In the third step, the position vector $\bar{\boldsymbol{p}}$ is determined from already known $\boldsymbol{U}$ and $\boldsymbol{R}$. With already known $\bar{\boldsymbol{g}}_i = \boldsymbol{R}\boldsymbol{U}\boldsymbol{g}_i$, we obtain the differential equations

$$\bar{\boldsymbol{p}}_{,i} = \bar{\boldsymbol{g}}_i \,. \tag{9}$$

Integrability conditions $\bar{\boldsymbol{p}}_{,ij} - \bar{\boldsymbol{p}}_{,ji} = \boldsymbol{0}$ of (9) lead to the conditions $(\bar{G}^k_{ij} - \bar{G}^k_{ji})\,\bar{\boldsymbol{g}}_k = \boldsymbol{0}$, which geometrically mean [4] that the torsion tensor $\bar{S}^k_{ij} = \bar{G}^k_{ij} - \bar{G}^k_{ji}$ of the deformed configuration should vanish. Since $\bar{G}^k_{ij}$ are calculated as in $(4)_1$, the conditions are identically satisfied for any symmetric E_{ij}.

At the subsequent parts of the integration path $P_0P'P''P$, the solution of (9) can be given through quadratures in the form

$$\bar{\boldsymbol{p}} = \bar{\boldsymbol{p}}_0 + \int_0^{\theta^1} \bar{\boldsymbol{g}}_1(\xi, 0, 0)\, \mathrm{d}\xi + \int_0^{\theta^2} \bar{\boldsymbol{g}}_2(\theta^1, \eta, 0)\, \mathrm{d}\eta + \int_0^{\theta^3} \bar{\boldsymbol{g}}_3(\theta^1, \theta^2, \zeta)\, \mathrm{d}\zeta \,. \tag{10}$$

Combining (10) and (8) together with $\boldsymbol{u}_0 \equiv \bar{\boldsymbol{p}}_0 - \boldsymbol{p}$, we obtain the final relation

$$\begin{aligned}\boldsymbol{u} = \boldsymbol{u}_0 + \boldsymbol{R}_0 \Bigg[&\int_0^{\theta^1} \boldsymbol{R}_1(\xi, 0, 0)\, \boldsymbol{U}(\xi, 0, 0)\, \boldsymbol{g}_1(\xi, 0, 0)\, \mathrm{d}\xi + \boldsymbol{R}_1 \int_0^{\theta^2} \boldsymbol{R}_2(\theta^1, \eta, 0)\, \boldsymbol{U}(\theta^1, \eta, 0)\, \boldsymbol{g}_2(\theta^1, \eta, 0)\, \mathrm{d}\eta + \\ &+ \boldsymbol{R}_1\boldsymbol{R}_2 \int_0^{\theta^3} \boldsymbol{R}_3(\theta^1, \theta^2, \zeta)\, \boldsymbol{U}(\theta^1, \theta^2, \zeta)\, \boldsymbol{g}_3(\theta^1, \theta^2, \zeta)\, \mathrm{d}\zeta \Bigg].\end{aligned} \tag{11}$$

The displacement field $\boldsymbol{u}$ is determined from the strain field $\boldsymbol{E}$ by the relation (11) to within a rigid-body translation $\boldsymbol{u}_0$ and a rigid-body rotation $\boldsymbol{R}_0$. In case of infinitesimal deformation the relation (11) can be reduced [9] to the formula of Cesaro [7].

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Hu-Washizu Variational Functional for the Lagrangian Geometrically Nonlinear Theory of Thin Elastic Shells

The classical version of the Lagrangian geometrically nonlinear theory of thin elastic shells [1—3] allowed global variational formulation, in terms of various free functionals, within moderate rotations only [4]. In order to allow global variational formulation of the Lagrangian nonlinear theory of shells undergoing unrestricted (finite) rotations, some modified variables should be introduced [5].

