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On non-classical forms of compatibility conditions in continuum mechanics

1. INTRODUCTION

Compatibility conditions in the classical continuum mechanics are usually understood as the conditions that should be imposed on the continuous strain field in order to ensure the existence of a single-valued and continuous displacement field.

In the case of finite deformations the strain-displacement relations constitute a complex system of six non-linear partial differential equations for three displacements. The direct derivation of the integrability conditions for such a system of non-linear equations has not yet been given. Instead, it is common to apply the geometric argument that during deformation the body remains in the three-dimensional Euclidian space. Since the Riemann-Christoffel curvature tensor of the space should identically vanish in the reference and deformed body configurations, this leads to six conditions for the strains, which are called the classical form of compatibility conditions for finite strains [1]. However, the problem of deciding in what sense the compatibility conditions, derived from geometrical considerations, are equivalent to the integrability conditions for the non-linear strain-displacement relations and, in particular, the implications of Bianchi identities, has not been fully explained in the literature.

Recently it has been shown [2] that the displacement field in the deformable continuum can be determined from the given strain field in three elementary steps. First, the stretch field was constructed using pure algebra. Then the rotation field was calculated from the strain and stretch fields by solving a system of linear first-order differential equations. Finally, the displacement field was determined through quadratures from the stretch and rotation fields. Under normal continuity and differentiability requirements for strains, the first and third steps can always be performed. As a result, the complex problem of integrability conditions for the non-linear strain-displacement relations was, in fact, reduced to the simple problem of integrability conditions for the system of linear first-order differential equations with respect to the rotation field.

In this report several different but equivalent forms of compatibility conditions in continuum mechanics are discussed. The conditions are derived as the integrability conditions for some definite systems of linear first-order partial differential equations, the solutions of which would allow the determination of the displacement field from a given strain field only through quadratures and algebraic operations. From this point of view, several different systems of linear equations are examined. It is explicitly shown that when the integrability conditions for the systems are satisfied, the Riemann-Christoffel curvature tensor (or the Ricci tensor) of the deformed body configuration vanishes as well.

Using the polar decomposition of deformation gradient tensor [3] and the convected system of curvilinear coordinates, two additional non-holonomic bases are introduced. The stretched basis is defined by pure stretch of the reference basis along the principal directions of the Lagrangian strain tensor, while the rotated basis is generated from the reference basis by its rigid-body rotation. The intermediate bases are very helpful in deriving various geometric relations describing the continuum deformation.

Particular attention is paid to the rotation field and its differential properties in the space. Along each of the curvilinear coordinate lines the rotation field is shown to satisfy a linear first-order differential equation with variable coefficients depending only upon strains and stretches. The equation may be written in two different but equivalent forms (3.9). Since the tensor coefficients may also be represented in two different but equivalent forms, this gives, in fact, four different but equivalent tensor forms of the linear first-order differential equations for the rotation field. The integrability conditions for each of the differential equations lead to four different but equivalent tensor forms (or two scalar forms) of compatibility conditions in continuum mechanics. The conditions $(4.7)_2$ can be reduced to those derived in Cartesian coordinates by Shield [4]; other forms are new.

In some applications it may be more convenient to use vector representation of tensor fields. Expressing the rotation tensor by means of an equivalent finite rotation vector, one can arrive at four different but equivalent forms of the first-order differential equations for the finite rotation vector. Integrability conditions for each vector differential equation again provide four different but equivalent vector forms of compatibility conditions in continuum mechanics. They are expressed in terms of vectors of change of curvatures of the coordinate lines. The relations analogous to (4.17) were given by Signorini [5], Guo [6] and Shamina [7]; other vector forms of compatibility conditions are new. They are of particular interest in the non-linear theories of thin bodies such as bars, plates and shells [8-13].

Original vector representatives (4.24)-(4.26) of the classical tensor forms of compatibility conditions are also given. They may be convenient to apply when solving some specific problems of continuum deformation. These vector forms were constructed from axial vectors of skew-symmetric parts of the tensors appearing in the linear differential equations for the deformation gradient field.

Finally, three forms of compatibility conditions referred to the principal directions of strain are derived. They may be of particular interest when solving special problems of continuum deformation in which the lines of principal directions constitute a relatively simple spatial grid.

All the geometric relations and various forms of compatibility conditions are derived here in the Lagrangian description, with respect to the undeformed state as the reference configuration. Similar dual relations and appropriate dual forms of compatibility conditions in the Eulerian description may be derived along the same lines, just by describing the continuum deformation with respect to the deformed state as the reference configuration, or by using appropriate transformation rules.

2. DEFORMATION AND COMPATIBILITY

During deformation of a material body $\mathcal{B}$ in the three-dimensional Euclidian space E the places occupied by a particle $X \in \mathcal{B}$ in the reference (undeformed) and deformed configurations are described by the respective position vectors

$$\underline{p} = x_a(\theta^i)\underline{i}_a, \quad \bar{\underline{p}} = y_a(\theta^i)\underline{i}_a = \underline{p} + \underline{u}, \quad a,i = 1,2,3. \tag{2.1}$$

Here θ^i are the curvilinear convected coordinates, $\underline{i}_a$ is the common orthonormal basis attached to an origin $0 \in E$, x_a and y_a are the Cartesian coordinates of $X \in \mathcal{B}$ and $\underline{u} = \underline{u}(\theta^i)$ is the displacement vector. The region $\mathcal{D}$, occupied by the body $\mathcal{B}$ in the reference configuration, is assumed to be simply connected with boundary $\partial\mathcal{D}$ consisting of the piecewise regular surfaces.

The position vectors define the base vectors $\underline{g}_i$, $\bar{\underline{g}}_i$ and components g_{ij}, $\bar{g}_{ij}$ of the metric tensor $\underline{1}$ in the reference and deformed configuration respectively:

$$\underline{g}_i = \frac{\partial\underline{p}}{\partial\theta^i} \equiv \underline{p}_{,i}, \quad \bar{\underline{g}}_i = \bar{\underline{p}}_{,i} = \underline{g}_i + \underline{u}_{,i},$$

$$g_{ij} = \underline{g}_i\cdot\underline{g}_j, \quad \bar{g}_{ij} = \bar{\underline{g}}_i\cdot\bar{\underline{g}}_j = g_{ij} + 2E_{ij}, \quad g_{ij}g^{ik} = \bar{g}_{ij}\bar{g}^{ik} = \delta^k_j, \tag{2.2}$$

$$\underline{1} = g_{ij}\underline{g}^i \otimes \underline{g}^j = \bar{g}_{ij}\bar{\underline{g}}^i \otimes \bar{\underline{g}}^j, \quad g = |g_{ij}|, \quad \bar{g} = |\bar{g}_{ij}|,$$

$$E_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i} + g^{mn}u_{m|i}u_{n|j})$$

where E_{ij} are components of the symmetric Lagrangian strain tensor $\underline{E} = E_{ij}\underline{g}^i \otimes \underline{g}^j$ and $(\ )_{|i}$ denotes the covariant derivative with respect to the reference metric g_{ij}.

