INSTITUT FÜR MECHANIK
RUHR-UNIVERSITÄT BOCHUM

WOJCIECH PIETRASZKIEWICZ

ON CONSISTENT APPROXIMATIONS IN THE GEOMETRICALLY NON-LINEAR THEORY OF SHELLS

MITTEILUNGEN AUS DEM INSTITUT FÜR MECHANIK
NR. 26
JUNI 1981

Editor:

Institut für Mechanik der Ruhr-Universität Bochum

ZUSAMMENFASSUNG

Die vorliegende Arbeit behandelt die Ableitung, Klassifizierung und Rechtfertigung von geometrisch nichtlinearen Theorien für dünne, elastische Schalen, die auf der sogenannten "ersten Approximation" der Verformungsenergiedichte beruhen. Ausgehend von der allgemeinen geometrisch nichtlinearen Kirchhoff-Love Schalentheorie, die für kleine Dehnungen und beliebige Rotationen gültig ist [1], wird eine Familie von vereinfachten Theorien hergeleitet, in denen die Rotationen als groß, mittelgroß oder klein vorausgesetzt werden. Insbesondere werden neue Theorien für solche Schalenprobleme angegeben, bei denen die Schalenelemente großen Rotationen um Tangenten zur Mittelfläche unterworfen werden, während die Rotationen um Normalen klein, mittelgroß oder ebenfalls groß sind. Außerdem wird gezeigt, wie sich aus der allgemeinen Kirchhoff-Love Schalentheorie mit Hilfe geeigneter, konsistenter Vereinfachungen die geometrisch nichtlineare Schalentheorie bei Auftreten von Rotationen mittlerer Größenordnung ableiten läßt, die kürzlich entwickelt worden ist.

SUMMARY

This report is concerned with the derivation, classification and justification of geometrically non-linear theories for thin, elastic shells which are based on the so called "first approximation" of the strain energy function. Starting from the general geometrically non-linear Kirchhoff-Love type theory of shells undergoing small strains and arbitrary rotations [1], a family of simplified theories is derived by restricting the magnitude of the rotations to be large, moderate or small. In particular new theories are given for such shell problems in which the shell material elements undergo large rotations about tangents to the shell middle surface, whereas the rotations about normals are small, moderate or even large, too. Furthermore it is shown that under appropriate consistent simplifications the general first-approximation shell theory reduces to the geometrically non-linear theory of shells undergoing moderate rotations, which has been established in the literature recently.

CONTENTS

ON CONSISTENT APPROXIMATIONS IN THE GEOMETRICALLY NON-LINEAR THEORY OF SHELLS

by

W. Pietraszkiewicz (Gdańsk)

1. INTRODUCTION

In the work [1] a set of equations for the non-linear theory of thin shells undergoing small strains and unrestricted rotations was derived. All shell relations were referred to the undeformed shell middle surface. A modified tensor of change of curvature and a new independent parameter describing the finite rotation of the shell boundary element were introduced. In the case of an elastic material and conservative surface and boundary loadings the theory allowed for proper variational formulation of geometrically non-linear shell problems.

The non-linear shell relations derived in [1] are still quite complex, since no kind of restrictions have been imposed on rotations of the shell material elements. For many engineering shell problems it is hardly necessary to allow rotations of any magnitude. Some shell structures would become unserviceable if really finite rotations were permitted to occure. Therefore, it is certainly worthwhile to discuss possible simplifications of complex non-linear shell relations obtained in [1] resulting from consistently restricted rotations.

Several approximation schemes leading to simplified sets of equations of geometrically non-linear theory of shells were proposed in the literature. In the works of Chien [26], Koiter [14], Pietraszkiewicz [2,4,18] and Simmonds [40] various restrictions were imposed on middle surface strains and changes of curvatures to derive various sets of approximate shell equations, among which the most important were membrane, bending and inextensional bending shell equations. Mushtari and Galimov [11], Galimov [12], Leonard [35], Sanders [13], Koiter [14], Shapovalov [31], Pietraszkiewicz [2] and Kabanov [29] restricted components of the linearized rotation vector and some of displacement

---

*) This work was prepared under an Agreement on Scientific Cooperation between the Institute of Fluid-Flow Machinery of the Polish Academy of Sciences in Gdańsk and the Institut für Mechanik of the Ruhr-Universität Bochum, FRG.

gradients as compared with strains, relative thickness and/or variability of deformation state and derived several simplified variants of shell relations under various names. Among the best known are approximate shell relations of medium bending theory given in [11], with moderately small rotations proposed in [13] and with small finite deflections discussed in [14]. A variety of simplified variants of non-linear shell equations derived by Duszek [28,36] followed from restricting displacements and their surface gradients in terms of several independent small parameters, while those given by Novotny [37] were obtained from three-dimensional equations through a formal asymptotic procedure.

Most of the simplified variants proposed in the works referred to above, were derived by omitting some terms in definitions of strain measures, in equilibrium equations, in boundary conditions and/or in compatibility conditions. The terms were usually omitted because of their relative smallness as compared with other terms in the same relations, but not because of their small contribution to the strain energy of a shell. Such procedures, although formally correct, not always lead to satisfactory sets of shell equations. They do not assure that the (local) simplifications of the equations are consistent within the assumed approximation to the (global) variational formulation of the shell problem. By applying such formal procedures some supposedly important terms are retained in the equations, although their contribution to the elastic shell strain energy may be negligibly small indeed. On the other hand, by discussing possible simplifications only of the equations we may construct such approximate system of shell relations, which cannot be derivable from any variational principle. In order to avoid this, only strain-displacement relations were simplified in [12 -14] by omitting terms which were small as compared with leading terms, but other shell equations were generated from the simplified strain measures using a principle of virtual displacements.

The lack of appropriate variational formulation is a serious disadvantage of all simplified procedures based on omission of supposedly small terms from the shell equations. The main reason is that the most powerful numerical procedures used nowadays, such as the finite element method or the finite difference energy method, may be applied only if the problem allows for a variational formulation. Otherwise, some special methods of solution, which are not so effective, should be used.

In this report we shall formulate approximate variants of geometrically non-linear equations of thin elastic shells undergoing small, moderate, large or finite rotations. The scheme of simplifications, proposed in our earlier works [3-5], is based on consistent restrictions of rotation angles of material elements during the shell deformation. Strains and rotations may be exactly separated from each other [16] by the polar decomposition of the shell deformation gradient tensor. Since in the geometrically non-linear theory of shells strains are restricted to be small everywhere, further possible approximations are most natural if they follow from consistently restricted rotations.

The restrictions assumed on the rotations are formulated in terms of one common small parameter θ defined in [19] and redefined in [20] using physical arguments. The rotations are said to be small, moderate, large or finite if the rotation angles allowed are of the order of θ^2, θ, $\sqrt{\theta}$ or 1, respectively. Order-of-magnitude discussion allows then to estimate orders of linearized rotations and linearized strains and, therefore, orders of all terms in the definitions of the surface strain tensor $\gamma_{\alpha\beta}$ and of the tensor of change of surface curvature $\chi_{\alpha\beta}$. Only these two strain measures appear in the first-approximation theory of thin elastic shells, for which the elastic strain energy function Σ becomes [21,22] a quadratic form, with respect to the shell strain measure to within a relative error $O(\theta^2)$ as compared with unity.

For each of the simplified variants of shell equations only those terms are omitted in the strain-displacement relations, whose contribution to the shell strain energy lies within the error margin already introduced to Σ within the first approximation theory. Other shell relations (equilibrium equations, geometric and static boundary and corner conditions) are generated from the simplified strain-displacement relations by applying the variational principle of virtual displacements. Therefore, the simplifications assumed in this work are consistent in the sense of an error introduced to the shell strain energy function and assure the existence of variational principles for each of the approximate variants of shell equations.

Many shell structures are manufactured to be quite rigid for in-surface deformation being flexible for out-of-surface deformation.

This feature of thin-walled structures is taken into account in our classification scheme by restricting not only the value of the rotation angle but also the direction of the rotation axis. Therefore, we assume different restricitons on particular components of the finite rotation vector $\underset{\sim}{\Omega}$ and associate the names "small, moderate, large or finite rotation" with the particular component of $\underset{\sim}{\Omega}$.

In the consistent theory of shells undergoing moderate rotations the tensor of change of curvature becomes a linear function of displacements. As a result, all transformations are much simpler than in the general case. In particular, the fourth boundary condition for a moment takes the same form as in the classical linear theory of shells. The consistent set of equations may be constructed either by direct transformations as given in [3-5], or by applying to this particular case the general scheme of derivation given in [1]. The latter approach is used here. The shell relations of the moderate rotation theory contain, as special cases, the equations of various simplified variants of the non-linear shell theory, which have been proposed in the literature [8-15]. The problem was discussed in detail in [6,38], where a variety of variational functionals was constructed.

As the principal new results of the work several variants of the consistent theory of shells undergoing large rotations are constructed. Only out-of-surface rotations are assumed to be always large, while in-surface rotations are supposed to be either small or moderate or large. This allows to discuss three particular cases of the shell equations with large/small or large/moderate or simply large rotations.

The variant of theory of shells undergoing large/small rotations represents the simplest case within the large rotation theory. It exhibits certain features of the general theory discussed in [1], leading at the same time to relatively simple shell relations. In the process of derivation it is shown, in particular, that approximate expressions for the shell strain measures, which are consistent to within indicated error in the strain energy function, may lead to some splitting of the boundary terms. In order to restore appropriate structure of the static boundary conditions some small terms, already neglected from strain-displacement relations, should be retained in boundary conditions. The boundary terms themselves are constructed by expanding the exact non-rational square-root functions given in [1] into series and retaining only terms within a desired accuracy. The relative error introduced into Σ does not exceed $O(\theta^2)$ compatible with the accuracy of the first-approximation theory.

Besides the main variant of the large/small rotation theory two consistently simplified variants are proposed. The simplified variants are constructed by allowing greater relative error in the strain energy to be $O(\theta\sqrt{\theta})$ or $O(\theta)$, respectively. Within the consistent approximations both strain measures become quadratic polynomials in displacements. Equilibrium equations and two of the static boundary and corner conditions are linear both in displacements and internal stress and moment resultants, while the remaining static boundary and corner conditions contain also some squares of the displacemental variables. The simplified variants of large/small rotation shell theory seem to be particularly suitable and convenient to apply in engineering calculations of various shell structures.

Finally, the Lagrangian shell equations are derived for the variant of theory of shells undergoing large rotations in all directions. The relations obtained are obviously more complicated than for the large/small rotation theory. As a special case appropriate shell relations for the large/moderate rotation theory are also derived. It is interesting to note that within the accuracy of the first-approximation theory some parameters at the shell boundary should be estimated here with a higher precision, directly from their exact definitions given in [1] for unrestricted strains. This proves once again that in geometrically non-linear theory it is not advisable to omit strains with respect to unity at too early stage of derivation of shell equations, since occasionally such procedure may lead to inaccurate results. Again, two simplified variants of the large rotation theory are constructed allowing for a greater error $O(\theta\sqrt{\theta})$ or $O(\theta)$ in the strain energy function. The simplified shell relations may be applied in engineering calculations, when the lower accuracy of Σ is regarded as satisfactory.

The shell relations derived here for the large rotation range have no counterpart in the literature. In our earlier works [3,4] the approximate strain measures and generated by them appropriate equilibrium equations were given, but at that time we failed to construct appropriate boundary conditions. In [14,31,32] the tensor of change of curvature was proposed in the form of quadratic polynomials of displacements by omitting some terms which were supposedly small with respect to other supposedly principal terms. Such procedure and boundary conditions given in [31] cannot be regarded as consistent from the variational point of view and cannot be compared with our consistently derived shell relations.

Possible simplifications of relations of the geometrically non-linear theory of shells resulting only from consistently restricted in-surface rotations are not discussed here. It has been found that even in the simplest case of the finite/small rotations, within relative accuracy $O(\theta^2)$ of Σ only few terms may be omitted from the exact definition of the tensor of change of curvature. It seems, therefore, that considerably simplified shell relations derived in [29,30] for finite/small rotations cannot be regarded as justified within the first-approximation theory of shells.

The classification of the simplified shell relations presented in this report assures, within the assumed error limits, the existence of the general Hu-Washizu variational principle [1,41] for each of the approximate versions of shell relations. It should be noted that even for the simplest version of the theory of shells undergoing large/small rotations the variationally derived definitions of physical quantities at the boundary still contain some non-rational square-root functions of the displacement parameters. In this report all the non-rational functions are expanded into series where only those terms which are important within the prescribed accuracy are taken into account. Such equivalent polynomial representations of the non-rational expressions are convenient for numerical calculations. Besides, the procedure clearly indicates which boundary terms are really important in the particular approximate version of the shell theory. In some theoretical considerations it may be more convenient to preserve the original non-rational structure of the boundary conditions of the exact theory [1] . Then some of the small terms which have been omitted here from the definitions of the shell strain measures, because of their small contribution to the elastic strain energy of a shell, should additionally be taken into account. Then the shell relations would become more complex.

2. BASIC RELATIONS FOR THIN SHELLS UNDERGOING SMALL ELASTIC STRAINS

The notation used in this paper follows that of [1-5]. In order to make the work self-contained let us remind, without derivation, some results given in [1] and in our earlier papers.

The deformation of the shell middle surface is described by a displacement vector $\underset{\sim}{u} = u^{\alpha}\underset{\sim}{a}_{\alpha} + w\underset{\sim}{n}$, where $\underset{\sim}{a}_{\alpha}$ are base vectors of the reference surface M and $\underset{\sim}{n}$ is a unit normal to M. The surface strain tensor $\gamma_{\alpha\beta}$ and the modified tensor of change of curvature $\chi_{\alpha\beta}$ can be presented in the following symmetric forms

$$\gamma_{\alpha\beta} = \frac{1}{2}\,(l^{\lambda}_{\cdot\alpha} l_{\lambda\beta} + \varphi_{\alpha}\varphi_{\beta} - a_{\alpha\beta})$$

$$\chi_{\alpha\beta} = \frac{1}{2}\,[l^{\lambda}_{\cdot\alpha}(m_{\lambda|\beta} - b_{\lambda\beta}m) + l^{\lambda}_{\cdot\beta}(m_{\lambda|\alpha} - b_{\lambda\alpha}m) + \tag{2.1}$$

$$+ \varphi_{\alpha}(m_{,\beta} + b^{\lambda}_{\beta}m_{\lambda}) + \varphi_{\beta}(m_{,\alpha} + b^{\lambda}_{\alpha}m_{\lambda})] + b_{\alpha\beta}(1 + \gamma^{\kappa}_{\kappa})$$

where

$$l_{\alpha\beta} = a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta}\quad , \qquad \varphi_{\alpha} = w_{,\alpha} + b^{\lambda}_{\alpha}u_{\lambda}$$

$$\theta_{\alpha\beta} = \frac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w\ ,\quad \omega_{\alpha\beta} = \frac{1}{2}\,(u_{\beta|\alpha} - u_{\alpha|\beta}) = \epsilon_{\alpha\beta}\varphi \tag{2.2}$$

$$m_{\lambda} = -\,(1 + \theta^{\kappa}_{\kappa})\varphi_{\lambda} + \varphi^{\mu}(\theta_{\mu\lambda} - \omega_{\mu\lambda}) = \sqrt{\frac{\bar{a}}{a}}\,n_{\lambda}$$

$$m = 1 + \theta^{\kappa}_{\kappa} + \frac{1}{2}\,(\theta^{\kappa}_{\kappa})^{2} - \frac{1}{2}\,\theta^{\kappa}_{\rho}\theta^{\rho}_{\kappa} + \frac{1}{2}\,\omega^{\kappa\rho}\omega_{\kappa\rho} = \sqrt{\frac{\bar{a}}{a}}\,n \tag{2.3}$$

$$\frac{\bar{a}}{a} = 1 + 2\gamma^{\alpha}_{\alpha} + 2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha})$$

The deformation of the shell boundary element is described by three displacements $\underset{\sim}{u} = u_{\nu}\underset{\sim}{\nu} + u_{t}\underset{\sim}{t} + w\underset{\sim}{n}$ and three components of the deformed unit normal $\bar{\underset{\sim}{n}} = n_{\nu}\underset{\sim}{\nu} + n_{t}\underset{\sim}{t} + n\underset{\sim}{n}$. Here $n_{\nu} = n_{\alpha}\nu^{\alpha}$ and $n_{t} = n_{\alpha}t^{\alpha}$, where $\underset{\sim}{t} = t^{\alpha}\underset{\sim}{a}_{\alpha}$ is a unit vector tangent to the reference boundary contour C and $\underset{\sim}{\nu} = \nu_{\alpha}\underset{\sim}{a}^{\alpha}$ is an outward unit vector normal to C.

