Flexible Shells

Theory and Applications

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On Entirely Lagrangian Displacemental Form of Non-Linear Shell Equations

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Summary

Equations of equilibrium and corresponding four geometric and static boundary conditions are derived for an entirely Lagrangian non-linear theory of thin shells. In case of a linearly elastic material and conservative external forces all shell relations are exactly derivable as stationarity conditions of the Hu - Washizu free functional. The set of equations is consistently reduced in the case of the geometrically non-linear theory of thin elastic shells undergoing large/small rotations.

1. Introduction

In the numerical analysis of flexible shell structures it is desirable to apply shell relations which are referred entirely to the undeformed shell geometry. Such entirely Lagrangian theory of shells should also be derivable from appropriately constructed variational principles.

Some forms of Lagrangian equilibrium equations for thin shells, but without associated boundary conditions, were given already in [1,2] . In [3,4] the equilibrium equations and three force boundary conditions were also referred to base vectors of the undeformed shell middle surface. However, in the fourth static boundary condition of [3,4] the resulting boundary couple was measured per unit length of the undeformed boundary contour but its axial vector was still tangent to the unknown boundary contour of the deformed shell. This caused difficulties in the construction of corresponding variational functionals even in the simplest case of dead loads applied to the shell boundary. Only recently [5] a complete set of entirely Lagrangian shell equations was derived which allowed for a proper formulation of corresponding Lagrangian variational principles [6,7]. In [5] a new Lagrangian displacemental parameter n_ν was introduced at the shell boundary and a modified tensor of change of curvature $\chi_{\alpha\beta}$ was used, which by definition was a third-degree polynomial in displacements.

In this report a different but equivalent to [5] version of the entirely Lagrangian theory of shells is presented in terms of the usual tensor of change of curvature $\kappa_{\alpha\beta}$ which is a non-rational function of displacements. The main reason for the development of the theory are simple transformation properties of $\kappa_{\alpha\beta}$ under the change of the reference shell configuration. This feature becomes of primary importance when superposed deformations and incremental formulations of shell equations are discussed in the total Lagrangian and in the updated Lagrangian descriptions [8,9] . In case of a linearly elastic material and when conservative loads are applied to the shell middle surface and the shell lateral boundary surface, the entirely Lagrangian shell equations are shown to be derivable exactly as stationarity conditions of the Hu - Washizu type free functional.

When strains are assumed to be small everywhere the shell relations derived here reduce exactly to those given in [5] for the geometrically non-linear theory. Additionally, rotations of the shell material elements may be restricted to be small, moderate or large, according to the classification scheme suggested in [10-12]. As a result, several consistently simplified versions of the entirely Lagrangian non-linear theory of shells may be constructed [13]. Here two consistent versions of equations of the non-linear theory of shells undergoing large/small rotations are developed. This shell theory describes accurately the behaviour of a majority of elastic flexible shell structures.

The sets of Lagrangian shell equations presented here are supposed to be solved in displacements as basic independent field variables. Since our shell equations are derivable from variational principles, mixed hybrid finite element methods may also be applied in which the strain and/or the stress fields appearing in corresponding variational functionals are discretized independently of the discretization of the displacement field.

2. Lagrangian Shell Equations

Within the Kirchhoff - Love type theory of shells the deformation of the three-dimensional thin shell-like body is described by the deformation of its middle surface. During the surface deformation components of the Lagrangian strain tensor and of the tensor of change of curvature are given by [5,12]

$$\begin{aligned} \gamma_{\alpha\beta} &= \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \frac{1}{2}(1^{\lambda}_{\cdot\alpha} 1_{\lambda\beta} + \phi_\alpha\phi_\beta - a_{\alpha\beta}) , \\ \kappa_{\alpha\beta} &= -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = 1_{\lambda\alpha}(n^{\lambda}|_{\beta} - b^{\lambda}_{\beta} n) + \phi_\alpha(n_{,\beta} + b^{\lambda}_{\beta} n_\lambda) + b_{\alpha\beta} . \end{aligned} \tag{2.1}$$

Here $a_{\alpha\beta}$, $\bar{a}_{\alpha\beta}$ and $b_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ are components of the surface metric and curvature tensors in its undeformed M and deformed $\bar{M}$ configurations, respectively,

$$\begin{aligned} 1_{\alpha\beta} &= a_{\alpha\beta} + \theta_{\alpha\beta} - \omega_{\alpha\beta} , \qquad \theta_{\alpha\beta} = \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta} w , \\ \omega_{\alpha\beta} &= \frac{1}{2}(u_{\beta|\alpha} - u_{\alpha|\beta}) = \epsilon_{\alpha\beta}\phi , \quad \phi_\alpha = w_{,\alpha} + b^{\lambda}_{\alpha} u_\lambda , \\ n_\mu &= \frac{1}{j} m_\mu = \frac{1}{j}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu}\phi_\alpha 1^{\lambda}_{\cdot\beta} , \qquad n = \frac{1}{j} m = \frac{1}{2j}\epsilon^{\alpha\beta}\epsilon_{\lambda\mu} 1^{\lambda}_{\cdot\alpha} 1^{\mu}_{\cdot\beta} , \\ j &= \sqrt{\frac{\bar{a}}{a}} , \qquad \frac{\bar{a}}{a} = 1 + 2\gamma^{\alpha}_{\alpha} + 2(\gamma^{\alpha}_{\alpha}\gamma^{\beta}_{\beta} - \gamma^{\alpha}_{\beta}\gamma^{\beta}_{\alpha}) , \end{aligned} \tag{2.2}$$

$$\bar{\underset{\sim}{a}}_{\alpha} = 1^{\lambda}_{\cdot\alpha}\underset{\sim}{a}_{\lambda} + \phi_\alpha \underset{\sim}{n} , \qquad \bar{\underset{\sim}{n}} = n^{\lambda}\underset{\sim}{a}_{\lambda} + n\underset{\sim}{n} \tag{2.3}$$

and $\underset{\sim}{u} = u_\alpha \underset{\sim}{a}^{\alpha} + w\underset{\sim}{n}$ is the displacement vector.