Let G_{kij}, G^k_{ij} and $\bar{G}_{kij}$, $\bar{G}^k_{ij}$ be the Christoffel symbols of the first and second kind in the reference and deformed configurations, respectively. Recall the known geometric relations [8, 14, 15]:

$$\underline{g}_{i,j} = G^k_{ij}\underline{g}_k, \quad \underline{g}^k_{\ ,i} = -G^k_{ij}\underline{g}^j, \quad \underline{g}_{i|j} = \underline{g}^i_{\ |j} = \underline{0},$$

$$\bar{\underline{g}}_{i,j} = \bar{G}^k_{ij}\bar{\underline{g}}_k, \quad \bar{\underline{g}}^k_{\ ,i} = -\bar{G}^k_{ij}\bar{\underline{g}}^j, \quad G^k_{ij} = g^{kl}G_{lij},$$

$$G_{kij} = \frac{1}{2}(g_{ki,j} + g_{kj,i} - g_{ij,k}), \quad \bar{G}_{kij} = (g_{kl} + 2E_{kl})G^l_{ij} + E_{kij}, \tag{2.3}$$

$$A^k_{ij} = \bar{G}^k_{ij} - G^k_{ij} = \bar{g}^{kl}E_{lij}, \quad E_{kij} = E_{ki|j} + E_{kj|i} - E_{ij|k},$$

$$\varepsilon_{ijk} = \underline{g}_i \cdot (\underline{g}_j \times \underline{g}_k), \quad \varepsilon^{ijk} = \bar{\underline{g}}^i \cdot (\bar{\underline{g}}^j \times \bar{\underline{g}}^k) = \sqrt{(g/\bar{g})}\,\varepsilon^{ijk},$$

$$\varepsilon^{ijk}\varepsilon_{imn} = \delta^{jk}_{mn} = \delta^j_m\delta^k_n - \delta^j_n\delta^k_m, \quad \varepsilon^{ijk}\varepsilon_{ijm} = 2\delta^k_m,$$

$$\bar{g}^{il} = \frac{1}{2}\frac{g}{\bar{g}}\varepsilon^{ijk}\varepsilon^{lmn}(g_{jm} + 2E_{jm})(g_{kn} + 2E_{kn}), \tag{2.4}$$

$$\frac{\bar{g}}{g} = \frac{1}{6}\varepsilon^{ijk}\varepsilon^{lmn}(g_{il} + 2E_{il})(g_{jm} + 2E_{jm})(g_{kn} + 2E_{kn}).$$

For other geometric relations in convected coordinates we refer to [8, 15].

Compatibility conditions in continuum mechanics are usually understood as the conditions that should be imposed on appropriately smooth and continuous components of the strain field $\underline{E}$ which would ensure the existence of a single-valued and continuous displacement field $\underline{u}$.

A direct way of deriving the conditions would be the elimination of three displacements u_i from six second-order differential equations $(2.2)_4$ by their partial differentiation and some algebraic operations. Unfortunately, such a direct procedure, commonly used in the linear theory of elasticity, becomes extremely involved in the general case of finite deformations.

An alternative geometric method for the derivation of compatibility conditions takes into account the fact that during deformation the material body remains in the Euclidian space. Therefore, the components R_{klij} and $\bar{R}_{klij}$ of the Riemann-Christoffel curvature tensors in the reference and deformed body configurations, respectively, should identically vanish. Equivalent definitions for the components of the R-C curvature tensor in the reference configuration are [1, 14]

$$R_{klij} = g_{kr}(G^r_{lj,i} - G^r_{li,j} + G^r_{mi}G^m_{lj} - G^r_{mj}G^m_{li})$$

$$= -\frac{1}{2}\delta^{mr}_{kl}\delta^{ns}_{ij}g_{mn,rs} + g^{mn}(G_{mkj}G_{nli} - G_{mki}G_{nlj}),$$

$$R_{klij} = -R_{lkij} = -R_{klji} = R_{ijkl}, \quad R_{klij} + R_{kijl} + R_{kjli} = 0. \tag{2.5}$$

From analogous definitions of $\bar{R}_{klij}$, after involved transformations performed with the help of (2.3), (2.4) and (2.5), we obtain

$$\bar{R}_{klij} = (g_{kr} + 2E_{kr})R^r_{.lij} + E_{klij}, \tag{2.6}$$

$$E_{klij} = E_{kj|li} - E_{ki|lj} - E_{lj|ki} + E_{li|kj}$$
$$+ \bar{g}^{mn}(E_{mkj}E_{nli} - E_{mki}E_{nlj}). \tag{2.7}$$

Now it is evident that with $R_{klij} \equiv 0$ the conditions $\bar{R}_{klij} = 0$ reduce to the classical form of compatibility conditions for finite strains [1]

$$E_{klij} = 0. \tag{2.8}$$

Since among 81 components $\bar{R}_{klij}$ of the R-C curvature tensor only six are algebraically independent and non-vanishing, it is sometimes more convenient to use the Ricci tensor with components

$$\bar{P}^{rs} = \frac{1}{4}\bar{\varepsilon}^{rkl}\bar{\varepsilon}^{sij}\bar{R}_{klij}, \quad \bar{P}^{rs} = \bar{P}^{sr}. \tag{2.9}$$

Then (2.8), together with (2.7) and (2.9), leads to the more compact form of the classical compatibility conditions [16]:

$$\bar{P}^{rs} = \bar{\varepsilon}^{rkl}\bar{\varepsilon}^{sij}(E_{kj|li} + \frac{1}{2}\bar{g}^{mn}E_{mkj}E_{nli}) = 0. \tag{2.10}$$

The geometric method of derivation of the compatibility conditions (2.8) and (2.10) may raise the question of in what sense the conditions are indeed equivalent to the integrability conditions for the system of six non-linear partial differential equations $(2.2)_4$ with respect to three displacements. Note that here six algebraically independent components $\bar{R}_{klij}$ or $\bar{P}^{rs}$ are subject to three differential Bianchi identities [14]. In the limiting case of infinitesimal deformation, when $E_{ij} \simeq e_{ij} = \frac{1}{2}(u_{i|j} + u_{j|i})$, the resulting six linear compatibility conditions $\varepsilon^{rkl}\varepsilon^{sij}e_{kj|li} = 0$ may be divided into two groups of three equations. Then using Bianchi identities it may be proved [29] that, when one of the groups of equations is satisfied in the interior domain $\mathcal{D}$ of the reference body configuration and all six conditions are satisfied on the reference boundary surface $\partial\mathcal{D}$, the remaining group of equations becomes identically satisfied in the interior domain $\mathcal{D}$ as well. Unfortunately, for a finite deformation the full implications of Bianchi identities for the solutions of (2.8) or (2.10) have not yet been explored.

In our opinion, the notion of compatibility conditions in continuum mechanics should be associated with the integrability conditions for some definite systems of linear first-order partial differential equations, the solutions of which would allow the determination of the displacement field from a given strain field only through quadratures and algebraic operations. From this point of view several different systems of linear first-order partial differential equations may be examined, which lead to several different but equivalent forms of compatibility conditions in the continuum mechanics.

Let us introduce [3, 8] the deformation gradient tensor

$$\underline{F} = \nabla\underline{\bar{p}} = \underline{\bar{g}}_i \otimes \underline{g}^i = \underline{1} + \underline{u}_{,i} \otimes \underline{g}^i,$$
$$\underline{F}^{-1} = \underline{g}_j \otimes \underline{\bar{g}}^j, \qquad 0 < \det \underline{F} < +\infty. \tag{2.11}$$

The tensor contains complete information as to the strains and rotations of all material elements in the neighbourhood of $X \in \mathcal{B}$.

Differentiating the field $\underline{F}$ along the coordinate lines we obtain

$$\underline{F}_{,j} = \underline{F}\underline{A}_j, \; \underline{F}_{,j} = \underline{B}_j\underline{F}, \tag{2.12}$$

$$\underline{A}_j = A^k_{lj}\underline{g}_k \otimes \underline{g}^l = \bar{g}^{mn}g_{mk}E_{nlj}\underline{g}^k \otimes \underline{g}^l \tag{2.13}$$

$$\underline{B}_j = A^k_{lj}\underline{\bar{g}}_k \otimes \underline{\bar{g}}^l = E_{klj}\underline{\bar{g}}^k \otimes \underline{\bar{g}}^l = \underline{F}\underline{A}_j\underline{F}^{-1}. \tag{2.14}$$

When the symbols G^k_{ij} and $\bar{G}^k_{ij}$ are supposed to be known, each of the relations (2.12) is equivalent to the system of 27 scalar linear first-order partial differential equations for nine components of $\underline{F}$. If the Christoffel symbols are connected through given strains according to $(2.3)_3$, then using (2.12) $\underline{u}$ may be determined from $\underline{E}$ in two steps. First, $\underline{F}$ is calculated from given $\underline{E}$, then $\underline{u}$ is determined from already known $\underline{F}$.

