Mit dem Ansatz

$$a(x) = a_0 \cdot \cos(\pi x/L) + \frac{a_1}{\mathrm{ch}\, hL} \cdot [\mathrm{ch}(2hx) - \cos(\pi x/L)] \quad \text{für} \quad |x| \leqq L/2\,,$$

$$= a_1 \cdot \exp[f \cdot (L/2 - |x|)] \quad \text{für} \quad |x| \geqq L/2 \tag{9}$$

mit

$$fL = 2hL \cdot \mathrm{th}(hL) - \pi \cdot \frac{a_0}{a_1} - \left(\frac{1}{\mathrm{ch}(hL)}\right)$$

wird Gleichung (8) näherungsweise gelöst; die drei Parameter a_0, a_1 und h werden iterativ so festgelegt, daß die Integralgleichung an den Stellen $x = 0$, $L/2$ und L erfüllt ist. Für ebene Querschnitte ($c \to \infty$) führt der Ansatz zur exakten Lösung. Mit dem derart bestimmten Biegeanteil $T_0 \cdot a(x)$ kann nun nach Gleichung (7) die Dehnungsverteilung ε_{xx} sowie die Extremwerte abhängig von T_0, q, c und L/B berechnet werden.

Für jeden Gurt läßt sich der zulässige Trumkraftbereich als Funktion von L/B angeben, in dem keine lokalen Stauchungen einzelner Zugträger (unterer Rand) und keine Überdehnung der Gurtunterkante (oberer Rand) auftreten. In Bild 2 sind diese Bereiche für einen Gewebegurt EP 800/4 und einen Stahlseilgurt St 1250 der Breite $B = 1{,}2$ m aufgetragen. Dabei wurde ε_{zul} zu $16{,}7^0/_{00}$ beim Gurt EP 800/4 sowie $2{,}5^0/_{00}$ beim St 1250 gesetzt. Ohne Querschnittsverwölbung schrumpft der zulässige Bereich für den Stahlseilgurt auf den gestrichelten Zipfel zusammen.

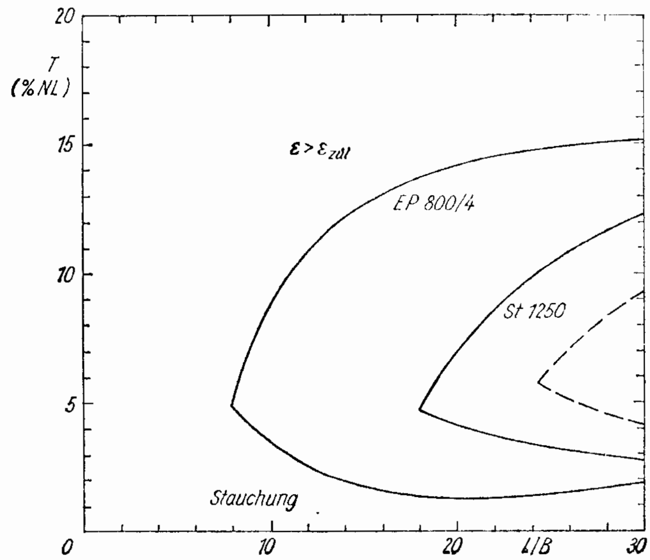

Bild 2. Zulässige Trumkräfte in der Wendung ($B = 1{,}2$ m)

Literatur

1 OEHMEN, K. H., Einfluß vertikaler und horizontaler Kurven auf die Dehnungsverteilung in Fördergurten — Theorie und Anwendung. BRAUNKOHLE **31**, S. 340—348 (1979).

2 OEHMEN, K. H., Berechnung der Dehnungsverteilung in Fördergurten infolge Muldungsübergang, Gurtwendung und Seilunterbrechung. BRAUNKOHLE **31**, S. 394—402 (1979).

