mathscr{M}$.

Let a shell with simply connected middle surface be in equilibrium under the surface force $\boldsymbol{p}$, per unit area of $\mathscr{M}$, and under the boundary force $\boldsymbol{F}$ and boundary couple $\boldsymbol{K}$, per unit length of the reference boundary $\mathscr{C}$. Then for any additional virtual displacement field $\delta\boldsymbol{u}$ subject to geometrical constraints there are symmetric Lagrangean stress and couple resultant tensors $\boldsymbol{N}=N^{\alpha\beta}\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_\beta$ and $\boldsymbol{M}=M^{\alpha\beta}\boldsymbol{a}_\alpha\otimes\boldsymbol{a}_\beta$ such that the Lagrangean virtual work principle IVW=EVW takes the form [2]

$$\iint_{\mathscr{M}}(N^{\alpha\beta}\delta\gamma_{\alpha\beta}+M^{\alpha\beta}\delta\kappa_{\alpha\beta})\,dA = \iint_{\mathscr{M}}\boldsymbol{p}\cdot\delta\boldsymbol{u}\,dA+\int_{\mathscr{C}}(\boldsymbol{F}\cdot\delta\boldsymbol{u}+\boldsymbol{K}\cdot\delta\boldsymbol{\Omega}_t)\,ds. \tag{4.1}$$

By applying the Stockes' theorem we obtain

$$\begin{aligned} -\iint_{\mathscr{M}}(\boldsymbol{G}\boldsymbol{N}^\beta)_{|\beta}\cdot\delta\boldsymbol{u}\,dA+\int_{\mathscr{C}}[\boldsymbol{P}_\nu\cdot\delta\boldsymbol{u}+\bar{M}_{\nu\nu}(\bar{\boldsymbol{a}}_t\cdot\delta\boldsymbol{\Omega}_t)]\,ds+\sum_{M_i}\Delta\bar{M}_{t\nu}(\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}) = \\ = \iint_{\mathscr{M}}\boldsymbol{p}\cdot\delta\boldsymbol{u}\,dA+\int_{\mathscr{C}}[\boldsymbol{R}\cdot\delta\boldsymbol{u}+\bar{K}_\nu(\bar{\boldsymbol{a}}_t\cdot\delta\boldsymbol{\Omega}_t)]\,ds+\sum_{M_i}\Delta\bar{K}_t(\bar{\boldsymbol{n}}\cdot\delta\boldsymbol{u}), \end{aligned} \tag{4.2}$$

where

$$\boldsymbol{N}^\beta = Q^{\alpha\beta}\boldsymbol{a}_\alpha+Q^\beta\boldsymbol{n}, \qquad \bar{\boldsymbol{a}}_t\cdot\delta\boldsymbol{\Omega}_t=\sqrt{1+2\gamma_{tt}}\,\delta\bar{\beta}_\nu,$$

$$Q^{\alpha\beta}=N^{\alpha\beta}-\bar{b}^\alpha_\lambda M^{\lambda\beta}, \qquad Q^\beta = M^{\alpha\beta}\big|_\alpha+\bar{a}^{\beta\kappa}(2\gamma_{\kappa\lambda|\mu}-\gamma_{\lambda\mu|\kappa})M^{\lambda\kappa},$$

$$P_\nu = GN^\beta \nu_\beta + \frac{d}{ds}(\bar{M}_{t\nu}\bar{n}), \quad R = F + \frac{d}{ds}(\bar{K}_t \bar{n}), \tag{4.3}$$

$$\bar{M}_{\nu\nu} = \frac{1}{1+2\gamma_{tt}}\sqrt{\frac{\bar{a}}{a}} M^{\alpha\beta}\nu_\alpha \nu_\beta, \quad \bar{M}_{t\nu} = \frac{1}{1+2\gamma_{tt}} M^{\alpha\beta}(\delta^\lambda_\alpha + 2\gamma^\lambda_\alpha)\, t_\lambda \nu_\beta,$$

$$\bar{K}_\nu = \frac{1}{1+2\gamma_{tt}} K \cdot \bar{a}_t, \quad \bar{K}_t = -\frac{1}{1+2\gamma_{tt}} K \cdot \bar{a}_\nu,$$

and at all corners M_t of $\mathscr{C}$ labelled by $s = s_i$, $i = 1, 2, \ldots, N$ we have

$$\Delta\bar{M}_{t\nu} = \bar{M}_{t\nu}(s_i+0) - \bar{M}_{t\nu}(s_i-0), \quad \Delta\bar{K}_t = \bar{K}_t(s_i+0) - \bar{K}_t(s_i-0). \tag{4.4}$$