The deformation of the shell boundary surface may be described [5] by two vectors defined at the boundary contour C of M

$$\begin{aligned} \underset{\sim}{u} &= \bar{\underset{\sim}{r}} - \underset{\sim}{r} = u_\nu \underset{\sim}{\nu} + u_t \underset{\sim}{t} + w\underset{\sim}{n} , \\ \underset{\sim}{\beta} &= \bar{\underset{\sim}{n}} - \underset{\sim}{n} = n_\nu \underset{\sim}{\nu} + n_t \underset{\sim}{t} + (n-1)\underset{\sim}{n} . \end{aligned} \tag{2.4}$$

When rotations of the shell boundary elements do not exceed $\pm\frac{\pi}{2}$ for $\bar{\underset{\sim}{n}}$ we have the unique vector representation

$$\begin{aligned} \bar{\underset{\sim}{n}} &= \frac{1}{c_t^2 + c^2}\left[n_\nu \bar{\underset{\sim}{a}}_t \times (\underset{\sim}{\nu} \times \bar{\underset{\sim}{a}}_t) + \sqrt{(1 + 2\gamma_{tt})(1 - n_\nu^2) - c_\nu^2}\; \underset{\sim}{\nu} \times \bar{\underset{\sim}{a}}_t \right], \\ \bar{\underset{\sim}{a}}_t &= 1 + \frac{d\underset{\sim}{u}}{ds} = c_\nu \underset{\sim}{\nu} + c_t \underset{\sim}{t} + c\underset{\sim}{n} , \qquad c_\nu = \frac{du_\nu}{ds} + \tau_t w - \kappa_t u_t , \\ c_t &= 1 + \frac{du_t}{ds} + \kappa_t u_\nu - \sigma_t w , \qquad c = \frac{dw}{ds} + \sigma_t u_t - \tau_t u_\nu . \end{aligned} \tag{2.5}$$

It follows from (2.5) that along the boundary contour $\bar{\underset{\sim}{n}}$ is described completely by $\underset{\sim}{u}$ and n_ν as independent parameters.

Consider now $\bar{M}$ to be a middle surface of a thin shell in an equilibrium state, under the Lagrangian surface force $\underset{\sim}{p} = p^{\alpha}\underset{\sim}{a}_{\alpha} + p\underset{\sim}{n}$, per unit area of M, and under the Lagrangian boundary force $\mathbf{T} = T_{\nu}\boldsymbol{\nu} + T_{t}\mathbf{t} + T\mathbf{n}$ and the boundary static moment $\mathbf{H} = H_{\nu}\boldsymbol{\nu} + H_{t}\mathbf{t} + H\mathbf{n}$, both per unit length of C, such that

$$\int_{C} \mathbf{T} ds = \iint_{\partial B} \mathbf{f} dA \quad , \qquad \int_{C} \mathbf{H} ds = \iint_{\partial B} \mathbf{f}\zeta dA \quad , \tag{2.6}$$

where $\mathbf{f}$ is the Lagrangian surface load, per unit area of the undeformed shell boundary surface ∂B, while ζ is the distance from M. Then for any additional virtual displacement field $\delta\mathbf{u} = \delta u_{\alpha}\mathbf{a}^{\alpha} + \delta w\mathbf{n}$, which is subject to geometric constraints, the principle of virtual displacements can be presented in the Lagrangian description to be

$$\iint_{M} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})dA = \iint_{M} \mathbf{p}\cdot\delta\mathbf{u} dA + \int_{C_f} (\mathbf{T}\cdot\delta\mathbf{u} + \mathbf{H}\cdot\delta\boldsymbol{\beta})ds \quad , \tag{2.7}$$

where $N^{\alpha\beta}$, $M^{\alpha\beta}$ are components of the symmetric second Piola - Kirchhoff stress resultant and stress couple tensors.
Taking into account that

$$\delta\left(\frac{1}{j}\right) = -\frac{1}{j}\,\frac{a}{\bar{a}}[(1 + 2\gamma^{\sigma}_{\sigma})a^{\alpha\beta} - 2\gamma^{\alpha\beta}]\delta\gamma_{\alpha\beta} \quad , \tag{2.8}$$

the left-hand side of (2.7) can be transformed into

$$IVW = -\iint_{M} \mathbf{T}^{\beta}\big|_{\beta}\cdot\delta\mathbf{u} + \int_{C_f} (\mathbf{T}^{\beta}\cdot\delta\mathbf{u} + M^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\cdot\delta\bar{\mathbf{n}})\nu_{\beta} ds \quad , \tag{2.9}$$

$$\mathbf{T}^{\beta} = T^{\lambda\beta}\mathbf{a}_{\lambda} + T^{\beta}\mathbf{n} \quad ,$$

$$\begin{aligned} T^{\lambda\beta} = &\Big\{N^{\alpha\beta} + \frac{a}{\bar{a}}[(1 + 2\gamma^{\sigma}_{\sigma})a^{\alpha\beta} - 2\gamma^{\alpha\beta}]\Big([(M^{\kappa\rho}1_{\mu\kappa})|_{\rho} - M^{\kappa\rho}\phi_{\kappa}b_{\mu\rho}]n^{\mu} + \\ &+ [(M^{\kappa\rho}\phi_{\kappa})|_{\rho} + M^{\kappa\rho}1_{\gamma\kappa}b^{\gamma}_{\rho}]n\Big)\Big\}1^{\lambda}_{\cdot\alpha} + M^{\alpha\beta}(n^{\lambda}|_{\alpha} - b^{\lambda}_{\alpha}n) + \\ &+ \frac{1}{j}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\Big\{[(M^{\kappa\rho}\phi_{\kappa})|_{\rho} + M^{\kappa\rho}1_{\gamma\kappa}b^{\gamma}_{\rho}]1_{\mu\alpha} - [(M^{\kappa\rho}1_{\mu\kappa})|_{\rho} - M^{\kappa\rho}\phi_{\kappa}b_{\mu\rho}]\phi_{\alpha}\Big\} \;, \\ T^{\beta} = &\Big\{N^{\alpha\beta} + \frac{a}{\bar{a}}[(1 + 2\gamma^{\sigma}_{\sigma})a^{\alpha\beta} - 2\gamma^{\alpha\beta}]\Big([(M^{\kappa\rho}1_{\mu\kappa})|_{\rho} - M^{\kappa\rho}\phi_{\kappa}b_{\mu\rho}]n^{\mu} + \\ &+ [(M^{\kappa\rho}\phi_{\kappa})|_{\rho} + M^{\kappa\rho}1_{\gamma\kappa}b^{\gamma}_{\rho}]n\Big)\Big\}\phi_{\alpha} + M^{\alpha\beta}(n_{,\alpha} + b^{\lambda}_{\alpha}n_{\lambda}) + \\ &+ \frac{1}{j}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}[(M^{\kappa\rho}1_{\mu\kappa})|_{\rho} - M^{\kappa\rho}\phi_{\kappa}b_{\mu\rho}]1_{\lambda\alpha} \;. \end{aligned} \tag{2.10}$$