Anschrift: Dr. K. H. OEHMEN, Universität Hannover, Institut für Mechanik Appelstr. 11, D-3000 Hannover 1, BRD

ZAMM 63, T 200 – T 202 (1983)

W. PIETRASZKIEWICZ

A Simplest Consistent Version of the Geometrically Non-Linear Theory of Elastic Shells Undergoing Large/Small Rotations

In the paper [1] a new formulation was given for the Lagrangian geometrically non-linear theory of thin elastic shells. In case of conservative surface and boundary loadings the theory allowed for a proper variational formulation within unrestricted rotations. Simplified versions of the theory were discussed in [2] by restricting the magnitude of the rotations to be small, moderate, large or finite, according to a classification scheme proposed in [3, 4], in terms of the small parameter $\theta = \max(h/d, h/L, \sqrt{h/R}, \sqrt{\eta})$. Here h is the shell thickness, d is the distance from the lateral shell boundary, L is the smallest wavelength of deformation patterns, R is the smallest principal radius of curvature of the undeformed shell middle surface $\mathcal{M}$ and η is the largest strain in the shell space.

In what follows a version of the non-linear theory of shells is constructed, in which the shell material elements may undergo large rotations about tangents to $\mathcal{M}$ while the rotations about normals to $\mathcal{M}$ are assumed to be small. The version may be described [2—4] by the following estimates of the linearized strains $\theta_{\alpha\beta}$ and linearized rotations φ_α, φ:

$$\left.\begin{aligned} &\theta_{\alpha\beta} = O(\theta)\,, \qquad \varphi_\alpha = O(\sqrt{\theta})\,, \qquad \varphi = O(\theta^2)\,,\\ &\theta_{\alpha\beta} = \tfrac{1}{2}\,(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w\,, \qquad \varphi_\alpha = w_{,\alpha} + b^\lambda_\alpha u_\lambda\,, \qquad \varphi = \tfrac{1}{2}\,\varepsilon^{\alpha\beta}u_{\beta|\alpha}\,. \end{aligned}\right\} \tag{1}$$

Here u_α, w, $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\varepsilon^{\alpha\beta}$ are components of the displacement vector $\boldsymbol{u}$, the metric, curvature and permutation tensors of $\mathcal{M}$, respectively.

According to the first-approximation theory of thin isotropic and elastic shells, the strain energy function may be approximated, to within a relative error $O(\theta^2)$, by the quadratic expression $\Sigma = \frac{1}{2}\,hH^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + h^2/12\chi_{\alpha\beta}\chi_{\lambda\mu})$. Here $H^{\alpha\beta\lambda\mu}$ are elasticities and $\gamma_{\alpha\beta}$ and $\chi_{\alpha\beta}$ are components of the surface strain tensor and the modified tensor of change of curvature [1], respectively. Within this error of Σ the components $\gamma_{\alpha\beta}$ can be given to within terms $O(\eta\theta^2)$ and $\chi_{\alpha\beta}$ to within terms $O(\eta\theta/\lambda)$, where $\lambda = h/\theta$ is used to estimate the order of covariant derivatives, $(\ )_{|\alpha} \sim$ $\sim (\ )/\lambda$. Such consistently simplified version of the theory of shells undergoing large/small rotations was discussed in detail in [2].

In engineering calculations the larger relative error $O(\theta)$ of Σ may be regarded as satisfactory. Within the theory of shells undergoing large/small rotations any numerical calculations are very complex anyway and numerical procedures applied in the calculations introduce themselves substantial round-off errors. Within the larger relative error $O(\theta)$ of Σ the surface strain measures may be taken in the symmetric approximate form

$$\left.\begin{aligned} &\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \tfrac{1}{2}\,\varphi_\alpha\varphi_\beta + \tfrac{1}{2}\,\theta^\lambda_\alpha\theta_{\lambda\beta} + O(\eta\theta)\,,\\ &\chi_{\alpha\beta} = \tfrac{1}{2}\,(n_{\alpha|\beta} + n_{\beta|\alpha}) - \tfrac{1}{2}\,(\theta^\lambda_\alpha\varphi_{\lambda|\beta} + \theta^\lambda_\beta\varphi_{\lambda|\alpha}) + \tfrac{1}{2}\,(\varphi_\alpha n_{,\beta} + \varphi_\beta n_{,\alpha}) + O(\eta/\lambda)\,,\\ &n_\lambda = -(1 + \theta^\varkappa_\varkappa)\,\varphi_\lambda + \varphi^\mu\theta_{\mu\lambda}\,, \qquad n = 1 + \theta^\varkappa_\varkappa\,. \end{aligned}\right\} \tag{2}$$

