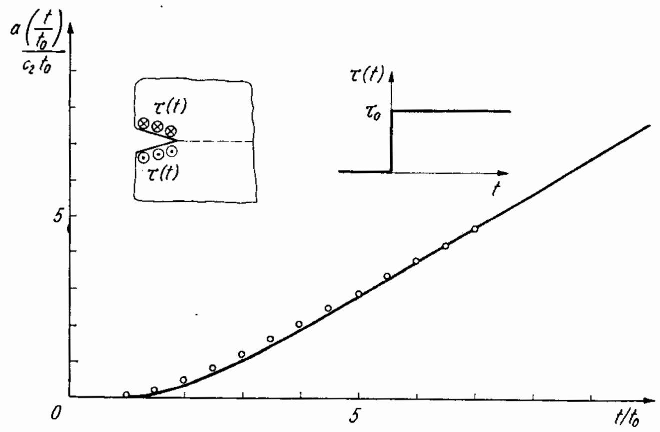

Abb. 3. Numerisch (○○○○○) und analytisch (———) berechnete zeitliche Änderung der Rißspitze für einen nichtebenen Schubspannungszustand ($\nu = 0.25$)

für den nichtebenen Schubspannungszustand. Der halbunendliche Riß wird in diesem Fall zur Zeit $t = 0$ längs der Rißflanken durch τ_0 belastet. Zur Zeit $t = t_0$ beginnt der Riß sich auszubreiten. Die Abbildung zeigt einen Vergleich mit den analytischen Ergebnissen von Kostrov [1]. Der Vergleich ist nur für Zeiten t kleiner $6t_0$ möglich, da bei den numerischen Ergebnissen, die an endlichen Proben bestimmt wurden, dann Randeffekte berücksichtigt werden müssen. Bei diesem Problem wurde das Griffithsche Bruchkriterium (4) unter Annahme eines konstanten Wertes

$$G_C = \frac{4(1+\nu)\,\tau_0^2 c_2 t_0}{\pi E} \tag{5}$$

verwendet.

Bestimmt man $G(t)$ aus der Nahfeldlösung des nichtebenen Schubspannungszustands in Abhängigkeit vom Spannungsintensitätsfaktor $K_{III}(t)$, so kann das Griffithsche Bruchkriterium (4) in ein Bruchkriterium der Form (3) überführt werden. Der zugehörige kritische Wert K_{IIIC} ist dann von der Rißgeschwindigkeit v abhängig:

$$K_{IIIC} = \frac{4}{\sqrt{2\pi}}\,\tau_0 \sqrt{c_2 t_0}\,\sqrt[4]{1 - \frac{v^2}{c_2^2}}\,. \tag{6}$$

Für die numerische Berechnung kann dann das Bruchkriterium in der Form (3) unter Berücksichtigung von (6) verwendet werden.

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Determination of Displacements from Given Strains in the Non-Linear Continuum Mechanics

According to Cauchy's theorem [1] the deformation near a continuum particle my be produced by successive superposition of a rigid-body translation $\boldsymbol{u}$, a pure stretch $\boldsymbol{U}$ and a rigid-body rotation $\boldsymbol{R}$. When the displacement field $\boldsymbol{u}$ is given, the fields $\boldsymbol{U}$, $\boldsymbol{R}$ and the Lagrangian strain field $\boldsymbol{E} = \frac{1}{2}(\boldsymbol{U}^2 - \boldsymbol{1})$ can easily be calculated. In this report we shall discuss the inverse problem: the determination of $\boldsymbol{u}$ from given $\boldsymbol{E}$.

Let $\boldsymbol{p} = x^k(\theta^i)\, \boldsymbol{i}_k$ and $\bar{\boldsymbol{p}} = y^k(\theta^i)\, \boldsymbol{i}_k = \boldsymbol{p} + \boldsymbol{u}$, $i, k = 1, 2, 3$, be position vectors of a continuum particle in the reference and deformed configurations, respectively. Here $\boldsymbol{i}_k$ is the common orthonormal basis, attached to an origin O of three-dimensional Euclidean space, while θ^i are the curvilinear convected coordinates. Let $\boldsymbol{g}_i = \partial \boldsymbol{p}/\partial\theta^i \equiv \boldsymbol{p}_{,i}$ and $\bar{\boldsymbol{g}}_i = \bar{\boldsymbol{p}}_{,i}$ be the natural base vectors while $g_{ij} = \boldsymbol{g}_i \cdot \boldsymbol{g}_j$ and $\bar{g}_{ij} = \bar{\boldsymbol{g}}_i \cdot \bar{\boldsymbol{g}}_j$ the covariant components of the metric tensors, with determinants $g = |g_{ij}|$ and $\bar{g} = |\bar{g}_{ij}|$, of the respective configurations. Components of the symmetric Lagrangian strain tensor are defined by

