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ON GEOMETRICALLY NON-LINEAR THEORY OF ELASTIC SHELLS DERIVED FROM PSEUDO-COSSERAT CONTINUUM WITH CONSTRAINED MICRO-ROTATIONS

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1. Introduction

The set of equations for the geometrically non-linear theory of thin elastic shells is usually expressed in terms of displacements as basic independent variables of the shell deformation. Various general and reduced displacemental forms of bending shell equations are summarized, for example, by MUSHTARI and GALIMOV [1], KOITER [2], PIETRASZKIEWICZ [3,4], SCHMIDT [5] and BAŞAR and KRÄTZIG [6], where further references may be found. When displacement field is determined from the shell equations, strains, rotations and stresses may be obtained by prescribed algebraic or differential procedures.

The displacemental form of non-linear shell equations is very complex even in the tensor notation. When strains are small, components of the strain tensor are quadratic and components of the tensor of the change of curvature are cubic polynomials in displacements and their surface derivatives [4]. Additionally, the tensor of change of curvature, even in the simplest linearized case, depends upon the second derivatives of the displacements. It means that the strain energy density for the geometrically non-linear theory of shells is the polynomial up to the sixth order in displacements as well as their first and second surface derivatives. In modern computerized structural analysis, based on finite elements or finite differences, the need for discretization of the second derivatives causes many problems associated with inter-element continuity as well as discrete formulation of the boundary conditions. As a result, higher-order shape functions and difference schemes are required, which lead to additional degrees of freedom, complex schemes of numerical integration, reformulated boundary conditions etc.

The complexities associated with the displacemental shell equations can make it more attractive an alternative approach to shell theory, based on the polar decomposition of the shell deformation gradient into the rigid-body rotation and the pure stretch along the principal directions of strain. Then the finite rotation field is used as an independent or intermediate variable of the shell theory. This approach, originated by ALUMÄE[7] for the general shell geometry and by REISSNER [8] for axisymmetric deformation of shells of revolution, has been developed by WEMPNER [9], SIMMONDS and DANIELSON [10,11], PIETRASZKIEWICZ [3,12-14], SHKUTIN [15], LIBAI and SIMMONDS [16], ATLURI [17], KAYUK and SAKHATSKIY[18] and MAKOWSKI and STUMPF [19]. In this approach the structure of the non-linear shell equations becomes similar to the structure of the Cosserat surface theory [20,21,22]. However, in the latter theory displacements and rotations are, by definition, two independent kinematic field variables and the surface strain energy density is postulated from two-dimensional considerations,without any reference to the three-dimensional continuum mechanics.

In this paper we propose a new procedure for the derivation of the non-linear shell equations directly from the three dimensional constrained elastic Cosserat continuum. It is known [20,23-26] that within the Cosserat continuum each material particle can translate and independently rotate. This micro-rotation field does not coincide, in general, with the macro-rotation of the particle's neighbourhood as calculated from the displacement field in the classical continuum mechanics. The stress state is described by two, generally non-symmetric, stress and couple stress tensors.

In this paper we assume that the couple-stress tensor vanishes everywhere, what leads to the so-called pseudo-Cosserat continuum [24,25]. Additionally, micro-rotations are constrained to coincide everywhere with the macro-rotations. As a result, the Cosserat elastic continuum with the two constraints becomes entirely equivalent to the classical non-polar non-linear elasticity, but written here in different form, in terms of the Cosserat field variables. In particular, with the help of Lagrangian multipliers the second constraint is explicitly introduced into the strain energy function of the elastic Cosserat body.

In order to describe such constrained thin Cosserat body by a two--dimensional shell theory, we introduce the Kirchhoff-Love kinematic constrains and take the strain energy density in the form used in the classical first-approximation theory of thin isotropic elastic shells [27]. As a result, in the case of dead external loads, the functional of the total potential energy is constructed. It depends upon three displacements, three rotations and three Lagrangian multipliers (the scew-symmetric part of the internal surface stress resultant tensor and two shearing forces) as independent variables subject to variation. The functional is linear in the Lagrangian multipliers and is rational function containing at most fourth-order polynomials in displacements, rotations and their first derivatives. The latter feature seams to be very attractive for the numerical applications. It allows to use the simplest shape functions or difference schemes which assure the high efficiency of the numerical analysis. The stationarity conditions of the functional lead to six equilibrium equations, three constraint conditions and appropriate static boundary and corner conditions for the nine unknowns to be determined in the solution process. All the relations are given through components with respect to the rotated basis. Some advantages, similarities and differences of such a formulation of the non--linear shell theory in comparison with the ones proposed in [7,11,15, 17] are discussed.

2. Some relations of the Cosserat continuum

Here we briefly discuss a deformation of the body $\mathcal{B}$, consisting of material particles X, Y, ..., in the three-dimensional Euclidean point space $\mathcal{E}$. Let $\mathcal{P}=\kappa(\mathcal{B})$ and $\bar{\mathcal{P}}=\bar{\kappa}(\mathcal{B})$ be regions of $\mathcal{E}$ occupied by the body $\mathcal{B}$ in the reference (undeformed) and in the actual (deformed) configuration, respectively. The places P and $\bar{P}$ occupied by the particle $X \in \mathcal{B}$ in both configurations are given by the respective position vectors

$$\underset{\sim}{p} = x^k(\theta^i)\underset{\sim}{i}_k \quad , \qquad \bar{\underset{\sim}{p}} = y^k(\theta^i)\underset{\sim}{i}_k = \chi(\underset{\sim}{p}) = \underset{\sim}{p} + \underset{\sim}{w} \quad , \qquad (2.1)$$

where θ^i, $i = 1,2,3$, are the curvilinear convected coordinates, $\underset{\sim}{i}_k$ is the common orthonormal basis attached to an origin $\mathcal{O} \in \mathcal{E}$, χ is the macro-deformation function and $\underset{\sim}{w}$ is the displacement vector.

