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FINITE ROTATIONS IN THE DESCRIPTION OF CONTINUUM DEFORMATION

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Abstract—Finite rotations in continuum mechanics are described by means of either a proper orthogonal tensor or finite rotation vectors. Some algebraic relations concerning the finite rotations are reviewed. Formulae expressing them in terms of displacements are given. Along each of the curvilinear coordinate lines the finite rotations are shown to satisfy some systems of the linear first-order differential equations. Each system of the equations is presented in four different but equivalent forms associated with an intermediate stretched basis or with an intermediate rotated basis. Integrability conditions of the system of equations provide various alternative forms of compatibility conditions in continuum mechanics. The displacement field is expressed through the stretch and rotation fields in the form of three successive line integrals. The formula describes the displacements to within a constant finite translation and a constant finite rotation. The procedure proposed here generalizes the formula derived by Cesàro (1906) within the classical linear theory of elasticity.

1. INTRODUCTION

THE DEFORMATION near a continuum particle may be produced by successive superposition of a rigid-body finite translation, a pure finite stretch along the principal directions and a rigid-body finite rotation[1–3].

In many fundamental papers and monographs concerning continuum mechanics, attention is focussed mainly on the displacement, the strain and the deformation gradient fields. Much less attention is paid to interpret the importance of the rotation field, although it is one of the three equivalently important elementary states of local deformation of the continuum. Only in a few papers the rotation field was used explicitly as an independent kinematic variable of the continuum deformation, which allowed to obtain some interesting original results.

In several Italian papers reviewed by Guo[4], and in those of Ferrarese[5], Shield[6] and Shamina[7] the finite rotations were used to derive the non-classical forms of compatibility conditions in the continuum mechanics. Beatty[8] investigated the effects of rigid-body rotations as virtual motions in the energy criteria of elastic stability. Fraeijs de Veubeke[9] derived the variational principle of complementary energy for finite elasticity, which is expressed in terms of the finite rotation field. Computer programs developed by Sander and Carnoy[10] as well as by Murakawa and Atluri[11] showed the efficiency of the use of the finite rotation field as an independent variable.

The rotational part of deformation is of particular importance in the non-linear theories of thin bodies such as bars, plates and shells, since in this case the finite rotations may appear within the infinitesimal strain theory. Within the non-linear Kirchhoff theory of thin rods an interesting analogy to the motion of a rigid body about a fixed point[12, 13] allowed to construct many exact solutions in closed form[14]. The general theory of finite rotations in shells developed in [15–19] led, among others, to several new formulations of the theory, to new forms of geometric boundary conditions and energetically compatible with them static boundary conditions as well as to a new classification of the approximate versions of the theory of shells undergoing consistently restricted rotations.

In this report we examine in more detail the importance of the finite rotation field and its spatial derivatives in the description of deformation of a continuum. Applying the polar decomposition theorem and the Lagrangian description of deformation, several representations for finite stretches and strains as well as for finite rotations are discussed. Besides the natural bases $\mathbf{g}_i$ and $\bar{\mathbf{g}}_i$ of the set of curvilinear convected coordinates in the reference and deformed body configurations, respectively, two additional intermediate, non-holonomic bases are introduced. The stretched basis $\mathbf{s}_i$ is obtained from $\mathbf{g}_i$ by its pure stretch along the principal directions of strain, while the rotated basis $\mathbf{r}_i$ is generated from $\mathbf{g}_i$ by its rigid-body rotation. The bases are shown to be very helpful in deriving various geometric relations of the continuum.

The rotation field is usually described by a proper orthogonal tensor $\mathbf{R}$[1–3] or by an equivalent finite rotation vectors $\mathbf{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$,[7, 15, 20–22]. The directions of the vectors are given by an eigenvector $\mathbf{e}$ corresponding to the real eigenvalue of $\mathbf{R}$ and their lengths are taken as $\sin\omega$, $2\text{tg}\,\omega/2$ or ω, respectively, where ω is the rotation angle about $\mathbf{e}$. We review here some algebraic relations concerning mathematical description of the finite rotations, their composition rules and their action on a vector. Several formulae expressing $\mathbf{R}$ and $\mathbf{\Omega}$ in terms of displacements are derived, among which (3.13), $(3.15)_2$ and (3.17) are new.

The spatial differentiation of the finite rotations is examined. Along each of the curvilinear coordinate lines the rotation tensor $\mathbf{R}$ is shown to satisfy a linear first-order differential equation. The equation may be written in two different but equivalent forms $(4.2)_1$ or $(4.10)_1$. Since $\mathbf{R}$ itself may be represented in two different but equivalent forms, associated with the intermediate bases $\mathbf{s}_i$ or $\mathbf{r}_i$, this leads to four different but equivalent forms of the linear first-order differential equation for $\mathbf{R}$ to be satisfied. The tensorial coefficients $\mathbf{K}_j$ and $\mathbf{L}_j$ of the differential equations are shown to be skew-symmetric and depend only upon stretches and strains. They describe that part of changes of curvatures of the coordinate lines during deformation which is caused by variations of $\mathbf{R}$ along the lines. The solution of each of the differential equations $(4.2)_1$ or $(4.10)_1$ allows to calculate displacements from the given strains through quadratures and algebraic operations only. The integrability conditions of the differential equations provide different but equivalent tensor forms (5.11) and (5.12) and component forms (5.16) of compatibility conditions in continuum mechanics. One form of (5.16) can be reduced to that given by Shield[6] in Cartesian coordinates, other forms are new.

In some applications it may be more convenient to use vectorial representations of the tensors. Expressing the orthogonal tensor $\mathbf{R}$ by equivalent finite rotation vectors $\mathbf{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$ and the skew-symmetric tensors $\mathbf{K}_j$ and $\mathbf{L}_j$ by their axial vectors of change of curvature $\mathbf{k}_j$ and $\mathbf{l}_j$, four different but equivalent forms of the first-order differential equations for $\mathbf{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$ are derived. Integrability conditions of each of the differential equations provide again various different but equivalent forms of compatibility conditions (5.20) and (5.21). In particular, such vectorial representations allow to derive simple differentation rules (4.8) and (4.18) of the intermediate bases $\mathbf{s}_i$ and $\mathbf{r}_i$. The compatibility conditions $(5.20)_2$ were originally given by Signorini[23] and within the non-linear theory of shells were applied by Pietraszkiewicz[15–19]. Other vector forms of compatibility conditions derived here are new.

It is explicitly shown how to determine the displacement vector $\mathbf{u}$ from the given Lagrangian strain tensor $\mathbf{E}$. The problem is divided into three steps. First, the right stretch tensor $\mathbf{U}$ is constructed from $\mathbf{E}$ with the use of purely algebraic transformations. Then the rotation tensor $\mathbf{R}$ is calculated from $\mathbf{U}$ and $\mathbf{E}$ by solving the appropriate linear first-order differential equation in terms of matricants along the specifically chosen integration path, consisting of three subsequent paths along the coordinate lines. The tensor $\mathbf{R}$ is obtained in the form of subsequent superpositon of the four orthogonal tensors. As a final step, the vector $\mathbf{u}$ is calculated from $\mathbf{R}$ and $\mathbf{U}$ through three subsequent quadratures along the integration path mentioned above. The final original formula (6.14) gives the displacement vector to within a constant translation and a constant rotation in the space. The formula (6.14) is valid for an arbitrary non-linear deformation and may be considered as the counterpart of the classical Cesàro[24] result which is valid only within the infinitesimal deformations.

2. CONTINUUM DEFORMATION

Let us describe the deformation of a body $\mathscr{B}$, consisting of material particles $X, Y, \ldots$, in a three-dimensional Euclidean space $\mathscr{E}$. The regions $\mathscr{D} \subset \mathscr{E}$ and $\bar{\mathscr{D}} \subset \mathscr{E}$, occupied by the body $\mathscr{B}$ in the reference (undeformed) and deformed configurations, are assumed here, for simplicity, to be simply connected with boundaries $\partial\mathscr{D}$ and $\partial\bar{\mathscr{D}}$, respectively, consisting of piecewise regular surfaces. The places occupied by the particle $X \in \mathscr{B}$ in the reference and deformed configurations are given by the respective position vectors

$$\mathbf{p} = x^k(\theta^i)\mathbf{i}_k, \quad \bar{\mathbf{p}} = y^k(\theta^i)\mathbf{i}_k = \chi(\mathbf{p}) = \mathbf{p} + \mathbf{u}. \tag{2.1}$$

Here θ^i, $i = 1, 2, 3$, are the curvilinear convected coordinates, $\mathbf{i}_k$ is the common orthonormal basis attached to an origin $0 \in \mathscr{E}$, χ is the deformation function and $\mathbf{u} = \mathbf{u}(\theta^i)$ is the displacement vector of the particle $X \in \mathscr{B}$.

The position vectors (2.1) define base vectors and the metric tensors in the reference and deformed configurations, respectively,

$$\mathbf{g}_i = \frac{\partial \mathbf{p}}{\partial \theta^i} = \mathbf{p},_i, \quad \bar{\mathbf{g}}_i = \bar{\mathbf{p}},_i = \mathbf{g}_i + \mathbf{u},_i, \tag{2.2}$$

$$\begin{aligned} \mathbf{1} &= g_{ij}\mathbf{g}^i \otimes \mathbf{g}^j, \quad g_{ij} = \mathbf{g}_i \cdot \mathbf{g}_j, \quad g = |g_{ij}|, \\ \bar{\mathbf{1}} &= \bar{g}_{ij}\bar{\mathbf{g}}^i \otimes \bar{\mathbf{g}}^j, \quad \bar{g}_{ij} = \bar{\mathbf{g}}_i \cdot \bar{\mathbf{g}}_j, \quad \bar{g} = |\bar{g}_{ij}|. \end{aligned} \tag{2.3}$$

Since

$$\begin{aligned} \mathrm{d}\bar{\mathbf{p}} &= \bar{\mathbf{g}}_i \mathrm{d}\theta^i = \mathbf{F} d\mathbf{p}, \quad \mathrm{d}\mathbf{p} = \mathbf{F}^{-1}\mathrm{d}\bar{\mathbf{p}}, \\ \mathbf{F} &= \mathbf{1} + \mathbf{u},_i \otimes \mathbf{g}^i = \bar{\mathbf{g}}_i \otimes \mathbf{g}^i, \quad \mathbf{F}^{-1} = \mathbf{g}_j \otimes \bar{\mathbf{g}}^j, \end{aligned} \tag{2.4}$$

the deformation gradient tensor $\mathbf{F}$, $0 < \det \mathbf{F} < \infty$, contains complete information as to the strains and rotations of all the line elements in the neighbourhood of $X \in \mathscr{B}$.