The principle (4.2) gives us the Lagrangean equilibrium equations

$$(GN^\beta)_{|\beta} + p = 0 \text{ in } \mathscr{M}, \tag{4.5}$$

the Lagrangean statical boundary conditions

$$P_\nu = R, \ \bar{M}_{\nu\nu} = \bar{K}_\nu \text{ on } \mathscr{C} \tag{4.6}$$

and the conditions $\Delta\bar{M}_{t\nu} = \Delta\bar{K}_t$ to be satisfied at each corner M_t.

When expressed in components along deformed basis $\bar{a}_\alpha$, $\bar{n}$ the relation (4.5) gives the following three equilibrium equations

$$Q^{\alpha\beta}|_\beta + \bar{a}^{\alpha\kappa}\gamma_{\kappa\lambda\beta} Q^{\lambda\beta} - \bar{b}^\alpha_\beta Q^\beta + \sqrt{\frac{\bar{a}}{a}}\,\bar{p}^\alpha = 0, \tag{4.7}$$

$$Q^\beta|_\beta + \bar{b}_{\alpha\beta} Q^{\alpha\beta} + \sqrt{\frac{\bar{a}}{a}}\,\bar{p} = 0,$$

where

$$\bar{p} = \sqrt{\frac{a}{\bar{a}}}\, p = \bar{p}^\alpha \bar{a}_\alpha + \bar{p}\bar{n} \tag{4.8}$$

is a surface force per unit area of the deformed shell middle surface.

Let F_ν and $B_\nu(O)$ be a total force and a total couple, with respect to an origin O in space, of all internal stress and couple resultants acting at the part of the shell boundary. By using the Lagrangean description these vectors are defined by

$$F_\nu = F^0_\nu + \int_{M_0}^{M} P_\nu\, ds, \quad B_\nu(O) = B^0_\nu(O) + \int_{M_0}^{M} (\bar{M}_{\nu\nu}\bar{a}_t + \bar{r} \times P_\nu)\, ds, \tag{4.9}$$

where F^0_ν and $B^0_\nu(O)$ are initial values of F_ν and $B^0_\nu(O)$ at $M = M_0$.

The total couple $B_\nu(\bar{M}) \equiv B_\nu$ with respect to a current point $\bar{M}$ of deformed boundary is calculated as follows

$$B_\nu = B_\nu(O) - \bar{r} \times F_\nu. \tag{4.10}$$

By differentiating (4.9) and (4.10) we obtain

$$\frac{d}{ds} F_\nu = P_\nu, \quad \frac{d}{ds} B_\nu = \bar{M}_{\nu\nu}\bar{a}_t - \bar{a}_t \times F_\nu. \tag{4.11}$$

Let us also differentiate $\delta\boldsymbol{u}$ and $\delta\boldsymbol{\Omega}_t$ with respect to $\bar{s}$ and take into account that $d\bar{s}=\sqrt{1+2\gamma_{tt}}ds$, which gives

$$\frac{d}{ds}\delta\boldsymbol{\Omega}_t=\sqrt{1+2\gamma_{tt}}\,\delta\boldsymbol{k}_t,$$
$$\frac{d}{ds}\delta\boldsymbol{u}=\delta\bar{\gamma}_{tt}\,\bar{\boldsymbol{a}}_t+\delta\boldsymbol{\Omega}_t\times\bar{\boldsymbol{a}}_t. \tag{4.12}$$

Here $\delta\boldsymbol{k}_t$ is the vector of virtual change of the boundary curvature and $\delta\bar{\gamma}_{tt}=\delta\bar{\gamma}_{\alpha\beta}\bar{t}^\alpha\bar{t}^\beta=\frac{1}{1+2\gamma_{tt}}\delta\gamma_{tt}$, where $\delta\gamma_{tt}=\delta\gamma_{\alpha\beta}t^\alpha t^\beta$.