Suppose, for example, that the system $(2.12)_1$ is integrated and the field $\underline{F}$ is constructed. Then $\underline{\bar{g}}_i = \underline{F}\underline{g}_i$ and the position vector $\underline{\bar{p}}$ (and, therefore, the displacement vector $\underline{u} = \underline{\bar{p}} - \underline{p}$) follows through quadratures [2] from the system of nine scalar linear first-order partial differential equations

$$\underline{\bar{p}}_{,i} = \underline{\bar{g}}_i. \tag{2.15}$$

The integrability conditions [17, 18] for such linear systems are

$$\underline{\bar{p}}_{,[ij]} \equiv \underline{\bar{p}}_{,ij} - \underline{\bar{p}}_{,ji} = (\bar{G}^k_{ij} - \bar{G}^k_{ji})\underline{\bar{g}}_k = \underline{0}. \tag{2.16}$$

Relations (2.16) are equivalent to six independent scalar conditions and ensure the existence of three components of $\underline{\bar{p}}$

satisfying (2.15). Since $\bar{G}^k_{ij} - \bar{G}^k_{ji} = \bar{S}^k_{ij}$ are components of the torsion tensor [14], conditions (2.16) may be interpreted geometrically as the vanishing of the torsion tensor in the deformed body configuration. For $\bar{G}^k_{ij}$ calculated from G^k_{ij} and E_{ij} according to $(2.3)_3$, conditions (2.16) are identically satisfied. Therefore, integration of (2.15) does not require any additional conditions for strains. As a result, the only conditions for strains which would allow the determination of $\underline{u}$ from given $\underline{E}$ are those required for integration of $(2.12)_1$.

To ensure the existence of an appropriate non-singular tensor field $\underline{F}$ of class C^2, which is the solution of (2.12), the following integrability conditions [17, 18] for (2.12) should be satisfied:

$$\underline{F}_{,[ji]} \equiv \underline{F}_{,ji} - \underline{F}_{,ij} = \underline{0}. \tag{2.17}$$

In terms of $\underline{A}_j$ or $\underline{B}_j$ this leads to

$$\underline{F}^{-1}\underline{F}_{,[ji]} = \underline{A}_{[j,i]} - \underline{A}_{[j}\underline{A}_{i]} = \underline{0}$$

$$= F^k_{.lij}\underline{g}_k \otimes \underline{g}^l = \underline{0}, \tag{2.18}$$

$$\underline{F}_{,[ji]}\underline{F}^{-1} = \underline{B}_{[j,i]} + \underline{B}_{[j}\underline{B}_{i]} = \underline{0}$$

$$= F^k_{.lij}\bar{\underline{g}}_k \otimes \bar{\underline{g}}^l = \underline{0}. \tag{2.19}$$

Substituting (2.13), (2.14) into (2.18), (2.19) and using $(2.3)_4$ together with (2.5) we obtain

$$F^k_{.lij} = A^k_{lj|i} - A^k_{li|j} + A^k_{mi}A^m_{lj} - A^k_{mj}A^m_{li} = \bar{R}^k_{.lij} - R^k_{.lij} \tag{2.20}$$

which leads to the following component form of (2.18) and (2.19):

$$F^k_{.lij} = 0. \tag{2.21}$$

Each of relations (2.18), (2.19) or (2.21) is equivalent to 18 independent scalar differential conditions which should be satisfied by $\bar{G}^k_{ij}$. If $\bar{G}^k_{ij}$ are expressed in terms of G^k_{ij} and E_{ij} according to $(2.3)_3$, then 12 of the conditions will become identities for any E_{ij}.

It is easy to see that relations (2.18), (2.19) or (2.21) are just the alternative forms of compatibility conditions in continuum mechanics, which are equivalent to those derived using geometric arguments (2.8). However, those conditions have been derived here directly as the integrability conditions for the systems of linear equations (2.12). The tensor form (2.18) was given in [2], while (2.19) was presented in [19]. The component form (2.21), however also appeared in [7].

If $\bar{G}^k_{ij}$ are expressed in terms of strains, the differential equations $(2.12)_1$ can be transformed into the alternative tensor form

$$\underline{F}_{,j} = \underline{F}\underline{C}^{-1}\underline{E}_j \tag{2.22}$$

$$\underline{E}_j = E_{klj}\underline{g}^k \otimes \underline{g}^l, \qquad \underline{C}^{-1} = \underline{F}^{-1}\underline{F}^{-T} = \bar{g}^{ij}\underline{g}_i \otimes \underline{g}_j,$$
$$\underline{C} = \underline{F}^T\underline{F} = \bar{g}_{ij}\underline{g}^i \otimes \underline{g}^j, \quad \underline{C}_{,j} = \underline{E}_j + \underline{E}_j^T. \tag{2.23}$$

Since $\underline{F}\underline{C}^{-1}$ is also non-singular, the integrability conditions for (2.22) lead to

$$\underline{E}_{[j,i]} + \underline{E}_{[j}^T\underline{C}^{-1}\underline{E}_{i]} = E_{klij}\underline{g}^k \otimes \underline{g}^l = \underline{0}. \tag{2.24}$$

Again, relation (2.24) is just the alternative tensor form of compatibility conditions in the continuum mechanics [20]. It is equivalent to the component form (2.8) which expresses the vanishing of the R-C curvature tensor in the deformed body configuration.

3. FINITE ROTATIONS

According to the polar decomposition of $\underline{F}$ we have [3]

$$\underline{F} = \underline{RU} = \underline{VR}, \tag{3.1}$$

where $\underline{U}$ and $\underline{V}$ are the right and left stretch tensors (positive definite and symmetric) and $\underline{R}$ is the rotation tensor (proper orthogonal). They are defined by the relations

$$\underline{U} = (\underline{F}^T\underline{F})^{1/2}, \qquad \underline{V} = (\underline{FF}^T)^{1/2} = \underline{RUR}^T,$$
$$\underline{R} = \underline{FU}^{-1} = \underline{V}^{-1}\underline{F},\ \underline{R}^{-1} = \underline{R}^T,\ \det \underline{R} = +1. \tag{3.2}$$

Let us introduce two non-holonomic intermediate stretched or rotated bases [8, 21]:

$$\underline{s}_i = \underline{Ug}_i = \underline{R}^T\bar{\underline{g}}_i,\ \underline{r}_i = \underline{Rg}_i = \underline{V}^{-1}\bar{\underline{g}}_i, \tag{3.3}$$

$$\underline{s}_i\cdot\underline{s}_j = \bar{g}_{ij}, \qquad \underline{r}_i\cdot\underline{r}_j = g_{ij}. \tag{3.4}$$

The two intermediate bases (3.3) are uniquely defined only in the convected system of coordinates used here. In the case of two independent coordinate systems [1] introduced in the reference and deformed configurations, there would appear four different non-holonomic intermediate bases. Then the description of the deformation in terms of such four bases would become more complex.