The parameters n_{t} and n can be expressed in terms of $\underset{\sim}{u}$ and n_{ν} by the formulae

$$n_{t} = -\,\frac{c_{\nu}c_{t}n_{\nu} - cD}{1 + 2\gamma_{tt} - c^{2}_{\nu}}\quad , \qquad n = -\,\frac{c_{\nu}cn_{\nu} + c_{t}D}{1 + 2\gamma_{tt} - c^{2}_{\nu}} \tag{2.4}$$

where

$$c_\nu = \theta_{\nu t} - \varphi \ , \quad c_t = 1 + \theta_{tt} \ , \quad c = \varphi_t = \varphi_\alpha t^\alpha \ , \quad \gamma_{tt} = \gamma_{\alpha\beta} t^\alpha t^\beta$$

$$\theta_{\nu t} = \theta_{\alpha\beta}\nu^\alpha t^\beta \ , \quad \theta_{tt} = \theta_{\alpha\beta} t^\alpha t^\beta \ , \quad D = -\sqrt{(1+2\gamma_{tt})(1-n_\nu^2) - c_\nu^2} \tag{2.5}$$

Let $N^{\alpha\beta}$ and $M^{\alpha\beta}$ be the Lagrangian symmetric internal force and couple resultant tensors of the shell in an equilibrium state. For any additional virtual displacement field $\delta \underset{\sim}{u} = \delta u_\alpha \underset{\sim}{a}^\alpha + \delta w \underset{\sim}{n}$, subject to geometric constraints, the internal virtual work, performed by the stress and couple resultant tensors on variations of corresponding strain measures, can be transformed as follows [1]:

$$IVW = \iint_M (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\chi_{\alpha\beta})dA =$$

$$= -\iint_M \underset{\sim}{T}^\beta|_\beta \cdot \delta\underset{\sim}{u}dA + \int_C [(\underset{\sim}{T}^\beta \nu_\beta)\cdot\delta\underset{\sim}{u} + R_{\nu\nu}\delta n_\nu + R_{t\nu}\delta n_t + R_\nu \delta n]ds =$$

$$= -\iint_M \underset{\sim}{T}^\beta|_\beta \cdot \delta\underset{\sim}{u}dA + \int_C (\underset{\sim}{P}\cdot\delta\underset{\sim}{u} + M\delta n_\nu)ds + \sum_k \underset{\sim}{F}_k \cdot \delta\underset{\sim}{u}_k \tag{2.6}$$

where

$$\underset{\sim}{T}^\beta = T^{\lambda\beta}\underset{\sim}{a}_\lambda + T^\beta \underset{\sim}{n}$$

$$T^{\lambda\beta} = 1^\lambda_{\cdot\alpha}(N^{\alpha\beta} + a^{\alpha\beta} b_{\kappa\rho}M^{\kappa\rho}) + (\mathbf{m}^\lambda|_\alpha - b^\lambda_\alpha \mathbf{m})M^{\alpha\beta} +$$

$$+ \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\{1_{\mu\alpha}[(\varphi_\kappa M^{\kappa\rho})|_\rho + b^\gamma_\kappa 1_{\gamma\rho}M^{\kappa\rho}] - \varphi_\alpha[(1_{\mu\kappa}M^{\kappa\rho})|_\rho - b_{\mu\kappa}\varphi_\rho M^{\kappa\rho}]\}$$

$$T^\beta = \varphi_\alpha(N^{\alpha\beta} + a^{\alpha\beta}b_{\kappa\rho}M^{\kappa\rho}) + (\mathbf{m}_{,\alpha} + b^\lambda_\alpha \mathbf{m}_\lambda)M^{\alpha\beta} +$$

$$+ \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}1_{\lambda\alpha}[(1_{\mu\kappa}M^{\kappa\rho})|_\rho - b_{\mu\kappa}\varphi_\rho M^{\kappa\rho}] \tag{2.7}$$

$$\underset{\sim}{P} = \underset{\sim}{T}^\beta\nu_\beta + \underset{\sim}{Q} \ , \quad M = R_{\nu\nu} + fR_{t\nu} + kR_\nu$$

$$\underset{\sim}{Q} = [d_\nu R_{t\nu} + h_\nu R_\nu - \frac{d}{ds}(g_\nu R_{t\nu} + r_\nu R_\nu)]\underset{\sim}{\nu} +$$

$$+ [d_t R_{t\nu} + h_t R_\nu - \frac{d}{ds}(g_t R_{t\nu} + r_t R_\nu)]\underset{\sim}{t} +$$

$$+ [dR_{t\nu} + hR_\nu - \frac{d}{ds}(gR_{t\nu} + rR_\nu)]\underset{\sim}{n} \tag{2.8}$$

$$\underset{\sim}{F} = (g_\nu R_{t\nu} + r_\nu R_\nu)\underset{\sim}{\nu} + (g_t R_{t\nu} + r_t R_\nu)\underset{\sim}{t} + (g R_{t\nu} + r R_\nu)\underset{\sim}{n}$$

$$\underset{\sim}{F}_k = \underset{\sim}{F}(s_k + 0) - \underset{\sim}{F}(s_k - 0)$$

$$d_\nu = \frac{n}{D}(\kappa_t n_t - \tau_t n) \ , \quad d_t = \frac{n}{D}(\sigma_t n - \kappa_t n_\nu) \ , \quad d = \frac{n}{D}(\tau_t n_\nu - \sigma_t n_t)$$

$$g_\nu = \frac{n_\nu n}{D} \ , \quad g_t = \frac{n_t n}{D} \ , \quad g = \frac{n^2}{D}$$

$$h_\nu = \frac{n_t}{D}(\tau_t n - \kappa_t n_t) \ , \quad h_t = \frac{n_t}{D}(\kappa_t n_\nu - \sigma_t n) \ , \quad h = \frac{n_t}{D}(\sigma_t n_t - \tau_t n_\nu) \tag{2.9}$$

$$r_\nu = -\frac{n_\nu n_t}{D} \ , \quad r_t = -\frac{n_t^2}{D} \ , \quad r = -\frac{n_t n}{D}$$

$$f = \frac{1}{D}(c_\nu n - c n_\nu) \ , \quad k = \frac{1}{D}(c_t n_\nu - c_\nu n_t)$$

$$\sigma_t = b_{\alpha\beta} t^\alpha t^\beta \ , \quad \tau_t = -b_{\alpha\beta} \nu^\alpha t^\beta \ , \quad \kappa_t = t_\alpha \nu^\alpha|_\beta t^\beta$$

$$K_{\nu\nu} = (1 + \theta_{\nu\nu})M_{\nu\nu} + (\theta_{\nu t} - \varphi)M_{t\nu} = \sqrt{\frac{a}{\bar{a}}}\, R_{\nu\nu}$$

$$K_{t\nu} = (\theta_{\nu t} + \varphi)M_{\nu\nu} + (1 + \theta_{tt})M_{t\nu} = \sqrt{\frac{a}{\bar{a}}}\, R_{t\nu} \tag{2.10}$$

$$K_\nu = \varphi_\nu M_{\nu\nu} + \varphi_t M_{t\nu} = \sqrt{\frac{a}{\bar{a}}}\, R_\nu$$

$$m_\nu = -\varphi_\nu(1 + \theta_{tt}) + \varphi_t(\theta_{\nu t} + \varphi) = \sqrt{\frac{\bar{a}}{a}}\, n_\nu$$

$$m_t = \varphi_\nu(\theta_{\nu t} - \varphi) - \varphi_t(1 + \theta_{\nu\nu}) = \sqrt{\frac{\bar{a}}{a}}\, n_t \tag{2.11}$$

$$m = 1 + \theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \theta_{\nu\nu}\theta_{tt} - \theta_{\nu t}^2 = \sqrt{\frac{\bar{a}}{a}}\, n$$

Let the shell be subjected to the conservative middle surface load $\underset{\sim}{p} = p^\alpha \underset{\sim}{a}_\alpha + p\underset{\sim}{n}$ and to the conservative resultant boundary force $\underset{\sim}{N} = N_\nu \underset{\sim}{\nu} + N_t \underset{\sim}{t} + N\underset{\sim}{n}$ and the boundary static moment $\underset{\sim}{H} = H_\nu \underset{\sim}{\nu} + H_t \underset{\sim}{t} + H\underset{\sim}{n}$. Then the external virtual work performed by $\underset{\sim}{p}$, $\underset{\sim}{N}$ and $\underset{\sim}{H}$ on variations of corresponding displacemental parameters, can be presented in analogous to (2.6) form [1]

$$\mathrm{EVW} = \iint_M \underset{\sim}{p} \cdot \delta\underset{\sim}{u}\, dA + \int_{C_f} (\underset{\sim}{N} \cdot \delta\underset{\sim}{u} + H_\nu \delta n_\nu + H_t \delta n_t + H\delta n)\, ds =$$

$$= \iint_M \underset{\sim}{p} \cdot \delta\underset{\sim}{u}\, dA + \int_{C_f} (\underset{\sim}{P}^* \cdot \delta\underset{\sim}{u} + M^*\delta n_\nu)\, ds + \sum_j \underset{\sim}{F}{}^*_j \cdot \delta\underset{\sim}{u}_j \quad . \tag{2.12}$$

Here starred quantities have exactly the same structure as those given in (2.8), only there $\underset{\sim}{T}^\beta \nu_\beta$ should be replaced by $\underset{\sim}{N}$ and $R_{\nu\nu}, R_{t\nu}, R_\nu$ by H_ν, H_t, H, respectively.

From IVW = EVW we have the following Lagrangian equilibrium equations and corresponding static and geometric boundary conditions:

$$\begin{aligned}
&\underset{\sim}{T}^\beta|_\beta + \underset{\sim}{p} = \underset{\sim}{0} && \text{in } M \\
&\underset{\sim}{P} = \underset{\sim}{P}^* \text{ and } M = M^* && \text{on } C_f \\
&\underset{\sim}{F}_j = \underset{\sim}{F}{}^*_j && \text{at each } M_j \in C_f \\
&\underset{\sim}{u} = \underset{\sim}{u}^* \text{ and } n_\nu = n^*_\nu && \text{on } C_u \\
&\underset{\sim}{u}_i = \underset{\sim}{u}{}^*_i && \text{at each } M_i \in C_u
\end{aligned} \tag{2.13}$$

Within the first-approximation theory of thin isotropic and elastic shells the strain energy function may be approximated [21,22] by the quadratic expression

$$\begin{aligned}
\Sigma &= \frac{h}{2} H^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12} \chi_{\alpha\beta}\chi_{\lambda\mu}) + O(Eh\, \eta^2\theta^2) \\
H^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)} (a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu} a^{\alpha\beta}a^{\lambda\mu})
\end{aligned} \tag{2.14}$$

where h is the small thickness, E is the Young's modulus and ν is the Poisson's ratio. The error of Σ at any point of the shell is expressed through the small parameter θ defined by [19,20,2,4]

$$\theta = \max\, (\frac{h}{L}, \frac{h}{d}, \sqrt{\frac{h}{R}}, \sqrt{\eta}\,) \tag{2.15}$$

where L is the smallest wavelength of deformation patterns at M, d is the distance of the point from the lateral shell boundary, R is the smallest principal radius of curvature of M and η is the largest strain in the shell space.

From $(2.14)_1$ we obtain the constitutive equations

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2}\,[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^{\kappa}_{\kappa}] + O(Eh\eta\theta^2)$$

$$M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\chi_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}\,[(1-\nu)\chi^{\alpha\beta} + \nu a^{\alpha\beta}\chi^{\kappa}_{\kappa}] + O(Eh^2\eta\theta^2) \qquad (2.16)$$

Deformation near a point of the shell middle surface may be exactly decomposed [3-5] into a rigid-body translation, a pure stretch along principal directions of strain and a rigid-body rotation of the principal directions. The rotations may be calculated through a finite rotation vector $\underset{\sim}{\Omega} = \underset{\sim}{e}\,\sin\omega$, where the unit vector $\underset{\sim}{e}$ describes direction of the rotation axis while ω is the rotation angle about the rotation axis. Within small strains but unrestricted rotations $\underset{\sim}{\Omega}$ may be approximated by [16]

$$\underset{\sim}{\Omega} \simeq \epsilon^{\beta\alpha}[\varphi_\alpha(1 + \tfrac{1}{2}\theta^{\kappa}_{\kappa}) - \tfrac{1}{2}\varphi^{\lambda}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})]\underset{\sim}{a}_\beta + \varphi\underset{\sim}{n} \qquad (2.17)$$

The relations (2.1) - (2.13) are exact and are valid for an arbitrary deformation of the shell middle surface M. One would expect that within the first approximation theory the mid-surface strains should always be neglected with respect to unity and that n_ν, n_t, n and $R_{\nu\nu}$, $R_{t\nu}$, R_ν should be identified with m_ν, m_t, m and $K_{\nu\nu}$, $K_{t\nu}$, K_ν, respectively. According to (2.15) the maximal value of $\gamma_{\alpha\beta}$ may reach $O(\theta^2)$, for example, in case of a very thin shell made of a composite or a polymer with relatively small Young's modulus. When discussing boundary conditions in the large rotation shell theory (see p. 5 and 6) it will be shown that within the desired accuracy terms $O(\theta^2)$ should be taken into account in the approximate expressions for n, D, n/D and $R_{\nu\nu}$. As a result, the mid-surface strains $\gamma_{\nu\nu}$ and γ_{tt} will explicitly appear in the approximate formulaee. However, for the shell structures made of steel, aluminum, concrete etc. $\eta \ll \theta^2$, as a rule, within the elastic range of shell deformation and strains may be omitted with respect to unity even if other terms $O(\theta^2)$ are taken into account.

3. CLASSIFICATION OF ROTATIONS

The shell relations given above were obtained by restricting strains to be small everywhere in the shell. This has led to consistently simplified relations of geometrically non-linear theory of thin shells [1].