Let the deformation of the shell middle surface be described by the usual surface strain tensor $\gamma_{\alpha\beta}$ and the modified tensor of change of curvature $\chi_{\alpha\beta}$ defined by

$$\gamma_{\alpha\beta} = \tfrac{1}{2}\,(\bar{a}_{\alpha\beta} - a_{\alpha\beta})\,, \qquad \chi_{\alpha\beta} = -\left(\sqrt{\frac{\bar{a}}{a}}\,\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right) + b_{\alpha\beta}\,\gamma^\varkappa_\varkappa \,. \tag{1}$$

Here $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ are components of metric and curvature tensors of the shell middle surface in the reference (undeformed) and in the deformed configurations $\mathcal{M}$ and $\bar{\mathcal{M}}$, respectively, with determinants $a = |a_{\alpha\beta}|$ and $\bar{a} = |\bar{a}_{\alpha\beta}|$. By definition (1), $\gamma_{\alpha\beta}$ are quadratic polynomials and $\chi_{\alpha\beta}$ are third-order polynomials in the displacements $\boldsymbol{u} = u^\alpha \boldsymbol{a}_\alpha + w\boldsymbol{n}$ where $\boldsymbol{a}_\alpha$ are base vectors of $\mathcal{M}$ and $\boldsymbol{n} = \frac{1}{\sqrt{a}}\,\boldsymbol{a}_1 \times \boldsymbol{a}_2$ is the unit normal to $\mathcal{M}$.

The deformation of the shell boundary element can be described entirely by the displacement vector $\boldsymbol{u} = u_\nu \boldsymbol{\nu} + u_t \boldsymbol{t} + w\boldsymbol{n}$ of the undeformed boundary curve $\mathcal{C}$ and by the components along $\mathcal{C}$ of the unit normal to $\bar{\mathcal{M}}$, the vector $\bar{\boldsymbol{n}} = n_\nu \boldsymbol{\nu} + n_t \boldsymbol{t} + n\boldsymbol{n}$, where $\boldsymbol{t}$ is the unit tangent to $\mathscr{C}$ and $\boldsymbol{\nu} = \boldsymbol{t} \times \boldsymbol{n}$. Under the KIRCHHOFF-LOVE constraints only $\boldsymbol{u}$, n_ν are independent variables, while $n_t = n_t(\boldsymbol{u}, n_\nu)$ and $n = n(\boldsymbol{u}, n_\nu)$ must be satisfied.

Let us consider now a thin shell in equilibrium. For any additional infinitesimal displacement field $\delta\boldsymbol{u} = \delta u^\alpha \boldsymbol{a}_\alpha + \delta w \boldsymbol{n}$ subject to geometric constraints, the internal virtual work (IVW) should be equal to the external virtual work (EVW). Here IVW is performed by the internal Lagrangian stress resultants $N^{\alpha\beta}$ and the stress couples $M^{\alpha\beta}$ on the variations of their conjugate strain measures. EVW is performed by the external surface load $\boldsymbol{p} = p^\alpha \boldsymbol{a}_\alpha + p\boldsymbol{n}$, the resultant boundary force $\boldsymbol{N} = N_\nu \boldsymbol{\nu} + N_t \boldsymbol{t} + N\boldsymbol{n}$ and the resultant boundary couple $\boldsymbol{k}$ on the variations of the corresponding displacement variables. From IVW = EVW we obtain the incremental Lagrangian principle of virtual displacements [1—3]

$$\iint_{\mathscr{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\chi_{\alpha\beta})\, \mathrm{d}A = \iint_{\mathscr{M}} \boldsymbol{p}\cdot\delta\boldsymbol{u}\, \mathrm{d}A + \int_{\mathscr{C}_f} (\boldsymbol{N}\cdot\delta\boldsymbol{u} + \boldsymbol{k}\cdot\delta\boldsymbol{\Omega}_t)\, \mathrm{d}s \tag{2}$$

where $\boldsymbol{\Omega}_t$ is the total finite rotation vector of the boundary.