Now it is possible to transform the line integral in $(4.2)_1$ as follows

$$I=\int_{M_0}^{M}[\boldsymbol{P}_\nu\cdot\delta\boldsymbol{u}+\bar{M}_{\nu\nu}(\bar{\boldsymbol{a}}_t\cdot\delta\boldsymbol{\Omega}_t)]\,ds=$$
$$=\int_{M_0}^{M}\left[\frac{d}{ds}(\boldsymbol{F}_\nu\cdot\delta\boldsymbol{u})-\boldsymbol{F}_\nu\cdot\frac{d}{ds}(\delta\boldsymbol{u})+\bar{M}_{\nu\nu}(\bar{\boldsymbol{a}}_t\cdot\delta\boldsymbol{\Omega}_t)\right]ds=$$
$$=\int_{M_0}^{M}\left[(\bar{M}_{\nu\nu}\bar{\boldsymbol{a}}_t-\bar{\boldsymbol{a}}_t\times\boldsymbol{F}_\nu)\cdot\delta\boldsymbol{\Omega}_t-\frac{\bar{\boldsymbol{a}}_t\cdot\boldsymbol{F}_\nu}{1+2\gamma_{tt}}\delta\gamma_{tt}\right]ds+\boldsymbol{F}_\nu\cdot\delta\boldsymbol{u}\Big|_{M_0}^{M}. \tag{4.13}$$

By introducing $(4.11)_2$ and $(4.12)_1$ into (4.13) we also obtain

$$I=-\int_{M_0}^{M}\left[\sqrt{1+2\gamma_{tt}}\,\boldsymbol{B}_\nu\cdot\delta\boldsymbol{k}_t+\frac{\bar{\boldsymbol{a}}_t\cdot\boldsymbol{F}_\nu}{1+2\gamma_{tt}}\delta\gamma_{tt}\right]ds+\boldsymbol{F}_\nu\cdot\delta\boldsymbol{u}\Big|_{M_0}^{M}+\boldsymbol{B}_\nu\cdot\delta\boldsymbol{\Omega}_t\Big|_{M_0}^{M}. \tag{4.14}$$

It follows now from (4.13) and (4.14) that some statical quantities work on variations of geometrical quantities which establish the deformed lateral boundary. Therefore, to each of geometrical boundary conditions at $\mathscr{C}$ there corresponds an energetically compatible statical boundary condition expressed by the following quantities:

$$\begin{aligned}
&1)\qquad \boldsymbol{u}\leftrightarrow\boldsymbol{P}_\nu,\qquad \beta_\nu\leftrightarrow\bar{M}_{\nu\nu}\sqrt{1+2\gamma_{tt}}\\
&2)\qquad \boldsymbol{\Omega}_t\leftrightarrow\bar{M}_{\nu\nu}\bar{\boldsymbol{a}}_t-\bar{\boldsymbol{a}}_t\times\boldsymbol{F}_\nu,\qquad \gamma_{tt}\leftrightarrow-\frac{\bar{\boldsymbol{a}}_t\cdot\boldsymbol{F}_\nu}{1+2\gamma_{tt}},\\
&3)\qquad \boldsymbol{k}_t\leftrightarrow-\sqrt{1+2\gamma_{tt}}\,\boldsymbol{B}_\nu,\qquad \gamma_{tt}\leftrightarrow-\frac{\bar{\boldsymbol{a}}_t\cdot\boldsymbol{F}_\nu}{1+2\gamma_{tt}}.
\end{aligned} \tag{4.15}$$

The constant terms appearing outside the integration in (4.13) and (4.14) are responsible for discontinuities of boundary values at the corner points of $\mathscr{C}$ and should also be taken into account.