By the polar decomposition theorem strains and rotations of the shell material elements have been exactly separated from each other in [3,4]. Therefore, further consistent simplifications of the geometrically non-linear shell relations may be achieved by imposing additional restrictions on the rotations of the shell material elements.

The basic parameter describing the magnitude of a finite rotation is the rotation angle ω. According to the exact theory of finite rotations in shells [5] the angle ω appears in many shell relations as an argument of trigonometric functions $\sin\omega$, $\cos\omega$, $2\cos\omega/2$ etc. Expanding the trigonometric functions into Taylor series in the vicinity of $\omega = 0$ we obtain, for example

$$\sin\omega = \omega - \frac{\omega^3}{3!} + \frac{\omega^5}{5!} - \dots \quad , \quad \cos\omega = 1 - \frac{\omega^2}{2!} + \frac{\omega^4}{4!} - \dots \qquad (3.1)$$

It is seen, that substantial simplification of shell relations may be achieved if the restrictions put on rotations permit to approximate the series (3.1) by their leading terms. Approximation of (3.1) by their two first terms lead also to some simplification of geometrically non-linear shell relations.

For a thin isotropic elastic shell undergoing small strains there exists a small parameter θ defined by (2.15). This parameter is used here to introduce the following classification of rotations:

$\omega \le O(\theta^2)$ - small rotations

$\omega = O(\theta)$ - moderate rotations

$\omega = O(\sqrt{\theta})$ - large rotations

$\omega \ge O(1)$ - finite rotations

Since terms of the order of θ^2, referred to as $O(\theta^2)$, are small they may be neglected as compared with unity. In the case of small and moderate rotations this allows to approximate all trigonometric functions of ω by their leading terms in Taylor series, while in the case of large rotations two first terms of the series should be retained.

Note that for $|\omega| < \pi/2$ we have $O(|\underset{\sim}{\Omega}|) = O(\sin\omega) = O(\omega)$. Therefore, the classification proposed above restricts only the magnitude of the finite rotation vector but not its direction. It is known, however, that many shell structures are manufactured to be quite rigid for in-surface deformation even if they are allowed to be flexible for out-of-surface deformation. The rotational parts of the both deformations may also be estimated by respective components $\Omega = \underset{\sim}{\Omega} \cdot \underset{\sim}{n}$ and $\Omega_\alpha = \underset{\sim}{\Omega} \cdot \underset{\sim}{a}_\alpha$ of the finite rotation vector. The names "small, moderate, large or finite rotations" may then be associated with the particular component of $\underset{\sim}{\Omega}$.

In what follows we shall discuss possible simplifications of shell strain measures (2.1), and other shell relations generated by them, resulting from consistently restricted rotations. The measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are defined directly in terms of linearized quantities $\theta_{\alpha\beta}$, φ_α and φ (or $\omega_{\alpha\beta}$). For any restrictions imposed on finite rotations estimates for φ_α and φ follow from (2.17). The estimate for $\theta_{\alpha\beta}$ may then be found by solving $(2.1)_1$ with respect to $\theta_{\alpha\beta}$ and taking into account that $\gamma_{\alpha\beta}$ are always small. In each case of restricted rotations we obtain estimates for linearized quantities according to the following scheme:

Restrictions on rotations		Estimate of the quantity		
Ω_α	Ω	φ_α	φ	$\theta_{\alpha\beta}$
small	small	θ^2	θ^2	η
moderate	small	θ	θ^2	θ^2
moderate	moderate	θ	θ	θ^2
large	small	$\sqrt{\theta}$	θ^2	θ
large	moderate	$\sqrt{\theta}$	θ	θ
large	large	$\sqrt{\theta}$	$\sqrt{\theta}$	θ
finite	small	1	θ^2	1
finite	moderate	1	θ	1
finite	large	1	$\sqrt{\theta}$	1
finite	finite	1	1	1

Table 1

From the approximate form of the shell strain energy function (2.14) it follows, that within the first-approximation theory $\gamma_{\alpha\beta}$ are already calculated with an error $O(\eta\theta^2)$ while $\chi_{\alpha\beta}$ with an error $O(\frac{\eta\theta^2}{h})$. Even if we would use better approximations for $\gamma_{\alpha\beta}$ or $\chi_{\alpha\beta}$ the accuracy of Σ could not be raised within the first-approximation theory discussed here. Such accuracy of Σ might be raised only if the second-approximation [22] to the elastic strain energy were used. Then, however, it would be necessary to introduce into the theory some additional strain and stress measures and the whole shell theory would become much more complex [5,23].

In the following parts of the paper estimates for linearized quantities given in the table 1 will be used to simplify $\gamma_{\alpha\beta}$ and $\chi_{\alpha\beta}$ within the error already introduced by using here the first-approximation theory of shells. We shall also discuss possible simplifications of strain measures and resulting shell equations to within larger error $O(Eh\,\eta^2\theta\sqrt{\theta})$ or even $O(Eh\,\eta^2\theta)$ of the strain energy function (2.14). In the estimation procedure covariant derivatives of various terms will be estimated dividing their maximal value by a parameter λ defined by

$$\lambda = \frac{h}{\theta} = \min\,(L, d, \sqrt{hR}, \frac{1}{\sqrt{\eta}}) \quad . \tag{3.2}$$

In a variety of shell problems each of the parameters appearing in definitions of θ or λ may assume different values, which in extreme cases may differ by one or even two orders from each other. Since in our classification scheme we are using only one common measure θ of various small quantities, the estimation procedure should take into account also such cases when a particular parameter plays a dominant role in the definition of θ. In order to assure this we assume here that $\gamma_{\alpha\beta} = O(\eta) = O(\theta^2)$, $h\chi_{\alpha\beta} = O(\eta) = O(\theta^2)$, $b_{\alpha\beta} = O(\frac{1}{R}) = O(\frac{\theta^2}{h}) = O(\frac{\theta}{\lambda})$, what allows to relate various terms of those orders to the common parameter θ.

Within small rotations estimates of all terms in the strain-displacement relations (2.1) allow to reduce definitions of the strain measures to the form known in the linear bending theory of shells. Since the linear shell theory is discussed in many monographs we shall not discuss the case here.

4. THEORY OF SHELLS UNDERGOING MODERATE ROTATIONS

Within the moderate rotation theory estimates of the linearized quantities given in tab. 1 and the identity $\omega_{\lambda\alpha|\beta} = \theta_{\alpha\beta|\lambda} - \theta_{\lambda\beta|\alpha} + b_{\alpha\beta}\varphi_\lambda - b_{\lambda\beta}\varphi_\alpha$ allow to reduce the shell strain measures to the form

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\underline{a_{\alpha\beta}\varphi^2 - \frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta^2)$$

$$\chi_{\alpha\beta} = -\frac{1}{2}[\varphi_{\alpha|\beta} + \varphi_{\beta|\alpha} + \underline{b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})}] + O(\frac{\eta\theta}{\lambda}) \qquad (4.1)$$

Here in $\chi_{\alpha\beta}$ terms $\frac{1}{2}(b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) = O(\frac{\theta^3}{\lambda}) = O(\frac{\eta\theta}{\lambda})$ and might be omitted as well in $(4.1)_2$ within the same accuracy of the strain energy function [5]. Note that these terms are linear in displacements and their derivatives. In the linear shell theory the linearized tensor of change of curvature $\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})$ is defined conventionally by the full expression $(4.1)_2$, while for the simplified expression a modified definition [21] of the tensor of change of curvature is used: $\rho_{\alpha\beta} = \kappa_{\alpha\beta} + \frac{1}{2}(b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha})$. Since our tensor $\chi_{\alpha\beta}$ was so defined in [1] as to have the same linear parts with $\kappa_{\alpha\beta}$, we follow the convention and keep here all linear terms indicated in $(4.1)_2$.

When $(4.1)_2$ is compared with $(2.1)_2$ it is easy to note that within moderate rotations the parameters m_α and m appear in $(4.1)_2$ in approximate degenerate forms: $\hat{m}_\alpha = -\varphi_\alpha$ and $\hat{m} = 1$. As a result, all transformations leading to consistent shell relations become much simpler than in the general case. In particular, if (4.1) is introduced into IVW then it transforms into the form

$$IVW = -\iint_M \underset{\sim}{T}^\beta|_\beta \cdot \delta\underset{\sim}{u}\,dA + \int_C [(\underset{\sim}{T}^\beta\nu_\beta) \cdot \delta\underset{\sim}{u} + M_{\nu\nu}\delta\hat{m}_\nu + M_{t\nu}\delta\hat{m}_t]ds \qquad (4.2)$$

where for components of $\underset{\sim}{T}^\beta$ we have

$$T^{\lambda\beta} = N^{\lambda\beta} - b^\lambda_\alpha M^{\alpha\beta} - \frac{1}{2}\omega^{\lambda\beta}N^\alpha_\alpha - \underline{\frac{1}{2}(\omega^{\lambda\alpha}N^\beta_\alpha + \omega^{\beta\alpha}N^\lambda_\alpha)} + \underline{\frac{1}{2}(\theta^{\lambda\alpha}N^\beta_\alpha - \theta^{\beta\alpha}N^\lambda_\alpha)}$$

$$T^\beta = \varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha \qquad (4.3)$$

Since in this case

$$\delta\hat{m}_\nu = \delta\hat{n}_\nu = -\delta\varphi_\nu \ , \ \delta\hat{m} = \delta\hat{n} = 0$$
$$\delta\hat{m}_t = \delta\hat{n}_t = -\delta\varphi_t = \tau_t \delta u_\nu - \sigma_t \delta u_t - \frac{d}{ds}\delta w \tag{4.4}$$

it is possible to perform direct transformation [4,5] of the last term in (4.2) and obtain an equivalent form of (4.2) compatible with $(2.6)_3$:

$$IVW = -\iint_M \underset{\sim}{T}^\beta|_\beta \cdot \delta\underset{\sim}{u} dA + \int_C (\underset{\sim}{P} \cdot \delta\underset{\sim}{u} + M\delta\hat{n}_\nu) ds + \sum_k \underset{\sim}{F}_k \cdot \delta\underset{\sim}{u}_k \tag{4.5}$$

where now

$$\underset{\sim}{P} = \tau_t M_{t\nu} \underset{\sim}{\nu} - \sigma_t M_{t\nu} \underset{\sim}{t} + \frac{dM_{t\nu}}{ds} \underset{\sim}{n} \equiv \frac{d}{ds}(M_{t\nu} \underset{\sim}{n})$$
$$M = M_{\nu\nu}, \ \underset{\sim}{F} = M_{t\nu}\underset{\sim}{n}, \ \underset{\sim}{F}_k = [M_{t\nu}(s_k + 0) - M_{t\nu}(s_k - 0)]\underset{\sim}{n}(s_k) \ . \tag{4.6}$$

The line integral in (4.2) is a counterpart of the line integral $(2.6)_2$ of the general theory. Therefore, in order to transform it into (4.5) we may also apply general approach discussed in [1]. The approximate strain measures (4.1) generate (4.3) and components of $\underset{\sim}{T}^\beta \nu_\beta$ in the line integral. The smallest terms containing moment resultants in $T^{\lambda\beta}$ are $O(\frac{\theta}{\lambda} M) = O(\frac{Eh^3}{12} \frac{\eta\theta}{\lambda})$, while in T^β they are $O(\frac{1}{\lambda} M)$, where M is the largest eigenvalue of $M^{\alpha\beta}$. Therefore, only terms of the same or lower order should be retained in $\underset{\sim}{P}$, M and $\underset{\sim}{F}$ of (2.8). When all parameters (2.9) ÷ (2.11) are estimated it follows that the leading terms in most of the parameters given in (2.8) are of a higher order, except in the following quantities

$$d_\nu = \tau_t + O(\frac{\theta^2}{\lambda}) \ , \ d_t = -\sigma_t + O(\frac{\theta^2}{\lambda}) \ , \ g = -1 + O(\theta)$$
$$R_{\nu\nu} = M_{\nu\nu} + O(\theta M) \ , \ R_{t\nu} = M_{t\nu} + O(\theta M) \ . \tag{4.7}$$

If now (4.7) is introduced into (2.8) we obtain, within assumed accuracy of the moderate rotation shell theory, the relations coinciding with (4.6).

It is interesting to note, in particular, that $R_\nu = O(\theta M)$ and does not appear at the shell boundary within the approximations (4.1) of the moderate rotation theory. As a result, within the same approximation only two components of the external static boundary moment $\underset{\sim}{H} = H_\nu \underset{\sim}{\nu} + H_t \underset{\sim}{t}$ may be assumed at C. Since the whole structure of EVW should be the same as of IVW described by (4.2) ÷ (4.6), definitions of starred quantities follow from (4.6), where H_ν and H_t should be introduced in place of $M_{\nu\nu}$ and $M_{t\nu}$, respectively. Then the Lagrangian equilibrium equations and appropriate static and geometric boundary and corner conditions follow immediately from specialization of (2.13). These shell relations were already discussed in detail in [4-6,38]. We remind their extended component form here just in order to make the paper complete:

$$\left.\begin{aligned}
&[\underline{N^{\lambda\beta} - b^\lambda_\alpha M^{\alpha\beta} - \tfrac{1}{2}\,\omega^{\lambda\beta} N^\alpha_\alpha - \tfrac{1}{2}\,(\omega^{\lambda\alpha} N^\beta_\alpha + \omega^{\beta\alpha} N^\lambda_\alpha)} \; + \\
&+ \underline{\tfrac{1}{2}\,(\theta^{\lambda\alpha} N^\beta_\alpha - \theta^{\beta\alpha} N^\lambda_\alpha)}\,]\,|_\beta - b^\lambda_\beta(\varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha) + p^\lambda = 0 \\
&(\varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha)\,|_\beta + b_{\lambda\beta}(N^{\lambda\beta} - \underline{b^\lambda_\alpha M^{\alpha\beta} - \omega^{\lambda\alpha} N^\beta_\alpha}) + p = 0
\end{aligned}\right\}\quad \text{in } M$$

$$\left.\begin{aligned}
&(N^{\lambda\beta} - \underline{b^\lambda_\alpha M^{\alpha\beta}} - \omega^{\lambda\alpha} N^\beta_\alpha)\,\nu_\lambda \nu_\beta + \tau_t M_{t\nu} = N_\nu + \tau_t H_t \\
&[\underline{N^{\lambda\beta} - b^\lambda_\alpha M^{\alpha\beta} - \tfrac{1}{2}\omega^{\lambda\beta} N^\alpha_\alpha - \tfrac{1}{2}\,(\omega^{\lambda\alpha} N^\beta_\alpha + \omega^{\beta\alpha} N^\lambda_\alpha)} \; + \\
&+ \underline{\tfrac{1}{2}\,(\theta^{\lambda\alpha} N^\beta_\alpha} - \theta^{\beta\alpha} N^\lambda_\alpha)\,]\,t_\lambda \nu_\beta - \sigma_t M_{t\nu} = N_t - \sigma_t H_t \\
&(\varphi_\alpha N^{\alpha\beta} + M^{\alpha\beta}|_\alpha)\,\nu_\beta + \frac{d}{ds} M_{t\nu} = N + \frac{d}{ds} H_t \\
&M_{\nu\nu} = H_\nu
\end{aligned}\right\}\quad \text{on } C_f$$

$$M_{t\nu}(s_j + 0) - M_{t\nu}(s_j - 0) = H_t(s_j + 0) - H_t(s_j - 0) \qquad \text{at each } M_j \in C_f$$

$$u_\nu = u^*_\nu \,, \quad u_t = u^*_t \,, \quad w = w^* \,, \quad \varphi_\nu = \varphi^*_\nu \qquad \text{on } C_u$$

$$w(s_i) = w^*(s_i) \qquad \text{at each } M_i \in C_u \qquad (4.8)$$

In many engineering shell structures only rotations Ω_α are allowed to be moderate while rotation Ω is supposed to be always small. Within such moderate/small rotation theory the relations (4.1) and (4.8) may be considerably simplified by omitting there terms underlined by a solid line.