$$E_{ij} = \tfrac{1}{2}(\bar{g}_{ij} - g_{ij}) = \tfrac{1}{2}(u_{i|j} + u_{j|i} + g^{mn} u_{i|m} u_{j|n}) \tag{1}$$

where $(\ )_{|j}$ means the covariant derivative in the reference metric.

Let all geometric quantities of the reference configuration be known. Assume also to have given six functions $E_{ij} = E_{ij}(\theta^k)$. In the reference configuration choose a point $P = P(\theta^i)$ and a reference point P_0 at which $\theta^i = 0$, for convenience. Then at P we have $\boldsymbol{E} = E_{ij} \boldsymbol{g}^i \otimes \boldsymbol{g}^j$. In what follows $\boldsymbol{u}$ is determined form $\boldsymbol{E}$ in three successive steps.

In the first step, $\boldsymbol{U}$ is calculated from $\boldsymbol{E}$ by pure algebra. Since $\boldsymbol{U}$ is positive definite, symmetric and co-axial with $\boldsymbol{E}$, it takes the diagonal form $\boldsymbol{U} = \sum_r \sqrt{1 + 2E_r}\, \boldsymbol{h}_r \otimes \boldsymbol{h}_r$ where E_r and $\boldsymbol{h}_r$ are three eigenvalues and eigenvectors of $\boldsymbol{E}$ satisfying the set of equations $\boldsymbol{E}\boldsymbol{h}_r = E_r \boldsymbol{h}_r$ (no sum over r).

In the second step, the field $\boldsymbol{R}$ is determined from $\boldsymbol{E}$ and $\boldsymbol{U}$. Note that during the deformation process $\boldsymbol{R}\boldsymbol{U} = \bar{\boldsymbol{g}}_i \otimes \boldsymbol{g}^i$ in convected coordinates. Differentiating the relation with respect to the coordinates, we obtain after ransformations that $\boldsymbol{R}$ should satisfy the following system of the linear first-order differential equations

$$\boldsymbol{R}_{,j} = \boldsymbol{R}\boldsymbol{K}_j\,, \qquad \boldsymbol{K}_j = (\boldsymbol{U}\boldsymbol{A}_j - \boldsymbol{U}_{,j})\, \boldsymbol{U}^{-1}\,, \qquad \boldsymbol{A}_j = (\bar{G}^k_{ij} - G^k_{ij})\, \boldsymbol{g}_k \otimes \boldsymbol{g}^i\,. \tag{2), (3}$$

Here G^k_{ij} and $\bar{G}^k_{ij}$ are the Christoffel symbols of the second kind associated with the reference and deformed metrics, respectively. The symbols $\bar{G}^k_{ij}$ are calculated in terms of E_{ij} and the reference metric according to [2]

$$\begin{aligned}
&\bar{G}^k_{ij} = G^k_{ij} + \bar{g}^{kl}(E_{li|j} + E_{lj|i} - E_{ij|l})\,, \qquad \bar{g}^{il} = \frac{1}{2}\frac{g}{\bar{g}}\, \varepsilon^{ijk} \varepsilon^{lmn} (g_{jm} + 2E_{jm})\,(g_{kn} + 2E_{kn})\,,\\
&\frac{\bar{g}}{g} = \frac{1}{2}\, \varepsilon^{ijk} \varepsilon^{lmn} (g_{il} + 2E_{il})\,(g_{jm} + 2E_{jm})\,(g_{kn} + 2E_{kn})
\end{aligned} \tag{4}$$

where ε^{ijk} are the components of the skew-symmetric Ricci permutation tensor. Also note that $\boldsymbol{K}_j$ are skew-symmetric, $\boldsymbol{K}_j^T = -\boldsymbol{K}_j$, which follows from the identity $(\boldsymbol{R}^T\boldsymbol{R})_{,j} = \boldsymbol{0}$ and (2).