Since

$$\delta\bar{\mathbf{n}} = \frac{1}{a_{\nu}}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})(\bar{\mathbf{n}}\cdot\frac{d}{ds}\delta\mathbf{u}) + \bar{\mathbf{a}}_{\nu}\delta n_{\nu}] \quad , \qquad a_{\nu} = (\bar{\mathbf{a}}_{t}\times\bar{\mathbf{n}})\cdot\boldsymbol{\nu} \;, \tag{2.11}$$

we can reduce to four the number of independent variations of displacemental parameters at the shell boundary, transforming the last term in (2.9) as follows

$$\int_{C_f} M^{\alpha\beta}\bar{\underset{\sim}{a}}_\alpha \cdot \delta\bar{\underset{\sim}{n}}\nu_\beta ds = \int_{C_f} \left(-\underset{\sim}{F}\cdot\frac{d}{ds}\delta\underset{\sim}{u} + M\delta n_\nu\right)ds =$$

$$= \int_{C_f}\left(\frac{d\underset{\sim}{F}}{ds}\cdot\delta\underset{\sim}{u} + M\delta n_\nu\right)ds + \sum_j \underset{\sim}{F}_j\cdot\delta\underset{\sim}{u}_j \quad , \tag{2.12}$$

where

$$\underset{\sim}{F} = -\frac{1}{a_\nu}[(\bar{\underset{\sim}{n}}\times\bar{\underset{\sim}{a}}_\alpha)\cdot\underset{\sim}{\nu}]M^{\alpha\beta}\nu_\beta\bar{\underset{\sim}{n}} =$$

$$= (g_\nu R_{t\nu} + r_\nu R_\nu)\underset{\sim}{\nu} + (g_t R_{t\nu} + r_t R_\nu)\underset{\sim}{t} + (gR_{t\nu} + rR_\nu)\underset{\sim}{n} \quad ,$$

$$M = \frac{1}{a_\nu}(\bar{\underset{\sim}{n}}\times\bar{\underset{\sim}{a}}_\alpha)\cdot\bar{\underset{\sim}{a}}_t M^{\alpha\beta}\nu_\beta = R_{\nu\nu} + fR_{t\nu} + kR_\nu \quad , \tag{2.13}$$

$$\underset{\sim}{F}_j = \underset{\sim}{F}(s_j + 0) - \underset{\sim}{F}(s_j - 0) \quad ,$$

$$R_{\nu\nu} = \nu^\lambda l_{\lambda\alpha}M^{\alpha\beta}\nu_\beta = (1+\theta_{\nu\nu})M_{\nu\nu} + (\theta_{\nu t} - \phi)M_{t\nu} \quad ,$$

$$R_{t\nu} = t^\lambda l_{\lambda\alpha}M^{\alpha\beta}\nu_\beta = (\theta_{\nu t} + \phi)M_{\nu\nu} + (1+\theta_{tt})M_{t\nu} \quad , \tag{2.14}$$

$$R_\nu = \phi_\alpha M^{\alpha\beta}\nu_\beta = \phi_\nu M_{\nu\nu} + \phi_t M_{t\nu} \quad ,$$

$$g_\nu = \frac{n_\nu n}{a_\nu} \quad , \quad g_t = \frac{n_t n}{a_\nu} \quad , \quad g = \frac{n^2}{a_\nu} \quad ,$$

$$r_\nu = \frac{n_\nu n_t}{a_\nu} \quad , \quad r_t = \frac{n_t^2}{a_\nu} \quad , \quad r = \frac{n_t n}{a_\nu} \quad , \tag{2.15}$$

$$f = \frac{1}{a_\nu}(cn_\nu - c_\nu n) \quad , \quad k = \frac{1}{a_\nu}(c_\nu n_t - c_t n_\nu) \quad .$$

Now the Lagrangian principle of virtual displacements (2.7) takes the final form

$$-\iint_M (\underset{\sim}{T}^\beta|_\beta + \underset{\sim}{p})\cdot\delta\underset{\sim}{u}dA + \int_{C_f}[(\underset{\sim}{P} - \underset{\sim}{P}^*)\cdot\delta\underset{\sim}{u} + (M - M^*)\delta n_\nu]ds + \sum_j(\underset{\sim}{F}_j - \underset{\sim}{F}_j^*)\cdot\delta\underset{\sim}{u}_j = 0 \quad , \tag{2.16}$$

where

$$\underset{\sim}{P} = \underset{\sim}{T}^\beta\nu_\beta + \frac{d\underset{\sim}{F}}{ds} \quad , \quad \underset{\sim}{P}^* = \underset{\sim}{T} + \frac{d\underset{\sim}{F}^*}{ds} \quad ,$$

$$\underset{\sim}{F}^* = -\frac{1}{a_\nu}[(\bar{\underset{\sim}{n}}\times\underset{\sim}{H})\cdot\underset{\sim}{\nu}]\bar{\underset{\sim}{n}} =$$

$$= (g_\nu H_t + r_\nu H)\underset{\sim}{\nu} + (g_t H_t + r_t H)\underset{\sim}{t} + (gH_t + rH)\underset{\sim}{n} \quad , \tag{2.17}$$

$$M^* = \frac{1}{a_\nu}(\bar{\underset{\sim}{n}}\times\underset{\sim}{H})\cdot\bar{\underset{\sim}{a}}_t = H_\nu + fH_t + kH \quad , \quad \underset{\sim}{F}_j^* = \underset{\sim}{F}^*(s_j + 0) - \underset{\sim}{F}^*(s_j - 0) .$$

From (2.16) follow the equilibrium equations and the corresponding static boundary conditions, together with already known geometric boundary conditions, of the entirely Lagrangian non-linear theory of thin shells

$$\underset{\sim}{T}^{\beta}\Big|_{\beta} + \underset{\sim}{p} = \underset{\sim}{0} \quad \text{in} \quad M \ ,$$
$$\underset{\sim}{P} = \underset{\sim}{P}^* \ , \ M = M^* \ \text{on} \ C_f \ \text{and} \ \underset{\sim}{F}_j = \underset{\sim}{F}_j^* \ \text{at each corner} \ M_j \in C_f \ , \tag{2.18}$$
$$\underset{\sim}{u} = \underset{\sim}{u}^* \ , \ n_\nu = n_\nu^* \ \text{on} \ C_u \ \text{and} \ \underset{\sim}{u}_i = \underset{\sim}{u}_i^* \ \text{at each corner} \ M_i \in C_u \ .$$

Note that, in general, the Lagrangian equilibrium equations and the Lagrangian static boundary conditions are linear in the stress measures $N^{\alpha\beta}$, $M^{\alpha\beta}$ but are non-rational in the displacemental parameters, since in the expressions (2.10), (2.13) and (2.17) there are square roots of polynomials of those parameters.