5. Bending shell equations in terms of strain measures

The various shell relations discussed in § 2 - 4 have a purely geometrical character, which follows from the assumption of Kirchhoff-Love constraints on deformation process of the shell. The relations still contain unrestricted strains and rotations and do not depend upon the shell material properties.

In what follows we shall discuss the simplified sets of shell equations in the case of a thin shell composed of an isotropic elastic material under the assumption that strains are small everywhere in the shell.

Within the small strains we are allowed to use the estimates

$$\frac{\bar{a}}{a}=1+O(\eta),\ \bar{a}_{\alpha\beta}=a_{\alpha\beta}+O(\eta),\ \bar{a}^{\alpha\beta}=a^{\alpha\beta}+O(\eta),$$
$$\check{\gamma}_{\alpha\beta}=\gamma_{\alpha\beta}+O(\eta^2),\ \bar{\varepsilon}_{\alpha\beta}=\varepsilon_{\alpha\beta}+O(\eta), \tag{5.1}$$

where $\eta \ll 1$ is the largest strain in the shell space.

The constitutive equations of the isotropic elastic shell have the following form

$$N^{\alpha\beta}=C[(1-\nu)\gamma^{\alpha\beta}+\nu a^{\alpha\beta}\gamma^{\lambda}_{\lambda}]+O(Eh\eta\vartheta^2),$$
$$M^{\alpha\beta}=D[(1-\nu)\kappa^{\alpha\beta}+\nu a^{\alpha\beta}\kappa^{\lambda}_{\lambda}]+O(Eh^2\eta\vartheta^2), \tag{5.2}$$

where

$$C=\frac{Eh}{1-\nu^2},\quad D=\frac{Eh^3}{12(1-\nu^2)}. \tag{5.3}$$

Here ϑ is the small parameter defined in [5, 2], h is the small shell thickness, E and ν are the Youngs modulus and the Poisson's ratio, respectively.

When the small strains in the shell space caused by stretching and bending of its middle surface are of comparable order in the whole shell region, the errors indicated in (5.1) and (5.2) allow us to reduce also other shell relations. We assume here that it is permissible to omit in all shell relations such terms which are of the same order as the error terms introduced to these relations from the approximate constitutive equations (5.2). As a result, the equilibrium equations (4.7) and the compatibility conditions (2.6) are reduced [2, 10] to

$$C[(1-\nu)\gamma^{\beta}_{\alpha}|_{\beta}+\nu\gamma^{\beta}_{\beta}|_{\alpha}]+\bar{p}_{\alpha}=O\left(Eh\frac{\eta\vartheta^2}{\lambda}\right),$$
$$D\kappa^{\alpha}_{\alpha}|^{\beta}_{\beta}+C(b^{\alpha}_{\beta}-\kappa^{\alpha}_{\beta})[(1-\nu)\gamma^{\beta}_{\alpha}+\nu\delta^{\beta}_{\alpha}\gamma^{\lambda}_{\lambda}]+\bar{p}=O\left(Eh^2\frac{\eta\vartheta^2}{\lambda^2}\right),$$
$$\kappa^{\beta}_{\alpha}|_{\beta}-\kappa^{\beta}_{\beta}|_{\alpha}=O\left(\frac{\eta\vartheta^2}{h\lambda}\right), \tag{5.4}$$
$$\gamma^{\beta}_{\alpha}|^{\alpha}_{\beta}-\gamma^{\alpha}_{\alpha}|^{\beta}_{\beta}-(b^{\beta}_{\alpha}\kappa^{\alpha}_{\beta}-b^{\alpha}_{\alpha}\kappa^{\beta}_{\beta})+\tfrac{1}{2}(\kappa^{\beta}_{\alpha}\kappa^{\alpha}_{\beta}-\kappa^{\alpha}_{\alpha}\kappa^{\beta}_{\beta})=O\left(\frac{\eta\vartheta^2}{\lambda^2}\right),$$

where $\lambda=h/\vartheta$ is a large parameter [5, 2].