When applying the polar decomposition theorem [1] the tensor $\mathbf{F}$ is represented as

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{2.5}$$

where $\mathbf{U}$ and $\mathbf{V}$ are the right and left stretch tensors, respectively, and $\mathbf{R}$ is the finite rotation tensor. The tensors $\mathbf{U}$ and $\mathbf{V}$ are symmetric and positive definite while $\mathbf{R}$ is the proper orthogonal tensor. They are defined by

$$\begin{aligned} \mathbf{U} &= (\mathbf{F}^T\boldsymbol{F})^{1/2}, \quad \mathbf{V} = (\mathbf{F}\mathbf{F}^T)^{1/2} = \mathbf{R}\mathbf{U}\mathbf{R}^T, \\ \mathbf{R} &= \mathbf{F}\mathbf{U}^{-1} = \mathbf{V}^{-1}\mathbf{F}, \quad \mathbf{R}^{-1} = \mathbf{R}^T, \quad \det \mathbf{R} = +1. \end{aligned} \tag{2.6}$$

From (2.4) and (2.5) it follows that

$$\begin{aligned} \bar{\mathbf{g}}_i &= \mathbf{F}\mathbf{g}_i = \mathbf{R}\mathbf{s}_i = \mathbf{V}\mathbf{r}_i, \quad \bar{\mathbf{g}}^i = (\mathbf{F}^{-1})^T\mathbf{g}^i = \mathbf{R}\mathbf{s}^i = \mathbf{V}^{-1}\mathbf{r}^i, \\ \mathbf{s}_i &= \mathbf{U}\mathbf{g}_i = \mathbf{R}^T\bar{\mathbf{g}}_i, \quad \mathbf{r}_i = \mathbf{R}\mathbf{g}_i = \mathbf{V}^{-1}\bar{\mathbf{g}}_i, \end{aligned} \tag{2.7}$$

where two additional intermediate non-holonomic bases are introduced. The stretched basis $\mathbf{s}_i$ is constructed from $\mathbf{g}_i$ by its pure stretch along principal directions of $\mathbf{U}$, while the rotated basis $\mathbf{r}_i$ is obtained from $\mathbf{g}_i$ by its finite rotation with the help of $\mathbf{R}$. The definitions of the bases $\mathbf{s}_i$ and $\mathbf{r}_i$ are closely connected with the convected system of coordinates used here, since only in convected coordinates the bases are defined uniquely either in the Lagrangian description (preferred in this paper) or in the Eulerian description. If two independent systems of curvilinear coordinates were used in the reference and deformed body configurations [1, 3], the stretched basis, defined by a pure stretch of the reference basis, would differ from the basis obtained by the inverse finite rotation of the deformed basis. Likewise for the rotated basis. As a result, for such general choice of the coordinate systems it would be necessary to deal with four intermediate non-holonomic bases and the whole description of deformation in terms of the bases would have become more complex.

In terms of the bases $\mathbf{s}_i$ and $\mathbf{r}_i$ we obtain simple absolute representations of various tensors

$$\begin{aligned} \mathbf{U} &= \mathbf{s}_i \otimes \mathbf{g}^i, \quad \mathbf{U}^{-1} = \mathbf{g}_i \otimes \mathbf{s}^i, \\ \mathbf{V} &= \bar{\mathbf{g}}_i \otimes \mathbf{r}^i, \quad \mathbf{V}^{-1} = \mathbf{r}_i \otimes \bar{\mathbf{g}}^i, \\ \mathbf{R} &= \bar{\mathbf{g}}_i \otimes \mathbf{s}^i = \mathbf{r}_i \otimes \mathbf{g}^i. \end{aligned} \tag{2.8}$$

Note that $\mathbf{s}_i$ and $\mathbf{r}_i$ are not the natural bases of the convected coordinates. They are connected with the reference and deformed metrics as follows

$$\mathbf{s}_i \cdot \mathbf{s}_j = \bar{g}_{ij}, \quad \mathbf{r}_i \cdot \mathbf{r}_j = g_{ij}. \tag{2.9}$$

For the description of strains it is convenient to introduce the Lagrangian strain tensors $\mathbf{E}$ and $\boldsymbol{\eta}$, both coaxial with $\mathbf{U}$, defined by

$$\begin{aligned}
\mathbf{E} &= \frac{1}{2}(\mathbf{F}^T\mathbf{F} - \mathbf{1}) = \frac{1}{2}(\mathbf{U}^2 - \mathbf{1}) = E_{ij}\mathbf{g}^i \otimes \mathbf{g}^j,\\
\boldsymbol{\eta} &= \mathbf{U} - \mathbf{1} = (\mathbf{1} + 2\mathbf{E})^{1/2} - \mathbf{1} = \eta_{ij}\mathbf{g}^i \otimes \mathbf{g}^j,\\
E_{ij} &= \frac{1}{2}(\bar{g}_{ij} - g_{ij}) = \frac{1}{2}(U^m_i U_{mj} - g_{ij}) = \frac{1}{2}(u_{i|j} + u_{j|i} + g^{mn}u_{m|i}u_{n|j}).
\end{aligned} \tag{2.10}$$

Using (2.10) and (2.7) for various quantities we obtain[15] the following transformation relations

$$\begin{aligned}
\bar{g}^{il} &= \frac{1}{2}\frac{g}{\bar{g}}\epsilon^{ijk}\epsilon^{lmn}(g_{jm} + 2E_{jm})(g_{kn} + 2E_{kn}),\\
\frac{\bar{g}}{g} &= \frac{1}{6}\epsilon^{ijk}\epsilon^{lmn}(g_{il} + 2E_{il})(g_{jm} + 2E_{jm})(g_{kn} + 2E_{kn}),
\end{aligned} \tag{2.11}$$

$$\begin{aligned}
&\bar{\epsilon}_{ijk} = \sqrt{\left(\frac{\bar{g}}{g}\right)}\epsilon_{ijk} = \epsilon_{lmn}U^l_i U^m_j U^n_k, \quad \bar{\epsilon}^{ijk} = \sqrt{\left(\frac{g}{\bar{g}}\right)}\epsilon^{ijk} = \frac{g}{\bar{g}}\epsilon^{lmn}U^i_l U^j_m U^k_n,\\
&\sqrt{\left(\frac{\bar{g}}{g}\right)} = \frac{1}{6}\epsilon^{ijk}\epsilon_{lmn}U^l_i U^m_j U^n_k, \quad \bar{\epsilon}^{lmn}\epsilon_{lmn} = \frac{g}{\bar{g}}\epsilon^{ijk}\bar{\epsilon}_{ijk},\\
&\bar{g}_{ij} = U^k_i U^l_j g_{kl}, \quad U^k_i = \delta^k_i + \eta^k_i, \quad \bar{g}^{il} = \frac{1}{2}\frac{g}{\bar{g}}\epsilon^{ijk}\epsilon^{lmn}U^p_j U^r_m U^s_k U^t_n g_{pr} g_{st},
\end{aligned} \tag{2.12}$$

$$\begin{aligned}
&\mathbf{g}^l = U^l_i \mathbf{s}^i, \quad \mathbf{s}_i = U^m_i \mathbf{g}_m,\\
&\mathbf{g}_l = \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\epsilon^{ijk}\epsilon_{lmn}U^m_j U^n_k \mathbf{s}_i, \quad \mathbf{s}^i = \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\epsilon^{ijk}\epsilon_{lmn}U^m_j U^n_k \mathbf{g}^l = (U^{-1})^i_l \mathbf{g}^l,
\end{aligned} \tag{2.13}$$

$$\begin{aligned}
&\bar{\mathbf{g}}_i = \mathbf{R}\mathbf{U}\mathbf{R}^T\mathbf{r}_i = U^m_i \mathbf{r}_m, \quad \mathbf{r}^l = U^l_i \bar{\mathbf{g}}^i,\\
&\bar{\mathbf{g}}^i = \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\epsilon^{ijk}\epsilon_{lmn}U^m_j U^n_k \mathbf{r}^l, \quad \mathbf{r}_l = \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\epsilon^{ijk}\epsilon_{lmn}U^m_j U^n_k \bar{\mathbf{g}}_i.
\end{aligned} \tag{2.14}$$

Note that $\mathbf{E}$ is exactly quadratic in terms of $\mathbf{u}$, but $(\bar{g}/g)^{1/2}$ is non-rational with respect to E_{ij}. On the other hand, $\mathbf{U}$ (or $\boldsymbol{\eta}$) is defined to depend upon $\mathbf{u}$ through the non-rational relation $(2.10)_2$, but $(\bar{g}/g)^{1/2}$ is the third-degree polynomial $(2.12)_1$ with respect to U_{ij} (or η_{ij}).

Let $\Gamma_{k\,ij}$, Γ^k_{ij} and $\bar{\Gamma}_{k\,ij}$, $\bar{\Gamma}^k_{ij}$ be Christoffel symbols of the first and second kind and let us denote by $(\ )_{|i}$ and $(\ )_{\|i}$ the covariant derivatives in the reference and deformed configurations, respectively. Then we have known relations:

$$\begin{aligned}
&\mathbf{g}_{i,j} = \Gamma^k_{ij}\mathbf{g}_k, \quad \mathbf{g}^k_{,j} = -\Gamma^k_{ij}\mathbf{g}^i, \quad \mathbf{g}_{i|j} = \mathbf{g}^i_{|j} = \mathbf{0},\\
&\bar{\mathbf{g}}_{i,j} = \bar{\Gamma}^k_{ij}\bar{\mathbf{g}}_k, \quad \bar{\mathbf{g}}^k_{,j} = -\bar{\Gamma}^k_{ij}\bar{\mathbf{g}}^i, \quad \bar{\mathbf{g}}_{i\|j} = \bar{\mathbf{g}}^i_{\|j} = \mathbf{0},
\end{aligned} \tag{2.15}$$

$$\Gamma_{k.ij} = \frac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}), \quad \Gamma^k_{ij} = g^{kl}\Gamma_{l.ij},$$

$$\bar{\Gamma}_{k.ij} = \bar{g}_{kl}\Gamma^l_{ij} + E_{kij}, \quad A^k_{ij} = \bar{\Gamma}^k_{ij} - \Gamma^k_{ij} = \bar{g}^{kl}E_{lij}, \tag{2.16}$$

$$E_{kij} = E_{ki|j} + E_{kj|i} - E_{ij|k} = E_{ki,j} + E_{kj,i} - E_{ij,k} - 2\Gamma^l_{ij}E_{kl}.$$

For other geometric relations in convected coordinates we refer to [15, 25, 26].

3. FINITE ROTATIONS

Any proper orthogonal tensor $\mathbf{R}$ has one real eigenvalue equal to $+1$ and two complex conjugate eigenvalues equal to $\cos\omega \pm i\sin\omega$. Let $\mathbf{e}$ be a unit vector satisfying $\mathbf{Re} = +\mathbf{e}$. If $\mathbf{e}_1 \perp \mathbf{e}$ and $\mathbf{e}_2 = \mathbf{e} \times \mathbf{e}_1$ are unit vectors of the remaining principal directions then $\mathbf{R}$ can be put in the form [21, 27–29]

$$\begin{aligned}\mathbf{R} &= \cos\omega\,(\mathbf{e}_1 \otimes \mathbf{e}_1 + \mathbf{e}_2 \otimes \mathbf{e}_2) - \sin\omega\,(\mathbf{e}_1 \otimes \mathbf{e}_2 - \mathbf{e}_2 \otimes \mathbf{e}_1) + \mathbf{e} \otimes \mathbf{e} \\ &= \cos\omega\,\mathbf{1} + \sin\omega\,\mathbf{e} \times \mathbf{1} + (1 - \cos\omega)\mathbf{e} \otimes \mathbf{e}.\end{aligned} \tag{3.1}$$

The direction defined by $\mathbf{e}$ is the axis of rotation of $\mathbf{R}$ and ω is the angle of rotation of $\mathbf{R}$ about the axis of rotation.