The set of shell relations (4.1) and (4.8) contains, as special cases, the equations of various simplified variants of the geometrically non-linear theory of shells which have been proposed in the literature [8-15]. Detailed review of those special cases was given in [6], where also many variational principles were constructed (see also [38]). An extensive discussion of shell stability problems within the simplified moderate rotation theory (without two last terms in $(4.1)_1$) has been presented by Stumpf [39].

5. THEORY OF SHELLS UNDERGOING LARGE/SMALL ROTATIONS

Within the theory of shells undergoing large rotations it is reasonable to assume that only Ω_α are always large, while Ω are allowed to be large, moderate or small. This leads to three different types of approximation within the large rotation shell theory. Our earlier papers [3,4,17] contain derivation of simplified forms of strain measures and of equilibrium equations for the large rotation shell theory, without discussion of appropriate boundary conditions. In what follows several complete sets of Lagrangian shell equations are presented.

The variant of geometrically non-linear theory of shells undergoing large/small rotations represents the simplest case within the large rotation theory. It exhibits certain important features of the general theory [1], such as the non-linear expression for $\chi_{\alpha\beta}$, necessity to express δn_t and δn in terms of $\delta \underset{\sim}{u}$ and δn_ν. At the same time it leads to shell relations which are still not too complex and, therefore, applicable in numerical calculations of shell structures. Keeping this in mind, the derivation of equations of the variant is presented in more detail, independently of the results which may be given for more advanced variants of the non-linear shell theory.

Using estimates of the linearized quantities given in table 1 for the shell strain measures we obtain approximate formulae

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\theta^\lambda_\alpha\theta_{\lambda\beta} - \frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha}) + O(\eta\theta^2)$$

$$\begin{aligned}\chi_{\alpha\beta} = {} & \frac{1}{2}\{(\delta^\lambda_\alpha + \theta^\lambda_\alpha)[-(1+\theta^\kappa_\kappa)\varphi_\lambda + \varphi^\mu\theta_{\mu\lambda}]|_\beta + (\delta^\lambda_\beta + \theta^\lambda_\beta)[-(1+\theta^\kappa_\kappa)\varphi_\lambda + \\ & + \varphi^\mu\theta_{\mu\lambda}]|_\alpha\} - \frac{1}{2}[(\varphi^\mu\omega_{\mu\alpha})|_\beta + (\varphi^\mu\omega_{\mu\beta})|_\alpha] + \frac{1}{2}(\omega_{\lambda\alpha}\varphi^\lambda|_\beta + \omega_{\lambda\beta}\varphi^\lambda|_\alpha) + \\ & + \frac{1}{2}\{\varphi_\alpha[1+\theta^\lambda_\lambda + \frac{1}{2}(\theta^\lambda_\lambda)^2 - \frac{1}{2}\theta^\lambda_\mu\theta^\mu_\lambda]|_\beta + \varphi_\beta[1+\theta^\lambda_\lambda + \frac{1}{2}(\theta^\lambda_\lambda)^2 - \frac{1}{2}\theta^\lambda_\mu\theta^\mu_\lambda]|_\alpha\} - \\ & - \frac{1}{2}(b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) - \frac{1}{2}(b^\lambda_\alpha\varphi_\beta + b^\lambda_\beta\varphi_\alpha)\varphi_\lambda + \frac{1}{2}b_{\alpha\beta}\varphi^\lambda\varphi_\lambda + O(\frac{\eta\theta}{\lambda})\end{aligned} \quad (5.1)$$

Note that within indicated error m appears in $(5.1)_2$ in an approximate form while m_λ are split into three separate parts, written in two first lines of $(5.1)_2$. When (5.1) is introduced into IVW, after some transformation we obtain

$$IVW = -\iint_{M} \underset{\sim}{T}^{\beta}|_{\beta} \cdot \delta\underset{\sim}{u} dA + \int_{C} \underset{\sim}{T}^{\beta}\nu_{\beta} \cdot \delta\underset{\sim}{u} ds + I_M$$

$$I_M = \int_C M^{\alpha\beta}[(\delta^{\lambda}_{\alpha} + \theta^{\lambda}_{\alpha})\delta\tilde{m}_{\lambda} + \omega^{\lambda}_{\cdot\alpha}\delta\varphi_{\lambda} - \delta(\varphi^{\mu}\omega_{\mu\alpha}) + \varphi_{\alpha}\delta\tilde{m}]\nu_{\beta} ds \tag{5.2}$$

where

$$\begin{aligned} T^{\lambda\beta} = {} & (\delta^{\lambda}_{\alpha} + \theta^{\lambda}_{\alpha})N^{\alpha\beta} - \frac{1}{2}(\omega^{\lambda\alpha}N^{\beta}_{\alpha} + \omega^{\beta\alpha}N^{\lambda}_{\alpha}) - \\ & - \frac{1}{2}[(b^{\lambda}_{\alpha} - \tilde{m}^{\lambda}|_{\alpha} + \varphi^{\lambda}|_{\alpha})M^{\alpha\beta} + (b^{\beta}_{\alpha} - \tilde{m}^{\beta}|_{\alpha} - \varphi^{\beta}|_{\alpha})M^{\alpha\lambda}] \\ & - \varphi^{\lambda}M^{\beta\rho}|_{\rho} - \frac{1}{2}[\varphi^{\lambda}(\theta^{\beta}_{\kappa}M^{\kappa\rho})|_{\rho} + \varphi^{\beta}(\theta^{\lambda}_{\kappa}M^{\kappa\rho})|_{\rho}] \\ & + a^{\lambda\beta}[\varphi_{\alpha}(\theta^{\alpha}_{\kappa}M^{\kappa\rho})|_{\rho} - \varphi_{\kappa}|_{\rho}M^{\kappa\rho}] - (a^{\lambda\beta}\theta^{\alpha}_{\alpha} - \theta^{\lambda\beta})(\varphi_{\kappa}M^{\kappa\rho})|_{\rho} \end{aligned} \tag{5.3}$$

$$\begin{aligned} T^{\beta} = {} & \varphi_{\alpha}N^{\alpha\beta} + (\tilde{m}_{,\alpha} - b^{\lambda}_{\alpha}\varphi_{\lambda})M^{\alpha\beta} + [\delta^{\beta}_{\alpha}(1 + \theta^{\lambda}_{\lambda}) - \theta^{\beta}_{\alpha}][(\delta^{\alpha}_{\kappa} + \theta^{\alpha}_{\kappa})M^{\kappa\rho}]_{|\rho} - \\ & - (b^{\beta}_{\kappa}\varphi_{\rho} + \omega^{\beta}_{\cdot\kappa|\rho} - b_{\kappa\rho}\varphi^{\beta})M^{\kappa\rho} \end{aligned}$$

$$\tilde{m}_{\lambda} = -(1 + \theta^{\kappa}_{\kappa})\varphi_{\lambda} + \varphi^{\mu}\theta_{\mu\lambda} \quad , \quad \tilde{m} = 1 + \theta^{\kappa}_{\kappa} + \frac{1}{2}(\theta^{\lambda}_{\lambda})^2 - \frac{1}{2}\theta^{\lambda}_{\mu}\theta^{\mu}_{\lambda} \tag{5.4}$$

The line integral $(5.2)_2$ is transformed further to obtain

$$I_M = \int_C (\tilde{K}_{\nu\nu}\delta\tilde{m}_{\nu} + K'_{\nu\nu}\delta\hat{m}_{\nu} + M_{\nu\nu}\delta m'_{\nu} + \tilde{K}_{t\nu}\delta\tilde{m}_{t} + K'_{t\nu}\delta\hat{m}_{t} + M_{t\nu}\delta m'_{t} + K_{\nu}\delta\tilde{m})ds \tag{5.5}$$

where the following abbreviations

$$\begin{aligned} & \tilde{K}_{\nu\nu} = M_{\nu\nu} + \theta_{\nu\nu}M_{\nu\nu} + \theta_{\nu t}M_{t\nu} \quad , \quad \tilde{K}_{t\nu} = M_{t\nu} + \theta_{t\nu}M_{\nu\nu} + \theta_{tt}M_{t\nu} \\ & K'_{\nu\nu} = -\varphi M_{t\nu} \quad , \quad K'_{t\nu} = +\varphi M_{\nu\nu} \quad , \\ & \hat{m}_{\nu} = -\varphi_{\nu} \quad , \quad \hat{m}_{t} = -\varphi_{t} \quad , \quad m'_{\nu} = \varphi_{t}\varphi \quad , \quad m'_{t} = -\varphi_{\nu}\varphi \end{aligned} \tag{5.6}$$

have been introduced. Since

$$K_{\nu\nu} = \tilde{K}_{\nu\nu} + K'_{\nu\nu} \quad , \quad K_{t\nu} = \tilde{K}_{t\nu} + K'_{t\nu}$$
$$m_{\nu} = \tilde{m}_{\nu} + m'_{\nu} \quad , \quad m_t = \tilde{m}_t + m'_t \tag{5.7}$$

We note that only some most important terms of the products $K_{\nu\nu}\delta m_{\nu}$ and $K_{t\nu}\delta m_t$ have appeared in (5.5). Other terms of the products have not appeared in (5.5), since their contributions to the elastic strain energy is negligibly small indeed within the assumptions of the first-approximation theory of shells undergoing large/small rotations. This has led to splitting of each of the boundary terms $K_{\nu\nu}\delta m_{\nu}$ and $K_{t\nu}\delta m_t$ into three separate parts.

The form of I_M given in (5.5) is not convenient for a proper formulation of static boundary and corner conditions. However, within the same approximation we are allowed to retain in (5.5) some small terms which have been omitted from (5.1) and (5.3) and write (5.5) in an alternative form, which is equivalent to (5.5) within the error of the first approximation theory:

$$I_M = \int_C (K_{\nu\nu}\delta m_{\nu} + K_{t\nu}\delta m_t + K_{\nu}\delta\tilde{m})\,ds = \int_C (R_{\nu\nu}\delta n_{\nu} + R_{t\nu}\delta n_t + R_{\nu}\delta\tilde{n})\,ds \tag{5.8}$$

where an identity $M^{\alpha\beta}(1_{\lambda\alpha}\delta m^{\lambda} + \varphi_{\alpha}\delta m) = \sqrt{\frac{\bar{a}}{a}}\, M^{\alpha\beta}(1_{\lambda\alpha}\delta n + \varphi_{\alpha}\delta n)$ has been used.

This formaly shorter representation of I_M can now be transformed into the final form (2.6). The appropriate transformations could be performed directly, expressing the approximate quantities n_t and $\tilde{n}$ in terms of m and $\underset{\sim}{u}$, similarly as it was performed with the exact quantities n_t and n in [1]. Such procedure would lead again to the non-rational expressions for the parameters (2.9) where the small terms, retained in n_t in order to present the boundary integral in the form (2.5), would also appear. As a result, such direct transformations would give us more complex formulae for $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M than it is necessary within the approximation of the large/small rotation theory.

In what follows we prefer to use an alternative approach, which has already been suggested in p.4. The approach is based on the consistent reduction of the exact formulae (2.8) to within the error already introduced into the approximate expressions (5.3) by using the simplifying assumptions of the large/small rotation theory. The approximate strain measures (5.1) generate consistently reduced expression $\underset{\sim}{T}^{\beta}\nu_{\beta}$ in $(5.2)_1$, where from (5.3) we note the accuracy $T^{\lambda\beta} = \ldots + O(\frac{\theta^2}{\lambda}M)$ and $T^{\beta} = \ldots + O(\frac{\theta^2\sqrt{\theta}}{\lambda}M)$. Therefore, within the large/small rotation theory appropriate components of the quantities $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M should be calculated with the same accuracy. The estimation procedure allows to simplify the formulae (2.8) by taking into account only the terms which are important within the desired degree of accuracy. Besides, the procedure leads to the equivalent polynomial representations for $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M , which are more convenient in applications.

Let us expand (2.9) into series and omit all terms within the indicat error to obtain

$$n_\nu = -\varphi_\nu - (\varphi_\nu\theta_{tt} - \varphi_t\theta_{\nu t}) + O(\theta^2\sqrt{\theta})$$

$$n_t = -\varphi_t + (\varphi_\nu\theta_{\nu t} - \varphi_t\theta_{\nu\nu}) + O(\theta^2\sqrt{\theta}) \qquad (5.9)$$

$$n = 1 + \theta_{\nu\nu} + \theta_{tt} + A + O(\theta^3) \ , \quad D = -(1 - \frac{1}{2}\varphi_\nu^2 + B) + O(\theta^3)$$

$$\frac{n_\nu}{D} = \varphi_\nu + (\varphi_\nu\theta_{tt} - \varphi_t\theta_{\nu t} + \frac{1}{2}\varphi_\nu^3) + O(\theta^2\sqrt{\theta})$$

$$\frac{n_t}{D} = \varphi_t - (\varphi_\nu\theta_{\nu t} - \varphi_t\theta_{\nu\nu} - \frac{1}{2}\varphi_\nu^2\varphi_t) + O(\theta^2\sqrt{\theta}) \qquad (5.10)$$

$$\frac{n}{D} = -[1 + (\theta_{\nu\nu} + \theta_{tt} + \frac{1}{2}\varphi_\nu^2) + C] + O(\theta^3)$$

$$A = \theta_{\nu\nu}\theta_{tt} - \theta_{\nu t}^2 - \gamma_{\nu\nu} - \gamma_{tt}$$

$$B = -\varphi_\nu(\varphi_\nu\theta_{tt} - \varphi_t\theta_{\nu t}) - \frac{1}{2}\theta_{\nu t}^2 - \frac{1}{8}\varphi_\nu^4 + \gamma_{tt} \qquad (5.11)$$

$$C = A - B + \frac{1}{2}\varphi_\nu^2(\theta_{\nu\nu} + \theta_{tt} + \frac{1}{2}\varphi_\nu^2)$$

The parameters n , D and n/D have been estimated here with a higher precision. This is performed in order to estimate the internal boundary force components Q , F and M with a desired accuracy.