Introducing (2) into the internal virtual work we obtain

$$\begin{aligned} &IVW = \iint_{\mathcal{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\chi_{\alpha\beta})\,\mathrm{d}A = -\iint_{\mathcal{M}} \boldsymbol{T}^\beta|_\beta\cdot\delta\boldsymbol{u}\,\mathrm{d}A + \int_{\mathcal{C}} \boldsymbol{T}^\beta\nu_\beta\cdot\delta\boldsymbol{u}\,\mathrm{d}s + I\,,\\ &I = \int_{\mathcal{C}} M^{\alpha\beta}(\delta n_\alpha - \theta^\lambda_\alpha\delta\varphi_\lambda + \varphi_\alpha\delta n)\,\nu_\beta\,\mathrm{d}s\,, \qquad \boldsymbol{T}^\beta = T^{\lambda\beta}\boldsymbol{a}_\lambda + T^\beta\boldsymbol{n}\,,\\ &T^{\lambda\beta} = N^{\lambda\beta} + \tfrac{1}{2}\,(\theta^\lambda_\alpha N^{\alpha\beta} + \theta^\beta_\alpha N^{\alpha\lambda}) - a^{\lambda\beta}\varphi_{\varkappa|\varrho}\,M^{\varkappa\varrho} - \tfrac{1}{2}\,(\varphi^\lambda M^{\beta\varrho} + \varphi^\beta M^{\lambda\varrho})\,|_\varrho\\ &T^\beta = \varphi_\alpha N^{\alpha\beta} + [(1 + \theta^\varkappa_\varkappa)\,M^{\beta\varrho}]|_\varrho + \theta^\beta_{\varkappa|\varrho}M^{\varkappa\varrho}\,. \end{aligned} \tag{3}$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are the Lagrangian symmetric stress and moment resultants, $\boldsymbol{a}_\lambda$, $\boldsymbol{n}$ is the basis on $\mathcal{M}$, $\boldsymbol{t} = t^\alpha\boldsymbol{a}_\alpha$ and $\boldsymbol{\nu} = \nu^\alpha\boldsymbol{a}_\alpha$ are the unit tangent and the outward unit normal to the boundary curve $\mathcal{C}$, respectively.

The form $(3)_1$ of IVW is the counterpart of the exact relation (4.7) of [1], only here in I some boundary terms appear in the splitted form. This is the result of the omission from $(2)_2$ all the terms whose contribution to Σ is negligibly small within the assumed relative error $O(\theta)$. However, within the assumed approximation it is allowed to keep some small terms in I, which were already omitted in $(2)_2$, and write I in an equivalent form

$$\left.\begin{aligned} &I = \int_{\mathcal{C}} (R_{\nu\nu}\,\delta n_\nu + R_{t\nu}\,\delta n_t + R_\nu\,\delta n)\,\mathrm{d}s\,, \qquad n_\nu = n_\alpha\nu^\alpha\,, \qquad n_t = n_\alpha t^\alpha\,,\\ &R_{\nu\nu} = (\nu_\alpha + \nu_\lambda\theta^\lambda_\alpha)\,M^{\alpha\beta}\nu_\beta\,, \qquad R_{t\nu} = (t_\alpha + t_\lambda\theta^\lambda_\alpha)\,M^{\alpha\beta}\nu_\beta\,, \qquad R_\nu = \varphi_\alpha M^{\alpha\beta}\nu_\beta\,. \end{aligned}\right\} \tag{4}$$

The structure of I in $(4)_1$ becomes now the same as in the exact theory, see (4.7) of [1]. Therefore, n_t and n can now be expressed approximately in terms of $\boldsymbol{u}$ and n_ν as independent variables. The appropriate approximation level of the boundary terms follows from the approximation error of $T^{\lambda\beta}$ and T^β, which is generated by the error of consistently approximated strain measures (2). Since $\chi_{\alpha\beta} = O(\eta/h)$, then $M^{\alpha\beta} = O(Eh^2\eta)$, where E is the YOUNG's modulus. As a result, in (3) $T^{\lambda\beta}$ is given to within terms $O(Eh^2\eta\theta/\lambda)$ and T^β to within $O(Eh^2\eta\theta\sqrt{\theta}/\lambda)$.