The action of $\mathbf{R}$ on a vector $\mathbf{w}$ may be interpreted by decomposing $\mathbf{w}$ into $\mathbf{w}_r$ directed along the rotation axis and $\mathbf{w}_p$ orthogonal to $\mathbf{e}$. If α is the angle between $\mathbf{w}_p$ and $\mathbf{e}_1$, while $w_p = |\mathbf{w}_p|$, then

$$\mathbf{w} = w_p\,(\cos\alpha\,\mathbf{e}_1 + \sin\alpha\,\mathbf{e}_2) + \mathbf{w}_r. \tag{3.2}$$

Now, the action of $\mathbf{R}$ on $\mathbf{w}$ leads to a new vector $\mathbf{w}^*$ which, in the same basis, takes according to (3.1) and (3.2) the form

$$\mathbf{w}^* = \mathbf{Rw} = w_p\,[\cos(\alpha + \omega)\,\mathbf{e}_1 + \sin(\alpha + \omega)\,\mathbf{e}_2] + \mathbf{w}_r. \tag{3.3}$$

It is evident that $\mathbf{R}$ rotates $\mathbf{w}$ through the angle ω about the axis of rotation defined by $\mathbf{e}$.

Sometimes it is more convenient to describe rotations by means of an equivalent finite rotation vector. The direction of the vector is defined by $\mathbf{e}$ and its length is taken as it is required. For example, Shamina[7] and Pietraszkiewicz[15] used the vector $\mathbf{\Omega} = \sin\omega\mathbf{e}$, Lur'e[20] used the vector $\boldsymbol{\theta} = 2\mathrm{tg}\omega/2\mathbf{e}$ while Chernykh[22] preferred to use the vector $\boldsymbol{\omega} = \omega\mathbf{e}$. These finite rotation vectors are connected by the relations

$$\mathbf{e} = \frac{1}{\sin\omega}\,\mathbf{\Omega} = \frac{1}{2\mathrm{tg}\omega/2}\,\boldsymbol{\theta} = \frac{1}{\omega}\,\boldsymbol{\omega}. \tag{3.4}$$

Each of the definitions has some advantages in specific applications: $\mathbf{\Omega}$ is particularly convenient to be expressed in terms of displacements, $\boldsymbol{\theta}$ leads to some relations which do not contain trigonometric functions, while $\boldsymbol{\omega}$ may be defined by logarithmic function of $\mathbf{R}$. In case of infinitesimal rotations all three definitions are reduced to the infinitesimal rotation vector. Gibb's definition of the finite rotation vector $\mathbf{g} = \mathrm{tg}\omega/2\mathbf{e}$ is also frequently used in algebra [31, 32], but it cannot be reduced to the infinitesimal rotation vector as it is defined in classical linear elasticity.

The formulae analogous to (3.3) for the action of each of the finite rotation vectors on a vector $\mathbf{w}$ are[15, 22, 26]:

$$\begin{aligned}\mathbf{w}^* &= \mathbf{w} + \boldsymbol{\Omega} \times \mathbf{w} + \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{w}) \\ &= \mathbf{w} + \frac{1}{1+\boldsymbol{\theta}^2/4}\boldsymbol{\theta} \times \left(\mathbf{w} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{w}\right) \\ &= \mathbf{w} + \frac{\sin\omega}{\omega}\boldsymbol{\omega} \times \mathbf{w} + \frac{1-\cos\omega}{\omega^2}\boldsymbol{\omega} \times (\boldsymbol{\omega} \times \mathbf{w}).\end{aligned} \tag{3.5}$$

Let us introduce skew-symmetric tensors $\mathbf{M}$, $\mathbf{T}$ and $\mathbf{W}$ whose axial vectors are $\boldsymbol{\Omega}$, $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$, respectively:

$$\begin{aligned}&\mathbf{M} = \boldsymbol{\Omega} \times \mathbf{1}, \quad \mathbf{T} = \boldsymbol{\theta} \times \mathbf{1}, \\ &\mathbf{W} = \boldsymbol{\omega} \times \mathbf{1} = \omega(\mathbf{e}_2 \otimes \mathbf{e}_1 - \mathbf{e}_1 \otimes \mathbf{e}_2).\end{aligned} \tag{3.6}$$

Then using (3.1) and (3.5) we obtain equivalent representations for the finite rotation tensor

$$\begin{aligned}\mathbf{R} &= \mathbf{1} + \mathbf{M} + \frac{1}{2\cos^2\omega/2}\mathbf{M}^2 = \mathbf{1} + \frac{1}{1+\boldsymbol{\theta}^2/4}\left(\mathbf{T} + \frac{1}{2}\mathbf{T}^2\right) \\ &= \mathbf{1} + \frac{\sin\omega}{\omega}\mathbf{W} + \frac{1-\cos\omega}{\omega^2}\mathbf{W}^2.\end{aligned} \tag{3.7}$$

The tensor $\mathbf{R}$ can also be represented[27, 29] by an exponential function

$$\mathbf{R} = \exp\mathbf{W} = \exp(\boldsymbol{\omega} \times \mathbf{1}), \tag{3.8}$$

which has an inversion

$$\begin{aligned}\mathbf{W} &= \ln\mathbf{R} = -\frac{\omega}{2\sin\omega}[(1+2\cos\omega)\mathbf{1} - 2(1+\cos\omega)\mathbf{R} + \mathbf{R}^2], \\ \boldsymbol{\omega} &= \frac{1}{2}\mathbf{g}_i \times \mathbf{W}\mathbf{g}^i = -\frac{1}{2}\boldsymbol{\epsilon} \cdot \mathbf{W} = -\frac{1}{2}\epsilon^{ijk}W_{jk}\mathbf{g}_i,\end{aligned} \tag{3.9}$$

where $\boldsymbol{\epsilon} = -\mathbf{1} \times \mathbf{1} = \epsilon^{ijk}\mathbf{g}_i \otimes \mathbf{g}_j \otimes \mathbf{g}_k$ is a skew-symmetric Ricci tensor of the third order. Since

$$\mathbf{R}^2 = \exp(2\boldsymbol{\omega} \times \mathbf{1}) = \cos 2\omega\mathbf{1} + \sin 2\omega\mathbf{e} \times \mathbf{1} + (1-\cos 2\omega)\mathbf{e} \otimes \mathbf{e} \tag{3.10}$$

for $\boldsymbol{\Omega}$ and $\boldsymbol{\theta}$ we obtain the following relations

$$\begin{aligned}\boldsymbol{\Omega} &= \frac{1}{4}\boldsymbol{\epsilon} \cdot [(1+2\cos\omega)\mathbf{1} - 2(1+\cos\omega)\mathbf{R} + \mathbf{R}^2] \\ &= \frac{1}{4}\boldsymbol{\epsilon} \cdot [(\operatorname{tr}\mathbf{R})\mathbf{1} - (1+\operatorname{tr}\mathbf{R})\mathbf{R} + \mathbf{R}^2], \\ \boldsymbol{\theta} &= \frac{1}{4}\boldsymbol{\epsilon} \cdot \left[\left(3 - \frac{1}{4}\boldsymbol{\theta}^2\right)\mathbf{1} - 4\mathbf{R} + \left(1 + \frac{1}{4}\boldsymbol{\theta}^2\right)\mathbf{R}^2\right].\end{aligned} \tag{3.11}$$

If $\mathbf{R}_1$ and $\mathbf{R}_2$ are proper orthogonal tensors describing two subsequent finite rotations then the tensor $\mathbf{R} = \mathbf{R}_2\mathbf{R}_1$ describes the total finite rotation. In terms of the finite rotation vectors the equivalent composition rules are [15, 20]

$$\mathbf{\Omega} = \left(1 - \frac{\mathbf{\Omega}_1 \cdot \mathbf{\Omega}_2}{4\cos^2\omega_1/2\cos^2\omega_2/2}\right)\left(\cos^2\omega_2/2\mathbf{\Omega}_1 + \cos^2\omega_1/2\mathbf{\Omega}_2 - \frac{1}{2}\mathbf{\Omega}_1 \times \mathbf{\Omega}_2\right)$$
$$\boldsymbol{\theta} = \frac{1}{1 - \frac{1}{4}\boldsymbol{\theta}_1 \cdot \boldsymbol{\theta}_2}\left(\boldsymbol{\theta}_1 + \boldsymbol{\theta}_2 - \frac{1}{2}\boldsymbol{\theta}_1 \times \boldsymbol{\theta}_2\right). \tag{3.12}$$

The composition rule for $\boldsymbol{\omega}$, as more complex, is not presented here.

For known displacements $\mathbf{u}$ the tensor $\mathbf{R}$ can be calculated according to (2.8), $(2.2)_2$ and $(2.13)_2$ to be

$$\mathbf{R} = \bar{g}^{ij}(\delta_j^m + \eta_j^m)(\mathbf{g}_i + \mathbf{u}_{,i}) \otimes \mathbf{g}_m$$
$$= \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\,\epsilon^{ijk}\epsilon_{lmn}(\delta_j^m + \eta_j^m)(\delta_k^n + \eta_k^n)(\mathbf{g}_i + \mathbf{u}_{,i}) \otimes \mathbf{g}^l. \tag{3.13}$$

In order to obtain an analogous relation for $\mathbf{\Omega}$ let us note that according to (2.7) and $(3.5)_1$

$$\bar{\mathbf{g}}_i = \mathbf{s}_i + \mathbf{\Omega} \times \mathbf{s}_i + \frac{1}{2\cos^2\omega/2}\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{s}_i),$$
$$\mathbf{r}_i = \mathbf{g}_i + \mathbf{\Omega} \times \mathbf{g}_i + \frac{1}{2\cos^2\omega/2}\mathbf{\Omega} \times (\mathbf{\Omega} \times \mathbf{g}_i). \tag{3.14}$$

This leads to simple relations for the finite rotation vector

$$\mathbf{\Omega} = \frac{1}{2}\mathbf{s}^i \times \bar{\mathbf{g}}_i = \frac{1}{2}\mathbf{g}_i \times \mathbf{r}^i. \tag{3.15}$$

Introducing $\bar{\mathbf{g}}_i = \mathbf{g}_i + \mathbf{u}_{,i}$ into $(3.15)_1$ we obtain

$$\mathbf{\Omega} = \frac{1}{2}\bar{\epsilon}^{ijk}(\mathbf{u}_{,i} \cdot \mathbf{s}_j)\mathbf{s}_k \tag{3.16}$$

which was given in [7, 15]. An equivalent new representation for $\mathbf{\Omega}$ follows from $(3.15)_2$ if $\mathbf{g}_i = \bar{\mathbf{g}}_i - \mathbf{u}_{,i}$ and $(2.13)_1$ is used, which leads to

$$\mathbf{\Omega} = \frac{1}{2}\epsilon^{ijk}(\mathbf{u}_{,i} \cdot \mathbf{r}_j)\mathbf{r}_k. \tag{3.17}$$

The tensor $\mathbf{R}$ and the vector $\mathbf{\Omega}$ are non-rational square-root functions of the displacements. According to (3.4), analogous expressions for $\boldsymbol{\theta} = \boldsymbol{\theta}(\mathbf{u})$ and $\boldsymbol{\omega} = \boldsymbol{\omega}(\mathbf{u})$ involve also some trigonometric functions of ω.