In $\mathcal{P}$ we introduce the base vectors $\underset{\sim}{g}_i = \underset{\sim}{p}_{,i}$, $\underset{\sim}{g}^i \cdot \underset{\sim}{g}_j = \delta^i_j$, the metric tensor $\underset{\sim}{1} = g_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j = g^{ij}\underset{\sim}{g}_i \otimes \underset{\sim}{g}_j$ with components $g_{ij} = \underset{\sim}{g}_i \cdot \underset{\sim}{g}_j$, $g^{ij} = \underset{\sim}{g}^i \cdot \underset{\sim}{g}^j$ and the scalar $g = |g_{ij}|$. Analogously defined functions in $\bar{\mathcal{P}}$ are marked by a dash: $\bar{\underset{\sim}{g}}_i$, $\bar{\underset{\sim}{g}}^j$, $\bar{\underset{\sim}{1}}$, $\bar{g}_{ij}$, $\bar{g}^{ij}$, $\bar{g}$ etc. All properties of the macro-deformation of $\mathcal{B}$ can be described entirely in terms of $\underset{\sim}{w}$ (see [28] for details).

Within the Cosserat theory it is assumed that during deformation each material particle $X \in \mathcal{B}$ can translate and independently rotate [20, 23,24,26]. The translation is described by the vector field $\underset{\sim}{w}$ while the rotation is described by an independent proper orthogonal tensor field $\underset{\sim}{R}$, such that $\underset{\sim}{R}^{-1} = \underset{\sim}{R}^{T}$, $\det \underset{\sim}{R} = +1$. The tensor field is called the micro-rotation of $X \in \mathcal{B}$. In order to make it more convenient let us introduce an aholonomic triad of vectors $\underset{\sim}{d}_i$ associated with each $X \in \mathcal{B}$, rigidly rotating with the particle during its deformation. Assuming, for convenience, that in the reference configuration those vectors coincide with the base vectors $\underset{\sim}{g}_i$, in the deformed configuration we obtain

$$\underset{\sim}{d}_j = \underset{\sim}{R}\underset{\sim}{g}_j \quad , \quad \underset{\sim}{R} = \underset{\sim}{d}_i \otimes \underset{\sim}{g}^i \quad , \quad \underset{\sim}{d}_i \cdot \underset{\sim}{d}_j = g_{ij} \ . \tag{2.2}$$

The complete information about deformation of the neighbourhood of the material particle $X \in \mathcal{B}$ contain now two fields: the deformation gradient $\underset{\sim}{F}$ and the micro-rotation gradient $\underset{\sim}{D}$,defined by

$$\begin{aligned} \underset{\sim}{F} &= \operatorname{Grad} \bar{\underset{\sim}{p}} = \bar{\underset{\sim}{g}}_j \otimes \underset{\sim}{g}^j \quad , \qquad 0 < \det \underset{\sim}{F} < \infty \quad , \\ \underset{\sim}{D} &= \operatorname{Grad} \underset{\sim}{R} = \underset{\sim}{d}_{i;j} \otimes \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \ . \end{aligned} \tag{2.3}$$

where $(\ )_{;j}$ is the covariant derivative with respect to the reference metric g_{ij} .

In classical continuum mechanics it is usual to apply the polar decomposition $\underset{\sim}{F} = \underset{\sim}{R}\underset{\sim}{U} = \underset{\sim}{V}\underset{\sim}{R}$, where $\underset{\sim}{U}$ and $\underset{\sim}{V}$ are the (symmetric) right and left stretch tensors and $\underset{\sim}{R}$ is the (proper orthogonal) rotation tensor of the macro-deformation χ . However, $\underset{\sim}{V}$, $\underset{\sim}{U}$ and $\underset{\sim}{R}$ do not describe properly the strains and rotations of the neighbourhood of the Cosserat material particle $X \in \mathcal{B}$.

The Lagrangian strain measures appropriate for the Cosserat continuum are defined by [20,23,24]

$$\underset{\sim}{\mathcal{U}} = \underset{\sim}{R}^T \underset{\sim}{F} = \mathcal{U}_{ij} \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \quad , \qquad \underset{\sim}{K} = \frac{1}{2} \underset{\sim}{\epsilon} \cdot (\underset{\sim}{R}^T \underset{\sim}{D}) = K_{ij} \underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \ , \tag{2.4}$$

where $\underset{\sim}{\epsilon} = -\underset{\sim}{1} \times \underset{\sim}{1} = \epsilon^{ijk} \underset{\sim}{g}_i \otimes \underset{\sim}{g}_j \otimes \underset{\sim}{g}_k$ and dot means double scalar product performed on the second-order tensors according to $\underset{\sim}{\epsilon} \cdot \underset{\sim}{A} = \epsilon^{ijk} A_{jk} \underset{\sim}{g}_i$.

In what follows it will be convenient to use another strain measures (in analogy to $\underset{\sim}{V}$ in the classical continuum mechanics) defined according to

$$\underset{\sim}{V} = \underset{\sim}{R}\underset{\sim}{\mathcal{U}}\underset{\sim}{R}^T = \mathcal{U}_{ij} \underset{\sim}{d}^i \otimes \underset{\sim}{d}^j \quad , \qquad \underset{\sim}{L} = \underset{\sim}{R}\underset{\sim}{K}\underset{\sim}{R}^T = K_{ij} \underset{\sim}{d}^i \otimes \underset{\sim}{d}^j \ . \tag{2.5}$$

Since $\underset{\sim}{R}^T \underset{\sim}{R}_{,j}$ and $\underset{\sim}{R}_{,j} \underset{\sim}{R}^T$ are scew-symmetric let us introduce their axial vectors [28]

$$\underset{\sim}{k}_j = -\frac{1}{2}\underset{\sim}{\epsilon} \cdot (\underset{\sim}{R}^T \underset{\sim}{R}_{,j}) \quad , \qquad \underset{\sim}{l}_j = -\frac{1}{2}\underset{\sim}{\epsilon} \cdot (\underset{\sim}{R}_{,j} \underset{\sim}{R}^T) = \underset{\sim}{R}\underset{\sim}{k}_j = K_{ij} \underset{\sim}{d}^i . \tag{2.6}$$

The relative Lagrangian strain measures are given by

$$\underset{\sim}{E} = \underset{\sim}{U} - \underset{\sim}{1} = E_{ij}\underset{\sim}{g}^i \otimes \underset{\sim}{g}^j \quad , \quad \underset{\sim}{H} = \underset{\sim}{V} - \underset{\sim}{1} = E_{ij}\underset{\sim}{d}^i \otimes \underset{\sim}{d}^j \ ,$$
$$\underset{\sim}{h}_j = E_{ij}\underset{\sim}{d}^i = \bar{\underset{\sim}{g}}_j - \underset{\sim}{d}_j = \underset{\sim}{g}_j + \underset{\sim}{w}_{,j} - \underset{\sim}{R}\underset{\sim}{g}_j \ . \tag{2.7}$$