Let us expand all non-rational boundary functions given in [1] into series in terms of the linearized quantities $(1)_2$ and the parameter n_ν and, using the estimates $(1)_1$, take into account only terms of the same order as those appearing in $T^{\lambda\beta}$. After involved expansion and estimation procedure suggested in [2], for the boundary parameters defined in [1] we obtain the following consistently simplified expressions

$$n_t = -\varphi_t\,, \qquad n = 1 + \theta_{tt} - \tfrac{1}{2}\,\varphi_\nu^2 + \varphi_\nu\theta_{\nu t}\,, \tag{5}$$

$$\left.\begin{aligned} &\boldsymbol{P} = \boldsymbol{T}^\beta\nu_\beta - \left(\frac{\mathrm{d}F_\nu}{\mathrm{d}s}\boldsymbol{\nu} + \frac{\mathrm{d}F_t}{\mathrm{d}s}\boldsymbol{t} + \frac{\mathrm{d}F}{\mathrm{d}s}\boldsymbol{n}\right), \qquad M = R_{\nu\nu} - (\varphi_\nu\varphi_t + \theta_{\nu t})\,M_{t\nu}\,,\\ &\boldsymbol{F} = \varphi_\nu M_{t\nu}\boldsymbol{\nu} + \varphi_t M_{t\nu}\boldsymbol{t} - [(1 + 3\theta_{tt} - \tfrac{1}{2}\,\varphi_\nu^2)\,M_{t\nu} + \theta_{t\nu}M_{\nu\nu}]\,\boldsymbol{n}\,,\\ &\boldsymbol{P}^* = \boldsymbol{N} - \left(\frac{\mathrm{d}F^*_\nu}{\mathrm{d}s}\boldsymbol{\nu} + \frac{\mathrm{d}F^*_t}{\mathrm{d}s}\boldsymbol{t} + \frac{\mathrm{d}F^*}{\mathrm{d}s}\boldsymbol{n}\right), \qquad M^* = H_\nu - (\varphi_\nu\varphi_t + \theta_{\nu t})\,H_t\,,\\ &\boldsymbol{F}^* = \varphi_\nu H_t\boldsymbol{\nu} + \varphi_t H_t\boldsymbol{t} - (1 + 2\theta_{tt} - \tfrac{1}{2}\,\varphi_\nu^2)\,H_t\boldsymbol{n}. \end{aligned}\right\} \tag{6}$$

Here $\boldsymbol{N}$ is the resultant boundary force and $\boldsymbol{H} = H_\nu \boldsymbol{\nu} + H_t \boldsymbol{t}$ is the resultant boundary static moment. Note that the parameter n_ν appears in (6) in the reduced form $-\varphi_\nu$ and both R_ν and $H = \boldsymbol{H} \cdot \boldsymbol{n}$ do not appear at all in the boundary terms (6) within the assumed error of approximation.

Now the consistently simplified equilibrium equations, the static boundary and corner conditions as well as the geometric boundary and corner conditions for the simplest version of the theory of shells undergoing large/small rotations can be presented in the general vector form

$$\left.\begin{aligned} &\boldsymbol{T}^\beta|_\beta + \boldsymbol{p} = \boldsymbol{0} \quad \text{in } \mathcal{M}\,, \\ &\boldsymbol{P} = \boldsymbol{P}^*\,, \quad M = M^* \quad \text{on } \mathcal{C}_f \quad \text{and} \quad \boldsymbol{F}_j = \boldsymbol{F}_j^* \quad \text{at each} \quad M_j \in \mathcal{C}_f\,, \\ &\boldsymbol{u} = \boldsymbol{u}^*\,, \quad n_\nu = n_\nu^* \quad \text{on } \mathcal{C}_u \quad \text{and} \quad \boldsymbol{u}_i = \boldsymbol{u}_i^* \quad \text{at each} \quad M_i \in \mathcal{C}_u\,, \end{aligned}\right\} \tag{7}$$

where $(2)_3$, (3) and (6) should be used.