4. DIFFERENTIATION OF FINITE ROTATIONS

Let us differentiate the deformation gradient tensor (2.4) with respect to convected coordinates, which leads to

$$\mathbf{F}_{,j} = \mathbf{F}\mathbf{A}_j, \quad \mathbf{A}_j = A^k_{mj}\mathbf{g}_k \otimes \mathbf{g}^m. \tag{4.1}$$

Introducing the first decomposition form of (2.5) into $(4.1)_1$ and making use of $(2.8)_1$, $(4.1)_2$, (2.15) and of (2.16) we obtain

$$\mathbf{R}_{,j} = \mathbf{R}\mathbf{K}_j, \quad \mathbf{K}_j = (\mathbf{U}\mathbf{A}_j - \mathbf{U}_{,j})\mathbf{U}^{-1}, \tag{4.2}$$

$$\mathbf{K}_j = (\bar{\Gamma}^k_{ij}\mathbf{s}_k - \mathbf{s}_{i,j}) \otimes \mathbf{s}^i = K_{klj}\mathbf{g}^k \otimes \mathbf{g}^l = L_{klj}\mathbf{s}^k \otimes \mathbf{s}^l, \tag{4.3}$$

$$K_{klj} = (U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_l,$$
$$L_{klj} = E_{klj} - g^{mn}U_{mk}U_{nl|j} = U^m_k U^n_l K_{mnj}. \tag{4.4}$$

It follows from the condition $(\mathbf{R}^T\mathbf{R})_{,j} = \mathbf{0}$ that the tensors $\mathbf{K}_j$ are skew-symmetric, $\mathbf{K}_j^T = -\mathbf{K}_j$. Therefore, there exist axial vectors $\mathbf{k}_j$ such that

$$\mathbf{K}_j = \mathbf{k}_j \times \mathbf{1}, \tag{4.5}$$

$$\mathbf{k}_j = \frac{1}{2}\mathbf{g}^i \times \mathbf{K}_j\mathbf{g}_i = \frac{1}{2}\epsilon^{ikl}(U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_i\mathbf{g}_l$$
$$= \frac{1}{2}\mathbf{s}^i \times \mathbf{K}_j\mathbf{s}_i = \bar{\epsilon}^{ikl}\left(E_{kj|i} - \frac{1}{2}g^{mn}U_{mk}U_{ni|j}\right)\mathbf{s}_i \tag{4.6}$$

$\mathbf{K}_j$ and $\mathbf{k}_j$ so defined may be called[7] the tensors and vectors of change of curvature of the curvilinear convected coordinate lines, respectively. From (4.3), (4.4), (4.6), (2.13) and $(2.14)_3$ it is evident that the tensors $\mathbf{K}_j$ and the vectors $\mathbf{k}_j$ are described entirely by strains and stretches.

It should be noted that $\mathbf{R}$ describes completely the finite rotations of only those material lines that coincide with the principal directions of strain. Any other material line may suffer an additional finite rotation, caused by the pure stretch along the principal directions. As a result, $\mathbf{K}_j$ or $\mathbf{k}_j$ describe, in general, only those parts of changes of curvatures of the coordinate lines that are caused by variations of $\mathbf{R}$ along the lines. When discussing the deformation of a specific material line it may be of some importance to replace the two finite rotations, mentioned above, by one equivalent total rotation. Then a skew-symmetric tensor, appropriately defined through strains and related to the total rotation tensor by the formula analogous to $(4.2)_2$, would describe the total change of curvature of the material line during deformation. In particular, such an approach was used in the non-linear theory of shells[15–19] in order to describe properly the deformation of the shell boundary line.

Using the identities

$$\mathbf{s}^k \cdot \mathbf{s}_{i,j} = -\mathbf{s}^k_{,j} \cdot \mathbf{s}_i, \quad \mathbf{s}^k(\mathbf{k}_j \times \mathbf{1}) = -\mathbf{k}_j \times \mathbf{s}^k \tag{4.7}$$

from $(4.3)_1$ and (4.5) we obtain the formulae for derivatives of the intermediate stretched base vectors

$$\mathbf{s}_{i,j} = \bar{\Gamma}^k_{ij}\mathbf{s}_k - \mathbf{k}_j \times \mathbf{s}_i, \quad \mathbf{s}^k_{,j} = -\bar{\Gamma}^k_{ij}\mathbf{s}^i - \mathbf{k}_j \times \mathbf{s}^k,$$
$$\mathbf{s}_{i|j} = A^k_{ij}\mathbf{s}_k - \mathbf{k}_j \times \mathbf{s}_i, \quad \mathbf{s}^k_{|j} = -A^k_{ij}\mathbf{s}^i - \mathbf{k}_j \times \mathbf{s}^k, \tag{4.8}$$
$$\mathbf{s}_{i\|j} = -\mathbf{k}_j \times \mathbf{s}_i, \quad \mathbf{s}^k_{\|j} = -\mathbf{k}_j \times \mathbf{s}^k.$$

The derivative of $\mathbf{F}$, with respect to convected coordinates, may also be presented in an alternative form

$$\mathbf{F}_{,j} = \mathbf{B}_j\mathbf{F}, \quad \mathbf{B}_j = A^k_{mj}\bar{\mathbf{g}}_k \otimes \bar{\mathbf{g}}^m = \mathbf{F}\mathbf{A}_j\mathbf{F}^{-1} \tag{4.9}$$

which is equivalent to (4.1). Introducing the first decomposition form of (2.5) into $(4.9)_1$ we obtain

$$\mathbf{R}_{,j} = \mathbf{L}_j\mathbf{R}, \quad \mathbf{L}_j = \mathbf{B}_j - \mathbf{R}\mathbf{U}_{,j}\mathbf{U}^{-1}\mathbf{R}^T, \tag{4.10}$$

$$\mathbf{L}_j = (\bar{\Gamma}^k_{ij} - \mathbf{s}^k \cdot \mathbf{s}_{i,j})\bar{\mathbf{g}}_k \otimes \bar{\mathbf{g}}^i = \mathbf{R}\mathbf{K}_j\mathbf{R}^T = K_{klj}\mathbf{r}^k \otimes \mathbf{r}^l = L_{klj}\bar{\mathbf{g}}^k \otimes \bar{\mathbf{g}}^l. \tag{4.11}$$

The tensors $\mathbf{L}_j$ of change of curvature are also skew-symmetric, $\mathbf{L}_j^T = -\mathbf{L}_j$ and have axial vectors $\mathbf{l}_j$ of change of curvature such that

$$\mathbf{L}_j = \mathbf{l}_j \times \mathbf{1}, \quad \mathbf{l}_j = \mathbf{R}\mathbf{k}_j, \tag{4.12}$$

$$\begin{aligned} \mathbf{l}_j &= \frac{1}{2}\mathbf{r}^i \times \mathbf{L}_j\mathbf{r}_i = \frac{1}{2}\epsilon^{ikl}(U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_l\mathbf{r}_l \\ &= \frac{1}{2}\bar{\mathbf{g}}^i \times \mathbf{L}_j\bar{\mathbf{g}}_i = \bar{\epsilon}^{ikl}\left(E_{kj\|i} - \frac{1}{2}g^{mn}U_{mk}U_{ni|j}\right)\bar{\mathbf{g}}_l. \end{aligned} \tag{4.13}$$

Note that the so defined tensors $\mathbf{K}_j$ and $\mathbf{L}_j$ and vectors $\mathbf{k}_j$ and $\mathbf{l}_j$ have identical components with respect to different bases $\mathbf{g}_i$ or $\mathbf{s}_i$ and $\mathbf{r}_i$ or $\bar{\mathbf{g}}_i$, respectively, which are related by a rigid-body finite rotation.

The relations $(4.3)_1$, $(4.6)_2$ and (4.8) have been expressed entirely in terms of the intermediate stretched basis $\mathbf{s}_i$ that describes the metric of deformed body configuration. In some applications it may be more convenient to make use of the intermediate rotated basis $\mathbf{r}_i$, since it describes the metric of the reference body configuration. For example, in some formulations of the non-linear theory of shells[30, 33, 34] the vectors $\mathbf{r}_i$ were used as the reference basis. In order to derive equivalent relations entirely in terms of $\mathbf{r}_i$ let us introduce the second form of the polar decomposition (2.5) into $(4.1)_1$ and $(4.9)_1$, which leads to the following alternative definitions of the tensors and vectors of change of curvature

$$\mathbf{K}_j = \mathbf{A}_j - \mathbf{R}^T\mathbf{V}^{-1}\mathbf{V}_{,j}\mathbf{R} = -(\Gamma^k_{ij} + \mathbf{r}_i \cdot \mathbf{r}^k_{,j})\mathbf{g}_k \otimes \mathbf{g}^i, \tag{4.14}$$

$$\mathbf{k}_j = -\frac{1}{2}\epsilon^{ikl}(g_{km}\Gamma^m_{ij} - \mathbf{r}_k \cdot \mathbf{r}_{i,j})\mathbf{g}_l, \tag{4.15}$$

$$\mathbf{L}_j = \mathbf{V}^{-1}(\mathbf{B}_j\mathbf{V} - \mathbf{V}_{,j}) = -\mathbf{r}_k \otimes (\Gamma^k_{ij}\mathbf{r}^i + \mathbf{r}^k_{,j}) = \mathbf{R}\mathbf{K}_j\mathbf{R}^T, \tag{4.16}$$

$$\mathbf{l}_j = -\frac{1}{2}\epsilon^{ikl}(g_{km}\Gamma^m_{ij} - \mathbf{r}_k \cdot \mathbf{r}_{i,j})\mathbf{r}_l = \mathbf{R}\mathbf{k}_j. \tag{4.17}$$

Note that again tensors $\mathbf{K}_j$ and $\mathbf{L}_j$ and vectors $\mathbf{k}_j$ and $\mathbf{l}_j$ defined in (4.14)–(4.17) have identical components but with reference to different bases which are related by a rigid-body finite rotation. The formulae (4.16) and (4.17) seem to be of particular interest, since they are entirely expressed in terms of $\mathbf{r}_i$. From them follow the rules of covariant differentiation of the intermediate rotated base vectors

$$\begin{aligned} &\mathbf{r}_{i,j} = \Gamma^k_{ij}\mathbf{r}_k + \mathbf{l}_j \times \mathbf{r}_i, \quad \mathbf{r}^k_{,j} = -\Gamma^k_{ij}\mathbf{r}^i + \mathbf{l}_j \times \mathbf{r}^k, \\ &\mathbf{r}_{i|j} = \mathbf{l}_j \times \mathbf{r}_i, \quad \mathbf{r}^k_{|j} = \mathbf{l}_j \times \mathbf{r}^k, \\ &\mathbf{r}_{i\|j} = -A^k_{ij}\mathbf{r}_k + \mathbf{l}_j \times \mathbf{r}_i, \quad \mathbf{r}^k_{\|j} = A^k_{ij}\mathbf{r}^i + \mathbf{l}_j \times \mathbf{r}^k. \end{aligned} \tag{4.18}$$

Using (4.13), $(2.14)_1$ and (2.12) for the vectors $\mathbf{l}_j$ we obtain the following equivalent representation

$$\mathbf{l}_j = \sqrt{\left(\frac{g}{\bar{g}}\right)}\,\epsilon^{ikl}\left(E_{kj\|i} - \frac{1}{2}g^{mn}\eta_{km}\eta_{in|j}\right)(\delta^p_l + \eta^p_l)\mathbf{r}_p\,. \tag{4.19}$$

It is evident from (4.19), $(2.10)_3$ and (2.12) that differentiation rules of the rotated base vectors $\mathbf{r}_i$ are entirely described by the reference metric and the strains η_{ij}, so they do not depend explicitly upon the finite rotations.