The micro-rotation $\underset{\sim}{R}$ can be performed with an equivalent finite rotation vector $\underset{\sim}{\theta} = 2\,\mathrm{tg}\,\omega/2\,\underset{\sim}{e}$, where $\underset{\sim}{e}$ is the unit vector of the rotation axis and ω is the angle of rotation about $\underset{\sim}{e}$ associated with $\underset{\sim}{R}$. Then [28]

$$\underset{\sim}{d}_j = \underset{\sim}{g}_j + \frac{1}{t}\underset{\sim}{\theta} \times (\underset{\sim}{g}_j + \frac{1}{2}\underset{\sim}{\theta} \times \underset{\sim}{g}_j) \quad , \quad t = 1 + \frac{1}{4}\underset{\sim}{\theta}\cdot\underset{\sim}{\theta} \ ,$$
$$\underset{\sim}{l}_j = \frac{1}{t}(\underset{\sim}{\theta}_{,j} - \frac{1}{2}\underset{\sim}{\theta}_{,j} \times \underset{\sim}{\theta}) \ . \tag{2.8}$$

The relations $(2.7)_2$, $(2.8)_2$ and (2.6) allow to express the strain measures E_{ij} and K_{ij} in terms of $\underset{\sim}{w}$ and $\underset{\sim}{\theta}$. Solving $(2.7)_2$ and $(2.8)_2$ for $\underset{\sim}{w}_{,j}$ and $\underset{\sim}{\theta}_{,j}$ and applying the integrability conditions $\underset{\sim}{w}_{,ji} - \underset{\sim}{w}_{,ij} = \underset{\sim}{0}$, $\underset{\sim}{\theta}_{,ji} - \underset{\sim}{\theta}_{,ij} = \underset{\sim}{0}$ we obtain the vector form of compatibility conditions of the Cosserat continuum [26]

$$\epsilon^{sij}(\underset{\sim}{h}_{i;j} + \underset{\sim}{l}_j \times \underset{\sim}{d}_i) = \underset{\sim}{0} \quad , \quad \epsilon^{sij}(\underset{\sim}{l}_{i;j} + \frac{1}{2}\underset{\sim}{l}_i \times \underset{\sim}{l}_j) = \underset{\sim}{0} \ . \tag{2.9}$$

Suppose that in the deformed configuration $\bar{P}$ there is an oriented differential area $d\bar{A}$ with a unit outward normal $\bar{\underset{\sim}{n}}$ at $\bar{\underset{\sim}{p}}$. Let the image at $\underset{\sim}{p}$ in P corresponding to $\bar{\underset{\sim}{n}}d\bar{A}$ is $\underset{\sim}{n}dA$. Then [29]

$$\bar{\underset{\sim}{n}}d\bar{A} = J\underset{\sim}{n}\underset{\sim}{F}^{-1}dA \quad , \quad J = \det\underset{\sim}{F} = \sqrt{\bar{g}/g} \ . \tag{2.10}$$

Let $\underset{\sim}{t}$ nad $\underset{\sim}{m}$ be the stress and couple-stress vectors acting on the oriented differential area $\bar{\underset{\sim}{n}}d\bar{A}$ in $\bar{P}$. Then the "true" Cauchy--type stress and couple-stress tensors $\underset{\sim}{\tau}$ and $\underset{\sim}{\mu}$ are given by

$$\underset{\sim}{t}d\bar{A} = \underset{\sim}{\tau}\bar{\underset{\sim}{n}}d\bar{A} \quad , \quad \underset{\sim}{m}d\bar{A} = \underset{\sim}{\mu}\bar{\underset{\sim}{n}}d\bar{A} \quad , \quad \underset{\sim}{\tau} = \tau^{ij}\bar{\underset{\sim}{g}}_i \otimes \bar{\underset{\sim}{g}}_j \ , \ \underset{\sim}{\mu} = \mu^{ij}\bar{\underset{\sim}{g}}_i \otimes \bar{\underset{\sim}{g}}_j \tag{2.11}$$

Using (2.10) and $\underset{\sim}{R}$ we can define alternative stress and couple-stress tensors referred to the undeformed configuration

$$\underset{\sim}{t}d\bar{A} = \underset{\sim}{T}_R\underset{\sim}{n}dA = \underset{\sim}{R}\underset{\sim}{T}_J\underset{\sim}{n}dA = \underset{\sim}{T}_B\underset{\sim}{R}\underset{\sim}{n}dA \quad ,$$
$$\underset{\sim}{m}d\bar{A} = \underset{\sim}{M}_R\underset{\sim}{n}dA = \underset{\sim}{R}\underset{\sim}{M}_J\underset{\sim}{n}dA = \underset{\sim}{M}_B\underset{\sim}{R}\underset{\sim}{n}dA \quad , \tag{2.12}$$

$$\underset{\sim}{T}_R = J\underset{\sim}{\tau}\underset{\sim}{F}^{-T} \quad , \quad \underset{\sim}{T}_J = J\underset{\sim}{R}^T\underset{\sim}{\tau}\underset{\sim}{F}^{-T} \quad , \quad \underset{\sim}{T}_B = J\underset{\sim}{\tau}\underset{\sim}{F}^{-T}\underset{\sim}{R}^T \quad ,$$
$$\underset{\sim}{M}_R = J\underset{\sim}{\mu}\underset{\sim}{F}^{-T} \quad , \quad \underset{\sim}{M}_J = J\underset{\sim}{R}^T\underset{\sim}{\mu}\underset{\sim}{F}^{-T} \quad , \quad \underset{\sim}{M}_B = J\underset{\sim}{\mu}\underset{\sim}{F}^{-T}\underset{\sim}{R}^T \ . \tag{2.13}$$

In analogy to the classical continuum, $\underset{\sim}{T}_R$ and $\underset{\sim}{M}_R$ are the first

Piola-Kirchhoff type, $\underset{\sim}{T}_J$ and $\underset{\sim}{M}_J$ are the Jaumann type and $\underset{\sim}{T}_B$ and $\underset{\sim}{M}_B$ are the Biot type stress and couple-stress tensors, respectively.