The geometrical boundary conditions are expressed in terms of the strain measures by using deformational variables k_t and γ_{tt}. Under small strains for the bending shell theory

(3.19) reduce to

$$k_{tt}=\kappa_{tt}+O\left(\frac{\eta\vartheta^2}{h}\right),\qquad k_{\nu t}=\kappa_{\nu t}+O\left(\frac{\eta\vartheta^2}{h}\right),$$

$$k_{nt}=2\frac{d\gamma_{\nu t}}{ds}-\frac{d\gamma_{tt}}{ds_\nu}+2\kappa_\nu\gamma_{\nu t}+\kappa_t(\gamma_{\nu\nu}-\gamma_{tt})+O\left(\frac{\eta\vartheta^3}{h}\right). \tag{5.5}$$

The appropriate statical boundary conditions may be obtained from the consistent reduction of quantities appearing in (4.14). Under small strains for the bending shell theory we obtain

$$\boldsymbol{P}_\nu=P_{\nu\nu}\bar{\boldsymbol{\nu}}+P_{t\nu}\bar{\boldsymbol{t}}+P_{n\nu}\bar{\boldsymbol{n}}+O(Eh\eta\vartheta^2), \tag{5.6}$$

$$P_{\nu\nu}=C(\gamma_{\nu\nu}+\nu\gamma_{tt})+O(Eh\eta\vartheta^2),$$

$$P_{t\nu}=C(1-\nu)\gamma_{\nu t}+O(Eh\eta\vartheta^2),$$

$$P_{n\nu}=D\left[\frac{d\kappa_{\nu\nu}}{ds_\nu}+\nu\frac{d\kappa_{tt}}{ds_\nu}+2(1-\nu)\frac{d\kappa_{t\nu}}{ds}\right]+$$
$$+D(1-\nu)[\kappa_t(\kappa_{\nu\nu}-\kappa_{tt})+2\kappa_\nu\kappa_{t\nu}]+O(Eh\eta\vartheta^3), \tag{5.7}$$

$$\bar{\boldsymbol{a}}_t=\bar{\boldsymbol{t}}+O(\eta),\qquad \bar{M}_{\nu\nu}=M_{\nu\nu}+O(Eh^2\eta\vartheta^2),\qquad \bar{M}_{t\nu}=M_{t\nu}+O(Eh^2\eta\vartheta^2). \tag{5.8}$$

Therefore, for the statical boundary conditions $(4.15)_3$ we have

$$\boldsymbol{B}_\nu=\boldsymbol{B}_\nu^0(O)+\int_{M_0}^{M}[D(\kappa_{\nu\nu}+\nu\kappa_{tt})\bar{\boldsymbol{t}}+\bar{\boldsymbol{r}}\times\boldsymbol{P}_\nu]\,ds-\bar{\boldsymbol{r}}\times\int_{M_0}^{M}\boldsymbol{P}_\nu\,ds,$$

$$\bar{\boldsymbol{t}}\cdot\boldsymbol{F}_\nu=\bar{\boldsymbol{t}}\cdot\boldsymbol{F}_\nu^0+\bar{\boldsymbol{t}}\cdot\int_{M_0}^{M}\boldsymbol{P}_\nu\,ds, \tag{5.9}$$

where $\boldsymbol{P}_\nu$ is defined by (5.6) and (5.7).

The relations (5.4) form a set of six equations for six strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ to be solved within $\mathscr{M}$. All the formulae for components of $\boldsymbol{k}_t$ as well as those containing statical variables at the shell boundary are linear in the strain measures. This shows that the structure of the relations for geometrically non-linear bending theory of shells is remarkably simple.

By solving the set of equations (5.4) with appropriate boundary conditions we obtain the strain and stress distribution within the shell space according to the constraints assumed and with accuracy of the first-approximation theory. The position of the shell in space is established with the accuracy up to a rigid-body motion in space.