The proper orthogonal tensor $\boldsymbol{R}$ is uniquely described only by three independent scalar functions, such as, for example, Euler angles. Thus, (2) are equivalent to nine independent scalar differential equations. Integrability conditions $\boldsymbol{R}_{,ij} - \boldsymbol{R}_{,ji} = \boldsymbol{0}$ of (2) take the form [8]

$$\boldsymbol{K}_{i,j} - \boldsymbol{K}_{j,i} + \boldsymbol{K}_j\boldsymbol{K}_i - \boldsymbol{K}_i\boldsymbol{K}_j = \boldsymbol{0}\,. \tag{5}$$

With the use of Cartesian coordinates it was shown in [3] that when (5) are satisfied, the Riemann-Christoffel tensor of the deformed configuration vanishes as well. The conditions (5), which are the alternative form of compatibility conditions for finite strains, are equivalent to six independent scalar differential equations and assure the existence of three scalar functions describing $\boldsymbol{R}$ which is the solution of (2).

Along a curve connecting P_0 and P, defined by $\theta^i = \theta^i(t)$ with $t = 0$ at P_0, from (2) we obtain

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} = \boldsymbol{R}\boldsymbol{K}\,. \tag{6}$$

The general solution of (6) can be constructed in analogy to the corresponding matrix differential equation [5] in the form $\boldsymbol{R} = \boldsymbol{R}_0\boldsymbol{R}_t$, where $\boldsymbol{R}_0$ is an arbitrary constant proper orthogonal tensor and $\boldsymbol{R}_t$ is the matrizant of (6) defined by the tensor series

$$\boldsymbol{R}_t = \boldsymbol{1} + \int_0^t \boldsymbol{K}(\tau)\,\mathrm{d}\tau + \int_0^t \left[\int_0^\tau \boldsymbol{K}(\tau_1)\,\mathrm{d}\tau_1\right] \boldsymbol{K}(\tau)\,\mathrm{d}\tau + \ldots \tag{7}$$

For some specific forms of $\boldsymbol{K} = \boldsymbol{K}(t)$ effective analytic and numerical methods were developed [5]. Also note that (6) has exactly the same structure as the equation describing the motion of a rigid body about a fixed point. Therefore, the methods of solutions developed in analytic mechanics may be of assistance in solving (6) for problems related to the continuum deformation. In particular, 20 cases are known [6] for which the exact general or particular solution of (6) can be constructed in a closed form.

According to (3), only the tensors $\boldsymbol{K}_j$ along the coordinate lines are calculated. Let us choose a specific integration path $P_0P'P''P$ connecting P_0 with P. The path consists of three subsequent parts of the coordinate lines: along P_0P' there is $\theta^2 = 0$ and $\theta^3 = 0$, along $P'P''$ there is $\theta^1 = \text{const}$ and $\theta^3 = 0$, while along $P''P$ there is $\theta^1 = \text{const}$ and $\theta^2 = \text{const}$. Solving (2) along the specific integration path we obtain

$$\boldsymbol{R} = \boldsymbol{R}_0\boldsymbol{R}_1\boldsymbol{R}_2\boldsymbol{R}_3 \tag{8}$$

where $\boldsymbol{R}_1 = \boldsymbol{R}_1(\theta^1, 0, 0)$, $\boldsymbol{R}_2 = \boldsymbol{R}_2(\theta^1, \theta^2, 0)$, $\boldsymbol{R}_3 = \boldsymbol{R}_3(\theta^i)$ are matrizants of the corresponding differential equations along the subsequent parts of the integration path. For some specific forms of the fields $\boldsymbol{K}_j$ the matrizants $\boldsymbol{R}_j$ can be constructed analytically or even in the closed form [5, 6].

In the third step, the position vector $\bar{\boldsymbol{p}}$ is determined from already known $\boldsymbol{U}$ and $\boldsymbol{R}$. With already known $\bar{\boldsymbol{g}}_i = \boldsymbol{R}\boldsymbol{U}\boldsymbol{g}_i$, we obtain the differential equations

$$\bar{\boldsymbol{p}}_{,i} = \bar{\boldsymbol{g}}_i \,. \tag{9}$$

Integrability conditions $\bar{\boldsymbol{p}}_{,ij} - \bar{\boldsymbol{p}}_{,ji} = \boldsymbol{0}$ of (9) lead to the conditions $(\bar{G}^k_{ij} - \bar{G}^k_{ji})\,\bar{\boldsymbol{g}}_k = \boldsymbol{0}$, which geometrically mean [4] that the torsion tensor $\bar{S}^k_{ij} = \bar{G}^k_{ij} - \bar{G}^k_{ji}$ of the deformed configuration should vanish. Since $\bar{G}^k_{ij}$ are calculated as in $(4)_1$, the conditions are identically satisfied for any symmetric E_{ij}.