In terms of those stress and strain measures various but equivalent sets of basic balance laws for the Cosserat continuum may be given [23,24]. In particular, the stress working rate is given in the following equivalent forms

$$\begin{aligned} \dot{U} &= \int_P \dot{W}\, dV = \int_P (T^{ij}\dot{E}_{ij} + M^{ij}\dot{K}_{ij})\, dV = \\ &= \int_P (\underset{\sim}{T}_J \cdot \dot{\underset{\sim}{E}} + \underset{\sim}{M}_J \cdot \dot{\underset{\sim}{K}})\, dV = \int_P (\underset{\sim}{T}_B \cdot \overset{\nabla}{\underset{\sim}{H}} + \underset{\sim}{M}_B \cdot \overset{\nabla}{\underset{\sim}{L}})\, dV \quad , \end{aligned} \tag{2.14}$$

where the superposed dot denotes the material time derivative, the superposed triangle is the co-rotational time derivative and T^{ij} and M^{ij} are components of the Jaumann stress measures given by

$$\begin{aligned} &\overset{\nabla}{\underset{\sim}{H}} = \dot{\underset{\sim}{H}} + \underset{\sim}{H}\underset{\sim}{\Omega} - \underset{\sim}{\Omega}\underset{\sim}{H} \quad , \quad \underset{\sim}{\Omega} = \dot{\underset{\sim}{R}}\underset{\sim}{R}^T = -\underset{\sim}{\Omega}^T \quad , \\ &T^{ij} = \underset{\sim}{g}^i \underset{\sim}{T}_J \underset{\sim}{g}^j = \underset{\sim}{d}^i \underset{\sim}{T}_B \underset{\sim}{d}^j = J(\underset{\sim}{d}^i \cdot \bar{\underset{\sim}{g}}_k)\tau^{kj} \quad , \\ &M^{ij} = \underset{\sim}{g}^i \underset{\sim}{M}_J \underset{\sim}{g}^j = \underset{\sim}{d}^i \underset{\sim}{M}_B \underset{\sim}{d}^j = J(\underset{\sim}{d}^i \cdot \bar{\underset{\sim}{g}}_k)\mu^{kj} \quad . \end{aligned} \tag{2.15}$$

The respective scalar products in (2.14) contain the work-conjugate pairs of stress and strain measures [30]. The pairs $\underset{\sim}{T}_B$, $\underset{\sim}{H}$ and $\underset{\sim}{M}_B, \underset{\sim}{L}$ are particularly interesting here, since they are defined by components in the rotated triad $\underset{\sim}{d}_i$.

It follows from (2.14) that for an elastic Cosserat body the strain energy density depends explicitly only upon E_{ij} and K_{ij} , i.e. $W = W(E_{ij}, K_{ij})$. However, in what follows we are interested in analysing the Cosserat elastic body with two additional constraints. The first constraint states that the couple-stress tensor identically vanishes during an arbitrary motion, i.e. $\underset{\sim}{\mu} = \underset{\sim}{M}_R = \underset{\sim}{M}_J = \underset{\sim}{M}_B \equiv \underset{\sim}{0}$. In such pseudo--Cosserat elastic body the strain energy density does not depend explicitly upon K_{ij} , i.e. $W = W(E_{ij})$. The second constraint requires the micro-rotation $\underset{\sim}{R}$ to coincide with the macro-rotation $\underset{\sim}{R}$ during an arbitrary motion. As a result $\underset{\sim}{U} = \underset{\sim}{R}^T\underset{\sim}{F} = \underset{\sim}{U} = \underset{\sim}{U}^T = \underset{\sim}{U}^T$ and, similarly, $\underset{\sim}{V} = \underset{\sim}{V}^T$, $\underset{\sim}{E} = \underset{\sim}{E}^T$, or $\epsilon^{ijk}E_{ij} = 0$. This constraint can be introduced into the strain energy density with the help of Lagrangian multipliers λ_k , redefining it as follows

$$E = W(E_{ij}) + \epsilon^{ijk}E_{ij}\lambda_k \quad . \tag{2.16}$$

The strain energy density (2.16) defines the constrained Cosserat elastic body which is equivalent to the classical non-polar non-linear elastic body, only written here in terms of the Cosserat strain measure $\underset{\sim}{E}$. From (2.16) follow the constitutive equations of the constrained Cosserat body

$$T^{ij} = \frac{\partial E}{\partial E_{ij}} = \frac{\partial W}{\partial E_{ij}} + \epsilon^{ijk}\lambda_k \ ,$$
$$M^{ij} = \frac{\partial E}{\partial K_{ij}} = 0 \ , \quad \frac{\partial E}{\partial \lambda_k} = \epsilon^{ijk}E_{ij} = 0 \qquad (2.17)$$

which incorporate explicitly the two constraints.

3. Deformation of a thin shell under K-L constraints

Let in the region $P \in E$, occupied by the constrained Cosserat body B in the reference configuration, the normal coordinate system $\{\theta^\alpha, \theta^3 \equiv \zeta\}$ $(\alpha = 1,2)$ is introduced, where $-h/2 \leq \zeta \leq h/2$ is the distance from the middle surface M of P and h is the thickness of P, assumed to be small.

The geometry of M is described by the position vector $\underset{\sim}{r} = \underset{\sim}{r}(\theta^\alpha)$. At each point $M \in M$, we have the natural surface base vectors $\underset{\sim}{a}_\alpha = \underset{\sim}{r}_{,\alpha}$, $\underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}_\beta = \delta^\alpha_\beta$, the components $a_{\alpha\beta} = \underset{\sim}{a}_\alpha \cdot \underset{\sim}{a}_\beta$ and $a^{\alpha\beta} = \underset{\sim}{a}^\alpha \cdot \underset{\sim}{a}^\beta$ of the surface metric tensor $\underset{\sim}{a} = a_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta = a^{\alpha\beta}\underset{\sim}{a}_\alpha \otimes \underset{\sim}{a}_\beta$ with the scalar $a = |a_{\alpha\beta}|$, the unit vector $\underset{\sim}{a}_3 \equiv \underset{\sim}{n} = \frac{1}{2}\epsilon^{\alpha\beta}\underset{\sim}{a}_\alpha \times \underset{\sim}{a}_\beta$ orthogonal to M and the covariant components $b_{\alpha\beta} = -\underset{\sim}{n}_{,\alpha} \cdot \underset{\sim}{a}_\beta$ of the curvature tensor $\underset{\sim}{b} = b_{\alpha\beta}\underset{\sim}{a}^\alpha \otimes \underset{\sim}{a}^\beta$. The boundary contour C of M is described by $\theta^\alpha = \theta^\alpha(s)$, where s is the length parameter along C. Along C we define the unit tangent vector $\underset{\sim}{t} = d\underset{\sim}{r}/ds = t^\alpha \underset{\sim}{a}_\alpha$ and the outward unit normal vector $\underset{\sim}{\nu} = \underset{\sim}{t} \times \underset{\sim}{n}$.