Within the first-approximation theory of thin isotropic and elastic shells the strain energy function may be approximated by the quadratic form [14]

$$\Sigma = \frac{h}{2} H^{\alpha\beta\lambda\mu}(\gamma_{\alpha\beta}\gamma_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}) + O(Eh\eta^2\theta^2) \ , \tag{2.19}$$

where $H^{\alpha\beta\lambda\mu}$ are components of the modified elasticity tensor. The error of Σ at any point of the shell is expressed through the small parameter θ defined by [15,16]

$$\theta = \max(h/d, \ h/L, \ h/L^*, \ \sqrt{h/R}, \ \sqrt{\eta}) \ , \tag{2.20}$$

where d is the distance of the point from the lateral shell boundary, L is the wave length of deformation patterns of M , L^* is the wave length of the curvature patterns of M , R is the smallest principal radius of curvature of M and η is the largest principal strain in the shell space. From (2.19) follow the constitutive equations

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}} = \frac{Eh}{1-\nu^2}[(1-\nu)\gamma^{\alpha\beta} + \nu a^{\alpha\beta}\gamma^\kappa_\kappa] + O(Eh\eta\theta^2) \ ,$$
$$M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}} = \frac{Eh^3}{12(1-\nu^2)}[(1-\nu)\kappa^{\alpha\beta} + \nu a^{\alpha\beta}\kappa^\kappa_\kappa] + O(Eh^2\eta\theta^2) \ . \tag{2.21}$$

In the case of external dead loads there exist potential functions $\Phi(\underset{\sim}{u}) = -\underset{\sim}{p}\cdot\underset{\sim}{u}$ and $\Psi(\underset{\sim}{u},\underset{\sim}{\beta}) = -(\underset{\sim}{T}\cdot\underset{\sim}{u} + \underset{\sim}{H}\cdot\underset{\sim}{\beta})$ which allow to transform (2.16) into the variational principle $\delta I = 0$ for the functional

$$I = \iint\limits_M [\Sigma(\gamma_{\alpha\beta},\kappa_{\alpha\beta}) - \underset{\sim}{p}\cdot\underset{\sim}{u}]dA - \int\limits_{C_f}[\underset{\sim}{T}\cdot\underset{\sim}{u} + \underset{\sim}{H}\cdot\underset{\sim}{\beta}]ds \tag{2.22}$$

with (2.1), (2.5) and $(2.18)_3$ as subsidiary conditions. Eliminating at C_f the parameters n_t and n with the use of $(2.5)_1$

and introducing other subsidiary conditions into the functional I with the help of Lagrange multipliers $N^{\alpha\beta}$, $M^{\alpha\beta}$, $\underset{\sim}{P}$ and M we obtain the free Hu - Washizu type functional

$$I_1 = \iint_M \Big\{\Sigma(\gamma_{\alpha\beta},\kappa_{\alpha\beta}) - \underset{\sim}{p}\cdot\underset{\sim}{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(\underset{\sim}{u})] - M^{\alpha\beta}[\kappa_{\alpha\beta} - \kappa_{\alpha\beta}(\underset{\sim}{u})]\Big\}dA -$$
$$- \int_{C_f} \Big\{\underset{\sim}{T}\cdot\underset{\sim}{u} + H\cdot[\bar{\underset{\sim}{n}}(\underset{\sim}{u},n_\nu) - \underset{\sim}{n}]\Big\}ds - \qquad (2.23)$$
$$- \int_{C_u} [\underset{\sim}{P}\cdot(\underset{\sim}{u} - \underset{\sim}{u}^*) + M(n_\nu - n_\nu^*)]ds - \sum_i \underset{\sim}{F}_i\cdot(\underset{\sim}{u}_i - \underset{\sim}{u}_i^*) \ .$$

The associated Hu - Washizu variational principle $\delta I_1 = 0$ states that among all possible values of independent fields indicated in (2.23), which are not restricted by any subsidiary conditions, the solution values render the functional I_1 stationary. It can be shown by direct calculations that the stationarity conditions of the functional I_1 are exactly the Lagrangian shell equations (2.18), the strain-displacement relations (2.1) and the constitutive equations (2.21) together with relations which identify the Lagrange multipliers with the functions described already by the symbols used in (2.23). In analogy to [6,7,17] many other free or constrained functionals and associated with them variational principles may be constructed for the entirely Lagrangian non-linear theory of thin isotropic elastic shells. The functionals form a solid basis for a computerized analysis of flexible shell structures.

The shell relations derived above are two-dimensionally exact for the shell middle surface. However, the relations are meaningful for shells only within small strains, since by using the Kirchhoff - Love constraints the effect of change of the shell thickness was ignored in the description of shell deformation. Such simplified approach is consistent within the first-approximation theory of elastic shells used here, but it would not be permissible if large strains in the shell space were allowed [18] .

Within small strains some shell relations may be simplified by omitting strains with respect to unity. In particular,

$$j \simeq 1 + \gamma_\alpha^\alpha \simeq 1 \quad , \quad n \simeq m(1 - \gamma_\alpha^\alpha) \simeq m \quad , \quad n_\mu \simeq m_\mu \quad ,$$

$$\kappa_{\alpha\beta} \simeq 1_{\lambda\alpha}(m^{\lambda}|_{\beta} - b^{\lambda}_{\beta}m) + \phi_{\alpha}(m_{,\beta} + b^{\lambda}_{\beta}m_{\lambda}) + b_{\alpha\beta}(1 + \gamma^{\kappa}_{\kappa}) \quad , \tag{2.24}$$

$$\underset{\sim}{\bar{n}} \simeq \frac{1}{1 - c^2_{\nu}}[n_{\nu}\underset{\sim}{\bar{a}}_t \times (\underset{\sim}{\nu} \times \underset{\sim}{\bar{a}}_t) + \sqrt{1 - n^2_{\nu} - c^2_{\nu}}\; \underset{\sim}{\nu} \times \underset{\sim}{\bar{a}}_t] \ .$$

If $(2.24)_2$ is used in the left-hand side of (2.7) then [5] it generates the following reduced definitions of (2.10)

$$\begin{aligned} T^{\lambda\beta} &= 1^{\lambda}_{\cdot\alpha}(N^{\alpha\beta} + a^{\alpha\beta}b_{\kappa\rho}M^{\kappa\rho}) + (m^{\lambda}|_{\alpha} - b^{\lambda}_{\alpha}m)M^{\alpha\beta} + \\ &\quad + \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}\Big\{1_{\mu\alpha}[(\phi_{\kappa}M^{\kappa\rho})|_{\rho} + 1_{\gamma\kappa}b^{\gamma}_{\rho}M^{\kappa\rho}] - \phi_{\alpha}[(1_{\mu\kappa}M^{\kappa\rho})|_{\rho} - \phi_{\kappa}b_{\mu\rho}M^{\kappa\rho}]\Big\} \ , \end{aligned} \tag{2.25}$$

$$T^{\beta} = \phi_{\alpha}(N^{\alpha\beta} + a^{\alpha\beta}b_{\kappa\rho}M^{\kappa\rho}) + (m_{,\alpha} + b^{\lambda}_{\alpha}m_{\lambda})M^{\alpha\beta} + \epsilon^{\alpha\beta}\epsilon^{\lambda\mu}1_{\lambda\alpha}[(1_{\mu\kappa}M^{\kappa\rho})|_{\rho} - \phi_{\kappa}b_{\mu\rho}N^{\kappa\rho}].$$

As a result of the simplified expressions (2.24) and (2.25), for the geometrically non-linear theory of shells the Lagrangian equilibrium equations become linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ and quadratic in u_{α} , w while the Lagrangian static boundary conditions are linear in $N^{\alpha\beta}$, $M^{\alpha\beta}$ but still remain non-rational functions of u_{α} , w and n_{ν} since in the approximate formula $(2.24)_3$ for $\underset{\sim}{\bar{n}}$ there is still the square-root function of the displacemental parameters. When (2.24) and (2.25) are introduced into the Hu - Washizu functional (2.23) , then analytically derived stationarity conditions of I_1 will not exactly coincide with the shell boundary parameters (2.13) and (2.17). However, since the error of (2.24) and (2.25) lies within an error margin of the first-approximation theory, the Lagrangian shell equations and the Hu - Washizu free functional may be regarded as corresponding equivalent descriptions of the same version of the geometrically non-linear theory of shells.