6. Bending shell equations in terms of finite rotations and strains

The finite rotation vector and its derivatives are decomposed in the reference basis $\boldsymbol{a}_\alpha, \boldsymbol{n}$ as follows

$$\boldsymbol{\Omega}=\Omega_\alpha\boldsymbol{a}^\alpha+\Omega\boldsymbol{n},\qquad \boldsymbol{\Omega}_{,\beta}=\Phi_{\lambda\beta}\boldsymbol{a}^\lambda+\Phi_\beta\boldsymbol{n},$$

$$\Phi_{\alpha\beta}=\Omega_{\alpha|\beta}-b_{\alpha\beta}\Omega,\qquad \Phi_\beta=\Omega_{,\beta}+b_\beta^\alpha\Omega_\alpha. \tag{6.1}$$

For the vectors $\boldsymbol{k}_\beta$ defined in (3.12) we obtain

$$\boldsymbol{k}_\beta = k^\mu_{.\beta}\,\boldsymbol{a}_\mu + k_\beta\,\boldsymbol{n}\,, \tag{6.2}$$

$$k^\mu_{.\beta} = \Phi^\mu_{.\beta} + \frac{1}{2\cos^2\omega/2}\,\varepsilon^{\lambda\mu}(\Omega_\lambda\Phi_\beta - \Phi_{\lambda\beta}\Omega) + \omega_{,\beta}\,\mathrm{tg}\,\omega/2\,\Omega^\mu\,,$$

$$k_\beta = \Phi_\beta + \frac{1}{2\cos^2\omega/2}\,\varepsilon^{\lambda\mu}\Phi_{\lambda\beta}\Omega_\mu + \omega_{,\beta}\,\mathrm{tg}\,\omega/2\,\Omega\,. \tag{6.3}$$

Since the rotations are allowed to be arbitrarily large here the relations cannot be simplified.

Within the first-approximation theory of shells the tensor of change of curvature is defined with an error $O(\eta\vartheta^2/h)$, [2, 13]. Terms of this order added to (or subtracted from) $\kappa_{\alpha\beta}$ do not change the accuracy of the approximate strain energy function [14]. Since $\boldsymbol{k}_\beta = O(\eta/h)$ it follows from (3.15) that

$$\kappa_{\alpha\beta} = \tfrac{1}{2}(\varepsilon_{\alpha\lambda}\,k^\lambda_{.\beta} + \varepsilon_{\beta\lambda}\,k^\lambda_{.\alpha}) + O\left(\frac{\eta\vartheta^2}{h}\right). \tag{6.4}$$

This gives the representation of $\kappa_{\alpha\beta}$ entirely in terms of the finite rotations.

If we introduce (6.4) and (6.3) into the set of equations (5.4) then it becomes expressed entirely in terms of $\gamma_{\alpha\beta}$ and $\boldsymbol{\Omega}$.

The geometrical boundary conditions may be expressed in terms of $\gamma_{\alpha\beta}$ and $\boldsymbol{\Omega}$ by using kinematical variables $\boldsymbol{\Omega}_t$ and γ_{tt} at $\mathscr{C}$. Under small strains but arbitrarily large rotations we obtain [9]

$$\boldsymbol{\Omega}_t = \check{\boldsymbol{\Omega}}_t\cos^2\omega^+/2 + \boldsymbol{\Omega} + \tfrac{1}{2}\boldsymbol{\Omega}^+\times\check{\boldsymbol{\Omega}}_t - \frac{\check{\boldsymbol{\Omega}}_t\cdot\boldsymbol{\Omega}^+}{4\cos^2\omega^+/2}\,\boldsymbol{\Omega}^+ + O(\eta^2) = \boldsymbol{\Omega} + O(\eta)\,, \tag{6.5}$$

where

$$\check{\boldsymbol{\Omega}}_t = -\gamma_{\nu t}\,\boldsymbol{n} + O(\eta^2),\quad \cos^2\omega^+/2 = \tfrac{1}{4}(\mathrm{tr}\,\boldsymbol{G}+1), \tag{6.6}$$

$$\boldsymbol{\Omega}^+ = \tfrac{1}{2}\varepsilon^{\lambda\mu}\{[(2+\vartheta^\kappa_\kappa)\,\varphi_\lambda - \varphi^\kappa(\vartheta_{\kappa\lambda} - \omega_{\kappa\lambda})]\,\boldsymbol{a}_\mu + \omega_{\lambda\mu}\,\boldsymbol{n}\}\,.$$

Therefore within the bending theory for geometrical boundary conditions it is allowed to assume values of $\boldsymbol{\Omega}$ and γ_{tt} at $\mathscr{C}$.