Analogously defined functions at $\bar{M} \in \bar{M}$, which is the image of the $M \in M$ in the deformed configuration, are marked by an additional dash: $\bar{\underset{\sim}{r}}$, $\bar{\underset{\sim}{a}}_\alpha$, $\bar{\underset{\sim}{a}}^\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$, $\bar{\underset{\sim}{a}}$, $\bar{a}$, $\bar{\underset{\sim}{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\underset{\sim}{b}}$ etc. The deformation of M into $\bar{M}$ and C into $\bar{C}$ is described in detail in [3,4,12] where further details may be found.

Let on the middle surface M of P the following fields are introduced

$$\underset{\sim}{a}_i = \underset{\sim}{g}_i(P)_{|\zeta=0} \ , \quad \underset{\sim}{r}_i = \underset{\sim}{d}_i(P)_{|\zeta=0} \ , \quad \bar{\underset{\sim}{a}}_i = \bar{\underset{\sim}{g}}_i(\bar{P})_{|\zeta=0} \ ,$$
$$\underset{\sim}{u} = \underset{\sim}{w}(P)_{|\zeta=0} \ , \quad \underset{\sim}{Q} = \underset{\sim}{R}(P)_{|\zeta=0} \ , \quad \underset{\sim}{G} = \underset{\sim}{F}(P)_{|\zeta=0} \ , \quad \underset{\sim}{\eta} = \underset{\sim}{H}(P)_{|\zeta=0} \ . \qquad (3.1)$$

Material fibres of the body B, initially normal to M, after deformation may become neither straight nor normal to $\bar{M}$. In particular, $\bar{\underset{\sim}{a}}_3 \neq \bar{\underset{\sim}{n}}$, in general. However, when discussing deformation of thin elastic shells undergoing small strains it is possible in all kinematic relations to approximate it by assuming, that material fibres which are normal to M , after deformation remain straight and normal to $\bar{M}$ and do not change their lengths. Under such Kirchhoff-Love type kinematic constraint $\bar{\mathbf{a}}_3 \equiv \bar{\mathbf{n}}$. The change of the shell thickness during deformation will, however, be taken into account in the constitutive equations, see (4.2) below.

The geometry of P in the normal system of coodinates is described by the geometry of the surface M . Under the K-L kinematic constraints the same applies to $\bar{P}$ and $\bar{M}$, respectively,

$$\begin{aligned} &\mathbf{p} = \mathbf{r} + \zeta\mathbf{n} \quad , \qquad \bar{\mathbf{p}} = \bar{\mathbf{r}} + \zeta\bar{\mathbf{n}} \quad , \\ &\mathbf{g} = \mathbf{1} - \zeta\mathbf{b} \quad , \qquad \bar{\mathbf{g}} = \bar{\mathbf{1}} - \zeta\bar{\mathbf{b}} \quad , \\ &\mathbf{g}_i = \mathbf{g}\mathbf{a}_i \quad , \quad \mathbf{g}^i = \mathbf{g}^{-1}\mathbf{a}^i \quad , \quad \bar{\mathbf{g}}_i = \bar{\mathbf{g}}\bar{\mathbf{a}}_i \quad , \quad \bar{\mathbf{g}}^i = \bar{\mathbf{g}}^{-1}\bar{\mathbf{a}}^i \ . \end{aligned} \tag{3.2}$$

Under the K-L kinematic constraints the kinematic parameters of the pseudo-Cosserat shell are given by

$$\begin{aligned} &\mathbf{w} = \mathbf{u} + \zeta(\bar{\mathbf{n}} - \mathbf{n}) \ , \qquad \mathbf{R} = \mathbf{Q} \quad , \\ &\mathbf{F} = \bar{\mathbf{g}}\mathbf{G}\mathbf{g}^{-1} = (\mathbf{G} - \zeta\bar{\mathbf{b}}\mathbf{G})\mathbf{g}^{-1} \quad , \\ &\mathbf{H} = (\boldsymbol{\eta} + \zeta\boldsymbol{\kappa})\mathbf{g}_r^{-1} \quad , \qquad \mathbf{g}_r^{-1} = \mathbf{Q}\mathbf{g}^{-1}\mathbf{Q} \quad , \end{aligned} \tag{3.3}$$

$$\begin{aligned} &\boldsymbol{\eta} = \eta_{ij}\mathbf{r}^i \otimes \mathbf{r}^j = \boldsymbol{\eta}_j \otimes \mathbf{r}^j \quad , \\ &\boldsymbol{\eta}_\beta = \eta_{i\beta}\mathbf{r}^i = \bar{\mathbf{a}}_\beta - \mathbf{r}_\beta \quad , \quad \boldsymbol{\eta}_3 = \eta_{i3}\mathbf{r}^i = \bar{\mathbf{n}} - \mathbf{r}_3 \quad , \end{aligned} \tag{3.4}$$

$$\begin{aligned} &\boldsymbol{\kappa} = \kappa_{i\beta}\mathbf{r}^i \otimes \mathbf{r}^\beta = \boldsymbol{\kappa}_\beta \otimes \mathbf{r}^\beta \quad , \\ &\boldsymbol{\kappa}_\beta = \bar{\mathbf{n}}_{,\beta} + b^\alpha_\beta\,\mathbf{r}_\alpha = \mathbf{l}_\beta \times \mathbf{r}_3 + \boldsymbol{\eta}_{3,\beta} \quad , \\ &\mathbf{l}_\beta = -\frac{1}{2}\boldsymbol{\epsilon}\cdot(\mathbf{Q}_{,\beta}\mathbf{Q}^T) = \frac{1}{t}(\boldsymbol{\theta}_{,\beta} - \frac{1}{2}\boldsymbol{\theta}_{,\beta} \times \boldsymbol{\theta}) \ . \end{aligned} \tag{3.5}$$