The formulae $(4.2)_1$ and $(4.10)_1$ connect the derivatives of $\mathbf{R}$ with tensors $\mathbf{K}_j$ and $\mathbf{L}_j$ or with vectors $\mathbf{k}_j$ and $\mathbf{l}_j$, which are entirely expressed in terms of strains. Using representations (3.1), (3.4) and some algebraic and trigonometric identities it is possible to transform $(4.2)_1$ and $(4.10)_1$

into appropriate equivalent vector formulae, which connect the derivatives of the finite rotation vectors $\mathbf{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$ with the vectors of change of curvature $\mathbf{k}_j$ or $\mathbf{l}_j$.

From (3.1) it follows that along the θ^j coordinate line

$$\mathbf{R},_j = \omega,_j[\sin\omega\,(\mathbf{e}\otimes\mathbf{e}-\mathbf{1}) + \cos\omega\,\mathbf{e}\times\mathbf{1}] + \sin\omega\,\mathbf{e},_j\times\mathbf{1}$$
$$+(1-\cos\omega)(\mathbf{e},_j\otimes\mathbf{e}+\mathbf{e}\otimes\mathbf{e},_j), \tag{4.20}$$
$$\mathbf{R}^T = \cos\omega\,\mathbf{1} - \sin\omega\,\mathbf{e}\times\mathbf{1} + (1-\cos\omega)\mathbf{e}\otimes\mathbf{e}.$$

Since $(\mathbf{e}\cdot\mathbf{e}),_j = 0$ then $\mathbf{e},_j = a_j\mathbf{e}_1 + b_j\mathbf{e}_2$ and, after elementary but involved transformations, from (4.20) and $(4.2)_1$ we obtain

$$\mathbf{K}_j = \mathbf{R}^T\mathbf{R},_j = [\sin\omega\,\mathbf{e},_j + (1-\cos\omega)\,\mathbf{e},_j\times\mathbf{e} + \omega,_j\mathbf{e}]\times\mathbf{1}. \tag{4.21}$$

Now from (4.5) and (4.21) follows formula for the vector of change of curvature of θ^j-line

$$\mathbf{k}_j = \sin\omega\,\mathbf{e},_j + (1-\cos\omega)\,\mathbf{e},_j\times\mathbf{e} + \omega,_j\mathbf{e}. \tag{4.22}$$

Solving (4.22) with respect to $\mathbf{e},_j$ we obtain

$$\mathbf{e},_j = \frac{1}{2}\mathbf{e}\times\mathbf{k}_j - \frac{1}{2\mathrm{tg}\,\omega/2}\mathbf{e}\times(\mathbf{e}\times\mathbf{k}_j) = \frac{1}{2\mathrm{tg}\,\omega/2}(\mathbf{k}_j - \omega,_j\mathbf{e}) + \frac{1}{2}\mathbf{e}\times\mathbf{k}_j. \tag{4.23}$$

Introducing (3.4) into (4.23) and (4.22) it is possible now to derive the formulae for derivatives of $\mathbf{\Omega}$, $\boldsymbol{\theta}$ and $\boldsymbol{\omega}$ and for various representations of $\mathbf{k}_j$. The appropriate transformations are elementary but involved and are omitted here. In what follows we present only the final results.

Choosing the vectors $\mathbf{\Omega}$ and $\mathbf{k}_j$ as corresponding variables, we obtain the relations equivalent to those derived in [7, 15]

$$\mathbf{\Omega},_j = \cos\omega\,\mathbf{k}_j + \frac{1}{2}\mathbf{\Omega}\times\mathbf{k}_j - \frac{1}{4\cos^2\omega/2}\mathbf{\Omega}\times(\mathbf{\Omega}\times\mathbf{k}_j)$$
$$= \cos^2\omega/2\,\mathbf{k}_j + \frac{1}{2}\mathbf{\Omega}\times\mathbf{k}_j - \frac{1}{2}\omega,_j\mathrm{tg}\,\omega/2\,\mathbf{\Omega}, \tag{4.24}$$
$$\mathbf{k}_j = \mathbf{\Omega},_j + \frac{1}{2\cos^2\omega/2}\mathbf{\Omega},_j\times\mathbf{\Omega} + \omega,_j\mathrm{tg}\,\omega/2\,\mathbf{\Omega}.$$

Similar relations for $\boldsymbol{\theta}$ and $\mathbf{k}_j$ as corresponding variables follow from (3.4), (4.23) and (4.22) to be

$$\boldsymbol{\theta},_j = \left(1+\frac{1}{4}\theta^2\right)\mathbf{k}_j + \frac{1}{2}\boldsymbol{\theta}\times\mathbf{k}_j + \frac{1}{4}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{k}_j)$$
$$= \mathbf{k}_j + \frac{1}{2}\boldsymbol{\theta}\times\mathbf{k}_j + \frac{1}{4}(\mathbf{k}_j\cdot\boldsymbol{\theta})\boldsymbol{\theta}, \tag{4.25}$$
$$\mathbf{k}_j = \frac{1}{1+\frac{1}{4}\theta^2}\left(\boldsymbol{\theta},_j + \frac{1}{2}\boldsymbol{\theta},_j\times\boldsymbol{\theta}\right).$$

The relations equivalent to $(4.25)_{2,3}$, but expressed in terms of Gibb's vector $\mathbf{g} = (1/2)\boldsymbol{\theta}$, were given in [23, 4, 5].

Using $\boldsymbol{\omega}$ and $\mathbf{k}_j$ as corresponding variables, we obtain

$$
\begin{aligned}
\boldsymbol{\omega},_j &= \frac{1}{\omega}\,\omega,_j\boldsymbol{\omega} + \frac{1}{2}\,\boldsymbol{\omega}\times\mathbf{k}_j - \frac{1}{2\omega\mathrm{tg}\,\omega/2}\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{k}_j)\\
&= \frac{\omega}{2\mathrm{tg}\,\omega/2}\,\mathbf{k}_j + \frac{1}{2}\,\boldsymbol{\omega}\times\mathbf{k}_j + \left(\frac{1}{\omega} - \frac{1}{2\mathrm{tg}\,\omega/2}\right)\omega,_j\boldsymbol{\omega},\\
\mathbf{k}_j &= \frac{\sin\omega}{\omega}\,\boldsymbol{\omega},_j + \frac{1-\cos\omega}{\omega^2}\,\boldsymbol{\omega},_j\times\boldsymbol{\omega} + \left(\frac{1}{\omega} - \frac{\sin\omega}{\omega^2}\right)\omega,_j\boldsymbol{\omega}.
\end{aligned}
\tag{4.26}
$$

Upon the use of (3.4) and of some identities, each of the formulae (4.24)–(4.26) may be transformed into another corresponding one. This proves the correctness of all the formulae.

In exactly the same manner the relations connecting derivatives of $\boldsymbol{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$ with the vectors of change of curvature $\mathbf{l}_j$ may also be derived. From $(4.10)_1$ and (4.20), after elementary but involved transformations, we obtain

$$
\mathbf{L}_j = \mathbf{R},_j\mathbf{R}^T = [\sin\omega\mathbf{e},_j - (1-\cos\omega)\,\mathbf{e},_j\times\mathbf{e} + \omega,_j\mathbf{e}]\times\mathbf{1},
\tag{4.27}
$$

and from $(4.12)_1$ it follows that

$$
\mathbf{l}_j = \sin\omega\mathbf{e},_j - (1-\cos\omega)\,\mathbf{e},_j\times\mathbf{e} + \omega,_j\mathbf{e}.
\tag{4.28}
$$

The relation (4.28) may be solved with respect to $\mathbf{e},_j$, which gives

$$
\mathbf{e},_j = -\frac{1}{2}\mathbf{e}\times\mathbf{l}_j - \frac{1}{2\mathrm{tg}\,\omega/2}\mathbf{e}\times(\mathbf{e}\times\mathbf{l}_j) = \frac{1}{2\mathrm{tg}\,\omega/2}(\mathbf{l}_j - \omega,_j\mathbf{e}) - \frac{1}{2}\mathbf{e}\times\mathbf{l}_j.
\tag{4.29}
$$

Note that (4.28) and (4.29) differ from (4.22) and (4.23) only by the sign of the single vector product. Since all the transformations of (4.28) and (4.29), performed with the help of (3.4), would leave the sign of the single vector product unchanged, the relations connecting $\boldsymbol{\Omega}$, $\boldsymbol{\theta}$, $\boldsymbol{\omega}$ and $\mathbf{l}_j$ may be written at once, in analogy to (4.24)–(4.26), to be

$$
\begin{aligned}
\boldsymbol{\Omega},_j &= \cos\omega\mathbf{l}_j - \frac{1}{2}\boldsymbol{\Omega}\times\mathbf{l}_j - \frac{1}{4\cos^2\omega/2}\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{l}_j)\\
&= \cos^2\omega/2\mathbf{l}_j - \frac{1}{2}\boldsymbol{\Omega}\times\mathbf{l}_j - \frac{1}{2}\,\omega,_j\mathrm{tg}\,\omega/2\boldsymbol{\Omega},\\
\mathbf{l}_j &= \boldsymbol{\Omega},_j - \frac{1}{2\cos^2\omega/2}\boldsymbol{\Omega},_j\times\boldsymbol{\Omega} + \omega,_j\mathrm{tg}\,\omega/2\boldsymbol{\Omega}.
\end{aligned}
\tag{4.30}
$$

$$
\begin{aligned}
\boldsymbol{\theta},_j &= \left(1+\frac{1}{4}\boldsymbol{\theta}^2\right)\mathbf{l}_j - \frac{1}{2}\boldsymbol{\theta}\times\mathbf{l}_j + \frac{1}{4}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{l}_j)\\
&= \mathbf{l}_j - \frac{1}{2}\boldsymbol{\theta}\times\mathbf{l}_j + \frac{1}{4}(\mathbf{l}_j\cdot\boldsymbol{\theta})\boldsymbol{\theta},\\
\mathbf{l}_j &= \frac{1}{1+\frac{1}{4}\boldsymbol{\theta}^2}\left(\boldsymbol{\theta},_j - \frac{1}{2}\boldsymbol{\theta},_j\times\boldsymbol{\theta}\right),
\end{aligned}
\tag{4.31}
$$

$$
\begin{aligned}
\boldsymbol{\omega},_j &= \frac{1}{\omega}\,\omega,_j\boldsymbol{\omega} - \frac{1}{2}\,\boldsymbol{\omega}\times\mathbf{l}_j - \frac{1}{2\omega\mathrm{tg}\,\omega/2}\,\boldsymbol{\omega}\times(\boldsymbol{\omega}\times\mathbf{l}_j)\\
&= \frac{1}{2\omega\mathrm{tg}\,\omega/2}\,\mathbf{l}_j - \frac{1}{2}\,\boldsymbol{\omega}\times\mathbf{l}_j + \left(\frac{1}{\omega} - \frac{1}{2\mathrm{tg}\,\omega/2}\right)\omega,_j\boldsymbol{\omega},\\
\mathbf{l}_j &= \frac{\sin\omega}{\omega}\,\boldsymbol{\omega},_j - \frac{1-\cos\omega}{\omega^2}\,\boldsymbol{\omega},_j\times\boldsymbol{\omega} + \left(\frac{1}{\omega} - \frac{\sin\omega}{\omega^2}\right)\omega,_j\boldsymbol{\omega}.
\end{aligned}
\tag{4.32}
$$

The relations (4.30)–(4.32), which can be expressed entirely with respect to the rotated basis $\mathbf{r}_i$, have not been discussed in the literature on continuum mechanics. The relations may be of particular interest in the non-linear theories of thin bodies such as bars, plates and shells.