It should be pointed out that the general definitions of the Lagrangian measures of change of curvature used in this paper and in [1-5,19] are equivalent to each other from the point of view of the error introduced into the strain energy within the first-approximation theory. However, the important qualitative differences appear in the displacemental forms of the measures. In the derivations of changes of curvature presented in [1-4,19] the representation $\bar{b}_{\alpha\beta} = \underset{\sim}{\bar{a}}_{\alpha}|_{\beta}\cdot\underset{\sim}{\bar{n}}$ was applied for the curvature tensor of $\bar{M}$, while in [5] and in deriving $(2.1)_2$ here a different (although mathematically equivalent) expression $\bar{b}_{\alpha\beta} = -\underset{\sim}{\bar{a}}_{\alpha}\cdot\underset{\sim}{\bar{n}}|_{\beta}$ was used. As a result, our line integral in (2.9) consisted of six terms containing $\delta\underset{\sim}{u}$ and $\delta\underset{\sim}{\bar{n}}$. Since $\underset{\sim}{\bar{n}} = \underset{\sim}{\bar{n}}(\underset{\sim}{u}, n_{\nu})$, those terms were reduced further to four terms containing only $\delta \underset{\sim}{u}$ and δn_ν as independent variations along the shell boundary contour. This allowed to construct the two-dimensionally exact and variationally derivable four natural static boundary and corner conditions associated with the equilibrium equations (2.18). On the other hand, when the displacemental forms of changes of curvatures given in [1-4,19] are introduced into IVW and Stokes' theorem is applied, the resulting line integral consists of six terms containing $\delta \underset{\sim}{u}$ and $\frac{d}{ds_\nu}\delta \underset{\sim}{u}$. Those six terms cannot be reduced further to only four terms containing $\delta \underset{\sim}{u}$ and one of components of $\frac{d}{ds_\nu}\delta \underset{\sim}{u}$ as independent variations of displacemental variables. Let us remind that $\frac{d}{ds_\nu}$ above means differentiation at C performed in the direction of $\underset{\sim}{\nu}$ orthogonal to C , [10-12]. As a result, no four variationally derivable static boundary and corner conditions can be associated with the Lagrangian equilibrium equations given in [1-4]. This was the reason why no boundary conditions were given in [1,2] for the general bending theory of shells while the static boundary conditions suggested in [3,4] were derived by transforming corresponding Eulerian parameters into the undeformed reference configuration but not by the direct variational procedure. However, such transformed static boundary conditions are not entirely Lagrangian and do not allow to construct a free variational functional of the Hu - Washizu type even in the simplest case of dead loads applied to the shell lateral boundary surface.

3. Restricted Rotations

By the polar decomposition theorem of the deformation gradient tensor [10-12] strains and rotations of the shell material elements were exactly separated from each other. Therefore, further consistent simplifications of the geometrically non-linear entirely Lagrangian shell equations may be achieved by imposing some restrictions upon the rotations.

The basic parameter describing the magnitude of a rotation is the angle of rotation ω about the rotation axis defined by the unit vector $\underset{\sim}{e}$. The angle may be used to classify rotations in terms of the small parameter (2.20) as follows [10-12]: $\omega \le O(\theta^2)$ - small rotations, $\omega = O(\theta)$ - moderate rotations, $\omega = O(\sqrt{\theta})$ - large rotations, $\omega \ge O(1)$ - finite rotations. Introducing the finite rotation vector $\underset{\sim}{\Omega} = \sin\omega\, \underset{\sim}{e}$ we may approximate it within small strains

$$\underset{\sim}{\Omega} \simeq \epsilon^{\beta\alpha}\left[\phi_\alpha\left(1+\frac{1}{2}\theta^\kappa_\kappa\right)-\frac{1}{2}\phi^\lambda(\theta_{\lambda\alpha}-\omega_{\lambda\alpha})\right]\underset{\sim}{a}_\beta + \phi\underset{\sim}{n} \ . \tag{3.1}$$

For any restriction imposed on ω from (3.1) follow estimates for ϕ_α, ϕ and from $(2.1)_1$ we obtain an estimate for $\theta_{\alpha\beta}$. Estimates of those linearized parameters allow to simplify consistently the shell strain measures (2.1) within the error of the strain energy function (2.19). When introduced into (2.7) the simplified strain measures generate corresponding entirely Lagrangian non-linear shell equations for each simplified version of the theory of shells.

Within small rotations $\gamma_{\alpha\beta} = \theta_{\alpha\beta} + O(\eta\theta^2)$, $\kappa_{\alpha\beta} = -\frac{1}{2}(\phi_{\alpha|\beta}+\phi_{\beta|\alpha}) + O(\eta\theta/\lambda)$, where $\lambda = h/\theta$, and the theory reduces to the bending linear theory of shells which is discussed in many monographs.

Within moderate rotations the shell strain measures (2.7) may be simplified [10,12] to the form

$$\begin{aligned}\gamma_{\alpha\beta} &= \theta_{\alpha\beta}+\frac{1}{2}\phi_\alpha\phi_\beta+\frac{1}{2}a_{\alpha\beta}\phi^2-\frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta}+\theta^\lambda_\beta\omega_{\lambda\alpha})+O(\eta\theta^2)\ ,\\ \kappa_{\alpha\beta} &= -\frac{1}{2}[\phi_{\alpha|\beta}+\phi_{\beta|\alpha}+b^\lambda_\alpha(\theta_{\lambda\beta}-\omega_{\lambda\beta})+b^\lambda_\beta(\theta_{\lambda\alpha}-\omega_{\lambda\alpha})]+O(\eta\theta/\lambda).\end{aligned} \tag{3.2}$$

The complete set of Lagrangian equations of the theory of shells undergoing moderate rotations was given in [12] . The theory contains as special cases the equations of various simpler versions of the Lagrangian theory of shells which were proposed in the literature. A detailed review of those versions was given in [17] where also many free and constrained functionals and associated with them variational principles were constructed. Stability equations for flexible shells based on (3.2) were derived in [8,20].