If, additionally, the micro-rotations $\mathbf{Q}$ are constrained to coincide with the macro-rotations $\mathbf{R}$ then

$$\begin{aligned} &\mathbf{r}_3 = \bar{\mathbf{n}} \quad , \quad \boldsymbol{\eta}_3 = \mathbf{0} \quad , \quad \eta_{3\beta} = 0 \quad , \quad \kappa_{3\beta} = 0 \quad , \\ &\boldsymbol{\eta}_\beta = \eta_{\alpha\beta}\mathbf{r}^\alpha = \mathbf{u}_{,\beta} - \frac{1}{t}\boldsymbol{\theta} \times (\mathbf{r}_\beta - \frac{1}{2}\boldsymbol{\theta} \times \mathbf{r}_\beta) \quad , \\ &\boldsymbol{\kappa}_\beta = \kappa_{\alpha\beta}\mathbf{r}^\alpha = \mathbf{l}_\beta \times \bar{\mathbf{n}} = \frac{1}{t}(\boldsymbol{\theta}_{,\beta} - \frac{1}{2}\boldsymbol{\theta}_{,\beta} \times \boldsymbol{\theta}) \times \bar{\mathbf{n}} \ . \end{aligned} \tag{3.6}$$

Solving $(3.6)_2$ and $(3.5)_3$ for $\mathbf{u}_{,\beta}$ and $\boldsymbol{\theta}_{,\beta}$ and applying the integrability conditions $\underset{\sim}{u}_{,\beta\alpha} - \underset{\sim}{u}_{,\alpha\beta} = \underset{\sim}{0}$, $\underset{\sim}{\theta}_{,\beta\alpha} - \underset{\sim}{\theta}_{,\alpha\beta} = \underset{\sim}{0}$ we obtain the vector form of compatibility conditions of the non-linear shell theory

$$\epsilon^{\alpha\beta}(\underset{\sim}{\eta}_{\alpha|\beta} + \underset{\sim}{l}_{\beta} \times \underset{\sim}{r}_{\alpha}) = \underset{\sim}{0} \quad , \quad \epsilon^{\alpha\beta}(\underset{\sim}{l}_{\alpha|\beta} + \tfrac{1}{2}\underset{\sim}{l}_{\alpha} \times \underset{\sim}{l}_{\beta}) = \underset{\sim}{0} \tag{3.7}$$

given in component form in [7] and in vector form in [15,16].

Note that the strain measures $\eta_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ defined in $(3.6)_{2,3}$ are rational (quadratic, at most) functions of displacements, rotations and their first surface derivatives. In general, $\eta_{\alpha\beta} \neq \eta_{\beta\alpha}$ and $\kappa_{\alpha\beta} \neq \kappa_{\beta\alpha}$. Linearizing $\underset{\sim}{\eta}_{\beta}$ and $\underset{\sim}{\kappa}_{\beta}$ with respect to $\underset{\sim}{u}$ and $\underset{\sim}{\theta}$ and taking into account that in the linear shell theory [3,12] $\underset{\sim}{\theta} = \underset{\sim}{\theta}(\underset{\sim}{u}) \simeq \epsilon^{\beta\alpha}\phi_{\alpha}\underset{\sim}{a}_{\beta} + \phi\underset{\sim}{n}$ we obtain

$$\begin{aligned} &\eta_{\alpha\beta} \simeq \frac{1}{2}(u_{\alpha|\beta} + u_{\beta|\alpha}) - b_{\alpha\beta}w \quad , \quad \kappa_{\alpha\beta} \simeq -\phi_{\alpha|\beta} + b^{\lambda}_{\beta}\epsilon_{\lambda\alpha}\phi \quad , \\ &\phi_{\alpha} = w_{,\alpha} + b^{\beta}_{\alpha}u_{\beta} \quad , \quad \phi = \frac{1}{2}\epsilon^{\alpha\beta}u_{\beta|\alpha} \quad , \end{aligned} \tag{3.8}$$

where $(\ )_{|\alpha}$ means the covariant differentiation in the reference metric $a_{\alpha\beta}$. The linearized components of $\eta_{\alpha\beta}$ and $\kappa_{(\alpha\beta)} \equiv \frac{1}{2}(\kappa_{\alpha\beta} + \kappa_{\beta\alpha})$ are the measures used in the "best" linear shell theory [31].

4. Variationally derivable non-linear shell equations

The two-dimensional strain energy density, per unit area of the undeformed middle surface M , for the constrained Cosserat shell is given in terms of (2.16) by

$$\Sigma = \int_{-h/2}^{h/2} E\sqrt{\frac{g}{a}}\, d\zeta = \Sigma_{o} + \Sigma_{con} \ . \tag{4.1}$$

Within the consistent first-approximation geometrically non-linear theory of thin isotropic elastic shells the first term in (4.1) can be approximated by the quadratic expression [27,13]

$$\begin{aligned} &\Sigma_{o} = \frac{1}{2} H^{\alpha\beta\lambda\mu}\left(\eta_{\alpha\beta}\eta_{\lambda\mu} + \frac{h^2}{12}\kappa_{\alpha\beta}\kappa_{\lambda\mu}\right) \quad , \\ &H^{\alpha\beta\lambda\mu} = \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right) \ . \end{aligned} \tag{4.2}$$

The second term in (4.1) appears as the result of constrained micro-rotations of the pseudo-Cosserat body and follows from direct integration of (2.16), with $(3.6)_1$ and the accuracy of $(4.2)_1$ taken into account,

$$\Sigma_{con} = \int_{-h/2}^{h/2} \epsilon^{ijk}E_{ij}\lambda_{k}\sqrt{\frac{g}{a}}\, d\zeta = \epsilon^{\alpha\beta}\underset{\sim}{r}_{\alpha}\cdot\underset{\sim}{n}_{\beta}N + \underset{\sim}{\bar{n}}\cdot\underset{\sim}{n}_{\beta}Q^{\beta} \quad , \tag{4.3$_1$}$$

$$N = \int_{-h/2}^{h/2} \lambda_3 \, d\zeta \quad , \qquad Q^\beta = \int_{-h/2}^{h/2} \epsilon^{\beta\alpha} \lambda_\alpha \, d\zeta \quad . \qquad (4.3)_2$$