5. COMPATIBILITY CONDITIONS

Compatibility conditions in the continuum mechanics are usually understood as the conditions that should be imposed on appropriately smooth and continuous components of the strain field $\mathbf{E}$ which would ensure the existence of a single-valued and continuous displacement field $\mathbf{u}$.

The compatibility conditions should follow as a result of the elimination of three displacements u_i from six second-order non-linear partial differential equations $(2.10)_3$ by their additional differentiations and algebraic operations. Unfortunately, such direct procedure, which is commonly used in the linear theory of elasticity, becomes extremely involved in the general case of finite deformations.

An alternative geometric method takes into account that during deformation the material body remains in the three-dimensional Euclidean space. Therefore, the components R_{klij} and $\bar{R}_{klij}$ of the Riemann–Christoffel curvature tensor in the reference and deformed body configurations, respectively, should identically vanish. Remind that R_{klij} are defined by[1]

$$R_{klij} = g_{kr}(\Gamma^r_{lj,i} - \Gamma^r_{li,j} + \Gamma^r_{mi}\Gamma^m_{lj} - \Gamma^r_{mj}\Gamma^m_{li}),$$
$$R_{klij} = -R_{lkij} = -R_{klji} = R_{ijkl}. \tag{5.1}$$

From analogous definitions of $\bar{R}_{klij}$, after involved transformations, we obtain

$$\bar{R}_{klij} = (g_{kr} + 2E_{kr})R^r_{\cdot lij} + H_{klij}, \tag{5.2}$$

$$H_{klij} = E_{kj|li} - E_{ki|lj} - E_{lj|ki} + E_{li|kj} + \bar{g}^{mn}(E_{mkj}E_{nli} - E_{mki}E_{nlj}),$$
$$H_{klij} = -H_{lkij} = -H_{klji} = H_{ijkl}. \tag{5.3}$$

Since among 81 components of the R–C curvature tensor only six are algebraically independent and non-vanishing, it is sometimes more convenient to use the Ricci curvature tensor

$$P^{rs} = \frac{1}{4}\epsilon^{rkl}\epsilon^{sij}R_{klij}, \quad P^{rs} = P^{sr}. \tag{5.4}$$

Then, from analogous definition of $\bar{P}^{rs}$, with the help of (5.2) and $(2.11)_4$, we obtain

$$\bar{P}^{rs} = \frac{1}{4}\bar{\epsilon}^{rkl}\bar{\epsilon}^{sij}\bar{R}_{klij} = \frac{g}{\bar{g}}[(1 + E^p_p)P^{rs} - E^r_p P^{ps}] + H^{rs}, \tag{5.5}$$

$$H^{rs} = \bar{\epsilon}^{rkl}\bar{\epsilon}^{sij}\left(E_{kj|li} + \frac{1}{2}\bar{g}^{mn}E_{mkj}E_{nli}\right). \tag{5.6}$$

Now it is evident that with $R_{klij} \equiv 0$ the conditions $\bar{R}_{klij} = 0$ and $\bar{P}^{rs} = 0$ reduce to the classical forms of compatibility conditions for finite strains[1, 35, 36]

$$H_{klij} = 0, \quad H^{rs} = 0. \tag{5.7}$$

The geometric method of derivation of (5.7) may lead to some doubts in what sense the geometric conditions are equivalent to the integrability conditions of the system of non-linear eqns $(2.10)_3$.

In our opinion, the notion of compatibility conditions in the continuum mechanics should rather be associated with the integrability conditions of some definite systems, the solutions of which would allow to determine the displacement field from a given strain field only through quadratures and algebraic operations. From this point of view, several different systems of linear first-order partial differential equations may be examined, which lead to several different but equivalent forms of compatibility conditions in the continuum mechanics. Note that for arbitrarily given symbols Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$ each of the relations $(4.1)_1$ or $(4.9)_1$ is equivalent to the system of 27 scalar linear first-order partial differential equations for nine components of $\mathbf{F}$. When $\mathbf{F}$ is calculated the position vector $\bar{\mathbf{p}}$ and, therefore, the displacement field $\mathbf{u}=\bar{\mathbf{p}}-\mathbf{p}$ can be constructed through quadratures from $\bar{\mathbf{p}}_{,i}=\mathbf{F}\mathbf{g}_i$, see (6.13), for example.

In order to ensure the existence of appropriate tensor field of the class C^2, which is the solution of $(4.1)_1$ or $(4.9)_1$, the integrability conditions $\mathbf{F}_{,ji}-\mathbf{F}_{,ij}=\mathbf{0}$ of these systems of linear equations should be satisfied. Since $\mathbf{F}$ is non-singular, the conditions are reduced to the forms

$$
\begin{aligned}
\mathbf{A}_{j,i}-\mathbf{A}_{i,j}-\mathbf{A}_j\mathbf{A}_i+\mathbf{A}_i\mathbf{A}_j &= F^k_{\cdot lij}\mathbf{g}_k\otimes\mathbf{g}^l=\mathbf{0},\\
\mathbf{B}_{j,i}-\mathbf{B}_{i,j}+\mathbf{B}_j\mathbf{B}_i-\mathbf{B}_i\mathbf{B}_j &= F^k_{\cdot lij}\bar{\mathbf{g}}_k\otimes\bar{\mathbf{g}}^l=\mathbf{0}.
\end{aligned}
\tag{5.8}
$$

Applying (2.16), $(4.1)_2$, $(4.9)_2$ and $(5.1)_1$ we obtain

$$
F^k_{\cdot lij}=A^k_{lj|i}-A^k_{li|j}+A^k_{mi}A^m_{lj}-A^k_{mj}A^m_{li}=\bar{R}^k_{\cdot lij}-R^k_{\cdot lij}
\tag{5.9}
$$

which leads to the component form of (5.8)

$$
F^k_{\cdot lij}=0.
\tag{5.10}
$$

Each of the relations (5.8) and (5.10) is equivalent to 18 scalar differential conditions which should be satisfied by the symbols Γ^k_{ij} and $\bar{\Gamma}^k_{ij}$. When $\bar{\Gamma}^k_{ij}$ are expressed in terms of known Γ^k_{ij} and E_{ij} according to $(2.16)_2$ twelve of the conditions become identities for any E_{ij}.

The relations (5.8) and (5.10) are just the alternative forms of compatibility conditions in the continuum mechanics. From (5.9) it is easy to note that the compatibility conditions are equivalent to the conditions (5.7) derived by geometric method. The tensor form $(5.8)_2$ was given in [37], while the component form (5.10) was obtained in [7] as the integrability conditions of the system $\mathbf{u}_{|ij}=A^k_{ij}(\mathbf{g}_k+\mathbf{u}_{,k})$ with respect to $\mathbf{u}_{,i}$.

Let us discuss now the differential relations $(4.2)_1$ and $(4.9)_1$. When strains are supposed to be known, these relations may be regarded as systems of linear first-order partial differential equations for the rotation field $\mathbf{R}$. Remind that the field $\mathbf{R}$ is entirely described by three scalar parameters (two angles describing the direction of $\mathbf{e}$ and the angle of rotation ω, or three Euler's angles). Therefore, for given strains, each of $(4.2)_1$ and $(4.9)_1$ is equivalent to the system of nine independent first-order scalar partial differential equations with respect to those three scalar parameters. When $\mathbf{R}$ is calculated from $(4.2)_1$ or $(4.9)_1$, $\bar{\mathbf{p}}$ and $\mathbf{u}$ follows through quadratures and algebraic operations (see Section 6 below).

Since $\mathbf{R}$ is non-singular, the integrability conditions $\mathbf{R}_{,ji}-\mathbf{R}_{,ij}=\mathbf{0}$ of $(4.2)_1$ and $(4.9)_1$ take the forms

$$
\mathbf{K}_{j,i}-\mathbf{K}_{i,j}-\mathbf{K}_j\mathbf{K}_i+\mathbf{K}_i\mathbf{K}_j=K_{klij}\mathbf{g}^k\otimes\mathbf{g}^l=L_{klij}\mathbf{s}^k\otimes\mathbf{s}^l=\mathbf{0},
\tag{5.11}
$$

$$
\mathbf{L}_{j,i}-\mathbf{L}_{i,j}+\mathbf{L}_j\mathbf{L}_i-\mathbf{L}_i\mathbf{L}_j=K_{klij}\mathbf{r}^k\otimes\mathbf{r}^l=L_{klij}\bar{\mathbf{g}}^k\otimes\bar{\mathbf{g}}^l=\mathbf{0},
\tag{5.12}
$$

where

$$
\begin{aligned}
K_{klij} &= K_{klj|i}-K_{kli|j}+\bar{g}^{mn}(K_{mkj}K_{nli}-K_{mki}K_{nlj}),\\
L_{klij} &= L_{klj|i}-L_{kli|j}+\bar{g}^{mn}(E_{mkj}E_{nli}-E_{mki}E_{nlj})\\
&\quad -g_{pq}\bar{g}^{pm}\bar{g}^{qn}(U_{mk|j}U_{nl|i}-U_{mk|i}U_{nl|j}),
\end{aligned}
\tag{5.13}
$$

$$L_{klij} = U_k^m U_l^n K_{mnij}, \tag{5.14}$$

$$\begin{aligned} L_{klij} &= -L_{lkij} = -L_{klji} = L_{ijkl}, \\ K_{klij} &= -K_{lkij} = -K_{klji} = K_{ijkl}. \end{aligned} \tag{5.15}$$

The tensor relations (5.11) and (5.12) represent four different but equivalent forms of compatibility conditions in the continuum mechanics. In components with respect to appropriate bases (5.11) and (5.12) lead to

$$K_{klij} = 0, \quad L_{klij} = 0. \tag{5.16}$$

The form $(5.16)_1$ written in Cartesian coordinates was derived by Shield[6]. It was shown in [6] that when $(5.16)_1$ are satisfied the R–C curvature tensor in the deformed body configuration vanishes as well.

Each of the compatibility conditions (5.11), (5.12) and (5.16) is equivalent to six independent scalar differential equations with respect to the strains. The conditions ensure the existence of three scalar parameters describing $\mathbf{R}$, which are the solutions of nine equations $(4.2)_1$ or $(4.9)_1$, respectively.

Taking into account the symmetry conditions (5.15) we can define the symmetric tensor components

$$\begin{aligned} K^{rs} &= \frac{1}{4}\epsilon^{rkl}\epsilon^{sij}K_{klij} = \frac{1}{2}\epsilon^{rkl}\epsilon^{sij}(K_{klj|i} + \bar{g}^{mn}K_{mkj}K_{nli}), \\ L^{rs} &= \frac{1}{4}\bar{\epsilon}^{rkl}\bar{\epsilon}^{sij}L_{klij} = \sqrt{\left(\frac{g}{\bar{g}}\right)}(U^{-1})^r_p K^{ps} \end{aligned} \tag{5.17}$$

which allow to present (5.16) in the more compact form

$$K^{rs} = 0, \quad L^{rs} = 0. \tag{5.18}$$

These are again two different but equivalent component forms of compatibility conditions in continuum mechanics. It is easy to verify[38] that when (5.18) are satisfied, the Ricci curvature tensor in the deformed body configuration vanishes as well.

