J. Makowski[1]
Assistant Professor.

W. Pietraszkiewicz[2]
Professor.

Ruhr Universität,
Lehrstuhl für Mechanik II,
D-4630 Bochum 1,
Federal Republic of Germany

Work-Conjugate Boundary Conditions in the Nonlinear Theory of Thin Shells

Work-conjugate boundary conditions for a class of nonlinear theories of thin shells formulated in terms of displacements of the reference surface are discussed. Applying theorems of the theory of differential forms it is shown that many of the sets of static boundary conditions which have been proposed in the literature do not possess work-conjugate geometric counterparts. The general form of four geometric boundary conditions and their work-conjugate static boundary conditions is constructed and three particular cases are analyzed. The boundary conditions given here are valid for unrestricted displacements, rotations, strains and/or changes of curvatures of the reference surface.

Introduction

Within the nonlinear Kirchhoff-Love theory of shells, Galimov (1950) reduced the external forces applied to the lateral boundary surface of the deformed shell to three statically equivalent effective force resultants and one bending couple resultant. In particular, he replaced the torsional couple resultant by additional force resultants by applying the same procedure which had earlier been used by Love (1927) in the classical linear theory of shells and by Thompson and Tait (1883) in the linear theory of plates. The rigorous validity of those four reduced static boundary quantities was later confirmed by Koiter (1964) on the basis of purely static arguments.

From variational considerations it follows that each effective force resultant should perform work on an appropriate translation of the boundary while the bending couple resultant should perform work on a scalar parameter which describes the rotational deformation of the boundary. Such work-conjugate sets of static and geometric quantities and their related static and geometric boundary conditions have been established for the classical linear theory of shells as well as for various versions of the first-approximation geometrically nonlinear theory of shells undergoing moderate rotations (cf., Schmidt and Pietraszkiewicz, 1981, and the references given there). When strains and/or rotations of the shell material elements are not restricted, however, the effective force and couple resultants derived by means of purely static considerations do not necessarily possess work-conjugate geometric counterparts.

On the other hand, Novozhilov and Shamina (1975) performed a purely geometric analysis of an arbitrary deformation of the shell lateral boundary surface subject to the Kirchhoff-Love constraints. They were able to show, in particular, that three translations and one scalar parameter ϑ_ν completely describe an arbitrary deformation of the shell boundary. Unfortunately, the corresponding work-conjugate static boundary conditions have not been given in the literature.

A similar geometric analysis performed by Pietraszkiewicz and Szwabowicz (1981) led to the conclusion that three translations and an additional scalar function n_ν of the displacement derivatives may also be used to describe an arbitrary deformation of the shell boundary. The four work-conjugate static boundary conditions were then constructed in terms of n_ν as the natural boundary conditions generated by the two-dimensional principle of virtual displacements.

In most other works on the nonlinear theory of thin shells, the four static boundary conditions have also been obtained as the natural boundary conditions of the two-dimensional principle of virtual displacements, but this has been done without explicit reference to the corresponding geometric boundary conditions. Instead, it is usually assumed that the *virtual* displacements and rotations should be kinematically admissible. As a result, in the transformed boundary line integral, the bending couple resultant performs virtual work on some variational expression that describes the *virtual* rotation of the boundary but not the *variation* of a scalar parameter describing the rotation itself. Various forms of the variational expression associated with different natural definitions of the bending couple have been proposed in the literature. In each case the question arises whether the variational expression, possibly multiplied by a scalar function, can be represented as the variation of some scalar function φ of displacement derivatives. Only if such a representation is possible, the four natural static boundary quantities possess work-conjugate geometric counterparts.

The aim of this paper is to investigate the problem of ex-

[1] On leave from the Technical University, ul. ZSP 5, 45-223 Opole, Poland.
[2] On leave from the Institute of Fluid Flow Machinery of the Polish Academy of Sciences, ul. Fiszera 14, 80-952 Gdańsk, Poland.