Within large rotations $\gamma_{\alpha\beta}$ can not be simplified while for the tensor of change of curvature we obtain [13]

$$\begin{aligned}\kappa_{\alpha\beta} = {} & \frac{1}{2}(1^{\lambda}_{\cdot\alpha}m_{\lambda|\beta}+1^{\lambda}_{\cdot\beta}m_{\lambda|\alpha})+\frac{1}{2}(\phi_\alpha m_{,\beta}+\phi_\beta m_{,\alpha}) - \\ & -\frac{1}{2}[b^\lambda_\alpha(\theta_{\lambda\beta}-\omega_{\lambda\beta})+b^\lambda_\beta(\theta_{\lambda\alpha}-\omega_{\lambda\alpha})]+\frac{1}{2}(b^\lambda_\alpha\omega_{\lambda\beta}+b^\lambda_\beta\omega_{\lambda\alpha})\left(\theta^\kappa_\kappa+\frac{1}{2}\omega^{\kappa\rho}\omega_{\kappa\rho}\right) - \\ & -\frac{1}{2}(b^\lambda_\alpha\phi_\beta+b^\lambda_\beta\phi_\alpha)(\phi_\lambda+\phi^\mu\omega_{\mu\lambda})+\frac{1}{2}b_{\alpha\beta}\phi^\lambda\phi_\lambda + O(\eta\theta/\lambda)\ .\end{aligned} \tag{3.3}$$

The complete set of entirely Lagrangian shell equations based on $(2.1)_1$ and (3.3) was derived in [13] . Two special cases of (3.3) were also discussed in [13] in which rotations associated with in-surface deformation were allowed to be small or moderate. Even for such large/small or large/moderate rotation shell theories the resulting Lagrangian shell equations were still very complex and hardly readable. However, in all three cases it was possible to get rid of the non-rational expressions at the shell boundary approximating consistently the square-root functions by polynomials of the displacemental parameters.

4. Simplified Theories of Shells Undergoing Large/Small Rotations

In the two simplified theories discussed here within the large/small rotation range of shell deformation a greater error $O(Eh\eta^2\theta\sqrt{\theta})$ or $O(Eh\eta^2\theta)$ is allowed in the strain energy function (2.19). The scheme of derivation and equilibrium equations for such simplified versions of shell theory were given already in [10,12], but at that time we failed to construct variationally derivable Lagrangian static boundary and corner conditions. Only when entirely Lagrangian theory of shells was developed [5] it became possible to reduce it consistently also within the large/small rotation range of deformation [13,21,22] and to formulate properly the corresponding static boundary and corner conditions. In what follows the relations given in [13,21,22] are modified further and presented in what is believed to be their canonical form.

When rotations about tangents to M are allowed to be large while rotations about normals to M are supposed to be always small then from (3.1) it follows that $\phi = O(\theta^2)$, $\phi_\alpha = O(\sqrt{\theta})$ and from $(2.1)_1$ we obtain $\theta_{\alpha\beta} = O(\theta)$.

Within the error $O(Eh\eta^2\theta\sqrt{\theta})$ of the strain energy function the shell strain measures (2.1) of the large/small rotation theory take the consistently reduced form [13]

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\phi_\alpha\phi_\beta + \frac{1}{2}\theta^\lambda_{\alpha\cdot}\theta_{\lambda\beta} - \frac{1}{2}(\theta^\lambda_\alpha\omega_{\lambda\beta} + \theta^\lambda_\beta\omega_{\lambda\alpha}) + O(\eta\theta\sqrt{\theta}) ,$$

$$\kappa_{\alpha\beta} = \frac{1}{2}[(\mathfrak{m}_{\alpha|\beta} + \mathfrak{m}_{\beta|\alpha}) - (\theta^\lambda_\alpha\phi_{\lambda|\beta} + \theta^\lambda_\beta\phi_{\lambda|\alpha}) + (\phi_\alpha\mathfrak{m}_{,\beta} + \phi_\beta\mathfrak{m}_{,\alpha}) -$$
$$- (b^\lambda_\alpha\theta_{\lambda\beta} + b^\lambda_\beta\theta_{\lambda\alpha}) - (b^\lambda_\alpha\phi_\beta + b^\lambda_\beta\phi_\alpha)\phi_\lambda + b_{\alpha\beta}\phi^\lambda\phi_\lambda] + O(\eta\sqrt{\theta}/\lambda),$$

$$\mathfrak{m}_\lambda = -(1+\theta^\kappa_\kappa)\phi_\lambda + \phi^\mu\theta_{\mu\lambda} , \qquad \mathfrak{m} = 1 + \theta^\kappa_\kappa . \qquad (4.1)$$

It follows from $(4.1)_1$ that within the same approximation $\theta_{\alpha\beta} = -\frac{1}{2}\phi_\alpha\phi_\beta + O(\theta^2)$, see [10,22], which introduced into $(4.1)_3$ allow to reduce the parameters into the simpler forms

$$\mathfrak{m}_{\lambda} = -\phi_{\lambda} + O(\theta^2\sqrt{\theta}) \quad , \qquad \mathfrak{m} = 1 - \frac{1}{2}\phi^{\kappa}\phi_{\kappa} + O(\theta^2) \quad ,$$
$$\bar{\underset{\sim}{n}} \cong -\phi_{\nu}\underset{\sim}{\nu} - \phi_t \underset{\sim}{t} + (1 - \frac{1}{2}\phi_{\nu}^2 - \frac{1}{2}\phi_t^2)\underset{\sim}{n} \; , \qquad \phi_t = \frac{dw}{ds} - \tau_t u_{\nu} + \sigma_t u_t \; . \tag{4.2}$$

This allows to present $\kappa_{\alpha\beta}$ by a simpler equivalent expression

$$\kappa_{\alpha\beta} = -\frac{1}{2}\Big\{[(\delta^{\lambda}_{\alpha} + \theta^{\lambda}_{\alpha})\phi_{\lambda|\beta} + (\delta^{\lambda}_{\beta} + \theta^{\lambda}_{\beta})\phi_{\lambda|\alpha}] + \frac{1}{2}[\phi_{\alpha}(\phi^{\kappa}\phi_{\kappa})_{,\beta} + \phi_{\beta}(\phi^{\kappa}\phi_{\kappa})_{,\alpha}] +$$
$$+ (b^{\lambda}_{\alpha}\theta_{\lambda\beta} + b^{\lambda}_{\beta}\theta_{\lambda\alpha}) + (b^{\lambda}_{\alpha}\phi_{\beta} + b^{\lambda}_{\beta}\phi_{\alpha})\phi_{\lambda} - b_{\alpha\beta}\phi^{\lambda}\phi_{\lambda}\Big\} + O(\eta\sqrt{\theta}/\lambda). \tag{4.3}$$

Using some identities and the estimate for $\theta_{\alpha\beta}$ given above (4.2) the expression (4.3) can be shown to be equivalent, within the assumed error, to the one proposed in [12] f.(5.3.7). However, for the reasons explained at the end of §2, the displacemental form used in [12] did not allow to construct four variationally derivable static boundary conditions. In [13,22] equivalent to (4.3) measures of the change of curvature were proposed as quadratic polynomials in displacements. However, four static boundary conditions were constructed in [13] approximating consistently the square-root functions of the exact theory [5] by polynomials of the displacemental parameters, while in the transformation of IVW performed in [22] an approximate expression $\theta^{\kappa}_{\kappa} = -\frac{1}{2}\phi^{\kappa}\phi_{\kappa} + O(\theta^2)$ had to be additionally applied in the corresponding line integral. On the other hand, our expression (4.3) is a third-degree polynomial in displacements but it allows to perform exactly all further transformations presented in the following part of this paper.