In terms of bases (3.3) we obtain the following Lagrangian representations of various tensors

$$\underline{U} = \underline{s}_i \otimes \underline{g}^i = U_{ij}\underline{g}^i \otimes \underline{g}^j = (U^{-1})^{ij}\underline{s}_i \otimes \underline{s}_j,$$
$$\underline{U}^{-1} = \underline{s}^i \otimes \underline{g}_i = (U^{-1})^{ij}\underline{g}_i \otimes \underline{g}_j = U_{ij}\underline{s}^i \otimes \underline{s}^j, \tag{3.5}$$

$$\underline{V} = \bar{\underline{g}}_i \otimes \underline{r}^i = (U^{-1})^{ij}\bar{\underline{g}}_i \otimes \bar{\underline{g}}_j = U_{ij}\underline{r}^i \otimes \underline{r}^j,$$
$$\underline{V}^{-1} = U_{ij}\bar{\underline{g}}^i \otimes \bar{\underline{g}}^j = (U^{-1})^{ij}\underline{r}_i \otimes \underline{r}_j = \bar{\underline{g}}^i \otimes \underline{r}_i, \tag{3.6}$$

$$\underline{R} = \bar{\underline{g}}_i \otimes \underline{s}^i = \underline{r}_i \otimes \underline{g}^i,\ \underline{R}^{-1} = \underline{s}_i \otimes \bar{\underline{g}}^i = \underline{g}_i \otimes \underline{r}^i, \tag{3.7}$$

$$(U^{-1})^{mn} = \frac{1}{2}\sqrt{\left(\frac{g}{\bar{g}}\right)}\,\varepsilon^{mij}\varepsilon^{nkl}U_{ik}U_{jl}. \tag{3.8}$$

As a result of the polar decomposition (3.1), the problem of determination of $\underline{F}$ from given $\underline{E}$ may again be solved in two steps. First, $\underline{U}$ may be constructed from $\underline{E}$, and then $\underline{R}$ may be calculated from $\underline{U}$ and $\underline{E}$. The first step is purely algebraic and its completion does not require any additional differential conditions on strains. Thus, the only differential conditions which would ensure the existence of $\underline{F}$ satisfying (2.12) are those ensuring the existence of $\underline{R}$ which satisfy an appropriate system of differential equations following from (2.12) and (3.1).

Let us introduce (3.1) into (2.12). Making use of (3.5) and (2.3) we obtain

$$\underline{R}_{,j} = \underline{RK}_j,\ \underline{R}_{,j} = \underline{L}_j\underline{R}, \tag{3.9}$$

$$\underline{K}_j = (\underline{UA}_j - \underline{U}_{,j})\underline{U}^{-1} = (\bar{G}^k_{ij}\underline{s}_k - \underline{s}_{i,j}) \otimes \underline{s}^i$$
$$= K_{klj}\underline{g}^k \otimes \underline{g}^l = L_{klj}\underline{s}^k \otimes \underline{s}^l, \tag{3.10}$$

$$\underline{L}_j = \underline{B}_j - \underline{RU}_{,j}\underline{U}^{-1}\underline{R}^T = (\bar{G}^k_{ij} - \underline{s}^k\cdot\underline{s}_{i,j})\bar{\underline{g}}_k \otimes \bar{\underline{g}}^i$$
$$= K_{klj}\underline{r}^k \otimes \underline{r}^l = L_{klj}\bar{\underline{g}}^k \otimes \bar{\underline{g}}^l = \underline{RK}_j\underline{R}^T, \tag{3.11}$$

$$K_{klj} = (U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_l,$$
$$L_{klj} = E_{klj} - g^{mn}U_{mk}U_{nl|j} = U^m_k U^n_l K_{mnj}. \tag{3.12}$$

It follows from $(\underline{R}^T\underline{R})_{,j} = 0$ and (3.9) that the tensors $\underline{K}_j$ and $\underline{L}_j$ are skew-symmetric: $\underline{K}_j = -\underline{K}_j^T$, $\underline{L}_j = -\underline{L}_j^T$. Therefore, there exist axial vectors $\underline{k}_j$ and $\underline{l}_j$ such that

$$\underline{K}_j = \underline{k}_j \times \underline{1},\ \underline{L}_j = \underline{l}_j \times \underline{1},\ \underline{l}_j = \underline{R}\underline{k}_j, \tag{3.13}$$

$$\underline{k}_j = \frac{1}{2}\underline{g}^i \times \underline{K}_j\underline{g}_i = \frac{1}{2}\varepsilon^{ikl}(U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_i\underline{g}_l$$

$$= \frac{1}{2}\underline{s}^i \times \underline{K}_j\underline{s}_i = \bar{\varepsilon}^{ikl}(E_{kj|i} - \frac{1}{2}g^{mn}U_{mk}U_{ni|j})\underline{s}_l, \tag{3.14}$$

$$\underline{l}_j = \frac{1}{2}\underline{r}^i \times \underline{L}_j\underline{r}_i = \frac{1}{2}\varepsilon^{ikl}(U_{kn}A^n_{mj} - U_{km|j})(U^{-1})^m_i\underline{r}_l$$

$$= \frac{1}{2}\bar{\underline{g}}^i \times \underline{L}_j\bar{\underline{g}}_i = \bar{\varepsilon}^{ikl}(E_{kj|i} - \frac{1}{2}g^{mn}U_{mk}U_{ni|j})\bar{\underline{g}}_l. \tag{3.15}$$

The tensors $\underline{K}_j$, $\underline{L}_j$ so defined and their axial vectors $\underline{k}_j$, $\underline{l}_j$ describe that part of change of the curvatures of the curvilinear coordinate lines which is caused by the variation of $\underline{R}$ along the lines. For this reason, they are called the tensors and the vectors of change of curvature of the coordinate lines. The tensors and vectors are described entirely by the strains and stretches, (3.12), (3.14) and (3.15), and do not depend explicitly upon the rotations. Note also that $\underline{L}_j$ and $\underline{l}_j$ have components identical to $\underline{K}_j$ and $\underline{k}_j$ but referred to different bases related to each other by a rigid-body rotation $\underline{R}$.

Alternative definitions of the tensors $\underline{K}_j$ and $\underline{L}_j$ may be found if the second polar decomposition $(3.1)_2$ is introduced into (2.12), which gives

$$\underline{K}_j = \underline{A}_j - \underline{R}^T\underline{V}^{-1}\underline{V}_{,j}\underline{R} = -(G^k_{ij} + \underline{r}_i\cdot\underline{r}^k_{,j})\underline{g}_k \otimes \underline{g}^i,$$
$$\underline{L}_j = \underline{V}^{-1}(\underline{B}_j\underline{V} - \underline{V}_{,j}) = -\underline{r}_k \otimes (G^k_{ij}\underline{r}^i + \underline{r}^k_{,j}). \tag{3.16}$$

According to (3.10), (3.11) and (3.16) each equation (3.9) can be presented in two different but equivalent forms.

Therefore, (3.9) represent, in fact, four different but equivalent forms of the system of linear first-order partial differential equations for the rotation field $\underline{R}$.

Relations (3.10)-(3.16) yield the rules for partial and covariant differentiation of the stretched or rotated base vectors

$$\underline{s}_{i,j} = \bar{G}^k_{ij}\underline{s}_k - \underline{k}_j \times \underline{s}_i,\ \underline{s}_{i||j} = -\underline{k}_j \times \underline{s}_i,$$
$$\underline{r}_{i,j} = G^k_{ij}\underline{r}_k + \underline{l}_j \times \underline{r}_i,\ \underline{r}_{i|j} = \underline{l}_j \times \underline{r}_i, \tag{3.17}$$

where $(\ )_{||j}$ denotes the covariant derivative with respect to the deformed metric $\bar{g}_{ij}$.