When (5.9) and (5.10) are introduced into (2.8) and (2.9), after carefu estimation of all terms we obtain

$$Q_\nu = (\tau_t + \kappa_t \varphi_t) M_{t\nu} - \frac{dF_\nu}{ds} + O(\frac{\theta^2}{\lambda} M)$$

$$Q_t = - (\sigma_t + \kappa_t \varphi_\nu) M_{t\nu} - \frac{dF_t}{ds} + O(\frac{\theta^2}{\lambda} M) \qquad (5.12)$$

$$Q = (\tau_t \varphi_\nu - \sigma_t \varphi_t) M_{t\nu} - \frac{dF}{ds} + O(\frac{\theta^2 \sqrt{\theta}}{\lambda} M)$$

$$F_\nu = (\varphi_\nu^2 \varphi_t + \varphi_\nu \theta_{\nu t}) M_{\nu\nu} +$$

$$+ (\varphi_\nu + 3\varphi_\nu \theta_{tt} - \varphi_t \theta_{\nu t} + \varphi_\nu \theta_{\nu\nu} + \frac{1}{2} \varphi_\nu^3 + \varphi_\nu \varphi_t^2) M_{t\nu} + O(\theta^2 M)$$

$$F_t = (\varphi_\nu \varphi_t^2 + \varphi_t \theta_{\nu t}) M_{\nu\nu} +$$

$$+ (\varphi_t - \varphi_\nu \theta_{\nu t} + 2\varphi_t \theta_{\nu\nu} + 2\varphi_t \theta_{tt} + \frac{1}{2} \varphi_\nu^2 \varphi_t + \varphi_t) M_{t\nu} + O(\theta^2 M)$$

$$F = - \{\varphi_\nu \varphi_t + \theta_{\nu t} + [\theta_{\nu t}(2\theta_{\nu\nu} + 2\theta_{tt} + \frac{1}{2} \varphi_\nu^2) + \varphi - \varphi_\nu^2 \theta_{\nu t} +$$

$$+ \varphi_\nu \varphi_t (2\theta_{\nu\nu} + \theta_{tt} + \frac{1}{2} \varphi_\nu^2)]\} M_{\nu\nu} - \qquad (5.13)$$

$$- \{1 + 2\theta_{\nu\nu} + 3\theta_{tt} + \frac{1}{2}\varphi_\nu^2 + \varphi_t^2 + [\theta_{\nu\nu}^2 - \theta_{\nu t}^2 + 3\theta_{tt}^2 + 5\theta_{\nu\nu}\theta_{tt} + C +$$

$$+ \frac{1}{2}\varphi_\nu^2 (\theta_{\nu\nu} + 2\theta_{tt}) + \varphi_t^2 (2\theta_{\nu\nu} + \theta_{tt} + \frac{1}{2}\varphi_\nu^2) - \varphi_\nu \varphi_t \theta_{\nu t}]\} M_{t\nu} + O(\theta^2 \sqrt{\theta} M)$$

$$M = M_{\nu\nu} + (\theta_{\nu\nu} + \varphi_\nu^2) M_{\nu\nu} + [- \theta_{\nu t}^2 + \varphi_\nu (2\varphi_\nu \theta_{tt} - 3\varphi_t \theta_{\nu t} + \frac{1}{2} \varphi_\nu^3) +$$

$$+ \gamma_{\nu\nu} + \gamma_{tt}] M_{\nu\nu} + [- \theta_{\nu t} (\theta_{\nu\nu} + 2\, \theta_{tt} + \frac{1}{2} \varphi_\nu^2) - \varphi_t^2 \theta_{\nu t}] M_{t\nu} + O(\theta^2 \sqrt{\theta} M)$$

In deriving $(5.13)_{3,4}$ more accurate estimates for $R_{\nu\nu}$ and $R_{t\nu}$ have been used

$$R_{\nu\nu} = (1 + \theta_{\nu\nu} + \gamma_{\nu\nu} + \gamma_{tt}) M_{\nu\nu} + (\theta_{\nu t} - \varphi) M_{t\nu} + O(\theta^2 \sqrt{\theta} M)$$

$$R_{t\nu} = (\theta_{\nu t} + \varphi) M_{\nu\nu} + (1 + \theta_{tt} + \gamma_{\nu\nu} + \gamma_{tt}) M_{t\nu} + O(\theta^2 \sqrt{\theta} M)$$

In exactly the same way EVW may be transformed in order to obtain compatible definitions for Q^*_ν , Q^*_t , Q^*, F^*_ν , F^*_t , F^* and M^* . In performing the transformations one should remember that the structure of the starred quantities should be exactly the same as the one for the unstarred quantiti only now $\underset{\sim}{N}$, H_ν , H_t and H should appear in place of $\underset{\sim}{T}^\beta \nu_\beta$, $R_{\nu\nu}$, $R_{t\nu}$ and R_ν , respectively. Taking this into account, for the starred quantitie the following compatible definitions have been obtained

$$
\begin{aligned}
Q^*_\nu &= (\tau_t + \kappa_t \varphi_t) H_t - \frac{dF^*_\nu}{ds} \\
Q^*_t &= -(\sigma_t + \kappa_t \varphi_\nu) H_t - \frac{dF^*_t}{ds} \\
Q^* &= (\tau_t \varphi_\nu - \sigma_t \varphi_t) H_t - \frac{dF^*}{ds}
\end{aligned}
\tag{5.14}
$$

$$
\begin{aligned}
F^*_\nu &= [\varphi_\nu + \varphi_\nu(\theta_{\nu\nu} + 2\theta_{tt} + \tfrac{1}{2}\varphi^2_\nu) - \varphi_t \theta_{\nu t}] H_t + \varphi_\nu \varphi_t H \\
F^*_t &= [\varphi_t - \varphi_\nu \theta_{\nu t} + \varphi_t(2\theta_{\nu\nu} + \theta_{tt} + \tfrac{1}{2}\varphi^2_\nu)] H_t + \varphi^2_t H \\
F^* &= -\{1 + (2\theta_{\nu\nu} + 2\theta_{tt} + \tfrac{1}{2}\varphi^2_\nu) + [(\theta_{\nu\nu} + \theta_{tt})(\theta_{\nu\nu} + \theta_{tt} + \tfrac{1}{2}\varphi^2_\nu) + \\
&\quad + A + C]\} H_t - [\varphi_t - \varphi_\nu \theta_{\nu t} + \varphi_t(2\theta_{\nu\nu} + \theta_{tt} + \tfrac{1}{2}\varphi^2_\nu)] H
\end{aligned}
\tag{5.15}
$$

$$
\begin{aligned}
M^* &= H_\nu + \{-(\theta_{\nu t} + \varphi_\nu \varphi_t) - [\theta_{\nu t}(\theta_{\nu\nu} + \theta_{tt} + \tfrac{1}{2}\varphi^2_\nu) - \varphi + \\
&\quad + \varphi_t(\varphi_\nu \theta_{tt} - \varphi_t \theta_{\nu t} + \tfrac{1}{2}\varphi^3_\nu)]\} H_t + [\varphi_\nu + (2\varphi_\nu \theta_{tt} - 2\varphi_t \theta_{\nu t} + \tfrac{1}{2}\varphi^3_\nu)] H
\end{aligned}
$$

The consistent set of Lagrangian shell equations for the large/small rotation theory takes now the general form (2.12), only in this particular case the components of $\underset{\sim}{T}^\beta$ are given by (5.3), the components of $\underset{\sim}{Q}$ and M are given by (5.12) and (5.13). The appropriate compatible definitions for the external boundary forces follow from (5.14) and (5.15).

For some shell structures undergoing large/small rotations we may be interested to apply simpler relations, allowing for some small loss of accuracy of the solution. Such simplified relations may be obtained if we allow for a slightly greater error $O(Eh\eta^2\theta\sqrt{\theta})$ in the strain energy function $(2.13)_1$. Within this slightly greater error the strain measures reduce to

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\varphi_\alpha\varphi_\beta + \frac{1}{2}\theta^\lambda_\alpha\,\theta_{\lambda\beta} - \underline{\frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha})} + O(\eta\theta\sqrt{\theta})$$

$$\chi_{\alpha\beta} = \frac{1}{2}(\tilde{m}_{\alpha|\beta} + \tilde{m}_{\beta|\alpha}) - \frac{1}{2}(\theta^\lambda_\alpha\varphi_{\lambda|\beta} + \theta^\lambda_\beta\varphi_{\lambda|\alpha}) + \frac{1}{2}(\varphi_\alpha\tilde{\tilde{m}}_{|\beta} + \varphi_\beta\tilde{\tilde{m}}_{|\alpha}) \qquad (5.16)$$

$$\underline{- \frac{1}{2}(b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) - \frac{1}{2}(b^\lambda_\alpha\varphi_\beta + b^\lambda_\beta\varphi_\alpha)\varphi_\lambda + \frac{1}{2}b_{\alpha\beta}\varphi^\lambda\varphi_\lambda} + O(\frac{\eta\sqrt{\theta}}{\lambda})$$

Note that within this slightly worse approximation both strain measures become quadratic polynomials in displacements and their surface derivatives. Again, $\tilde{m}_\alpha$ is split here into two separate parts written in the first line of $(5.16)_2$. When (5.16) are introduced into IVW we obtain $(5.2)_1$, only now

$$I_M = \int_C M^{\alpha\beta}(\delta\tilde{m}_\alpha - \theta^\lambda_\alpha\delta\varphi_\lambda + \varphi_\alpha\delta\tilde{\tilde{m}})\nu_\beta ds \quad , \quad \tilde{\tilde{m}} = 1 + \theta^\lambda_\lambda \qquad (5.17)$$

$$T^{\lambda\beta} = N^{\lambda\beta} + \frac{1}{2}(\theta^\lambda_\alpha N^{\alpha\beta} + \theta^\beta_\alpha N^{\alpha\lambda}) + \underline{\frac{1}{2}(\theta^\lambda_\alpha N^{\alpha\beta} - \theta^\beta_\alpha N^{\alpha\lambda})} -$$

$$\underline{- \frac{1}{2}(\omega^{\lambda\alpha}N^\beta_\alpha + \omega^{\beta\alpha}N^\lambda_\alpha) - \frac{1}{2}(b^\lambda_\alpha M^{\alpha\beta} + b^\beta_\alpha M^{\alpha\lambda})} - a^{\lambda\beta}\varphi_{\kappa|\rho}M^{\kappa\rho} \qquad (5.18)$$

$$- \frac{1}{2}(\varphi^\lambda M^{\beta\rho} + \varphi^\beta M^{\lambda\rho})|_\rho$$

$$T^\beta = \varphi_\alpha N^{\alpha\beta} + [(1 + \theta^\kappa_\kappa)M^{\beta\rho}]|_\rho + \theta^\beta_{\kappa|\rho}M^{\kappa\rho} - \underline{b^\lambda_\alpha\varphi_\lambda M^{\alpha\beta} - b^\beta_\kappa\varphi_\rho M^{\kappa\rho} + b_{\kappa\rho}\varphi^\beta M^{\kappa\rho}}$$

The definitions (5.18) become now much simpler as compared with (5.3) for the unsimplified large/small rotation theory.

Again we note, that each of the exact product terms $R_{\nu\nu}\delta n_\nu$ and $R_{t\nu}\delta n_t$ is described in (5.17) by two most important terms. Other terms do not appear at all and the boundary integral (5.17) may be written as

$$I_M = \int_C (\tilde{K}_{\nu\nu}\delta\tilde{m}_\nu + \tilde{K}_{t\nu}\delta\tilde{m}_t + K_\nu \delta\tilde{\tilde{m}})\,ds \tag{5.19}$$

where the simplified definitions $(5.4)_1$, $(5.6)_1$ and $(5.17)_2$ are used.

Now the expressions (5.18) are given with the following accuracy: $T^{\lambda\beta} = \ldots + O(\frac{\theta\sqrt{\theta}}{\lambda}M)$ and $T^{\beta} = \ldots + O(\frac{\theta^2}{\lambda}M)$. Within the accuracy it is possible now to identify the parameters $\tilde{K}_{\nu\nu}$, $\tilde{K}_{t\nu}$, K_ν and $\tilde{m}_\nu$, $\tilde{m}_t$, $\tilde{\tilde{m}}$ with approximate values of the respective parameters $R_{\nu\nu}$, $R_{t\nu}$, R_ν and n_ν, n_t, n of the exact theory. Upon the elimination of $\delta\tilde{m}_t$ and $\delta\tilde{\tilde{m}}$ the appropriate definitions for the boundary quantities follow directly from (5.11) - (5.13), in which some small terms should be omitted:

$$\begin{aligned}
\underset{\sim}{Q} &= (\tau_t M_{t\nu} - \frac{dF_\nu}{ds})\underset{\sim}{\nu} - (\sigma_t M_{t\nu} + \frac{dF_t}{ds})\underset{\sim}{t} + \\
&\quad + [(\tau_t\varphi_\nu - \sigma_t\varphi_t)M_{t\nu} - \frac{dF}{ds}]\underset{\sim}{n} \\
\underset{\sim}{F} &= \varphi_\nu M_{t\nu}\underset{\sim}{\nu} + \varphi_t M_{t\nu}\underset{\sim}{t} - \\
&\quad - [(1 + 2\theta_{\nu\nu} + 3\theta_{tt} + \tfrac{1}{2}\varphi_\nu^2 + \varphi_t^2)M_{t\nu} + (\theta_{\nu t} + \varphi_\nu\varphi_t)M_{\nu\nu}]\underset{\sim}{n} \\
M &= M_{\nu\nu} + (\theta_{\nu\nu} + \varphi_\nu^2)M_{\nu\nu}
\end{aligned} \tag{5.20}$$

Appropriate definitions for starred quantities follow from (5.14) and (5.20) to be

$$\begin{aligned}
\underset{\sim}{Q}^* &= (\tau_t H_\nu - \frac{dF^*_\nu}{ds})\underset{\sim}{\nu} - (\sigma_t H_\nu + \frac{dF^*_t}{ds})\underset{\sim}{t} + \\
&\quad + [(\tau_t\varphi_\nu - \sigma_t\varphi_t)H_\nu - \frac{dF^*}{ds}]\underset{\sim}{n} \\
\underset{\sim}{F}^* &= \varphi_\nu H_t\underset{\sim}{\nu} + \varphi_t H_t\underset{\sim}{t} - \\
&\quad - [(1 + 2\theta_{\nu\nu} + 2\theta_{tt} + \tfrac{1}{2}\varphi_\nu^2)H_t + \varphi_t H]\underset{\sim}{n} \\
M^* &= H_\nu - (\theta_{\nu t} + \varphi_\nu\varphi_t)H_t + \varphi_\nu H
\end{aligned} \tag{5.21}$$