The structure of the shell relations is relatively simple. Both strain measures (2) are quadratic polynomials in displacements. The equilibrium equations $(7)_1$ with (3) and two of the static boundary and corner conditions $(7)_2$ are linear both in $N^{\alpha\beta}$, $M^{\alpha\beta}$ and in displacements. The remaining two static boundary conditions $(7)_2$ and the third static corner condition are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but quadratic in displacements. Also the fourth geometric boundary condition $(7)_3$ for n_ν is expressed as the quadratic function of displacements.

From the relations of the simplest large/small rotation shell theory given above it is easy to construct the free Hu-Washizu variational functional [1]. Then, following [5], a number of other free and constrained functionals and associated variational principles may be generated.

References

1 Pietraszkiewicz, W.; Szwabowicz, M. L., Entirely Lagrangian nonlinear theory of thin shells, Archives of Mechanics, **33**, 2' 273–288 (1981).
2 Pietraszkiewicz, W., On consistent approximations in the geometrically non-linear theory of shells, Ruhr-Universität, Mitt. Inst. f. Mech. Nr. 26., Bochum 1981.
3 Pietraszkiewicz, W., Introduction to the non-linear theory of shells, Ruhr-Universität, Mitt. Inst. für Mech. Nr. 10, Bochum 1977.
4 Pietraszkiewicz, W., Finite rotations in the non-linear theory of thin shells, in: Thin Shell Theory, Ed. by W. Olszak., Springer-Verlag, Wien—New York 1980.
5 Schmidt, R.; Pietraszkiewicz, W., Variational principles in the geometrically non-linear theory of shells undergoing moderate rotations, Ingenieur-Archiv, **50**, 3, 187–201 (1981).

Anschrift: Doc. dr habil. inż. Wojciech Pietraszkiewicz, Instytut Maszyn Przepływowych PAN, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland

ZAMM 63, T 202 (1983)

G. Rieder

Zum Plattenparadoxon von Sapondzhyan und Babuška

Es handelt sich um die Weiterführung einer Arbeit, die der Autor auf dem IUTAM-Symposium in Moskau 1972 vorgetragen hat und über die später kurz berichtet wurde [1]. Die als paradox empfundene Tatsache, daß die Lösung des Randwertproblems der drehbar gelagerten Polygonplatte in der Kirchhoffschen Theorie beim Grenzübergang nicht zur Lösung für die gekrümmte glatte Berandung führt, soll verständlicher gemacht werden. Zugleich soll nachgewiesen werden, daß die Kappung der Singularitäten oder, wie bereits von Babuška bemerkt, die Anwendung einer Plattentheorie Reissnerschen Typs das Paradox auflöst.

Das Ergebnis bedeutet eine allgemeinere, strengere und vertiefte Fassung der Folgerungen von Hanuška [2], Rajaiah und Rao [3], die auf die versteifende Wirkung der Eckensingularitäten hinweisen. Die Hilfsmittel sind (1) die Einteilung in eine dem Problem angepaßtes System von finiten Elementen mit genügend formfügigen (conforming) Funktionen und (2) die Konstruktion zweier Folgen von Vergleichslösungen nach der Hyperkreismethode von Prager und Synge einerseits mit dem üblichen Verschiebungsansatz und andererseits mit einem Mindlin-Schaefer-schen Spannungsfunktionenansatz.

Die Arbeit „On the Plate Paradox" wird in der Festschrift zum achtzigsten Geburtstag von Akademiker W. D. Kupradze in Tbilisi erscheinen.

Literatur

1 Rieder, G., On the Plate Paradox of Sapondzhyan and Babuška, Mech. Res. Comm. **1** (1974), S. 51–53.
2 Hanuška, A., On the Validity of the Solution of Simply Supported Uniformly Loaded Polygonal Plates, USTARCH-SAV, Bratislava 1969, nicht veröffentlicht.
3 Rajaiah, K.; Rao, A. K., On the Polygon-Circle Paradox, J. Appl. Mech. 48 (1981), S. 195–196.

Anschrift: Prof. Dr. rer. nat. G. Rieder, Institut für Technische Mechanik, RWTH Aachen, Templergraben 64, D-5100 Aachen, BRD