The appropriate statical boundary conditions in terms of $\boldsymbol{\Omega}$ and $\gamma_{\alpha\beta}$ follow from the reduction of quantities appearing in (4.13). Under small strains and bending theory we should assume at $\mathscr{C}$ the following quantities

$$M_{\nu\nu}\bar{\boldsymbol{t}} - \bar{\boldsymbol{t}}\times\boldsymbol{F}_\nu = M_{\nu\nu}\bar{\boldsymbol{t}} - \bar{\boldsymbol{t}}\times\Big(\boldsymbol{F}^0_\nu + \int_{M_0}^{M}\mathbf{P}_\nu\,ds\Big),$$

$$-\bar{\boldsymbol{t}}\cdot\boldsymbol{F}_\nu = -\bar{\boldsymbol{t}}\cdot\Big(\boldsymbol{F}^0_\nu + \int_{M_0}^{M}\mathbf{P}_\nu\,ds\Big). \tag{6.7}$$

Here for $\boldsymbol{P}_\nu$ we should use (5.6) and introduce there (6.4) and (6.3) in order to express $\kappa_{\alpha\beta}$ in terms of $\boldsymbol{\Omega}$.

The resulting set of the Lagrangean shell relations is quite complex with respect to $\boldsymbol{\Omega}$. Note that according to (3.11) and (6.3) the vectors $\boldsymbol{k}_\beta$ depend upon $\boldsymbol{\Omega}$ through trigonometric functions. Thus, the resulting shell relations will also contain trigonometric dependency upon $\boldsymbol{\Omega}$.

It seems to be more rewarding to divide the solution into two steps. First we may solve the shell problem in terms of strain measures as suggested in § 5. This allows to calculate the vectors $\boldsymbol{k}_\beta$ according to

$$\boldsymbol{k}_\beta = \varepsilon^{\lambda\mu}(\kappa_{\beta\lambda}\boldsymbol{a}_\mu + \gamma_{\beta\mu|\lambda}\boldsymbol{n}) + O\left(\frac{\eta\vartheta^2}{h}\right). \tag{6.8}$$

The vector $\boldsymbol{\Omega}$ may be found in the second step by integrating the two vector differential equations (3.10). Note that the structure of equations (3.10) is the same as that known in analytical mechanics of rigid-body motion about a fixed point, where the time derivative of a finite rotation vector is calculated in this way from an angular velocity vector. The solution methods developed in analytical mechanics may then be helpful in solving the shell problems in terms of $\boldsymbol{\Omega}$.

By solving the set of shell equations with respect to $\boldsymbol{\Omega}$ and $\gamma_{\alpha\beta}$ the position of the shell is established with accuracy up to a rigid-body translation in space.

7. Bending shell equations in terms of displacements

Within an error of the first-approximation theory the formula (2.4) for $\gamma_{\alpha\beta}$ cannot be simplified. In the formula (2.5) for $\kappa_{\alpha\beta}$ we are able to simplify $\boldsymbol{n}$ and $\boldsymbol{n}_\mu$ according to

$$\begin{aligned}
\boldsymbol{n} &= [1+\vartheta^\kappa_\kappa + \tfrac{1}{2}(\vartheta^\kappa_\kappa)^2 - \tfrac{1}{2}\vartheta^\kappa_\mu\vartheta^\mu_\kappa + \varphi^2][1-\gamma^\lambda_\lambda + O(\eta^2)],\\
\boldsymbol{n}_\mu &= [-(1-\vartheta^\kappa_\kappa)\varphi_\mu + \varphi^\lambda(\vartheta_{\lambda\mu}-\omega_{\lambda\mu})][1+O(\eta)],\\
\gamma^\lambda_\lambda &= \vartheta^\lambda_\lambda + \tfrac{1}{2}\vartheta^\lambda_\kappa\vartheta^\mu_\lambda + \tfrac{1}{2}\varphi^\lambda\varphi_\lambda + \varphi^2.
\end{aligned} \tag{7.1}$$

In effect $\kappa_{\alpha\beta}$ becomes a fifth-order polynomial in terms of displacements and their derivatives.