The tensor relations $(5.11)_1$ and $(5.12)_1$ may also be expressed entirely in terms of the vectors of change of curvature. Introducing (4.5) and $(4.12)_1$ into $(5.11)_1$ and $(5.12)_1$, respectively, after transformations we obtain

$$\begin{aligned} (\mathbf{k}_{j,i} - \mathbf{k}_{i,j} - \mathbf{k}_j \times \mathbf{k}_i) \times \mathbf{1} &= \mathbf{0}, \\ (\mathbf{l}_{j,i} - \mathbf{l}_{i,j} + \mathbf{l}_j \times \mathbf{l}_i) \times \mathbf{1} &= \mathbf{0}. \end{aligned} \tag{5.19}$$

According to (4.6), the vectors $\mathbf{k}_j$ may be represented either in the reference basis $\mathbf{g}_i$ or in the stretched basis $\mathbf{s}_i$. Then using $(5.19)_1$ and (5.17), after transformations we obtain

$$\begin{aligned} K^{rs}\mathbf{g}_r &= \epsilon^{sij}\left(\mathbf{k}_{i,j} - \frac{1}{2}\mathbf{k}_i \times \mathbf{k}_j\right) = \mathbf{0}, \\ L^{rs}\mathbf{s}_r &= \bar{\epsilon}^{sij}\left(\mathbf{k}_{i,j} - \frac{1}{2}\mathbf{k}_i \times \mathbf{k}_j\right) = \mathbf{0}. \end{aligned} \tag{5.20}$$

Similarly, representing the vectors $\mathbf{l}_j$ in the rotated basis $\mathbf{r}_i$ or in the deformed basis $\bar{\mathbf{g}}_i$ we obtain

$$\begin{aligned} K^{rs}\mathbf{r}_r &= \epsilon^{sij}\left(\mathbf{l}_{i,j} + \frac{1}{2}\mathbf{l}_i \times \mathbf{l}_j\right) = \mathbf{0}, \\ L^{rs}\bar{\mathbf{g}}_r &= \bar{\epsilon}^{sij}\left(\mathbf{l}_{i,j} + \frac{1}{2}\mathbf{l}_i \times \mathbf{l}_j\right) = \mathbf{0}. \end{aligned} \tag{5.21}$$

The relations (5.20) and (5.21) are different but equivalent vector forms of compatibility conditions. The conditions $(5.20)_2$ were derived by Signorini[23] as integrability conditions of the system analogous to $(4.25)_1$ but expressed in terms of the Gibb's finite rotation vector $\mathbf{g} = (1/2)\,\boldsymbol{\theta}$. Shamina[7] derived $(5.20)_2$ as integrability conditions of $(4.24)_1$. The vector compatibility conditions $(5.20)_1$ and (5.21) seem to be new in the literature. It is easy to verify by direct calculations that they are the integrability conditions of the respective systems of differential equations $(4.20)_1$–$(4.23)_1$, if appropriate representations for $\mathbf{k}_i$, $\mathbf{l}_i$, $\boldsymbol{\Omega}_{,i}$, $\boldsymbol{\theta}_{,i}$ and $\boldsymbol{\omega}_{,i}$ are used. The form $(5.21)_1$, which is expressed entirely with reference to the rotated basis $\mathbf{r}_i$, may be of particular interest for some applications in the theories of bars, plates and shells.

Further non-classical forms of compatibility conditions in the continuum mechanics are given in [38].

6. CALCULATION OF DISPLACEMENTS FROM GIVEN STRAINS

From an arbitrarily given continuous displacement vector $\mathbf{u}$, it is easy to calculate tensors $\mathbf{E}$, $\boldsymbol{\eta}$, $\mathbf{U}$, $\mathbf{V}$, $\mathbf{R}$ and vectors $\boldsymbol{\Omega}$, $\boldsymbol{\theta}$ or $\boldsymbol{\omega}$ applying only algebraic and differential operations presented in Sections 3 and 4.

More complex is an inverse problem: the calculation of the displacement field $\mathbf{u}$ from a given strain field $\mathbf{E}$. Approaching the problem directly, it requires the solution of six partial differential equations $(2.10)_3$ for the three unknowns u_i. For the existence of three single-valued and continuous scalar displacements u_i integrability conditions of the equations $(2.10)_3$ should be imposed on six strains E_{ij}.

In the particular case of infinitesimal deformations the infinitesimal displacement field can be calculated from the infinitesimal strain field only through quadratures and algebraic operations, according to the formula of Cesàro[24, 39, 40]. The formula allows to calculate the infinitesimal displacement field to within a rigid-body infinitesimal translation and a rigid-body infinitesimal rotation.

In the general case of finite deformations it is not possible to determine $\mathbf{u}$ from given $\mathbf{E}$ only through quadratures and algebraic operations. However, the finite displacement field should still be given to within a rigid-body finite translation and a rigid-body finite rotation.

A formula concerning the problem was proposed by Zak[37]. In our opinion the procedure applied in [37] is questionable, since in the process of derivation the deformed base vectors $\bar{\mathbf{g}}_i$ were used, although they were not supposed to be known in advance. The displacement vector $\mathbf{u}$ was given to within a constant non-singular tensor (but not to within a rigid-body finite rotation), which can neither be regarded as a satisfactory form of the result. In what follows we derive a general formula for the displacement field $\mathbf{u}$ which is free from the shorthcomings of this kind, [41].

Let all geometric quantities to the reference body configuration be known. Also assume to have six functions $E_{ij} = E_{ji} = E_{ij}(\theta^k)$ of class C^2 for the components of the Lagrangian strain tensor $(2.10)_1$. In the reference configuration choose arbitrarily a point $P_0 = P(0, 0, 0)$ where naught values of curvilinear coordinates are taken for convenience. An arbitrary point $P(\theta^1, \theta^2, \theta^3)$ may be reached from P_0 along a curve $\mathscr{C}$ described by the set of equations $\theta^i = \theta^i(t)$, where t is a parameter along $\mathscr{C}$, with $t = 0$ at P_0.

After the deformation of the body $\mathscr{B}$, the point P_0 move into $\bar{P}_0$, P into $\bar{P}$ etc. Since an arbitrary deformation of the body has been decomposed into the translation $\mathbf{u}$, the rotation $\mathbf{R}$ and the stretch $\mathbf{U}$ let us calculate in the reverse order these elementary deformation states in the neighbourhood of the particle $X \in \mathscr{B}$ coinciding with P in the reference configuration.

The calculation of $\mathbf{U}$ from $\mathbf{E}$ at P is a purely algebraic problem. Let E_r and $\mathbf{h}_r$ be eigenvalues and eigenvectors of $\mathbf{E}$, respectively, satisfying $\mathbf{E}\mathbf{h}_r = E_r\mathbf{h}_r$ (no sum over r), in terms of which the symmetric tensor $\mathbf{E}$ can be presented in the diagonal form $\mathbf{E} = \sum_r E_r\mathbf{h}_r \otimes \mathbf{h}_r$. Since $\mathbf{U} = \mathbf{1} + 2\mathbf{E}$ is positive definite, symmetric and co-axial with $\mathbf{E}$, its square root can also be represented in a similar diagonal form $\mathbf{U} = \sum_r \sqrt{(1 + 2E_r)}\mathbf{h}_r \otimes \mathbf{h}_r$. This representation of $\mathbf{U}$, in terms of $\mathbf{E}$, is unique[42, 43], although the construction may not be trivial in case of multiple eigenvalues of $\mathbf{E}$. Having determined $\mathbf{U}$ we also have $\boldsymbol{\eta} = \mathbf{U} - \mathbf{1}$ and the stretched basis $\mathbf{s}_i$ may be calculated according to $(2.7)_2$.

The tensor $\mathbf{R}$ can now be determined as a solution of the linear first-order differential equations $(4.2)_1$, provided the integrability conditions (5.11) of the equations are satisfied. Indeed, from the known E_{ij} we are able to calculate $\bar{g}_{ij}$, $\bar{\Gamma}^k_{ij}$ and A^k_{ij} according to (2.16), then using $(4.1)_2$ we can define $\mathbf{A}_j$, which together with the already known $\mathbf{U}$ allows to calculate $\mathbf{K}_j$ as in $(4.2)_2$.

Along a curve $\mathscr{C}$ connecting P_0 and P the eqn $(4.2)_1$ takes the form

$$\frac{d\mathbf{R}}{dt} = \mathbf{RK}, \quad \mathbf{R}^T\mathbf{R} = \mathbf{RR}^T = \mathbf{1}, \tag{6.1}$$

where $\mathbf{K}$ is a skew-symmetric tensor of change of curvature of $\mathscr{C}$.

The general solution of (6.1) can be given in analogy to corresponding matrix differential equations[43] in the form

$$\mathbf{R} = \mathbf{R}_0\mathbf{R}_t, \tag{6.2}$$

where $\mathbf{R}_0$ is an arbitrary constant proper orthogonal tensor and $\mathbf{R}_t$ is the matricant of (6.1) defined by the tensor series

$$\mathbf{R}_t = \mathbf{1} + \int_0^t \mathbf{K}(\tau)\,d\tau + \int_0^t \left[\int_0^\tau \mathbf{K}(\tau_1)\,d\tau_1\right]\mathbf{K}(\tau)\,d\tau + \dots. \tag{6.3}$$

If $\mathbf{K}$ commutes with its integral,

$$\mathbf{K}\int_0^t \mathbf{K}(\tau)\,d\tau = \left[\int_0^t \mathbf{K}(\tau)\,d\tau\right]\mathbf{K} \tag{6.4}$$

the matricant $\mathbf{R}_t$ can be written in the shorter exponential form

$$\mathbf{R}_t = \exp\left[\int_0^t \mathbf{K}(\tau)\,d\tau\right]. \tag{6.5}$$

The somewhat formal solution (6.2) and (6.3) of (6.1) follows from the general method of successive approximations, which may be used to solve an analogous matrix differential equation for any continuous function $\mathbf{K} = \mathbf{K}(t)$. For some specific forms of the function effective analytic and numerical methods of constructing the solution of (6.1) were developed which are now presented in monographs[43–45], where further references may be found. In particular, effective analytic solutions were constructed for $\mathbf{K} = \text{const}$, for $\mathbf{K}(t+T) = \mathbf{K}(t)$, $T = \text{const}$, for $\mathbf{K}(t) = \sum_m \mathbf{K}_m/(t-a_m) + \sum_n \mathbf{K}_n t^n$, $a_m = \text{const}$, and for many other specific forms of $\mathbf{K}(t)$.