The Lagrangian shell equations for the simplified variant of shell theory with large/small rotations may now be given in the general vector form (2.13), where (5.18), (5.20) and (5.21) should be used. Let us present them in an extended component form:

equilibrium equations in M

$$\begin{aligned}
&[N^{\lambda\beta} + \tfrac{1}{2}(\theta^{\lambda}_{\alpha}N^{\alpha\beta} + \theta^{\beta}_{\alpha}N^{\alpha\lambda}) + \underline{\tfrac{1}{2}(\theta^{\lambda}_{\alpha}N^{\alpha\beta} - \theta^{\beta}_{\alpha}N^{\alpha\lambda}) - \tfrac{1}{2}(\omega^{\lambda\alpha}N^{\beta}_{\alpha} + \omega^{\beta\alpha}N^{\lambda}_{\alpha})} -\\
&- \underline{\tfrac{1}{2}(b^{\lambda}_{\alpha}M^{\alpha\beta} + b^{\beta}_{\alpha}M^{\alpha\lambda})} - a^{\lambda\beta}\varphi_{\kappa|\rho}M^{\kappa\rho} - \tfrac{1}{2}(\varphi^{\lambda}M^{\beta\rho} + \varphi^{\beta}M^{\lambda\rho})|_{\rho}]|_{\beta} -\\
&- b^{\lambda}_{\beta}\{\varphi_{\alpha}N^{\alpha\beta} + [(1+\theta^{\kappa}_{\kappa})M^{\beta\rho}]|_{\rho} + \theta^{\beta}_{\kappa|\rho}M^{\kappa\rho} - \underline{b^{\kappa}_{\alpha}\varphi_{\kappa}M^{\alpha\beta} - b^{\beta}_{\kappa}\varphi_{\rho}M^{\kappa\rho} + b_{\kappa\rho}\varphi^{\beta}M^{\kappa\rho}}\} + p^{\lambda} = 0
\end{aligned} \tag{5.22}$$

$$\begin{aligned}
&\{\varphi_{\alpha}N^{\alpha\beta} + [(1+\theta^{\kappa}_{\kappa})M^{\beta\rho}]|_{\rho} + \theta^{\beta}_{\kappa|\rho}M^{\kappa\rho} - \underline{b^{\kappa}_{\alpha}\varphi_{\kappa}M^{\alpha\beta} - b^{\beta}_{\kappa}\varphi_{\rho}M^{\kappa\rho} + b_{\kappa\rho}\varphi^{\beta}M^{\kappa\rho}}\}|_{\beta} +\\
&+ b_{\lambda\beta}(N^{\lambda\beta} + \theta^{\lambda}_{\alpha}N^{\alpha\beta} - \underline{\omega^{\lambda\alpha}N^{\beta}_{\alpha} - b^{\lambda}_{\alpha}M^{\alpha\beta}} - a^{\lambda\beta}\varphi_{\kappa|\rho}M^{\kappa\rho} - \varphi^{\lambda}M^{\beta\rho}|_{\rho}) + p = 0
\end{aligned}$$

static boundary conditions on C_f

$$\begin{aligned}
&(N^{\lambda\beta} + \theta^{\lambda}_{\alpha}N^{\alpha\beta} - \underline{\omega^{\lambda\alpha}N^{\beta}_{\alpha} - b^{\lambda}_{\alpha}M^{\alpha\beta}} - a^{\lambda\beta}\varphi_{\kappa|\rho}M^{\kappa\rho} - \varphi^{\lambda}M^{\beta\rho}|_{\rho})\nu_{\lambda}\nu_{\beta} + \underline{\tau_t}M_{t\nu} - \frac{d}{ds}(\varphi_{\nu}M_{t\nu}) =\\
&\qquad = N_{\nu} + \underline{\tau_t}H_{\nu} - \frac{d}{ds}(\varphi_{\nu}H_t)\\
&[N^{\lambda\beta} + \tfrac{1}{2}(\theta^{\lambda}_{\alpha}N^{\alpha\beta} + \theta^{\beta}_{\alpha}N^{\alpha\lambda}) + \underline{\tfrac{1}{2}(\theta^{\lambda}_{\alpha}N^{\alpha\beta} - \theta^{\beta}_{\alpha}N^{\alpha\lambda}) - \tfrac{1}{2}(\omega^{\lambda\alpha}N^{\beta}_{\alpha} + \omega^{\beta\alpha}N^{\lambda}_{\alpha})} -\\
&- \underline{\tfrac{1}{2}(b^{\lambda}_{\alpha}M^{\alpha\beta} + b^{\beta}_{\alpha}M^{\alpha\lambda})} - a^{\lambda\beta}\varphi_{\kappa|\rho}M^{\kappa\rho} - \tfrac{1}{2}(\varphi^{\lambda}M^{\beta\rho} + \varphi^{\beta}M^{\lambda\rho})|_{\rho}]t_{\lambda}\nu_{\beta} -\\
&- \underline{\sigma_t}M_{t\nu} - \frac{d}{ds}(\varphi_t M_{t\nu}) = N_t - \underline{\sigma_t}H_{\nu} - \frac{d}{ds}(\varphi_t H_t)
\end{aligned} \tag{5.23}$$

$$\begin{aligned}
&\{\varphi_{\alpha}N^{\alpha\beta} + [(1+\theta^{\kappa}_{\kappa})M^{\beta\rho}]|_{\rho} + \theta^{\beta}_{\kappa|\rho}M^{\kappa\rho} - \underline{b^{\kappa}_{\alpha}\varphi_{\kappa}M^{\alpha\beta} - b^{\beta}_{\kappa}\varphi_{\rho}M^{\kappa\rho} + b_{\kappa\rho}\varphi^{\beta}M^{\kappa\rho}}\}\nu_{\beta} +\\
&+ \frac{d}{ds}[(1 + 2\theta_{\nu\nu} + 3\theta_{tt} + \tfrac{1}{2}\varphi_{\nu}^{2} + \varphi_{t}^{2})M_{t\nu} + (\theta_{\nu t} + \varphi_{\nu}\varphi_{t})M_{\nu\nu}] =\\
&= N + \frac{d}{ds}[(1 + 2\theta_{\nu\nu} + 2\theta_{tt} + \tfrac{1}{2}\varphi_{\nu}^{2})H_t + \varphi_t H]\\
&\quad [1 + (\theta_{\nu\nu} + \varphi_{\nu}^{2})]M_{\nu\nu} = H_{\nu} - (\varphi_{\nu}\varphi_t + \theta_{\nu t})H_t + \varphi_{\nu}H
\end{aligned}$$

static corner conditions at each $M_j \in C_f$:

$$\underset{\sim}{F}(s_j + 0) - \underset{\sim}{F}(s_j - 0) = \underset{\sim}{F}^*(s_j + 0) - \underset{\sim}{F}^*(s_j - 0) \qquad (5.24)$$

geometric boundary conditions on C_u:

$$u_\nu = u^*_\nu , \quad u_t = u^*_t , \quad w = w^* , \quad \tilde{n}_\nu = \tilde{n}^*_\nu \qquad (5.25)$$

geometric boundary conditions at each $M_i \in C_u$:

$$\underset{\sim}{u}(s_i) = \underset{\sim}{u}^*(s_i) \qquad (5.26)$$

The structure of shell relations (5.16) and (5.22) - (5.26) is relatively simple as far as the theory of shells undergoing large rotations is concerned. Both strain measures are quadratic polynomials in u_α, w. Equilibrium equations are linear both in $N^{\alpha\beta}$, $M^{\alpha\beta}$ and u_α, w. Also two of the static boundary conditions (5.23) are linear both in $N^{\alpha\beta}$, $M^{\alpha\beta}$ and u_α, w at C, while the remaining two $(5.23)_{3,4}$ and one of (5.24) are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but contain some quadratic terms in u_α, w as well. Also $\tilde{n}_\nu$ in (5.25) is a quadratic polynomial in displacements.

In some engineering applications we may be interested in using even simpler relations which are applicable within the theory of shells undergoing large/small rotations. This goal may be achieved only at the expence of a larger loss in accuracy of the strain energy function. Let us then allow the error $O(Eh\eta^2\theta)$ in $(2.14)_1$. Within this larger error the strain measures may be taken in an extremely simple form following from (5.16) by omitting there terms marked by dots. As a result of the simplification terms marked by dots do not appear also in (5.18), (5.22) and (5.23).

This extremely simple variant of the non-linear theory of shells may be applied to those engineering shell problems, in which the relative accuracy $O(\theta)$ in the strain energy function is regarded as satisfactory. Within the large rotation range any numerical calculations are very complex anyway and numerical procedures used may themselves introduce substantial round-off errors. For these reasons the simplest variant of the non-linear shell equations should prove to be popular in engineering calculations of nonlinear shell problems within the large rotation range.

6. THEORY OF SHELLS UNDERGOING LARGE ROTATIONS

Under simplifying assumptions of large/large rotation theory estimates of the linearized quantities given in tab. 1 do not permit to simplify the strain tensor $(2.1)_1$. The tensor of change of curvature $(2.1)_2$ may be approximated by

$$\varkappa_{\alpha\beta} = \frac{1}{2}\left(1^{\lambda}_{\cdot\alpha} m_{\lambda}|_{\beta} + 1^{\lambda}_{\cdot\beta} m_{\lambda}|_{\alpha}\right) + \frac{1}{2}\left(\varphi_{\alpha} m_{,\beta} + \varphi_{\beta} m_{,\alpha}\right) -$$

$$- \frac{1}{2}\left[b^{\lambda}_{\alpha}(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^{\lambda}_{\beta}(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})\right] + \underline{\frac{1}{2}\left(b^{\lambda}_{\alpha}\omega_{\lambda\beta} + b^{\lambda}_{\beta}\omega_{\lambda\alpha}\right)\left(\theta^{\kappa}_{\kappa} + \frac{1}{2}\omega^{\kappa\rho}\omega_{\kappa\rho}\right)}$$

$$- \frac{1}{2}\left(b^{\lambda}_{\alpha}\varphi_{\beta} + b^{\lambda}_{\beta}\varphi_{\alpha}\right)\left(\varphi_{\lambda} + \underline{\varphi^{\mu}\omega_{\mu\lambda}}\right) + \frac{1}{2} b_{\alpha\beta}\varphi^{\lambda}\varphi_{\lambda} + O\left(\frac{\eta\theta}{\lambda}\right) \qquad (6.1)$$

Note that terms in the first line of (6.1) are exact. Since those terms are responsible for the proper form of moment boundary conditions, the transformation of IVW with $(2.1)_1$ and (6.1) leads to $(2.6)_2$, where now

$$T^{\lambda\beta} = 1^{\lambda}_{\cdot\alpha}N^{\alpha\beta} + \left(m^{\lambda}|_{\alpha} - b^{\lambda}_{\alpha}\right)M^{\alpha\beta} - \underline{\frac{1}{2}\left(b^{\lambda}_{\alpha}M^{\alpha\beta} - b^{\beta}_{\alpha}M^{\alpha\lambda}\right)\left(\theta^{\kappa}_{\kappa} + \frac{1}{2}\omega^{\kappa\rho}\omega_{\kappa\rho}\right)} -$$

$$- \left[(1 + \theta^{\alpha}_{\alpha})a^{\lambda\beta} - \theta^{\lambda\beta} + \omega^{\lambda\beta}\right]\left(\varphi_{\kappa}M^{\kappa\rho}\right)|_{\rho} + \underline{b^{\alpha}_{\kappa}\left(a^{\lambda\beta} - \omega^{\lambda\beta}\right)\omega_{\alpha\rho}M^{\kappa\rho}} -$$

$$- \varphi^{\lambda}\left(1^{\beta}_{\cdot\kappa}M^{\kappa\rho}\right)|_{\rho} - \underline{\frac{1}{2}\left(b^{\lambda}_{\kappa}\varphi^{\beta} - b^{\beta}_{\kappa}\varphi^{\lambda}\right)\varphi_{\rho}M^{\kappa\rho}} + a^{\lambda\beta}\varphi_{\alpha}\left(1^{\alpha}_{\cdot\kappa}M^{\kappa\rho}\right)|_{\rho} \qquad (6.2)$$

$$T^{\beta} = \varphi_{\alpha}N^{\alpha\beta} + \left[m_{,\alpha} - b^{\lambda}_{\alpha}\left(\varphi_{\lambda} + \underline{\varphi^{\mu}\omega_{\mu\lambda}}\right)\right]M^{\alpha\beta} + \left(1 + \theta^{\alpha}_{\alpha}\right)\left(1^{\beta}_{\cdot\kappa}M^{\kappa\rho}\right)|_{\rho} -$$

$$- b^{\beta}_{\kappa}\varphi_{\rho}M^{\kappa\rho} - \left(\theta^{\beta\alpha} - \omega^{\beta\alpha}\right)\left(1_{\alpha\kappa}M^{\kappa\rho}\right)|_{\rho} + \underline{b_{\lambda\kappa}\omega^{\lambda\beta}\varphi_{\rho}M^{\kappa\rho}} + b_{\kappa\rho}\varphi^{\beta}M^{\kappa\rho}$$

The quantities (6.2) are generated from (6.1) by the principle of virtual displacements and have definite accuracy. Let

$$T^{\lambda\beta} = T^{(\lambda\beta)} + T^{[\lambda\beta]}$$

$$T^{(\lambda\beta)} = \frac{1}{2}\left(T^{\lambda\beta} + T^{\beta\lambda}\right) \quad , \quad T^{[\lambda\beta]} = \frac{1}{2}\left(T^{\lambda\beta} - T^{\beta\lambda}\right) \qquad (6.3)$$

It follows from (6.2) that $T^{(\lambda\beta)}$ are calculated to within $O(\frac{\theta^2}{\lambda} M)$, and $T^{[\lambda\beta]}$ and T^{β} to within $O(\frac{\theta^2\sqrt{\theta}}{\lambda} M)$. The quantities $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M should be calculated within the same accuracy.

Some parameters appearing in (2.9) may be approximated, to within indicated errors, by the following expansions:

$$\begin{aligned}
n_\nu &= -\varphi_\nu + \varphi_t\varphi - \varphi_\nu\theta_{tt} + \varphi_t\theta_{\nu t} + O(\theta^2\sqrt{\theta}) \\
n_t &= -\varphi_t - \varphi_\nu\varphi + \varphi_\nu\theta_{\nu t} - \varphi_t\theta_{\nu\nu} + O(\theta^2\sqrt{\theta}) \\
n &= 1 + \theta_{\nu\nu} + \theta_{tt} + \varphi^2 + A + O(\theta^3) \\
D &= -[1 - \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2) + \varphi(\varphi_\nu\varphi_t + \theta_{\nu t}) + B_1] + O(\theta^2\sqrt{\theta})
\end{aligned} \qquad (6.4)$$

$$\begin{aligned}
\frac{n_\nu}{D} &= \varphi_\nu - \varphi_t\varphi + \varphi_\nu\theta_{tt} - \varphi_t\theta_{\nu t} + \tfrac{1}{2}\varphi_\nu(\varphi_\nu^2 + \varphi^2) + O(\theta^2) \\
\frac{n_t}{D} &= \varphi_t + \varphi_\nu\varphi - \varphi_\nu\theta_{\nu t} + \varphi_t\theta_{\nu\nu} + \tfrac{1}{2}\varphi_t(\varphi_\nu^2 + \varphi^2) + O(\theta^2) \\
\frac{n}{D} &= -[1 + \theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2) - \varphi(\varphi_\nu\varphi_t + \theta_{\nu t}) + C_1] + O(\theta^2\sqrt{\theta})
\end{aligned} \qquad (6.5)$$

$$\begin{aligned}
A &= \theta_{\nu\nu}\theta_{tt} - \theta_{\nu t}^2 - \gamma_{\nu\nu} - \gamma_{tt} \\
B_1 &= -\tfrac{1}{8}(\varphi_\nu^2 + \underline{\varphi^2})^2 - \tfrac{1}{2}[\varphi_t^2\varphi^2 + \theta_{\nu t}^2 - 2\varphi_\nu(\varphi_t\theta_{\nu t} - \varphi_\nu\theta_{tt}) - 2\gamma_{tt}] \\
C_1 &= A - B_1 + \tfrac{1}{2}(\varphi_\nu^2 + \underline{\varphi^2})[\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2} + \tfrac{1}{2}(\varphi_\nu^2 + \underline{\varphi^2})]
\end{aligned} \qquad (6.6)$$