Note, that $\boldsymbol{k}$ is by definition tangent to the deformed middle surface along the boundary curve $\bar{\mathscr{C}}$ and therefore depends upon the boundary deformation. In the Lagrangian shell theory it is more convenient [5] to replace $\boldsymbol{k}$ by the equivalent external boundary static moment $\boldsymbol{H} = H_\nu \boldsymbol{\nu} + H_t \boldsymbol{t} + H\boldsymbol{n}$, where $\boldsymbol{k} = \bar{\boldsymbol{n}} \times \boldsymbol{H}$. Then, after transformations, (2) takes the form

$$-\iint_{\mathscr{M}} (\boldsymbol{T}^\beta|_\beta + \boldsymbol{p})\cdot\delta\boldsymbol{u}\, \mathrm{d}A + \int_{\mathscr{C}_f} [(\boldsymbol{P} - \boldsymbol{P}^*)\cdot\delta\boldsymbol{u} + (M - M^*)\,\delta n_\nu]\, \mathrm{d}s + \sum_k (\boldsymbol{F}_k - \boldsymbol{F}_k^*)\cdot\delta\boldsymbol{u}_k = 0 \tag{3}$$

where expressions for $\boldsymbol{T}^\beta = \boldsymbol{T}^\beta(N^{\alpha\beta}, M^{\alpha\beta}, \boldsymbol{u})$, $\boldsymbol{P} = \boldsymbol{P}(N^{\alpha\beta}, M^{\alpha\beta}, \boldsymbol{u}, n_\nu)$, $M = M(M^{\alpha\beta}, \boldsymbol{u}, n_\nu)$, $\boldsymbol{F}_k = \boldsymbol{F}_k(M^{\alpha\beta}, \boldsymbol{u}, n_\nu)$, $\boldsymbol{P}^* = \boldsymbol{P}^*(\boldsymbol{N}, \boldsymbol{H}, \boldsymbol{u}, n_\nu)$, $M^* = M^*(\boldsymbol{H}, \boldsymbol{u}, n_\nu)$ and $\boldsymbol{F}_k^* = \boldsymbol{F}_k^*(\boldsymbol{H}, \boldsymbol{u}, n_\nu)$ are given in [5].

From (3) follow the equilibrium equations $\boldsymbol{T}^\beta|_\beta + \boldsymbol{p} = \boldsymbol{0}$ in $\mathscr{M}$, the static boundary conditions $\boldsymbol{P} = \boldsymbol{P}^*$ and $M = M^*$ along $\mathscr{C}_f$ and the static corner conditions $\boldsymbol{F}_j = \boldsymbol{F}_j^*$ in each corner point M_j of $\mathscr{C}_f$. The relations are satisfied for unrestricted strains and rotations, an arbitrary material behaviour and arbitrary external surface and boundary loads.

Under small elastic strains and within the first-approximation shell theory there exists a quadratic shell strain energy function $\Sigma = \frac{1}{2}H^{\alpha\beta\lambda\mu}\left(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\chi_{\alpha\beta}\chi_{\lambda\mu}\right)$, where $H^{\alpha\beta\lambda\mu}$ are the components of the modified elasticity tensor [2]. In this case the specific internal virtual work in (2) can be expressed as a variation of the shell strain energy function: $N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\chi_{\alpha\beta} = \delta\Sigma\,(\gamma_{\alpha\beta}, \chi_{\alpha\beta})$. Let us also assume $\boldsymbol{p}, \boldsymbol{N}$ and $\boldsymbol{H}$ to be dead, i.e. their directions to be constant during the shell deformation. In this case there exist potentials [5, 6] of the external loads $\Phi(\boldsymbol{u}) = -\boldsymbol{p}\cdot\boldsymbol{u}$ and $\Psi(\boldsymbol{u}, n_\nu) = -\boldsymbol{N}\cdot\boldsymbol{u} - \boldsymbol{H}\cdot(\bar{\boldsymbol{n}} - \boldsymbol{n})$ such that their variations constitute the specific external virtual work. Therefore, the incremental Lagrangian principle of virtual displacements (2) can be transformed into the global variational principle $\delta I = 0$ for the functional