Sometimes it is more convenient to use a vector representation of the rotations. Note that any proper orthogonal tensor $\underline{R}$ may be described by the unit vector $\underline{e}$ of the rotation axis and the rotation angle ω about $\underline{e}$ according to [9, 10, 22]

$$\underline{R} = \cos\omega\,\underline{1} + \sin\omega\,\underline{e} \times \underline{1} + (1 - \cos\omega)\,\underline{e} \otimes \underline{e}. \tag{3.18}$$

Substitution of (3.18) into (3.9) and use of (3.13) results, after involved transformations [21], into

$$\underline{k}_j = \sin\omega\,\underline{e}_{,j} + (1 - \cos\omega)\,\underline{e}_{,j} \times \underline{e} + \omega_{,j}\underline{e},$$
$$\underline{l}_j = \sin\omega\,\underline{e}_{,j} - (1 - \cos\omega)\,\underline{e}_{,j} \times \underline{e} + \omega_{,j}\underline{e}. \tag{3.19}$$

The inverse formulae are

$$\underline{e}_{,j} = \frac{1}{2}\underline{e} \times \underline{k}_j - \frac{1}{2\mathrm{tg}\omega/2}\underline{e} \times (\underline{e} \times \underline{k}_j),$$
$$\underline{e}_{,j} = -\frac{1}{2}\underline{e} \times \underline{l}_j - \frac{1}{2\mathrm{tg}\omega/2}\underline{e} \times (\underline{e} \times \underline{l}_j). \tag{3.20}$$

Several different definitions of the finite rotation vectors may be found in the literature. Some of them may be more convenient than others for specific purposes. In particular, the vector $\underline{\Omega} = \sin\omega\,\underline{e}$ as used in [7-11] may be expressed conveniently in terms of displacements, the vector $\underline{\theta} = 2\,\mathrm{tg}\,\omega/2\,\underline{e}$ as used in [23] leads to differential relations which do not explicitly contain trigonometric functions, while the vector $\underline{\omega} = \omega\underline{e}$, used in [24], may be represented as a logarithmic function of $\underline{R}$. In case of infinitesimal rotations all three definitions given above reduce to the infinitesimal rotation vector as defined in classical linear elasticity.

Substitution of

$$\underline{e} = \frac{1}{2\,\mathrm{tg}\,\omega/2}\,\underline{\theta}$$

into (3.23)-(3.27) leads, after elementary but involved transformations, to

$$\underline{R} = \underline{1} + \frac{1}{1+\underline{\theta}^2/4}\left[\underline{\theta}\times\underline{1} + \frac{1}{2}\,\underline{\theta}\times(\underline{\theta}\times\underline{1})\right],\quad t = 1 + \frac{1}{4}\,\underline{\theta}^2, \tag{3.21}$$

$$\underline{k}_j = \frac{1}{t}\left(\underline{\theta}_{,j} + \frac{1}{2}\,\underline{\theta}_{,j}\times\underline{\theta}\right),$$
$$\underline{l}_j = \frac{1}{t}\left(\underline{\theta}_{,j} - \frac{1}{2}\,\underline{\theta}_{,j}\times\underline{\theta}\right), \tag{3.22}$$

$$\underline{\theta}_{,j} = t\underline{k}_j + \frac{1}{2}\,\underline{\theta}\times\underline{k}_j + \frac{1}{4}\,\underline{\theta}\times(\underline{\theta}\times\underline{k}_j),$$
$$\underline{\theta}_{,j} = t\underline{l}_j - \frac{1}{2}\,\underline{\theta}\times\underline{l}_j + \frac{1}{4}\,\underline{\theta}\times(\underline{\theta}\times\underline{l}_j). \tag{3.23}$$

Equivalent expressions in terms of $\underline{\Omega}$, $\underline{\omega}$ or any other finite rotation vector may easily be found by similar direct transformations, [21].

Again, since each of $\underline{k}_j$ and $\underline{l}_j$ can be given in two different but equivalent forms (3.14) and (3.15), relations (3.23) give us four different but equivalent forms of the

system of linear first-order partial differential equations for the finite rotation vector $\underline{\theta}$.

The explicit investigation of differential properties of the rotations associated with an arbitrary deformation of the classical continuum exhibits some formal analogies with the description of deformation of the Cosserat continuum [25-27]. However, for the Cosserat continuum both displacements and rotations are independent variables of deformation, while in the classical continuum mechanics only the displacement field is an independent variable. The rotation field in this case is an internal parameter of deformation entirely described by the displacement field.

4. NON-CLASSICAL FORMS OF COMPATIBILITY CONDITIONS

When strains and stretches are supposed to be known, relations (3.9) may be regarded as systems of the linear first-order partial differential equations for the rotation field. The field $\underline{R}$ is entirely described by three scalar parameters only, for example, the rotation angle ω and two angles of $\underline{e}$ in the Cartesian frame, or three Euler angles. Thus, each form of tensor equations (3.9) is equivalent to nine independent scalar differential equations. If $\underline{R}$ is found from $(3.9)_1$ (see [2, 21]) then $\underline{F} = \underline{RU}$ and $\bar{\underline{p}}$ (or $\underline{u}$) follows through quadratures from (2.15).

The integrability conditions $\underline{R}_{,[ji]} = \underline{0}$ for each system (3.9) lead to the following tensor relations

$$\underline{R}^T\underline{R}_{,[ji]} = \underline{K}_{[j,i]} - \underline{K}_{[j}\underline{K}_{i]} = \underline{0}$$
$$= K_{klij}\,\underline{g}^k\otimes\underline{g}^l = L_{klij}\,\underline{s}^k\otimes\underline{s}^l = \underline{0}, \tag{4.1}$$

$$\underline{R}_{,[ji]}\underline{R}^T = \underline{L}_{[j,i]} + \underline{L}_{[j}\underline{L}_{i]} = \underline{0}$$

$$= K_{klij}\underline{r}^k \otimes \underline{r}^l = L_{klij}\underline{\bar{g}}^k \otimes \underline{\bar{g}}^l = \underline{0}. \qquad (4.2)$$

Substitution of (3.10) and (3.11) into (4.1) and (4.2) leads to

$$K_{klij} = K_{klj|i} - K_{kli|j} + \bar{g}^{mn}(K_{mkj}K_{nli} - K_{mki}K_{nlj}), \qquad (4.3)$$

$$L_{klij} = L_{klj\|i} - L_{kli\|j} - \bar{g}^{mn}(L_{mkj}L_{nli} - L_{mki}L_{nlj})$$

$$= L_{klj|i} - L_{kli|j} + \bar{g}^{mn}(E_{mkj}E_{nli} - E_{mki}E_{nlj})$$

$$- g_{pq}\bar{g}^{pm}\bar{g}^{qn}(U_{mk|j}U_{nl|i} - U_{mk|i}U_{nl|j}), \qquad (4.4)$$

$$L_{klij} = U^m_k U^n_l K_{mnij}, \qquad (4.5)$$

$$K_{klij} = -K_{lkij} = -K_{klji} = K_{ijkl},$$
$$L_{klij} = -L_{lkij} = -L_{klji} = L_{ijkl}. \qquad (4.6)$$

Now tensor relations (4.1) and (4.2) may be written in component forms as follows:

$$K_{klij} = 0, \; L_{klij} = 0. \qquad (4.7)$$

Because of the symmetry conditions (4.6) each of relations (4.7) is equivalent to six independent differential equations for strains. They ensure the existence of three scalar parameters of $\underline{R}$ satisfying (3.9).

From the polar decomposition (3.1) we obtain

$$\underline{F}_{,[ji]} = \underline{R}_{,[ji]}\underline{U} + \underline{RU}_{,[ji]} = \underline{VR}_{,[ji]} + \underline{V}_{,[ji]}\underline{R}. \qquad (4.8)$$

It follows from (4.8) that the integrability conditions (2.18) and (2.19) are satisfied if (4.1) and (4.2) together with

$$\underline{U}_{,[ji]} = \underline{0}, \; \underline{V}_{,[ji]} = \underline{0} \qquad (4.9)$$

are satisfied. Each of the tensor relations (4.9) is equivalent to twelve independent scalar differential conditions. For given strains the conditions (4.9) become identities. As a result, each of the six conditions (4.1), (4.2) or (4.7) which ensures the existence of the rotation field $\underline{R}$ derivable from the given strain field $\underline{E}$ implies at the same time the existence of the compatible displacement field $\underline{u}$.