In (6.4) and (6.5) parameters n, D and n/D have been estimated with a higher precision from their exact expressions derived for finite mid-surface strains [1]. The higher accuracy of those parameters is necessary in order to estimate F with a desired accuracy following from the accuracy of $\underset{\sim}{T}^{\beta}\nu_\beta$. Now it is possible to estimate all terms appearing in $(2.8)_2$ and obtain the following approximate definitions for the boundary functions

$$
\begin{aligned}
Q_\nu &= (\kappa_t\varphi_t + \tau_t)R_{t\nu} - \frac{dF_\nu}{ds} + O(\frac{\theta^2}{\lambda}M) \\
Q_t &= -\{\sigma_t\underline{[1 + 2\theta_{\nu\nu} + 2\theta_{tt} + 2\varphi^2 + \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2)]} + \kappa_t(\varphi_\nu - \underline{\varphi_t\varphi})\}R_{t\nu} - \\
&\quad - \underline{\sigma_t\varphi_t R_\nu} - \frac{dF_t}{ds} + O(\frac{\theta^2\sqrt{\theta}}{\lambda}M) \\
Q &= [\tau_t(\varphi_\nu - \underline{\varphi_t\varphi}) - \sigma_t(\varphi_t + \underline{\varphi_\nu\varphi})]R_{t\nu} - \frac{dF}{ds} + O(\frac{\theta^2\sqrt{\theta}}{\lambda}M)
\end{aligned}
\tag{6.7}
$$

$$
\begin{aligned}
F_\nu &= [\varphi_\nu - \varphi_t\varphi - \varphi_t\theta_{\nu t} + \varphi_\nu\theta_{tt} + \tfrac{1}{2}\varphi_\nu(\varphi_\nu^2 + \varphi^2) + \varphi_\nu(\theta_{tt} + \theta_{\nu\nu} + \varphi^2)]R_{t\nu} + \\
&\quad + \varphi_\nu\varphi_t R_\nu + O(\theta^2 M) \\
F_t &= \{\varphi_t + \varphi_\nu\varphi + \varphi_t[\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2)] - \varphi_\nu\theta_{\nu t} + \varphi_t\theta_{\nu\nu} - \\
&\quad - \underline{\varphi_t\varphi(\varphi_\nu\varphi_t + \theta_{\nu t}) + \varphi_\nu\varphi[\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2)]}\}R_{t\nu} + \\
&\quad + (\varphi_t^2 + \underline{2\varphi_\nu\varphi_t\varphi})R_\nu + O(\theta^2\sqrt{\theta}M) \\
F &= -\{1 + 2\theta_{\nu\nu} + 2\theta_{tt} + 2\varphi^2 + \tfrac{1}{2}(\varphi_\nu^2 + \varphi^2) - \varphi(\varphi_\nu\varphi_t + \theta_{\nu t}) + \\
&\quad + (\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2})[\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2} + \tfrac{1}{2}(\varphi_\nu^2 + \underline{\varphi^2})] + A + C_1\}R_{t\nu} - \\
&\quad - [\varphi_t + \varphi_\nu\varphi - \varphi_\nu\theta_{\nu t} + \varphi_t\theta_{\nu\nu} + \tfrac{1}{2}\varphi_t(\varphi_\nu^2 + \underline{\varphi^2}) + \varphi_t(\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2})]R_\nu + O(\theta^2\sqrt{\theta}M)
\end{aligned}
\tag{6.8}
$$

$$
\begin{aligned}
M &= R_{\nu\nu} + \{\varphi - (\varphi_\nu\varphi_t + \theta_{\nu t}) + \varphi[\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2} + \tfrac{1}{2}(\varphi_\nu^2 + \underline{\varphi^2}) + \varphi_t^2] - \\
&\quad - \theta_{\nu t}[\theta_{\nu\nu} + \theta_{tt} + \underline{\varphi^2} + \tfrac{1}{2}(\varphi_\nu^2 + \underline{\varphi^2})] - \underline{\varphi^2(\varphi_\nu\varphi_t + \theta_{\nu t})} - \\
&\quad - \varphi_t[\varphi_\nu\theta_{tt} - \varphi_t\theta_{\nu t} + \tfrac{1}{2}\varphi_\nu(\varphi_\nu^2 + \underline{\varphi^2})]\}R_{t\nu} + \\
&\quad + [\varphi_\nu + 2\varphi_\nu\theta_{tt} - 2\varphi_t\theta_{\nu t} + \tfrac{1}{2}\varphi_\nu(\varphi_\nu^2 + \underline{\varphi^2}) + \varphi_\nu\underline{\varphi^2}]R_\nu + O(\theta^2\sqrt{\theta}M)
\end{aligned}
\tag{6.9}
$$

Introducing (2.10) into (6.7) - (6.9) it is easy to obtain those definitions expressed in terms of $M_{\nu\nu}$ and $M_{t\nu}$ and the displacement parameters

$$Q_\nu = \tau_t \varphi M_{\nu\nu} + (\tau_t + \kappa_t \varphi_t) M_{t\nu} - \frac{dF_\nu}{ds} + O(\frac{\theta^2}{\lambda} M)$$

$$Q_t = - [\sigma_t \varphi + \sigma_t (\theta_{\nu t} + \varphi_\nu \varphi_t) + \kappa_t \varphi_\nu \varphi] M_{\nu\nu} - \{\sigma_t + \sigma_t [2\theta_{\nu\nu} + 3\theta_{tt} + 2\varphi^2 + \varphi_t^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2)] + \kappa_t (\varphi_\nu - \varphi_t \varphi)\} M_{t\nu} - \frac{dF_t}{ds} + O(\frac{\theta^2 \sqrt{\theta}}{\lambda} M) \qquad (6.10)$$

$$Q = (\tau_t \varphi_\nu - \sigma_t \varphi_t) \varphi M_{\nu\nu} + [\tau_t (\varphi_\nu - \varphi_t \varphi) - \sigma_t (\varphi_t + \varphi_\nu \varphi)] M_{t\nu} - \frac{dF}{ds} + O(\frac{\theta^2 \sqrt{\theta}}{\lambda} M)$$

$$F_\nu = (\varphi_\nu \varphi + \varphi_\nu^2 \varphi_t - \varphi_t \varphi^2 + \varphi_\nu \theta_{\nu t}) M_{\nu\nu} + \{\varphi_\nu - \varphi_\nu \varphi - \varphi_t \theta_{\nu t} + \varphi_\nu [\theta_{\nu\nu} + 3\theta_{tt} + \varphi_t^2 + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2)]\} M_{t\nu} + O(\theta^2 M)$$

$$F_t = \{\varphi_t \varphi + \varphi_\nu (\varphi_t^2 + \varphi^2) + \varphi_t \theta_{\nu t} + \varphi_t \varphi [2\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \frac{1}{2}(3\varphi_\nu^2 + \varphi^2)]\} M_{\nu\nu} + \{\varphi_t + \varphi_\nu \varphi + \varphi_t [2\theta_{\nu\nu} + 2\theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2) + \varphi_t^2] - \varphi_\nu \theta_{\nu t} - \varphi_\nu \varphi (\varphi_\nu \varphi_t + \theta_{\nu t}) + \varphi_\nu \varphi [\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2)] + 2\varphi_\nu \varphi_t^2 \varphi\} M_{t\nu} + O(\theta^2 \sqrt{\theta} M) \qquad (6.11)$$

$$F = - \{\varphi + \varphi_\nu \varphi_t + \theta_{\nu t} + (\varphi + \theta_{\nu t})[2\theta_{\nu\nu} + 2\theta_{tt} + 2\varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2)] + \varphi_\nu^2 \varphi - \varphi^2 (\varphi_\nu \varphi_t + \theta_{\nu t}) + \varphi_\nu [-\varphi_\nu \theta_{\nu t} + \varphi_t \theta_{\nu\nu} + \frac{1}{2}\varphi_t (\varphi_\nu^2 + \varphi^2) + \varphi_t (\theta_{\nu\nu} + \theta_{tt} + \varphi^2)]\} M_{\nu\nu} - \{1 + \varphi_t^2 + \theta_{tt} + (1 + \theta_{tt})[2\theta_{\nu\nu} + 2\theta_{tt} + 2\varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2) + \varphi_\nu \varphi_t \varphi - \varphi(\varphi_\nu \varphi_t + \theta_{\nu t}) + (\theta_{\nu\nu} + \theta_{tt} + \varphi^2)[\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2) + \varphi_t^2] + A + C_1 + \gamma_{\nu\nu} + \gamma_{tt} + \varphi_t [-\varphi_\nu \theta_{\nu t} + \varphi_t \theta_{\nu\nu} + \frac{1}{2}\varphi_t (\varphi_\nu^2 + \varphi^2)]\} M_{t\nu} + O(\theta^2 \sqrt{\theta} M)$$

$$M = \{1 + \theta_{\nu\nu} + \varphi_\nu^2 + \varphi^2 - \varphi_\nu \varphi_t \varphi + \varphi^2 [\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2) + \varphi_t^2] - \theta_{\nu t}(\varphi_\nu \varphi_t + \theta_{\nu t}) + \varphi_\nu [2\varphi_\nu \theta_{tt} - 2\varphi_t \theta_{\nu t} + \frac{1}{2}\varphi_\nu (\varphi_\nu^2 + \varphi^2) + \varphi_\nu \varphi^2] + \gamma_{\nu\nu} + \gamma_{tt}\} M_{\nu\nu} + \{(\varphi - \theta_{\nu t})[\theta_{\nu\nu} + 2\theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi_\nu^2 + \varphi^2)] + \varphi_t^2 \varphi - \varphi^2 (\varphi_\nu \varphi_t + 2\theta_{\nu t}) - \varphi_t^2 \theta_{\nu t}\} M_{t\nu} + O(\theta^2 \sqrt{\theta} M) \qquad (6.12)$$

Again, in obtaining the formulae $(6.11)_3$ and (6.12) more accurate estimates for $R_{\nu\nu}$ and $R_{t\nu}$ have been used

$$R_{\nu\nu} = (1 + \theta_{\nu\nu} + \gamma_{\nu\nu} + \gamma_{tt})M_{\nu\nu} + (\theta_{\nu t} - \varphi)\, M_{t\nu} + O(\theta^2\sqrt{\theta}M)$$
$$R_{t\nu} = (\theta_{\nu t} + \varphi)\, M_{\nu\nu} + (1 + \theta_{tt} + \gamma_{\nu\nu} + \gamma_{tt})M_{t\nu} + O(\theta^2\sqrt{\theta}M) \qquad (6.13)$$

Since the structure of starred quantities $\underset{\sim}{P}^*$, $\underset{\sim}{Q}^*$, $\underset{\sim}{F}^*$ and M^* is exactly the same as of $\underset{\sim}{P}$, $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M, respectively, definitions for the starred quantities follow directly from $(2.8)_1$, (6.7), (6.8) and (6.9), where $\underset{\sim}{N}$, H_ν, H_t, H should be but in place of $\underset{\sim}{T}^\beta \nu_\beta$, $R_{\nu\nu}$, $R_{t\nu}$, R_ν, respectively.

The consistent Lagrangian shell relations for the large rotation theory can now be presented in the general vector form (2.12), where (6.2), (6.10) - (6.12) and appropriate expressions for starred quantities should be introduced. The component forms of the Lagrangian shell relations are easy to write as well. These component relations are quite complex and we do not present them explicitly here.

Again, the shell relations just derived may be simplified further at the expence of some loss in accuracy of the strain energy function. Let us then assume a slightly greater error $O(Eh\eta^2\theta\sqrt{\theta})$ in $(2.13)_1$. Within this error the strain tensor $(2.1)_1$ cannot be simplified, while (6.1) is reduced to

$$\begin{aligned}\chi_{\alpha\beta} = {} & \frac{1}{2}\,[(\delta^\lambda_\alpha - \omega^\lambda_{\cdot\alpha})m_{\lambda|\beta} + (\delta^\lambda_\beta - \omega^\lambda_{\cdot\beta})m_{\lambda|\alpha}] + \frac{1}{2}\,(\theta^\lambda_\alpha \bar{m}_{\lambda|\beta} + \theta^\lambda_\beta \bar{m}_{\lambda|\alpha}) + \\ & + \frac{1}{2}\,(\varphi_\alpha \bar{m}_{,\beta} + \varphi_\beta \bar{m}_{,\alpha}) - \frac{1}{2}\,[b^\lambda_\alpha(\theta_{\lambda\beta} - \omega_{\lambda\beta}) + b^\lambda_\beta(\theta_{\lambda\alpha} - \omega_{\lambda\alpha})] - \\ & - \frac{1}{2}\,(b^\lambda_\alpha\varphi_\beta + b^\lambda_\beta\varphi_\alpha)\varphi_\lambda + \frac{1}{2}\,b_{\alpha\beta}\varphi^\lambda\varphi_\lambda + O(\frac{\eta\sqrt{\theta}}{\lambda})\end{aligned} \qquad (6.14)$$

where

$$\bar{m}_\lambda = -\,\varphi_\lambda - \underline{\varphi^\mu\omega_{\mu\lambda}}\quad , \quad \bar{m} = 1 + \theta^\kappa_\kappa + \underline{\frac{1}{2}\,\omega^{\kappa\rho}\omega_{\kappa\rho}} \qquad (6.15)$$

Here $\bar{m}$ appears in an approximated form, while $\bar{m}_\lambda$ are splitted into two separate parts indicated in the first line of (6.14). When $(2.1)_1$ and (6.14) are introduced into IVW, after transformations it takes the form $(5.2)_1$, where now

$$I_M = \int_C M^{\alpha\beta}[(\delta^\lambda_\alpha - \omega^\lambda_{\cdot\alpha})\delta m_\lambda + \theta^\lambda_\alpha \delta\bar{m}_\lambda + \varphi_\alpha \delta\bar{m}]\nu_\beta ds \tag{6.16}$$

$$\begin{aligned} T^{\lambda\beta} = {} & 1^\lambda_{\cdot\alpha}N^{\alpha\beta} + \frac{1}{2}(\bar{m}^\lambda|_\alpha M^{\alpha\beta} + \bar{m}^\beta|_\alpha M^{\alpha\lambda}) + \frac{1}{2}(m^\lambda|_\alpha M^{\alpha\beta} - m^\beta|_\alpha M^{\alpha\lambda}) - \\ & - b^\lambda_\alpha M^{\alpha\beta} - (a^{\lambda\beta} - \omega^{\lambda\beta})(\varphi_\kappa M^{\kappa\rho})|_\rho - \varphi^\lambda[(\delta^\beta_\kappa - \omega^\beta_{\cdot\kappa})M^{\kappa\rho}]|_\rho - \\ & - \frac{1}{2}[\varphi^\lambda(\theta^\beta_\kappa M^{\kappa\rho})|_\rho - \varphi^\beta(\theta^\lambda_\kappa M^{\kappa\rho})|_\rho] + a^{\lambda\beta}\varphi_\alpha[(\delta^\alpha_\kappa - \omega^\alpha_{\cdot\kappa})M^{\kappa\rho}]|_\rho \end{aligned} \tag{6.17}$$

$$\begin{aligned} T^\beta = {} & \varphi_\alpha N^{\alpha\beta} + (\bar{m}_{,\alpha} - b^\lambda_\alpha\varphi_\lambda)M^{\alpha\beta} + (1 + \theta^\alpha_\alpha)[(\delta^\beta_\kappa - \omega^\beta_{\cdot\kappa})M^{\kappa\rho}]|_\rho + b_{\kappa\rho}\varphi^\beta M^{\kappa\rho} + \\ & + (\theta^\beta_\kappa M^{\kappa\rho})|_\rho - b^\beta_\kappa\varphi_\rho M^{\kappa\rho} - (\theta^{\beta\alpha} - \omega^{\beta\alpha})[(a_{\alpha\kappa} - \omega_{\alpha\kappa})M^{\kappa\rho}]|_\rho + \omega^{\beta\alpha}(\theta_{\alpha\kappa}M^{\kappa\rho})|_\rho \end{aligned}$$

Here $T^{(\lambda\beta)}$ are given to within terms $O(\frac{\theta\sqrt{\theta}}{\lambda}M)$ and $T^{[\lambda\beta]}$, T^β to within $O(\frac{\theta^2}{\lambda}M)$, what is lower by a factor $\sqrt{\theta}$ as compared with analogous quantities (6.2) of the unsimplified large rotation theory. Again, within the same error (6.16) can be given in an alternative equivalent form

$$I_M = \int_C (R_{\nu\nu}\delta n_\nu + R_{t\nu}\delta n_t + R_\nu \delta\bar{n})ds \tag{6.18}$$

in which δn_t and $\delta\bar{n}$ should be expressed in terms of $\delta\underset{\sim}{u}$ and δn_ν as in (2.4) .