$$I = \iint_{\mathscr{M}} [\Sigma\,(\gamma_{\alpha\beta}, \chi_{\alpha\beta}) - \boldsymbol{p}\cdot\boldsymbol{u}]\, \mathrm{d}A - \int_{\mathscr{C}_f} [\boldsymbol{N}\cdot\boldsymbol{u} + \boldsymbol{H}\cdot(\bar{\boldsymbol{n}} - \boldsymbol{n})]\, \mathrm{d}s \tag{4}$$

where the strain-displacement relations $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\boldsymbol{u})$, $\chi_{\alpha\beta} = \chi_{\alpha\beta}(\boldsymbol{u})$ in $\mathscr{M}$, geometric boundary conditions $\boldsymbol{u} = \boldsymbol{u}^*$, $n_\nu = n_\nu^*$ along $\mathscr{C}_u$, geometric corner conditions $\boldsymbol{u}_i = \boldsymbol{u}_i^*$ in each corner M_i of $\mathscr{C}_u$ and the geometric constraint relations $n_t = n_t(\boldsymbol{u}, n_\nu)$, $n = n(\boldsymbol{u}, n_\nu)$ along $\mathscr{C}_f$ have to be imposed as subsidiary conditions.

Introducing the subsidiary conditions of I into the functional itself by means of the Lagrange multiplier method we obtain the free functional

$$\begin{aligned} I_1 = &\iint_{\mathscr{M}} \{\Sigma\,(\gamma_{\alpha\beta}, \chi_{\alpha\beta}) - \boldsymbol{p}\cdot\boldsymbol{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(\boldsymbol{u})] - M^{\alpha\beta}[\chi_{\alpha\beta} - \chi_{\alpha\beta}(\boldsymbol{u})]\}\, \mathrm{d}A - \\ &- \int_{\mathscr{C}_f} \{\boldsymbol{N}\cdot\boldsymbol{u} + \boldsymbol{H}\cdot(\bar{\boldsymbol{n}} - \boldsymbol{n}) - \lambda_t[n_t - n_t(\boldsymbol{u}, n_\nu)] - \lambda[n - n(\boldsymbol{u}, n_\nu)]\}\, \mathrm{d}s - \\ &- \int_{\mathscr{C}_u} [\boldsymbol{P}\cdot(\boldsymbol{u} - \boldsymbol{u}^*) + M(n_\nu - n_\nu^*)]\, \mathrm{d}s - \sum_i \boldsymbol{F}_i\cdot(\boldsymbol{u}_i - \boldsymbol{u}_i^*)\,. \end{aligned} \tag{5}$$

The functional I_1 is defined in terms of the following independent free variables subject to variations: three displacements $\boldsymbol{u}$ in $\mathscr{M}$, four displacement variables $\boldsymbol{u}$, n_ν along $\mathscr{C}_u$, three displacements $\boldsymbol{u}_i$ in each corner M_i of $\mathscr{C}_u$, six strain components $\gamma_{\alpha\beta}$ and $\chi_{\alpha\beta}$ in $\mathscr{M}$, six displacement variables $\boldsymbol{u}$, n_ν, n_t, n along $\mathscr{C}_f$, six Lagrange multipliers $N^{\alpha\beta}$, $M^{\alpha\beta}$ in $\mathscr{M}$, two Lagrange multipliers λ_t and λ along $\mathscr{C}_f$, four Lagrange multipliers $\boldsymbol{P}$ and M along $\mathscr{C}_u$, and three Lagrange multipliers $\boldsymbol{F}_i$ in each corner point M_i of $\mathscr{C}_u$. The associated variational principle $\delta I_1 = 0$ states that among all the possible values of the variables, which are not restricted by any subsidiary conditions, the actual solution renders the functional I_1 stationary.