Using (2.18), (2.20), $(4.8)_1$, $(4.9)_1$ and (3.5) one can relate the expressions L_{klij} to components of the R-C curvature tensors according to

$$\bar{R}_{klij} = L_{klij} + \bar{g}_{km}R^m_{.lij}. \qquad (4.10)$$

From (4.10) and (4.5) it is evident that with $R_{klij} \equiv 0$ conditions (4.7) are equivalent to $\bar{R}_{klij} = 0$.

Tensor relations (4.1), (4.2) and their component representations (4.7) are the alternative forms of compatibility conditions in the continuum mechanics. For given strains they are fully equivalent to the compatibility conditions derived in Section 2 based on geometric considerations. It means that, in fact, the existence of a rotation field derivable from a given strain field is sufficient for the compatibility of deformation of a continuum. The form $(4.7)_2$ was derived by Shield [4] in Cartesian coordinates; other forms (4.1), (4.2) and $(4.7)_1$ of the compatibility conditions are new, [21].

Taking into account the symmetry conditions (4.6) and (3.8) we may introduce the symmetric tensors with components

$$K^{rs} = \frac{1}{4}\varepsilon^{rkl}\varepsilon^{sij}K_{klij} = \frac{1}{2}\varepsilon^{rkl}\varepsilon^{sij}(K_{klj|i} + \bar{g}^{mn}K_{mkj}K_{nli}). \tag{4.11}$$

$$L^{rs} = \frac{1}{4}\bar{\varepsilon}^{rkl}\bar{\varepsilon}^{sij}L_{klij} = \frac{1}{2}\bar{\varepsilon}^{rkl}\bar{\varepsilon}^{sij}(L_{klj||i} - \bar{g}^{mn}L_{mkj}L_{nli})$$

$$= \sqrt{\left(\frac{g}{\bar{g}}\right)}(U^{-1})^{r}_{p}K^{ps}, \tag{4.12}$$

which allow us to write (4.7) in the more compact form

$$K^{rs} = 0, \; L^{rs} = 0. \tag{4.13}$$

It follows from (4.10), (4.5) and (2.9) that the compatibility conditions (4.13) are equivalent to (2.10) which require vanishing of the Ricci tensor in the deformed body configuration.

The tensor forms of compatibility conditions (4.1) and (4.2) may be expressed in terms of the vectors of change of curvature $\underline{k}_j$ and $\underline{l}_j$. Note that for any vectors $\underline{v}$ and $\underline{w}$ we have the identity

$$(\underline{v} \times \underline{1})(\underline{w} \times \underline{1}) - (\underline{w} \times \underline{1}(\underline{v} \times \underline{1}) = (\underline{v} \times \underline{w}) \times \underline{1}. \tag{4.14}$$

Introducing representations (3.13) into (4.1) and (4.2) and using (4.14), we obtain

$$\begin{aligned} \underline{R}^T\underline{R}_{,[ji]} &= (\underline{k}_{j,i} - \underline{k}_{i,j} - \underline{k}_j \times \underline{k}_i) \times \underline{1}, \\ \underline{R}_{,[ji]}\underline{R}^T &= (\underline{l}_{j,i} - \underline{l}_{i,j} + \underline{l}_j \times \underline{l}_i) \times \underline{1}. \end{aligned} \tag{4.15}$$

The vectors $\underline{k}_j$ may be represented either in the reference basis $\underline{g}_i$ according to $(3.14)_1$ or in the stretched basis $\underline{s}_i$ according to $(3.14)_2$. Then (4.1) and $(4.15)_1$ lead to

$$\begin{aligned} K^{rs}\underline{g}_r &= \frac{1}{4}\varepsilon^{sij}\underline{\varepsilon}\cdot(\underline{R}^T\underline{R}_{,[ji]}) \\ &= \varepsilon^{sij}(\underline{k}_{i,j} - \frac{1}{2}\underline{k}_i \times \underline{k}_j) = \underline{0}, \end{aligned} \tag{4.16}$$

$$\begin{aligned} L^{rs}\underline{s}_r &= \frac{1}{4}\bar{\varepsilon}^{sij}\underline{\bar{\varepsilon}}\cdot(\underline{R}^T\underline{R}_{,[ji]}) \\ &= \bar{\varepsilon}^{sij}(\underline{k}_{i,j} - \frac{1}{2}\underline{k}_i \times \underline{k}_j) = \underline{0}. \end{aligned} \tag{4.17}$$

where $\underline{\varepsilon} = -\underline{1} \times \underline{1} = \varepsilon^{rkl}\underline{g}_r \otimes \underline{g}_k \otimes \underline{g}_l$ and $\underline{\varepsilon}\cdot\underline{T} = \varepsilon^{rkl}T_{kl}\underline{g}_r$.

Similarly, representing the vectors $\underline{l}_j$ either in the rotated basis $\underline{r}_i$ according to $(3.15)_1$ or in the deformed basis $\bar{\underline{g}}_i$ as in $(3.15)_2$, we have from relations (4.2) and $(4.15)_2$

$$K^{rs}\underline{r}_r = \varepsilon^{sij}(\underline{l}_{i,j} + \frac{1}{2}\underline{l}_i \times \underline{l}_j) = \underline{0}, \tag{4.18}$$

$$L^{rs}\bar{\underline{g}}_r = \bar{\varepsilon}^{sij}(\underline{l}_{i,j} + \frac{1}{2}\underline{l}_i \times \underline{l}_j) = \underline{0}. \tag{4.19}$$

Relations (4.16)-(4.19) are vector forms of compatibility conditions in the continuum mechanics, which are equivalent to (4.13) and, therefore, to (2.10). The compatibility conditions analogous to (4.17) were given by Signorini [5], Guo [6] and Shamina [7] as integrability conditions for the system of linear differential equations with respect to the finite rotation vectors, analogous to $(3.23)_1$. In [5,6] the Gibbs' finite rotation vector $\underline{g} = \mathrm{tg}\,\omega/2\;\underline{e}$ was used, while in [7] the vector $\underline{\Omega} = \sin\omega\;\underline{e}$ was applied. The remaining three vector forms of compatibility conditions (4.17)-(4.19) are new in the literature, [21]. It is easy to verify by direct transformations that they are indeed the integrability conditions for the respective systems of linear differential equations (3.23) for the finite rotation vector $\underline{\theta}$.

The vector forms of compatibility conditions (4.16)-(4.19) are of particular interest in the non-linear theories of thin bodies such as bars, plates and shells. In fact, the two-dimensional counterpart of (4.17) was used in [8]-[11], while the two-dimensional compatibility conditions derived in [12, 13] are equivalent to the two-dimensional counterpart of (4.18), although not written explicitly in terms of vectors of change of curvature.