At the subsequent parts of the integration path $P_0P'P''P$, the solution of (9) can be given through quadratures in the form

$$\bar{\boldsymbol{p}} = \bar{\boldsymbol{p}}_0 + \int_0^{\theta^1} \bar{\boldsymbol{g}}_1(\xi, 0, 0)\, \mathrm{d}\xi + \int_0^{\theta^2} \bar{\boldsymbol{g}}_2(\theta^1, \eta, 0)\, \mathrm{d}\eta + \int_0^{\theta^3} \bar{\boldsymbol{g}}_3(\theta^1, \theta^2, \zeta)\, \mathrm{d}\zeta \,. \tag{10}$$

Combining (10) and (8) together with $\boldsymbol{u}_0 \equiv \bar{\boldsymbol{p}}_0 - \boldsymbol{p}$, we obtain the final relation

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{R}_0 \Bigg[&\int_0^{\theta^1} \boldsymbol{R}_1(\xi, 0, 0)\, \boldsymbol{U}(\xi, 0, 0)\, \boldsymbol{g}_1(\xi, 0, 0)\, \mathrm{d}\xi + \boldsymbol{R}_1 \int_0^{\theta^2} \boldsymbol{R}_2(\theta^1, \eta, 0)\, \boldsymbol{U}(\theta^1, \eta, 0)\, \boldsymbol{g}_2(\theta^1, \eta, 0)\, \mathrm{d}\eta \,+ \\ &+ \boldsymbol{R}_1\boldsymbol{R}_2 \int_0^{\theta^3} \boldsymbol{R}_3(\theta^1, \theta^2, \zeta)\, \boldsymbol{U}(\theta^1, \theta^2, \zeta)\, \boldsymbol{g}_3(\theta^1, \theta^2, \zeta)\, \mathrm{d}\zeta \Bigg].\end{aligned} \tag{11}$$

The displacement field $\boldsymbol{u}$ is determined from the strain field $\boldsymbol{E}$ by the relation (11) to within a rigid-body translation $\boldsymbol{u}_0$ and a rigid-body rotation $\boldsymbol{R}_0$. In case of infinitesimal deformation the relation (11) can be reduced [9] to the formula of CESARO [7].

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Hu-Washizu Variational Functional for the Lagrangian Geometrically Nonlinear Theory of Thin Elastic Shells

The classical version of the Lagrangian geometrically nonlinear theory of thin elastic shells [1–3] allowed global variational formulation, in terms of various free functionals, within moderate rotations only [4]. In order to allow global variational formulation of the Lagrangian nonlinear theory of shells undergoing unrestricted (finite) rotations, some modified variables should be introduced [5].

Let the deformation of the shell middle surface be described by the usual surface strain tensor $\gamma_{\alpha\beta}$ and the modified tensor of change of curvature $\chi_{\alpha\beta}$ defined by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\,(\bar{a}_{\alpha\beta} - a_{\alpha\beta})\,, \qquad \chi_{\alpha\beta} = -\left(\sqrt{\frac{\bar{a}}{a}}\,\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right) + b_{\alpha\beta}\,\gamma^{\varkappa}_{\varkappa}. \tag{1}$$

Here $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ are components of metric and curvature tensors of the shell middle surface in the reference (undeformed) and in the deformed configurations $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, with determinants $a = |a_{\alpha\beta}|$ and $\bar{a} = |\bar{a}_{\alpha\beta}|$. By definition (1), $\gamma_{\alpha\beta}$ are quadratic polynomials and $\chi_{\alpha\beta}$ are third-order polynomials in the displacements $\boldsymbol{u} = u^\alpha \boldsymbol{a}_\alpha + w\boldsymbol{n}$ where $\boldsymbol{a}_\alpha$ are base vectors of $\mathcal{M}$ and $\boldsymbol{n} = \frac{1}{\sqrt{a}}\,\boldsymbol{a}_1 \times \boldsymbol{a}_2$ is the unit normal to $\mathcal{M}$.

The deformation of the shell boundary element can be described entirely by the displacement vector $\boldsymbol{u} = u_\nu \boldsymbol{\nu} + u_t \boldsymbol{t} + w\boldsymbol{n}$ of the undeformed boundary curve $\mathscr{C}$ and by the components along $\mathscr{C}$ of the unit normal to $\bar{\mathcal{M}}$, the