The eqn (6.1) can also be presented along $\mathscr{C}$ in an equivalent vector form analogous to $(4.25)_1$

$$\frac{d\boldsymbol{\theta}}{dt} = \left(1 + \frac{1}{4}\boldsymbol{\theta}^2\right)\mathbf{k} + \frac{1}{2}\boldsymbol{\theta}\times\mathbf{k} + \frac{1}{4}\boldsymbol{\theta}\times(\boldsymbol{\theta}\times\mathbf{k}), \tag{6.6}$$

where $\mathbf{k} = -(1/2)\boldsymbol{\epsilon}\cdot\mathbf{K}$ is the vector of change of curvature of $\mathscr{C}$. The structure of (6.6) is similar to the one describing the motion of a rigid body about a fixed point[20], where $\mathbf{k}$ means angular velocity vector. Within analytical mechanics, many specific analytic and numerical methods of integration of (6.6) were developed for various classes of functions $\mathbf{k} = \mathbf{k}(t)$. In particular, 20 different general or particular exact solutions of (6.6) were constructed in a closed form (these are summarized in [46]). Therefore, the results already obtained in analytical mechanics may be of great help also in solving (6.6) and (6.1) for analogous problems of the non-linear continuum deformation. For example, the analogy allowed to notice[12] that the displacement field can be obtained through quadratures from the given strain field only in two simple cases: for the three-dimensional infinitesimal deformations and for the two-dimensional finite deformations, [6]. It was also used to construct all exact closed-form solutions for the non-linear deformations of rods [14].

The formulae (6.2) and (6.3) would give us the desired solution for $\mathbf{R}$ provided the tensor $\mathbf{K}$ along $\mathscr{C}$ were calculated. Since according to $(4.2)_2$ only tensors $\mathbf{K}_j$ along curvilinear coordinate lines are supposed to be calculated, let us choose a specific integration path $P_0P'P''P$. The path consists of three subsequent parts of curvilinear coordinate lines: along P_0P' there is $\theta^2 = 0$, $\theta^3 = 0$, along $P'P''$ there is $\theta^1 = \text{const}$, $\theta^3 = 0$ and along $P''P$ there is $\theta^1 = \text{const}$, $\theta^2 = \text{const}$. Solving $(4.2)_2$ along the subsequent parts of the integration path and taking into account (6.2) and (6.3), we obtain

$$\mathbf{R}' = \mathbf{R}_0\mathbf{R}_1 \quad \text{along} \quad P_0P'$$
$$\mathbf{R}_1 = \mathbf{1} + \int_0^{\theta^1} \mathbf{K}_1(\xi, 0, 0)\,\mathrm{d}\xi + \int_0^{\theta^1}\left[\int_0^{\xi} \mathbf{K}_1(\xi_1, 0, 0)\,\mathrm{d}\xi_1\right]\mathbf{K}_1(\xi, 0, 0)\mathrm{d}\xi + \ldots \tag{6.7}$$

$$\mathbf{R}'' = \mathbf{R}'\mathbf{R}_2 \quad \text{along} \quad P'P''$$
$$\mathbf{R}_2 = \mathbf{1} + \int_0^{\theta 2} \mathbf{K}_2(\theta^1, \eta, 0)\,\mathrm{d}\eta + \int_0^{\theta 2}\left[\int_0^{\eta} \mathbf{K}_2(\theta^1, \eta_1, 0)\,\mathrm{d}\eta_1\right]\mathbf{K}_2(\theta^1, \eta, 0)\,\mathrm{d}\eta + \ldots \tag{6.8}$$

$$\mathbf{R} = \mathbf{R}''\mathbf{R}_3 \quad \text{along} \quad P''P$$
$$\mathbf{R}_3 = \mathbf{1} + \int_0^{\theta_3} \mathbf{K}_3(\theta^1, \theta^2, \zeta)\,\mathrm{d}\zeta + \int_0^{\theta^3}\left[\int_0^{\zeta} \mathbf{K}_3(\theta^1, \theta^2, \zeta_1)\,\mathrm{d}\zeta_1\right]\mathbf{K}_3(\theta^1, \theta^2, \zeta)\,\mathrm{d}\zeta + \ldots \tag{6.9}$$

or shortly

$$\mathbf{R} = \mathbf{R}_0\mathbf{R}_1\mathbf{R}_2\mathbf{R}_3, \tag{6.10}$$

where all tensors are proper orthogonal. The formula (6.10) allows to determine $\mathbf{R}$ from the known $\mathbf{E}$ and $\mathbf{U}$ to within a constant finite rotation $\mathbf{R}_0$. For those of $\mathbf{K}_j$ which commute with their integrals the corresponding matricants $\mathbf{R}_j$ can be presented in shorter exponential forms as in (6.5).

With the already known $\mathbf{R}$ and $\mathbf{U}$, from $(2.7)_1$ it is possible to construct the deformed base vectors $\bar{\mathbf{g}}_i = \mathbf{R}\mathbf{U}\mathbf{g}_i$. The position vector $\bar{\mathbf{p}}$ of the deformed body follows now as the solution of the differential equations

$$\bar{\mathbf{p}}_{,i} = \bar{\mathbf{g}}_i \tag{6.11}$$

whose integrability conditions $\bar{\mathbf{p}}_{,ij} - \bar{\mathbf{p}}_{,ji} = \mathbf{0}$ lead to

$$\bar{\mathbf{g}}_{i,j} - \bar{\mathbf{g}}_{j,i} = (\bar{\Gamma}^k_{ij} - \bar{\Gamma}^k_{ji})\bar{\mathbf{g}}_k = \mathbf{0}. \tag{6.12}$$

The geometric interpretation of (6.12) shows [28], that (6.11) are integrable if and only if the set of curvilinear coordinate lines $\{\theta^i\}$, generated by the fields of tangent vectors $\bar{\mathbf{g}}_i$, is holonomic. Note that $\bar{\Gamma}^k_{ij} - \bar{\Gamma}^k_{ji} = \bar{S}^k_{ij}$ are components of the torsion tensor [47]. Therefore, the conditions (6.12) may also be interpreted as the vanishing of the torsion tensor for the deformed body configuration. In our case, when $\bar{\Gamma}^k_{ij}$ are calculated according to $(2.16)_2$, the conditions (6.12) are identically satisfied.

With the help of $(2.4)_1$ the solution of (6.11) can be obtained through quadratures. Let us integrate (6.11) along the subsequent parts of the specific integration path $P_0P'P''P$, which give

$$\begin{aligned}\bar{\mathbf{p}} &= \bar{\mathbf{p}}_0 + \int_{P_0}^{P} \mathbf{R}\mathbf{U}\,\mathrm{d}\mathbf{p} \\ &= \bar{\mathbf{p}}_0 + \int_0^{\theta^1} \bar{\mathbf{g}}_1(\xi, 0, 0)\,\mathrm{d}\xi + \int_0^{\theta^2} \bar{\mathbf{g}}_2(\theta^1, \eta, 0)\,\mathrm{d}\eta + \int_0^{\theta^3} \bar{\mathbf{g}}_3(\theta^1, \theta^2, \zeta)\,\mathrm{d}\zeta.\end{aligned} \tag{6.13}$$

The formula (6.13) allows to determine $\bar{\mathbf{p}}$ from the known $\mathbf{R}$ and $\mathbf{U}$ to within an arbitrary constant position vector $\bar{\mathbf{p}}_0$.

The formula $(6.13)_2$, together with (6.7)–(6.10), and $(2.1)_2$ leads to the final relation

$$\bar{\mathbf{p}} = \bar{\mathbf{p}}_0 + \mathbf{R}_0\left[\int_0^{\theta^1} \mathbf{R}_1(\xi, 0, 0)\mathbf{U}(\xi, 0, 0)\mathbf{g}_1(\xi, 0, 0)\,\mathrm{d}\xi \right.$$

$$+ \mathbf{R}_1 \int_0^{\theta^2} \mathbf{R}_2(\theta^1, \eta, 0)\mathbf{U}(\theta^1, \eta, 0)\mathbf{g}_2(\theta^1, \eta, 0)\,\mathrm{d}\eta \tag{6.14}$$

$$\left. + \mathbf{R}_1\mathbf{R}_2 \int_0^{\theta^3} \mathbf{R}_3(\theta^1, \theta^2, \zeta)\mathbf{U}(\theta^1, \theta^2, \zeta)\mathbf{g}_3(\theta^1, \theta^2, \zeta)\,\mathrm{d}\zeta\right].$$

Here $\mathbf{R}_1$, $\mathbf{R}_2$ and $\mathbf{R}_3$ are expressed in terms of $\mathbf{E}$ according to (6.7)–(6.9), $(4.1)_2$ and (2.16).

By (6.14) the displacement vector $\mathbf{u} = \bar{\mathbf{p}} - \mathbf{p}$ is determined from the strain tensor $\mathbf{E}$ to within a rigid-body finite translation $\mathbf{u}_0 = \bar{\mathbf{p}}_0 - \mathbf{p}_0$ and a rigid-body finite rotation $\mathbf{R}_0$.

In case of infinitesimal deformation

$$E_{ij} \approx e_{ij}, \quad \mathbf{U} \approx \mathbf{1} + \mathbf{e}, \quad \mathbf{R} \approx \mathbf{1} + \boldsymbol{\phi}, \quad \boldsymbol{\phi} = \boldsymbol{\phi}_0 + \boldsymbol{\phi}_1 + \boldsymbol{\phi}_2 + \boldsymbol{\phi}_3,$$

$$e_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i}), \quad \phi_{ij} = \frac{1}{2}(u_{i,j} - u_{j,i}) \ . \tag{6.15}$$

$$\mathbf{K}_j \approx \boldsymbol{\phi}_{,j} - (e_{ij|k} - e_{kj|i})\mathbf{g}^i \otimes \mathbf{g}^k, \quad \mathbf{k}_j \approx \epsilon^{ikl} e_{kj|i}\mathbf{g}_l,$$

and (6.14) can be reduced to

$$\bar{\mathbf{p}} \approx \bar{\mathbf{p}}_0 + (\mathbf{1} + \boldsymbol{\phi}_0)\left[\int_0^{\theta^1} (\mathbf{1} + \boldsymbol{\phi}_1)(\mathbf{1} + \mathbf{e})\mathbf{g}_1\,\mathrm{d}\xi \right.$$

$$\left. +(\mathbf{1} + \boldsymbol{\phi}_1)\int_0^{\theta^2} (\mathbf{1} + \boldsymbol{\phi}_2)(\mathbf{1} + \mathbf{e})\mathbf{g}_2\,\mathrm{d}\eta + (\mathbf{1} + \boldsymbol{\phi}_1)(\mathbf{1} + \boldsymbol{\phi}_2)\int_0^{\theta^3} (\mathbf{1} + \boldsymbol{\phi}_3)(\mathbf{1} + \mathbf{e})\mathbf{g}_3\,\mathrm{d}\zeta\right] \tag{6.16}$$

or, to within the first-order terms,

$$\bar{\mathbf{p}} \approx \bar{\mathbf{p}}_0 + \int_{P_0}^{P} \mathrm{d}\mathbf{p} + \int_{P_0}^{P} (\mathbf{e} + \boldsymbol{\phi})\,\mathrm{d}\mathbf{p}. \tag{6.17}$$

Taking into account that along the $\mathscr{C}$ connecting P_0 and P we may write $\mathrm{d}\mathbf{p} = (\partial/\partial\tau^j)[\mathbf{p}(\tau^r) - \mathbf{p}]\mathrm{d}\tau^j$, after integration of (6.16), with the use of $(6.15)_3$, we obtain

$$\mathbf{u} \approx \mathbf{u}_0 + \boldsymbol{\phi}_0(\mathbf{p} - \mathbf{p}_0) + \int_{P_0}^{P} [\mathbf{p}\cdot\mathbf{g}^k(\tau^r) - p^k(\tau^r)](e_{ij|k} - e_{kj|i})\mathbf{g}^i\,\mathrm{d}\tau^j. \tag{6.18}$$

This is exactly the formula of Cesàro[24, 39, 40], written here in the curvilinear coordinate system of the reference configuration.

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