Appropriate definitions for $\underset{\sim}{Q}$, $\underset{\sim}{F}$ and M follow immediately from (6.7) - (6.9) with slightly larger error to be

$$\begin{aligned} Q_\nu &= \underline{-\tau_t R_{t\nu}} - \frac{dF_\nu}{ds} + O(\frac{\theta\sqrt{\theta}}{\lambda}M) \\ Q_t &= -(\sigma_t + \underline{\kappa_t\varphi_\nu})R_{t\nu} - \frac{dF_t}{ds} + O(\frac{\theta^2}{\lambda}M) \\ Q &= \underline{(\tau_t\varphi_\nu - \sigma_t\varphi_t)R_{t\nu}} - \frac{dF}{ds} + O(\frac{\theta^2}{\lambda}M) \end{aligned} \tag{6.19}$$

$$\begin{aligned} F_\nu &= (\varphi_\nu - \underline{\varphi_t\varphi})R_{t\nu} + O(\theta\sqrt{\theta}M) \\ F_t &= \{\varphi_t + \underline{\varphi_\nu\varphi + \varphi_t[\theta_{\nu\nu} + \theta_{tt} + \varphi^2 + \frac{1}{2}(\varphi^2_\nu + \varphi^2)] - \varphi_\nu\theta_{\nu t} + \varphi_t\theta_{\nu\nu}}\}R_{t\nu} + \\ & \quad + \underline{\varphi^2_t R_\nu} + O(\theta^2 M) \end{aligned} \tag{6.20}$$

$$= - [1 + 2\theta_{\nu\nu} + 2\theta_{tt} + 2\varphi^2 + \frac{1}{2} (\varphi_\nu^2 + \varphi^2) - \underset{\cdots\cdots\cdots\cdots}{\varphi(\varphi_\nu\varphi_t + \theta_{\nu t})}]R_{t\nu} -$$

$$- (\varphi_t + \underset{\cdots}{\varphi_\nu\varphi}) R_\nu + O(\theta^2 M)$$

$$= R_{\nu\nu} + \{\varphi - (\varphi_\nu\varphi_t + \theta_{\nu t}) + \underset{\cdots\cdots\cdots\cdots\cdots\cdots}{\varphi[\theta_{\nu\nu} + \theta_{tt} + \varphi^2 +}$$

$$\underset{\cdots\cdots\cdots\cdots\cdots\cdots}{+ \frac{1}{2} (\varphi_\nu^2 + \varphi^2) + \varphi_t^2]}\}R_{t\nu} + \varphi_\nu R_\nu + O(\theta^2 M) . \qquad (6.21)$$

The Lagrangian shell equations (2.13) with (6.17) and (6.19)-(6.21) become much simpler than those of the unsimplified large rotation theory. They may be applied to shell problems with all rotations allowed to be large.

At the expence of a greater error $O(Eh\eta^2\theta)$ in the strain energy function (2.14)$_1$ we may approximate (6.14) by

$$= \frac{1}{2} (m_{\alpha|\beta} + m_{\beta|\alpha}) - \frac{1}{2} (\omega_{\lambda\alpha}\bar{m}^\lambda|_\beta + \omega_{\lambda\beta}\bar{m}^\lambda|_\alpha) + \frac{1}{2} (\theta^\lambda_\alpha\hat{m}_{\lambda|\beta} + \theta^\lambda_\beta\hat{m}_{\lambda|\alpha})$$

$$+ \frac{1}{2} (\varphi_\alpha\bar{m}_{,\beta} + \varphi_\beta\bar{m}_{,\alpha}) + \frac{1}{2} (b^\lambda_\alpha\omega_{\lambda\beta} + b^\lambda_\beta\omega_{\lambda\alpha}) + O(\frac{\eta}{\lambda}) . \qquad (6.22)$$

Then (6.17) and (6.18) reduce to

$$= \int_C M^{\alpha\beta} (\delta m_\alpha - \omega_{\lambda\alpha}\delta\bar{m}^\lambda + \theta_{\lambda\alpha}\delta\hat{m}^\lambda + \varphi_\alpha\delta\bar{m}) \nu_\beta ds \qquad (6.23)$$

$$= 1^\lambda_{\cdot\alpha} N^{\alpha\beta} - \frac{1}{2}(\varphi^\lambda|_\alpha M^{\alpha\beta} + \varphi^\beta|_\alpha M^{\alpha\lambda}) + \frac{1}{2}(\bar{m}^\lambda|_\alpha M^{\alpha\beta} - \bar{m}^\beta|_\alpha M^{\alpha\lambda}) -$$

$$- \frac{1}{2}(b^\lambda_\alpha M^{\alpha\beta} - b^\beta_\alpha M^{\alpha\lambda}) - (a^{\lambda\beta} - \omega^{\lambda\beta}) (\varphi_\kappa M^{\kappa\rho})|_\rho - \varphi^\lambda M^{\beta\rho}|_\rho +$$

$$+ \frac{1}{2} [\varphi^\lambda (\omega^\beta_{\cdot\kappa} M^{\kappa\rho})|_\rho - \varphi^\beta (\omega^\lambda_{\cdot\kappa} M^{\kappa\rho})|_\rho] + a^{\lambda\beta}\varphi_\alpha M^{\alpha\rho}|_\rho \qquad (6.24)$$

$$= \varphi_\alpha N^{\alpha\beta} + \bar{m}_{,\alpha} M^{\alpha\beta} + (1 + \theta^\alpha_\alpha) M^{\beta\rho}|_\rho + (\theta^\beta_\kappa - \omega^\beta_{\cdot\kappa})|_\rho M^{\kappa\rho} -$$

$$- \omega^{\beta\alpha} (\omega_{\alpha\kappa} M^{\kappa\rho})|_\rho .$$

Appropriate definitions for boundary quantities follow from (6.19)-(6.21) by omitting terms marked by a dotted line.

7. THEORY OF SHELLS UNDERGOING LARGE/MODERATE ROTATIONS

This is an intermediate variant of the theory of shells undergoing large rotations. All shell relations can easily be constructed by appropriate simplifications of the large rotation theory discussed in the section 6. In particular, when Ω is supposed to be moderate all relations given in (6.1), (6.2), (6.6) - (6.12) are simplified by omitting there terms underlined by a solid line. The resulting shell relations assure the relative accuracy $O(\theta^2)$ of the strain energy function $(2.14)_1$ of the elastic shell.

In exactly the same way two other simplified variants of the theory of shells undergoing large/moderate rotations may be constructed. We should just omit some terms in appropriate relations of sec. 6, which are small within the relative accuracy $O(\theta\sqrt{\theta})$ or $O(\theta)$ of the shell strain energy $(2.14)_1$. We do not elaborate those variants here, since the reader can easily construct them himself, if necessary.

REFERENCES

W. PIETRASZKIEWICZ, M.L. SZWABOWICZ, Lagrangian nonlinear theory of thin shells, Archives of Mechanics, 1981 (to appear).

W. PIETRASZKIEWICZ, Nieliniowe teorie cienkich powłok sprężystych, w: "Konstrukcje powłokowe, teoria i zastosowania, pod. red. J. Orkisza i Z. Waszczyszyna, tom 1, Mat. Symp., Kraków 1974; PWN, Warszawa 1978, 27-50.

W. PIETRASZKIEWICZ, Introduction to the non-linear theory of shells, Ruhr-Universität Bochum, Mitt. Inst. für Mech., No. 10, Mai 1977, 1-154.

W. PIETRASZKIEWICZ, Finite rotations in the non-linear theory of thin shells, Lectures for CISM course "Thin Shells", Udine, Italy, October 1977 (in print by Springer-Verlag, Wien).

W. PIETRASZKIEWICZ, Finite rotations and Lagrangean description in the non-linear theory of shells, Polish Scientific Publishers, Warszawa - Poznań 1979.

R. SCHMIDT, W. PIETRASZKIEWICZ, Variational principles in the geometrically non-linear theory of shells undergoing moderate rotations, Ingenieur-Archiv, 50, 1981, 187-201.

L. Ya. AYNOLA, Variational problems in the non-linear theory of elastic shells (in Russian), Prikl. Mat. Mekh., 21, 3, 1957, 399 - 405.

L. H. DONNELL, Stability of thin-walled tubes under torsion, NACA TR 479, 1933.

K. MARGUERRE, Zur Theorie der gekrümmten Platte großer Formänderung, Proc. 5th Intern. Congr. of Appl. Mech, Camridge, Mass. 1938; Wiley and Sons, New York 1939, 93 - 101.

V. Z. VLASOV, General theory of shells and its applications in engineering, NASA TT F-99, 1964; Translation of "Obščaja teorija oboloček i eë priloženija v technike", Moscow-Leningrad 1949.

K. M. MUSHTARI, K.Z. GALIMOV, Non-linear theory of thin elastic shells, The Israel Program for Sci. Transl., Jerusalem 1961; Translation of "Nelineinaya teoriya uprugikh obolochek", Tatknigoizdat, Kazan' 1957.

K. Z. GALIMOV, Foundations of the non-linear theory of shells (in Russian), Kazan' University Press, Kazan' 1975.

J. L. SANDERS, Nonlinear theories for thin shells, Quart. Appl. Math., 21, 1963, 21 - 36.

[14] W. T. KOITER, On the nonlinear theory of thin elastic shells, Proc. Koninkl. Ned. Ak. Wet., ser. B., 69, 1966, 1 - 54.

[15] T. von KÁRMAN, Festigkeitsprobleme im Maschinenbau, Encykl. der Math. Wiss., Vol. IV/4, Teubner, Leipzig 1910, 311 - 385.

[16] W. PIETRASZKIEWICZ, Finite rotations in shells, in: "Theory of Shells", Ed. by W.T. Koiter and G.K. Mikhailov, Proc. 3rd IUTAM Symp., Tbilisi 1978; North-Holland P. Co., Amsterdam 1980, 445 - 471.

[17] W. PIETRASZKIEWICZ, Niektóre problemy nieliniowej teorii powłok, Mech. Teoret. Stos., 2, 1980, 169 - 190.

[18] W. PIETRASZKIEWICZ, Simplified equations for the geometrically non-linear theory of thin elastic shells, Trans. Inst. Fluid-Flow Mach., PASci Gdańsk, No. 75, 1978, 165 - 173.

[19] F. JOHN, Estimates for the derivatives of the stresses in a thin shell and interior shell equations, Comm. Pure and Appl. Math., 18, 1965, 235 - 267.

[20] W. T. KOITER, J. G. SIMMONDS, Foundations of shell theory, in: "Theoretical and Applied Mechanics", Proc. 13th IUTAM Congr., Moscow 1972; Springer-Verlag, Berlin-Heidelberg-New York 1973, 150 - 175.

[21] W. T. KOITER, A consistent first approximation in the general theory of thin elastic shells, in: "Theory of Thin Shells", Proc. IUTAM Symp., Delft 1959; North-Holland P. Co., Amsterdam 1960, 12 - 33.

[22] W. PIETRASZKIEWICZ, Consistent second approximation to the elastic strain energy of a shell, ZAMM, 59, 5, 1979, T206 - T208.

[23] W. PIETRASZKIEWICZ, Some relations of the non-linear Reissner-type theory of shells (in Russian), Vestnik Leningradskogo Un-ta, Ser. Mat., Mekh. i Astron., No. 1, 1979.

[24] M. L. SZWABOWICZ, Potential boundary loads in the Lagrangian nonlinear theory of shells, Prace IMP PAN (to appear).

[25] B. BUDIANSKY, J. L. SANDERS, On the "best" first-order linear shell theory, in: "Progress in Applied Mechanics", vol. 192, Macmillan, New York 1963, 129 - 140.

[26] W. Z. CHIEN, The intrinsic theory of thin shells and plates, (Parts I, II, III), Quart. Appl. Math., 1, 1944, 297 - 327; 2, 1944, 43 - 59; 3, 1944, 120 - 135.

[27] G. A. WEMPNER, Mechanics of solids, McGraw-Hill, New York 1973.

[28] M. DUSZEK, Equations of the large bending theory for plastic shells (in Polish), Rozprawy Inż., 20, 1972, 389 - 407.

[29] V. V. KABANOV, Equations of thin shells for strongly inhomogeneous stress and deformation states (in Russian), Izv. AN SSSR, Mekh. Tv. Tela, 4, 1979, 155 - 161.

[30] L. A. SHAPOVALOV, Equations of thin elastic shells for non-symmetric deformation (in Russian), Izv. AN SSSR, Mekh. Tv. Tela, 3, 1976.

[31] L. A. SHAPOVALOV, On a simplest variant of equations of geometrically non-linear theory of thin shells (in Russian), Inzh. Zhurnal, Mekh. Tv. Tela, 1, 1968, 56 - 62.

[32] C. H. TSAO, Strain-displacement relations in large displacement theory of shells, AIAA J., 2, 11, 1964,

[33] E. I. GRIGOLJUK, V. V. KABANOV, Stability of shells (in Russian), Nauka, Moscow 1978.

[34] N. A. ALUMÄE, Theory of elastic shells and plates (in Russian), in: "Mechanics in USSR in 50 years, vol. 3, Nauka, Moscow 1972.

[35] R. W. LEONARD, Nonlinear first approximation thin shell and membrane theory, Thesis, Virginia Polytechnic Inst., June 1961.

[36] M. K. DUSZEK, A systematic study of kinematics of shells at large strains and displacements, Bull.Acad. Polon.Sci., serie sci. techn., 26, 1978, 39 - 47.

[37] B. NOVOTNY, On the asymptotic integration of the three-dimensional non-linear equations of thin elastic shells and plates, Int. J. Sol. Str., 6, 1970, 433 - 451.

[38] R. SCHMIDT, Systematic derivation of variational principles for geometrically non-linear theories of shells undergoing moderate rotations. PhD Thesis, Ruhr-Universität Bochum, Institut für Mechanik, Bochum 1980.

[39] H. STUMPF, On the boundaries of stability and buckling of thin shells (to be published).

[40] J. G. SIMMONDS, Special cases of the nonlinear shell equations, in: "Trends in Solid Mechanics", Symposium to honor Prof. Dr. W.T. Koiter, Delft 1979;

[41] W. PIETRASZKIEWICZ, M. SZWABOWICZ, Hu - Washizu variational functional for the Lagrangian geometrically nonlinear theory of thin elastic shells, ZAMM, 1982 (to appear)