If we introduce (2.4) and (2.5) with (7.1) into the equilibrium equations $(5.4)_{1,2}$ they become expressed entirely in terms of displacements. Two equations $(5.4)_1$ are quadratic and the one $(5.4)_2$ is of the seventh-order polynomial in terms of u_α, w and their derivatives.

The displacemental boundary conditions are those given in $(4.15)_1$ where now $\beta_\nu = -\varphi_\alpha[\nu^\alpha + O(\eta)]$. When applying the statical boundary conditions we should express $\boldsymbol{P}_\nu$ and $M_{\nu\nu}$ in terms of displacements by using the constitutive equations (5.2) and the strain-displacement relations (2.4) and (2.5) with (7.1).

The solution of a shell problem in terms of displacements gives us the complete information about the shell behaviour within the accuracy of the first-approximation theory. Unfortunately, even within the bending theory the resulting set of shell relations in displacements is still quite complex and nobody, as yet, has tried to solve such displacemental equations without any additional simplification.

For some shell problems the displacement field may be obtained by dividing the solution process into three steps. In the first step the strains and changes of curvatures are calculated according to § 5. In the second step the finite rotation vector is found from the known strain measures as suggested in § 6. Taking into account that $\bar{\boldsymbol{a}}_\alpha = \boldsymbol{a}_\alpha + \boldsymbol{u}_{,\alpha}$ and using (3.9) we obtain

$$\boldsymbol{u}_{,\beta} = \breve{\gamma}_{\alpha\beta}\, \boldsymbol{a}^\alpha + (\delta^\lambda_\beta + \breve{\gamma}^\lambda_\beta)\left[\boldsymbol{\Omega} \times \boldsymbol{a}_\lambda + \frac{1}{2\cos^2 \omega/2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{a}_\lambda)\right], \tag{7.2}$$

or within the first-approximation theory

$$\boldsymbol{u}_{,\beta} = \gamma_{\alpha\beta}\, \boldsymbol{a}^\alpha + \boldsymbol{\Omega} \times \boldsymbol{a}_\beta + \frac{1}{2\cos^2 \omega/2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \boldsymbol{a}_\beta). \tag{7.3}$$

These are the first-order differential equations for $\boldsymbol{u}$ to be found in the third step from known $\boldsymbol{\Omega}$ and $\gamma_{\alpha\beta}$.

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Trzy postacie równań geometrycznie nieliniowej zgięciowej teorii powłok

Streszczenie

Deformację powłoki z więzami Kirchhoffa-Love'a rozłożono na sztywne przesunięcie, czyste rozciągnięcie wzdłuż głównych kierunków odkształcenia oraz obrót skończony kierunków głównych. Podano ogólne wzory na obroty skończone w powłokach. Równania równowagi powłoki wyrażono poprzez wielkości Lagrange'owskie. Skonstruowano trzy warianty geometrycznych warunków brzegowych oraz energetycznie spójne z nimi warianty statycznych warunków brzegowych. Podstawowe równania teorii powłok konsekwentnie uproszczono przy założeniu małych sprężystych odkształceń oraz teorii zgięciowej. Te podstawowe zależności wyrażono poprzez trzy grupy zmiennych niezależnych: odkształcenia i zmiany krzywizn, odkształcenia i obroty skończone lub przemieszczenia.

Три виды уравнений геометрически нелинейной изгибиой теории оболочек

Резюме

Диформация оболочки со связями Кирхгоффа-Лява разложена на жесткое перемещение, чистое растяжение вдоль главных осей деформации и конечный поворот этих главных осей. Получены общие формулы для конечных поворотов в оболочке. Выведены уравнения равновесия оболочки выраженные через Лагранжевые величины. Сформулированы три варианты геометрических краевых условий и энергетически согласованные с ними варианты статических краевых условий. Основные зависимости теории оболочек упрощены при предположении малых упругих деформаций и изгибной теории. Эти основные уравнения представлены через три группы независимых переменных: деформации и изменения кривизны, деформации и конечные повороты или перемещения.