If we introduce the symmetric surface internal stress resultant and stress couple tensors

$$N^{\alpha\beta} = \frac{\partial \Sigma_o}{\partial \eta_{\alpha\beta}} = H^{\alpha\beta\lambda\mu} \eta_{\lambda\mu} \quad , \qquad M^{\alpha\beta} = \frac{\partial \Sigma_o}{\partial \kappa_{\alpha\beta}} = \frac{h^2}{12} H^{\alpha\beta\lambda\mu} \kappa_{\lambda\mu} \qquad (4.4)$$

then the strain energy (4.1) with (4.3) generates the following constitutive equations for the internal stress resultant and stress couple vectors defined with reference to the rotated basis

$$\begin{aligned} \underset{\sim}{N}^\beta &= \frac{\partial \Sigma}{\partial \underset{\sim}{\eta}_\beta} = (N^{\alpha\beta} + \epsilon^{\alpha\beta} N) \underset{\sim}{r}_\alpha + Q^\beta \underset{\sim}{\bar{n}} \quad , \\ \underset{\sim}{K}^\beta &= \frac{\partial \Sigma}{\partial \underset{\sim}{\kappa}_\beta} = M^{\alpha\beta} \underset{\sim}{r}_\alpha \quad , \qquad \underset{\sim}{M}^\beta = \underset{\sim}{\bar{n}} \times \underset{\sim}{K}^\beta = \epsilon_{\alpha\lambda} M^{\alpha\beta} \underset{\sim}{r}^\lambda \quad . \end{aligned} \qquad (4.5)$$

Note that the vector $\underset{\sim}{M}^\beta$ in $(4.5)_2$ is given only through symmetric components $M^{\alpha\beta}$. This means that within the accuracy of the first--approximation non-linear shell theory only symmetric part $\kappa_{(\alpha\beta)}$ of the tensor of change of curvature enters explicitly into the shell equations. The scew-symmetric part $\kappa_{[\alpha\beta]}$ of this tensor does not contribute to the elastic strain energy density and, therefore, may be ignored in all shell relations.

The formulae (4.4) reveal the physical meaning of the Lagrangian multipliers N and Q^β to be just the scew-symmetric part of the internal surface stress resultant tensor and the shearing forces, respectively. Moreover, the tensor fields $\underset{\sim}{N} = \underset{\sim}{N}^\beta \otimes \underset{\sim}{r}_\beta$ and $\underset{\sim}{K} = \underset{\sim}{K}^\beta \otimes \underset{\sim}{r}_\beta$ are seen to be the two-dimensional counterparts of the Biot-type stress measures of the Cosserat continuum (2.13).

Let the shell be loaded by an external surface load $\underset{\sim}{p}$, per unit area of M, and by external boundary load $\underset{\sim}{f}$, per unit area of the reference boundary surface $C \times (-\frac{h}{2}, \frac{h}{2})$. For simplicity of further results let us assume $\underset{\sim}{p}$ and $\underset{\sim}{f}$ to be conservative and dead-load type. Then the total potential energy of the shell is given by the functional [4]

$$\begin{aligned} I &= \iint_M [\Sigma(\underset{\sim}{u}, \underset{\sim}{\theta}, N, Q^\beta) - \underset{\sim}{p} \cdot \underset{\sim}{u}] \, dA - \int_C [\underset{\sim}{T} \cdot \underset{\sim}{u} + \underset{\sim}{H} \cdot (\underset{\sim}{\bar{n}} - \underset{\sim}{n})] \, ds \quad , \\ \underset{\sim}{T} &= \int_{-h/2}^{h/2} \underset{\sim}{f} \sqrt{\frac{g}{a}} \, d\zeta \quad , \qquad \underset{\sim}{H} = \int_{-h/2}^{h/2} \underset{\sim}{f} \zeta \sqrt{\frac{g}{a}} \, d\zeta \quad . \end{aligned} \qquad (4.6)$$

The variational principle $\delta I = 0$ states that among all possible values of independent fields $\underset{\sim}{u}$, $\underset{\sim}{\theta}$, N and Q^β, which are subject to the geometric boundary conditions, the actual solution renders the functional stationary.

Let us find the stationarity conditions of I. Taking into account that $\delta \underset{\sim}{r}_\alpha = \delta \underset{\sim}{\omega} \times \underset{\sim}{r}_\alpha$, $\delta \underset{\sim}{\bar{n}} = \delta \underset{\sim}{\omega} \times \underset{\sim}{\bar{n}}$, $\delta \underset{\sim}{\bar{a}}_\beta = \delta \underset{\sim}{u}_{,\beta}$ and $\delta \underset{\sim}{l}_\beta = \delta \underset{\sim}{\omega}_{,\beta} - \underset{\sim}{l}_\beta \times \delta \underset{\sim}{\omega}$ then in the rotated basis

$$\delta \underset{\sim}{\omega} = (1 + \tfrac{1}{4}\underset{\sim}{\theta}\cdot\underset{\sim}{\theta})^{-1}(\delta\underset{\sim}{\theta} + \tfrac{1}{2}\underset{\sim}{\theta}\times\delta\underset{\sim}{\theta}) \quad ,$$

$$\begin{aligned} &\delta\eta_{\alpha\beta} = \delta(\underset{\sim}{r}_\alpha\cdot\underset{\sim}{\bar{a}}_\beta) = \underset{\sim}{r}_\alpha\cdot\delta\eta_{\kappa\beta}\underset{\sim}{r}^\kappa \ , \quad \delta\kappa_{\alpha\beta} = \delta[\underset{\sim}{r}_\alpha\cdot(\underset{\sim}{l}_\beta\times\underset{\sim}{\bar{n}})] = \underset{\sim}{r}_\alpha\cdot\delta\kappa_{\kappa\beta}\underset{\sim}{r}^\kappa \ , \\ &\delta\eta_{\alpha\beta}\underset{\sim}{r}^\alpha = \delta\underset{\sim}{u}_{,\beta} + \underset{\sim}{\bar{a}}_\beta\times\delta\underset{\sim}{\omega} \ , \qquad \delta\kappa_{\alpha\beta}\underset{\sim}{r}^\alpha = \delta\underset{\sim}{\omega}_{,\beta}\times\underset{\sim}{\bar{n}} \ . \end{aligned} \qquad (4.7)$$