In some applications it may be convenient to introduce vector representations of the compatibility conditions (2.24), (2.18) or (2.19), which would be analogous to the vector compatibility conditions (4.16)-(4.19). The axial vectors may be constructed from skew-symmetric parts of the tensors $\underline{E}_j$, $\underline{A}_j$ or $\underline{B}_j$. Then

$$\underline{e}_j = \frac{1}{4} \underline{g}^r \times (\underline{E}_j - \underline{E}_j^T)\underline{g}_r = \varepsilon^{rkl} E_{lj|k}\underline{g}_r,$$

$$\underline{a}_j = \frac{1}{4} \underline{g}^r \times (\underline{A}_j - \underline{A}_j^T)\underline{g}_r = \frac{1}{2} \varepsilon^{rkl} g_{lm}\bar{g}^{mn} E_{nkj}\underline{g}_r, \qquad (4.20)$$

$$\underline{b}_j = \frac{1}{4} \underline{\bar{g}}^r \times (\underline{B}_j - \underline{B}_j^T)\underline{\bar{g}}_r = \bar{\varepsilon}^{rkl} E_{lj|k}\underline{\bar{g}}_r,$$

$$\underline{E}_j = \underline{E}_j^* + \underline{e}_j \times \underline{1}, \quad \underline{A}_j = \underline{A}_j^* + \underline{a}_j \times \underline{1}, \quad \underline{B}_j = \underline{B}_j^* + \underline{b}_j \times \underline{1}, \qquad (4.21)$$

$$\underline{E}_j^* = \frac{1}{2} \underline{C}_{,j}, \quad \underline{A}_j^* = \frac{1}{2} \underline{C}^{-1}\underline{C}_{,j}, \quad \underline{B}_j^* = \frac{1}{2} \underline{F}^{-T}\underline{C}_{,j}\underline{F}^{-1}. \qquad (4.22)$$

Now, from (2.24) and (2.9) it follows that

$$\frac{\bar{g}}{g} \bar{p}^{rs}\underline{g}_r = \varepsilon^{sij}[\underline{e}_{i,j} - \frac{1}{2} \underline{\varepsilon}\cdot(\underline{E}_i^T\underline{C}^{-1}\underline{E}_j)] = \underline{0}. \qquad (4.23)$$

After transformation of the last term of (4.23), with the help of $(4.20)_1$ and $(4.21)_1$ we obtain

$$\varepsilon^{sij}[\underline{e}_{i,j} - \frac{1}{2} \underline{C}^{-1}(\underline{e}_i \times \underline{e}_j) - \frac{1}{2} \underline{\varepsilon}\cdot(\underline{E}_i^*\underline{C}^{-1}\underline{E}_j^*)$$
$$- (\underline{C}^{-1}\underline{E}_i^* - \mathrm{tr}\underline{C}^{-1}\underline{E}_i^*)\underline{e}_j] = \underline{0} \qquad (4.24)$$

Similar transformations give us the following forms of (2.18) and (2.19)

$$\frac{\bar{g}}{g} \bar{p}^{rs}\underline{g}_r = \varepsilon^{sij} \{\frac{1}{2} (\underline{C}-\mathrm{tr}\underline{C})\underline{a}_{j,i} + \frac{1}{2} (\underline{C}\underline{a}_j) \times \underline{a}_i + \frac{1}{4} \underline{\varepsilon}\cdot(\underline{C}\underline{A}_{j,i}^*)$$
$$- \frac{1}{2} \underline{\varepsilon}\cdot[\underline{C}\underline{A}_j^*(\underline{a}_i \times \underline{1}) + \underline{C}(\underline{a}_j \times \underline{1})\underline{A}_i^*]\} = \underline{0}, \qquad (4.25)$$

$$\bar{p}^{rs}\underline{\bar{g}}_r = \bar{\varepsilon}^{sij}[\underline{b}_{i,j} + \frac{1}{2} (\underline{b}_i \times \underline{b}_j)$$
$$- \frac{1}{2} \underline{\bar{\varepsilon}}\cdot(\underline{B}_i^*\underline{B}_j^*) + (\underline{B}_i^* - \mathrm{tr}\underline{B}_i^*)\underline{b}_j] = \underline{0}. \qquad (4.26)$$

Relations (4.23)-(4.26) are again non-classical vector forms of compatibility conditions in continuum mechanics. Each of them is equivalent to the requirement of vanishing of the Ricci tensor in the deformed body configuration, (2.10). The form (4.23) was given by Ferrarese [20, 28]; other forms (4.24)-(4.26) are new.

5. COMPATIBILITY IN PRINCIPAL DIRECTIONS

Special types of continuum deformation may be described by assuming simple grids of lines of principal directions of strain. In such problems it may be more convenient to present the system of linear equations and their integrability conditions relative to the orthonormal bases associated with the principal directions. In what follows the compatibility conditions (4.16), (4.18) and (4.23) are presented relative to the principal directions. Proceeding along similar lines one can, if necessary, present other forms of compatibility conditions, as given in Sections 2 and 3, relative to the principal directions.

Since $\underline{U}^2 = \underline{C} = \underline{1} + 2\underline{E}$, all three tensors are co-axial. According to the spectral decomposition theorem [3] they can be put in the diagonal form

$$\underline{U} = U_a \underline{c}_a \otimes \underline{c}_a, \quad \underline{C} = C_a \underline{c}_a \otimes \underline{c}_a, \quad \underline{E} = E_a \underline{c}_a \otimes \underline{c}_a,$$
$$U_a = \sqrt{C_a} = \sqrt{(1 + 2E_a)}, \quad U_a > 0, \quad C_a > 0, \quad a = 1,2,3, \tag{5.1}$$

where the usual summation convention over three repeated indices is applied. In (5.1) U_a, C_a and E_a are eigenvalues of the respective tensors and $\underline{c}_a$ are eigenvectors of $\underline{C}$ satisfying

$$\underline{c}_a \cdot \underline{c}_b = \delta_{ab}, \quad \underline{C}\underline{c}_a = C_a \underline{c}_a \quad \text{(no sum over a)}. \tag{5.2}$$

The basis $\underline{c}_a$ is supposed to be given through a rigid-body rotation of the common orthonormal basis $\underline{i}_a$ of the Cartesian frame (2.1) according to

$$\underline{c}_a = \underline{Q}\underline{i}_a, \quad \underline{Q} = \underline{c}_a \otimes \underline{i}_a,$$
$$\underline{Q} = \underline{1} + \frac{1}{1 + \underline{q}^2/4}[\underline{q} \times \underline{1} + \frac{1}{2}\underline{q} \times (\underline{q} \times \underline{1})], \tag{5.3}$$

where $\underline{Q}$ is a proper orthogonal tensor and $\underline{q} = q_a \underline{c}_a$ is its equivalent finite rotation vector; see (3.21).

Similar diagonal representation for $\underline{V}$ takes, according to (3.2) and $(5.1)_1$, the form

$$\underline{V} = U_a \underline{d}_a \otimes \underline{d}_a, \quad \underline{d}_a \cdot \underline{d}_b = \delta_{ab},$$
$$\underline{d}_a = \underline{R}\underline{c}_a, \qquad \underline{R} = \underline{d}_a \otimes \underline{c}_a, \tag{5.4}$$

where $\underline{d}_a$ are eigenvectors of $\underline{V}$ satisfying $\underline{V}\underline{d}_a = U_a \underline{d}_a$ (no sum).

In this section it is convenient to assume that all quantities and equations are functions of the Cartesian coordinates x_a of the reference configuration, so that

$$(\ )_{,a} \equiv \frac{\partial}{\partial x_a}(\ ).$$

If necessary, all the following formulae may easily be

recalculated with respect to the general curvilinear coordinates $\theta^i(x_a)$ according to the chain rule $(\ )_{,a} = (\ )_{,i}\,\partial\theta^i/\partial x_a$.

Since $\underline{Q}_{,b}\underline{Q}^T$ is also skew-symmetric, let us introduce axial vectors referred to $\underline{c}_a$ or $\underline{d}_a$ bases:

$$\underline{\gamma}_b = \frac{1}{2}\underline{c}_a \times (\underline{Q}_{,b}\underline{Q}^T)\underline{c}_a = \gamma_{ab}\underline{c}_a,$$
$$\underline{\lambda}_b = \frac{1}{2}\underline{d}_a \times (\underline{R}\underline{Q}_{,b}\underline{Q}^T\underline{R}^T)\underline{d}_a = \underline{R}\underline{\gamma}_b = \gamma_{ab}\underline{d}_a. \tag{5.5}$$

Here γ_{ab} are described by three scalar parameters of $\underline{q}$ and may be found from $(5.3)_1$ and $(5.5)_1$ to be

$$\gamma_{ab} = \frac{1}{1 + \underline{q}^2/4}(q_{a,b} + \frac{1}{2}\varepsilon_{acd}q_{c,b}q_d), \tag{5.6}$$

where ε_{acd} is the permutation symbol.