When $(4.1)_1$ and (4.3) are introduced into the principle of virtual displacements, after appropriate transformations it can be reduced exactly to the form (2.16), only now

$$T^{\lambda\beta} = (\delta^{\lambda}_{\alpha} + \theta^{\lambda}_{\alpha})N^{\alpha\beta} - \frac{1}{2}(\omega^{\lambda\alpha}N^{\beta}_{\alpha} + \omega^{\beta\alpha}N^{\lambda}_{\alpha}) - \frac{1}{2}[(b^{\lambda}_{\alpha} + \phi^{\lambda}|_{\alpha})M^{\alpha\beta} + (b^{\beta}_{\alpha} + \phi^{\beta}|_{\alpha})M^{\alpha\lambda}] \; ,$$
$$T^{\beta} = \phi_{\alpha}N^{\alpha\beta} + [(\delta^{\beta}_{\lambda} + \theta^{\beta}_{\lambda})M^{\lambda\alpha}]_{|\alpha} + (\phi_{\lambda}M^{\lambda\alpha})_{|\alpha}\phi^{\beta} - \phi^{\lambda}|_{\alpha}\phi_{\lambda}M^{\alpha\beta} -$$
$$- (b^{\lambda}_{\alpha}M^{\alpha\beta} + b^{\beta}_{\alpha}M^{\alpha\lambda})\phi_{\lambda} + b_{\alpha\lambda}\phi^{\beta}M^{\alpha\lambda} \quad ,$$
$$R_{\nu\nu} = (1 + \theta_{\nu\nu})M_{\nu\nu} + \theta_{\nu t}M_{t\nu} \; , \qquad R_{t\nu} = \theta_{t\nu}M_{\nu\nu} + (1 + \theta_{tt})M_{t\nu} , \tag{4.4}$$
$$R_{\nu} = \phi_{\nu}M_{\nu\nu} + \phi_t M_{t\nu} \; , \quad M = R_{\nu\nu} + \phi_{\nu}R_{\nu} = (1 + \theta_{\nu\nu} + \phi_{\nu}^2)M_{\nu\nu} + (\theta_{\nu t} + \phi_{\nu}\phi_t)M_{t\nu} ,$$
$$\underset{\sim}{F} = F\underset{\sim}{n} \quad , \qquad F = R_{t\nu} + \phi_t R_{\nu} = (\theta_{\nu t} + \phi_{\nu}\phi_t)M_{\nu\nu} + (1 + \theta_{tt} + \phi_t^2)M_{t\nu} ,$$
$$\underset{\sim}{F}^* = F^*\underset{\sim}{n} \quad , \qquad F^* = H_t + \phi_t H \quad , \qquad M^* = H_{\nu} + \phi_{\nu}H \; .$$

From (2.16) with (4.4) follow corresponding Lagrangian shell equations (2.18), in which $(4.2)_2$ and (4.4) should be used.

In some engineering applications we may be interested in the use of even simpler but consistently reduced shell relations which follow when a larger error $O(Eh\eta^2\theta)$ in the strain energy function is assumed to be permissible. Within this larger error the strain measures of such simplest large/small rotation theory of shells take the extremely simple form

$$\gamma_{\alpha\beta} = \theta_{\alpha\beta} + \frac{1}{2}\phi_\alpha\phi_\beta + \frac{1}{2}\theta^\lambda_\alpha\theta_{\lambda\beta} + O(\eta\theta) \quad ,$$
$$\kappa_{\alpha\beta} = -\frac{1}{2}[(\delta^\lambda_\alpha + \theta^\lambda_\alpha + \phi^\lambda\phi_\alpha)\phi_{\lambda|\beta} + (\delta^\lambda_\beta + \theta^\lambda_\beta + \phi^\lambda\phi_\beta)\phi_{\lambda|\alpha}] + O(\eta/\lambda) . \tag{4.5}$$

Again, within the assumed accuracy the strain measures (4.5) are equivalent to those proposed in [12,13,21,22] where various different displacemental expressions for $\kappa_{\alpha\beta}$ were suggested. However, only the expression $(4.5)_2$ given here allows to perform exactly all further transformations. When introduced into (2.7) the measures (4.5) lead exactly to (2.16) with the following corresponding definitions of the static field parameters

$$T^{\lambda\beta} = N^{\lambda\beta} + \frac{1}{2}(\theta^\lambda_\alpha N^{\alpha\beta} + \theta^\beta_\alpha N^{\alpha\lambda}) - \frac{1}{2}(\phi^\lambda|_\alpha M^{\alpha\beta} + \phi^\beta|_\alpha N^{\alpha\lambda}) \quad ,$$
$$T^\beta = \phi_\alpha N^{\alpha\beta} + [(\delta^\beta_\lambda + \theta^\beta_\lambda)M^{\lambda\alpha}]|_\alpha + (\phi_\lambda M^{\lambda\alpha})|_\alpha\phi^\beta - \phi^\lambda\phi_{\lambda|\alpha}M^{\alpha\beta} \quad , \tag{4.6}$$

while corresponding static boundary parameters remain identical with those given in $(4.4)_{3-6}$ and $(4.2)_2$.

From (2.16) we obtain the following component form of the entirely Lagrangian equations for both simplified versions of the theory of shells undergoing large/small rotations:
the equilibrium equations in M

$$T^{\lambda\beta}|_\beta - b^\lambda_\beta T^\beta + p^\lambda = 0 \;, \quad T^\beta|_\beta + b_{\lambda\beta}T^{\lambda\beta} + p = 0 \quad ; \tag{4.7}$$

the static boundary conditions on C_f

$$T^{\lambda\beta}\nu_\lambda\nu_\beta + \tau_t(R_{t\nu} + \phi_t R_\nu) = T_\nu + \tau_t(H_t + \phi_t H) \quad ,$$
$$T^{\lambda\beta}t_\lambda\nu_\beta - \sigma_t(R_{t\nu} + \phi_t R_\nu) = T_t - \sigma_t(H_t + \phi_t H) \quad ,$$
$$T^\beta\nu_\beta + \frac{d}{ds}(R_{t\nu} + \phi_t R_\nu) = T + \frac{d}{ds}(H_t + \phi_t H) \quad ,$$
$$R_{\nu\nu} + \phi_\nu R_\nu = H_\nu + \phi_\nu H \quad ; \tag{4.8}$$

the static corner conditions at each corner $M_j \in C_f$

$$F(s_j + 0) - F(s_j - 0) = F^*(s_j + 0) - F^*(s_j - 0) \quad ; \tag{4.9}$$

the geometric boundary conditions on C_u

$$u_\nu = u_\nu^* \ , \quad u_t = u_t^* \ , \quad w = w^* \ , \quad \phi_\nu = \phi_\nu^* \quad ; \qquad (4.10)$$

the geometric corner condition at each corner $M_i \in C_u$

$$w_i = w_i^* \ . \qquad (4.11)$$

Introducing appropriate definitions $(4.4)_{1,2}$ or (4.6) into (4.7) and (4.8) extended representations of the equilibrium equations and the static boundary conditions in terms of $N^{\alpha\beta}, M^{\alpha\beta}$ and u_α, w, ϕ_ν may easily be derived.