The variation of $(4.6)_1$, performed with the help of (4.5),(4.7) and Stokes' theorem, leads to

$$\begin{aligned} \delta I = &- \iint_M [(\underset{\sim}{N}^\beta|_\beta + \underset{\sim}{p})\cdot\delta\underset{\sim}{u} + (\underset{\sim}{M}^\beta|_\beta + \underset{\sim}{\bar{a}}_\beta\times\underset{\sim}{N}^\beta)\cdot\delta\underset{\sim}{\omega} - \\ &- \epsilon^{\alpha\beta}\underset{\sim}{\bar{r}}_\alpha\cdot\underset{\sim}{n}_\beta\delta N - \underset{\sim}{\bar{n}}\cdot\underset{\sim}{n}_\beta\delta Q^\beta\,]\,dA + \\ &+ \int_C \{\ (\underset{\sim}{N}^\beta\nu_\beta - \underset{\sim}{T})\cdot\delta\underset{\sim}{u} + [\underset{\sim}{\bar{n}}\times(\underset{\sim}{K}^\beta\nu_\beta - \underset{\sim}{H})]\cdot\delta\underset{\sim}{\omega}\ \}\ ds \quad . \end{aligned} \qquad (4.8)$$

The variations $\delta\underset{\sim}{u}$ and $\delta\underset{\sim}{\theta}$ at the boundary contour C are not independent, since the micro-rotations have been assumed to coincide with the macro-rotations. This constraint condition has been explicitly taken into account in the internal part of the shell. It has not been taken into account at the shell lateral boundary yet. Let $\underset{\sim}{H} = H_\nu\underset{\sim}{\bar{\nu}} + H_t\underset{\sim}{\bar{t}} + H\underset{\sim}{\bar{n}}$. Note also that for small strains $\underset{\sim}{\bar{t}} \simeq \underset{\sim}{\bar{a}}_t = \underset{\sim}{\bar{a}}_\alpha t^\alpha = \underset{\sim}{t} + d\underset{\sim}{u}/ds$, $\underset{\sim}{\bar{\nu}} = \underset{\sim}{\bar{t}}\times\underset{\sim}{\bar{n}} \simeq \underset{\sim}{\bar{a}}_t\times\underset{\sim}{\bar{n}}$. Then $(4.8)_3$ can be transformed further into

$$\int_C [(\underset{\sim}{P} - \underset{\sim}{P}^*)\cdot\delta\underset{\sim}{u} + (M - M^*)\underset{\sim}{\bar{t}}\cdot\delta\underset{\sim}{\omega}]\,ds + \sum_j(\underset{\sim}{F}_j - \underset{\sim}{F}_j^*)\cdot\delta\underset{\sim}{u}_j \quad , \qquad (4.9)$$

where

$$\begin{aligned} &\underset{\sim}{F} = M^{\alpha\beta}\nu_\alpha t_\beta\underset{\sim}{\bar{n}} \quad , \quad \underset{\sim}{F}^* = H_t\underset{\sim}{\bar{n}} \quad , \quad M = M^{\alpha\beta}\nu_\alpha\nu_\beta \quad , \quad M^* = H_\nu \quad , \\ &\underset{\sim}{P} = \underset{\sim}{N}^\beta\nu_\beta + \frac{d}{ds}\underset{\sim}{F} \quad , \quad \underset{\sim}{P}^* = \underset{\sim}{T} + \frac{d}{ds}\underset{\sim}{F}^* \ , \quad \underset{\sim}{F}_j = \underset{\sim}{F}(s_j+0) - \underset{\sim}{F}(s_j-0) \quad . \end{aligned} \qquad (4.10)$$

Now from $\delta I = 0$, together with (4.8),(4.9) and (4.10), we obtain the following stationarity conditions of I , which consist of equilibrium equations, constraint conditions as well as static boundary and corner conditions :

$$\left.\begin{aligned} &\underset{\sim}{N}^\beta|_\beta + \underset{\sim}{p} = \underset{\sim}{0} \quad , \qquad \underset{\sim}{M}^\beta|_\beta + \underset{\sim}{\bar{a}}_\beta\times\underset{\sim}{N}^\beta = \underset{\sim}{0} \\ &\epsilon^{\alpha\beta}\underset{\sim}{r}_\alpha\cdot\underset{\sim}{n}_\beta = 0 \quad , \qquad \underset{\sim}{\bar{n}}\cdot\underset{\sim}{n}_\beta = 0 \end{aligned}\right\}\ \text{in } M \qquad (4.11)$$

$$\underset{\sim}{P} = \underset{\sim}{P}^* \ , \quad M = M^* \quad \text{on } C_f \quad \text{and} \quad \underset{\sim}{F}_j = \underset{\sim}{F}_j^* \quad \text{at each corner } M_j \in C_f .$$

All field variables in (4.11) are given by components in the rotated basis $\underset{\sim}{r}_i$ and are understood to be expressed in terms of $\underset{\sim}{u}$, $\underset{\sim}{\theta}$, N and Q^{β} as independent variables.

5. Discussion

The non-linear shell theory based on the functional (4.6) or on its incremental form (4.8) has some interesting properties. The functional depends explicitly only upon displacements $\underset{\sim}{u}$, rotations $\underset{\sim}{\theta}$ and Lagrangian multipliers N, Q^{β} as nine basic independent variables to be discretized. The functional is linear in N, Q^{β} and is rational function containing at most fourth-order polynomials in $\underset{\sim}{u}$, $\underset{\sim}{\theta}$ and their first surface derivatives. The latter property is of great importance for the computerized numerical analysis of the flexible shell structures. It allows to apply the simplest shape functions in the finite-element analysis or the simplest difference schemes in the finite-difference analysis, which assure the high efficiency of numerical algorithms applied and better convergence to the accurate results for highly non-linear problems of flexible shells.

The equilibrium equations $(4.11)_1$ were derived already in the pioneering work of ALUMÄE [7], who suggested to solve them in the intrinsic form, together with the compatibility conditions (3.7). For shallow shells the set of equations was solved in [7] in terms of two scalar stress and displacement functions. SIMMONDS and DANIELSON [10,11] expressed the basic set of shell equations in terms of finite rotations and the internal stress resultants (or the stress function vector) as basic independent variables and constructed an appropriate variational functional [10]. Several variational principles, involving the finite rotation tensor and the stress function vector, were also discussed by ATLURI [17]. In the approach used in [10,11,17] displacement field is understood to be calculated, it necessary, by additional quadratures. SHKUTIN [15] proposed to solve the equilibrium equations $(4.11)_1$ in terms of displacements and rotations, but rotations were still expressed through displacements by additional differential relations. The primary difference between the present approach and that of [15] (apart from unusual notation applied in [15]) is that in our approach the constraint conditions put on the rotations in the internal shell space have been introduced explicitly into the variational principle. As a result three additional physically meaningful independent parameters N, Q^{β} have explicitly appeared in the shell equations.

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