It follows from $(5.3)_1$, $(5.4)_3$ and (5.5) that the differentiation of $\underline{c}_a$ and $\underline{d}_a$ along the Cartesian coordinate lines is performed according to

$$\underline{c}_{a,b} = \underline{\gamma}_b \times \underline{c}_a = \Gamma_{hab}\underline{c}_h,$$
$$\underline{d}_{a,b} = (\underline{1}_b + \underline{\lambda}_b) \times \underline{d}_a, \tag{5.7}$$

where

$$\Gamma_{hab} = \varepsilon_{hca}\gamma_{cb}, \quad \underline{1}_b = \frac{1}{2}\underline{d}_a \times (\underline{R}_{,b}\underline{R}^T)\underline{d}_a. \tag{5.8}$$

With the help of $(5.7)_1$ the differentiation of the tensor field $\underline{U}$ leads to

$$\underline{U}_{,b} = (U_a\delta_{ad})_{;b}\underline{c}_a \otimes \underline{c}_d,$$
$$(U_a\delta_{ad})_{;b} = U_{a,b}\delta_{ad} + (U_a - U_d)\gamma_{cb}\varepsilon_{cad}. \tag{5.9}$$

Writing (3.10) in the Cartesian coordinates x_a and introducing $\underline{A}_b = \underline{C}^{-1}\underline{E}_b$ we obtain

$$\underline{K}_b = (\underline{U}^{-1}\underline{E}_b - \underline{U}_{,b})\underline{U}^{-1}, \tag{5.10}$$

where, with reference to $\underline{c}_a$,

$$\underline{E}_b = \frac{1}{2}[\underline{C}_{,b} + (\underline{c}_a \times \underline{C}_{,a}\underline{c}_b) \times \underline{1}]. \tag{5.11}$$

Substitution of (5.11) and $(5.1)_1$ into (5.10) leads, after transformations, to

$$\underline{K}_b = K_{dab}\underline{c}_d \otimes \underline{c}_a,$$

$$K_{dab} = \frac{1}{U_a}U_{d,a}\delta_{db} - \frac{1}{U_d}U_{a,d}\delta_{ab} + \frac{1}{2}\left[\left(\frac{U_d}{U_a} - \frac{U_a}{U_d}\right)\gamma_{cb}\varepsilon_{cda} + \left(\frac{U_d}{U_a} - \frac{U_b^2}{U_dU_a}\right)\gamma_{ca}\varepsilon_{cdb} - \left(\frac{U_a}{U_d} - \frac{U_b^2}{U_aU_d}\right)\gamma_{cd}\varepsilon_{cab}\right]. \tag{5.12}$$

Now the vector of change of curvature $\underline{k}_b$ is given by

$$\underline{k}_b = \frac{1}{2}\underline{c}_a \times \underline{K}_b\underline{c}_a = k_{gb}\underline{c}_g, \tag{5.13}$$

$$k_{gb} = \frac{1}{2}\varepsilon_{gad}K_{dab} = \varepsilon_{gad}\left[\frac{1}{U_a}U_{d,a}\delta_{db} + \frac{1}{4}\left(\frac{U_d}{U_a} - \frac{U_a}{U_d}\right)\gamma_{cd}\varepsilon_{cda} + \left(\frac{U_d}{U_a} - \frac{U_b^2}{U_dU_a}\right)\gamma_{ca}\varepsilon_{cdb}\right]. \tag{5.14}$$

The result seems to be equivalent to that given in [20].

Integrability conditions for equations $(3.23)_1$, written in Cartesian coordinates, are

$$\varepsilon_{hab}(\underline{k}_{a,b} - \frac{1}{2}\underline{k}_a \times \underline{k}_b) = \underline{0}. \tag{5.15}$$

If

$$\underline{k}_{a,b} = k_{ga;b}\underline{c}_g, \quad k_{ga;b} = k_{ga,b} + k_{da}\varepsilon_{gcd}\gamma_{cb}, \tag{5.16}$$

then (5.15) takes the form

$$\varepsilon_{hab}(k_{ga;b} - \frac{1}{2}\varepsilon_{gcd}k_{ca}k_{db})\underline{c}_g = \underline{0}. \tag{5.17}$$

Similarly, integrability conditions for equations $(3.23)_2$, written in Cartesian coordinates, are

$$\varepsilon_{hab}(\underline{1}_{a,b} + \frac{1}{2}\underline{1}_a \times \underline{1}_b) = \underline{0}. \tag{5.18}$$

According to $(5.5)_2$, $(5.7)_2$ and $(3.13)_3$

$$\underline{1}_{a,b} = (k_{ga;b} + \varepsilon_{gdc}k_{ca}k_{db})\underline{d}_g \tag{5.19}$$

which, together with (5.18), leads to

$$\varepsilon_{hab}(k_{ga;b} - \frac{1}{2}\varepsilon_{gcd}k_{ca}k_{db})\underline{d}_g = \underline{0}. \tag{5.20}$$

Relations (5.17) and (5.20) are non-classical vector forms of the compatibility conditions in continuum mechanics. Each of the forms is equivalent to six scalar differential conditions imposed on three stretches U_a and three scalar parameters describing $\underline{q}$, and ensures the existence of three parameters describing the finite rotation field $\underline{\theta}$ satisfying nine scalar first-order partial differential equations (3.23).

Finally, the vector compatibility conditions (4.23) in the Cartesian coordinates x_a take the form

$$\varepsilon_{hab}[\underline{e}_{a,b} - \frac{1}{2}\underline{\varepsilon}\cdot(\underline{E}_a^T\underline{C}^{-1}\underline{E}_b)] = \underline{0}. \tag{5.21}$$

In order to write (5.21) relative to the Lagrangian principal directions $\underline{c}_g$, note that

$$\underline{e}_a = e_{ga}\underline{c}_g, \quad \underline{e}_{a,b} = e_{ga;b}\underline{c}_g,$$

$$e_{ga} = \frac{1}{2}\varepsilon_{gcd}(C_d\delta_{da})_{;c}$$

$$= \varepsilon_{gcd}[U_dU_{d,c}\delta_{da} + \frac{1}{2}(U_d^2 - U_a^2)\varepsilon_{fda}\gamma_{fc}],$$

$$e_{ga;b} = e_{ga,b} + \varepsilon_{gcd}\gamma_{cd}e_{da}, \quad \underline{E}_b = \frac{1}{2} C_{mcb}\underline{c}_m \otimes \underline{c}_b,$$

$$\begin{aligned} C_{mcb} &= (C_m\delta_{mc})_{;b} + (C_m\delta_{mb})_{;c} - (C_c\delta_{cb})_{;m} \\ &= 2U_mU_{m,b}\delta_{mc} + (U_m^2-U_c^2)\varepsilon_{fmc}\gamma_{fb} + 2U_mU_{m,c}\delta_{mb} \\ &+ (U_m^2-U_b^2)\varepsilon_{fmb}\gamma_{fc} - 2U_cU_{c,m}\delta_{cb} - (U_c^2-U_b^2). \end{aligned} \tag{5.22}$$

This allows us to present (5.21) in the form

$$\varepsilon_{hab}[e_{ga;b} + \frac{1}{8}\varepsilon_{gcd}\frac{1}{U_m^2}\delta_{mn}C_{mcb}C_{nda}]\underline{c}_g = \underline{0}. \tag{5.23}$$

Relations (5.23) are equivalent to six scalar differential conditions for three stretches U_m and three scalar parameters describing $\underline{q}$. They ensure the existence of $\underline{F}$ satisfying $\underline{F}_{,b} = \underline{FC}^{-1}\underline{E}_b$.

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