The structure of the Lagrangian shell relations given above is relatively simple. The strain tensors $(4.1)_1$ or $(4.5)_1$ are quadratic in displacements while the tensors of change of curvature (4.3) or $(4.5)_2$ are cubic in displacements , where cubic terms are expressed only through ϕ_α . The equilibrium equations (4.7) are linear in $N^{\alpha\beta}, M^{\alpha\beta}$ and quadratic in u_α, w only through squares of ϕ_α . The static boundary (4.8) and corner (4.9) conditions are linear in $N^{\alpha\beta}, M^{\alpha\beta}$ and quadratic in displacemental parameters again through ϕ_ν and ϕ_t . All four geometric boundary conditions (4.10) and the geometric corner conditions (4.11) are linear in displacements, what is very important when a numerical solution of a non-linear shell problem is constructed using finite elements in order to discretize the displacement field.

The Hu - Washizu free functional corresponding to the shell relations presented above takes the form

$$I_1 = \iint_M \Big\{ \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta}) - \underset{\sim}{p}\cdot\underset{\sim}{u} - N^{\alpha\beta}[\gamma_{\alpha\beta} - \gamma_{\alpha\beta}(\underset{\sim}{u})] - M^{\alpha\beta}[\kappa_{\alpha\beta} - \kappa_{\alpha\beta}(\underset{\sim}{u})]\Big\} dA -$$

$$- \int_{C_f} \Big\{ \underset{\sim}{T}\cdot\underset{\sim}{u} + \underset{\sim}{H}\cdot[\underset{\sim}{\bar{n}}(u, \phi_\nu) - \underset{\sim}{n}]\Big\} ds - \qquad (4.12)$$

$$- \int_{C_u} [\underset{\sim}{P}\cdot(\underset{\sim}{u} - \underset{\sim}{u}^*) - M(\phi_\nu - \phi_\nu^*)] ds - \sum_i F_i (w_i - w_i^*) \quad ,$$

where (2.19), $(4.1)_1$ and (4.3) or (4.5), $(4.2)_2$, $(4.4)_{3-6}$ and left-hand sides of (4.8) should be used. As stationarity conditions of I_1 we obtain exactly all relations of the respective simplified versions of the entirely Lagrangian theory of shells undergoing large/small rotations. From I_1 , following [17,6,7], a number of other free or constrained functionals and associated variational principles may be constructed. Appropriate stability equations for the large/small rotation theory of shells may be derived by specialization of those given in [9,23,24].

Let us remind some simplified versions of the non-linear theory of thin shells for which non-linear expressions for changes of curvatures were suggested. Koiter [19] f.(12.2) proposed a quadratic expression for $\rho_{\alpha\beta} \equiv -\kappa_{\alpha\beta} - \frac{1}{2}(b^{\lambda}_{\alpha}\gamma_{\lambda\beta} + b^{\lambda}_{\beta}\gamma_{\lambda\alpha})$ in the case of "moderate deflections", Başar [25] derived a quadratic expression for $\kappa_{\alpha\beta}$ in the case of "moderately large rotations", from Galimov [4] f.(3.38) follows a quadratic expression for $\kappa_{\alpha\beta}$ in the case of "strong bending". When compared with corresponding expression (5.3.7) of [12] for $\kappa_{\alpha\beta}$ with greater error, which is equivalent to our (4.3) in the sense of error, the lack of terms $\frac{1}{4}\phi^{\lambda}\phi_{\lambda}(\phi_{\alpha|\beta} + \phi_{\beta|\alpha})$ was noted in the measure [19] and in transformed version of the measure [25] obtained using an identity (3.34) of [19], while terms $\phi^{\lambda}(\theta_{\lambda\alpha|\beta} + \theta_{\lambda\beta|\alpha} - \theta_{\lambda\alpha|\beta})$ were missed in the resulting measure of [4]. According to our estimates, those terms are $O(\theta\sqrt{\theta}/\lambda)$ and should be taken into account even within the simplest large/small rotation shell theory, see $(5.3.9)_2$ of [12]. Apart from that, for the reasons explained at the end of §2, the expressions suggested in [4,19, 25] for the changes of curvatures do not allow to construct variationally derivable four Lagrangian static boundary conditions. Shapovalov [26] f.(1.9) proposed an extremely simple quadratic theory of shells in which $\gamma_{\alpha\beta}$ contain two first terms of (4.1), κ_{12} is linear while $\kappa_{\alpha\alpha} = -\phi_{\alpha|\alpha} - \frac{1}{2}b_{\alpha\alpha}\phi^2_{\beta}$, $\alpha \neq \beta$. The quadratic terms in $\kappa_{\alpha\alpha}$ result from the second line of (4.3) and are $O(\theta^2/\lambda)$. Since other terms $O(\theta^2)$ were omitted in $\gamma_{\alpha\beta}$ and even more important terms $O(\theta\sqrt{\theta}/\lambda)$ were omitted in $\kappa_{\alpha\beta}$, the version of [26] can not be regarded as consistent within the large-rotation theory of shells. In a refined version [27] a theory of shells undergoing finite/small rotations was given. At the shell boundary contour a "vector of elastic rotation" was introduced which had no geometric meaning of a finite rotation vector [10-12]. It was assumed that only one of the three components of the vector was independent but explicit transformation formulae for other two components were not given. The work performed by the Eulerian boundary couple on the "vector of elastic rotation" was assumed as a scalar product of the vectors what may not be correct in the general case. As a result, the boundary conditions constructed in [27] can not be regarded as entirely Lagrangian and their physical and geometrical meaning as well as the range of applicability is open to discussion. Finally, let us remind that already in [28,29] it was suggested to take into account all quadratic terms in $\kappa_{\alpha\beta}$. However, the non-linear theory of shells generated by such formal quadratic strain measures can not be regarded as consistent from the point of view of an error introduced into the shell strain energy function. Besides, such measures would not allow to construct variationally derivable four Lagrangian static boundary conditions.

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