Taking the variation of I_1 we obtain as its stationarity conditions all the basic relations: equilibrium equations, static boundary and corner conditions, strain-displacement relations, geometric boundary and corner conditions together with additional relations for the Lagrange multipliers: constitutive equations $N^{\alpha\beta} = hH^{\alpha\beta\lambda\mu}\gamma_{\lambda\mu}$, $M^{\alpha\beta} = \frac{h^2}{12}H^{\alpha\beta\lambda\mu}\chi_{\lambda\mu}$ in $\mathscr{M}$, the definitions for $\boldsymbol{P}$ and M along $\mathscr{C}_u$, the definitions for $\boldsymbol{F}_i$ in M_i of C_u, and the definitions $\lambda_t = H_t$, $\lambda = H$ along $\mathscr{C}_f$.

The functional I_1 is the HU-WASHIZU variational functional for the Lagrangian geometrically nonlinear theory of thin elastic shells undergoing unrestricted rotations. Following [4] a number of other free functionals and associated Lagrangian variational principles may be generated.

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Gegenbeispiel von I. Müller, hyperbolische Wärmeleitungsgleichung und Objektivität in 4-dimensionaler Kontinuumstheorie

Dreidimensionale Wärmeleitung

Es wird ein isotropes Material in einem Inertialsystem betrachtet. Für einen materiell angehefteten Ruhbeobachter $\mathscr{B}$ ist daher die Matrix der Materialabbildung ein Vielfaches der Einheitsmatrix

$$L^{ik} = \lambda(x^\alpha)\,\delta^{ik}\,, \qquad \alpha = 1, 2, 3\,. \tag{1}$$

Wird das betrachtete Material gegen das Inertialsystem beschleunigt bewegt, so ermittelt ein materiell angehefteter Ruhbeobachter B die Materialmatrix L'^{*ik}. Falls nun die Hooke-Poisson-Cauchy-Form des Prinzips der Objektivität gilt, folgt [1]

$$L'^{*ik} = L^{ik}\,, \tag{2}$$

d. h. beide Beobachter stellen isotropes Materialverhalten fest. Nun hat I. Müller [2, 3] an einem mit einem idealen Gas gefüllten Zylinder, dessen Achse geheizt wird, durch kinetische Überlegungen gezeigt, daß die Wärmestromdichte für $\mathscr{B}$ nicht mehr parallel zum Temperaturgradienten sein kann (es tritt eine Azimutalkomponente der Wärmestromdichte auf). Daher ist die HPC-Form der Objektivität im Gegensatz zur ZJ-Form (Kovarianz) [1] nicht gültig. Auch in einer 4-dimensionalen Formulierung läßt sich wegen der komponentenweisen Gleichheit in (2) die HPC-Objektivität nicht retten.

Vierdimensionale Wärmeleitung

Die Motivierung zu einer 4-dimensionalen Formulierung der Kontinuumsphysik besteht in ihrer Kovarianz, da sich die ZJ-Objektivität nur in kovarianten Theorien formulieren läßt [1]. Während z. B. die Geschwindigkeit im Dreidimensionalen gegen Euklidsche Transformation bekanntlich nicht kovariant ist,

$$v^{*\alpha} = Q^\alpha_\beta v^\beta + \dot{Q}^\alpha_\beta x^\beta + \dot{c}^\alpha\,, \tag{3}$$

gilt dies aber im Vierdimensionalen

$$\begin{pmatrix} v^* \\ 1 \end{pmatrix} = \begin{pmatrix} Q & \dot{Q}X + \dot{C} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v \\ 1 \end{pmatrix} \tag{4}$$

oder in Komponenten für die 4-Geschwindigkeit

$$z^a = Q^a_b z^b\,, \qquad a, b = 1, \dots, 4\,. \tag{5}$$

Für einen Ruhbeobachter $\mathscr{B}$ wird wegen $v^\alpha = 0$ die 4-Geschwindigkeit gemäß (4)

$$\mathscr{B} : \boldsymbol{z} = \boldsymbol{e}_4\,. \tag{6}$$

Mit

$$\nabla = (\partial/\partial x^\alpha, \partial/\partial t) \tag{7}$$