Contributed by the Applied Mechanics Division of THE AMERICAN SOCIETY OF MECHANICAL ENGINEERS for publication in the JOURNAL OF APPLIED MECHANICS.

Discussion on this paper should be addressed to the Editorial Department, ASME, United Engineering Center, 345 East 47th Street, New York, N.Y. 10017, and will be accepted until two months after final publication of the paper itself in the JOURNAL OF APPLIED MECHANICS. Manuscript received by ASME Applied Mechanics Division, April 26, 1988; final revision, August 17, 1988.

istence and to derive the general form of the work-conjugate sets of static and geometric boundary conditions for the nonlinear theory of thin shells expressed in terms of displacements as basic independent field variables. In the analysis it is only assumed that the deformation of the shell as a three-dimensional body is completely determined by the stretching and bending of its reference surface. This assumption is far less restrictive than the usual Kirchhoff-Love constraints. In particular, the deformation of the shell in the direction of the normal to the reference surface is not restricted by this assumption.

A new and entirely general approach to the problem of work-conjugate boundary conditions is developed here. It is shown that at any point of the boundary, each of the variational expressions associated with the bending couple resultant may be regarded as a differential 1-form ω on a suitably defined six-dimensional manifold of displacement derivatives. Then the theorem of Poincaré provides the necessary condition for ω to be exact, i.e., of the form $\omega = \delta\varphi$, and the theorem of Frobenius provides the necessary condition for ω to be integrable, i.e., of the form $\mu\omega = \delta\varphi$, where μ is an integrating factor. Applying those theorems to various variational expressions proposed in the literature, their exactness and integrability is established. In particular, it is proved that the variational expression used originally by Galimov (1951) and in various different but equivalent forms in many subsequent papers, is *not integrable*. In such formulations of the nonlinear theory of shells the four natural static boundary quantities do not possess work-conjugate geometric counterparts.

The general procedure is worked out for the transformation of a nonintegrable 1-form into an integrable 1-form for which the primitive is obtained using a method of integration of total differential equations. This primitive is an *arbitrary* scalar function φ of the displacement derivatives. Associated general expressions for the natural force resultants and bending couple resultant, which perform virtual work on variations of the respective displacement components and of the function φ, are derived. Three particular definitions of φ are discussed, and the work-conjugate static boundary and corner conditions corresponding to the geometric boundary conditions of Novozhilov and Shamina (1975) are established.

Notation and Basic Relations

In this paper we largely rely on notation used by Koiter (1966) and Pietraszkiewicz (1977, 1979).

The position vectors of the undeformed and deformed reference surface M and $\bar{M}$ of the shell are denoted by $\mathbf{r}(\Theta^\alpha)$ and $\bar{\mathbf{r}}(\Theta^\alpha)$, respectively, where Θ^α, $\alpha = 1, 2$, are convected (material) surface coordinates. At each point $M \in M$ we have the natural base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\Theta^\alpha \equiv \mathbf{r},_\alpha$, the unit normal vector $\mathbf{n} = 1/2\epsilon^{\alpha\beta}\mathbf{a}_\alpha \times \mathbf{a}_\beta$, the covariant metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with its determinant $a = \det a_{\alpha\beta} > 0$, the curvature tensor $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n},_\beta$, and the permutation tensor $\epsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$. The reciprocal base vectors $\mathbf{a}^\alpha$ and the contravariant metric tensor $a^{\alpha\beta}$ are then defined by $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$, where δ^α_β denotes the Kronecker symbol. In what follows Greek indices always refer to the coordinates Θ^α, and for a diagonally repeated index the summation convention will be invoked.

The boundary C of M is assumed to consist of a finite set of piecewise smooth curves with the position vector $\mathbf{r}(s) = \mathbf{r}[\Theta^\alpha(s)]$, where s is the arc length along C. At each regular point $M \in C$ we denote the unit tangent vector by $\mathbf{t} = \mathbf{r}' = \mathbf{a}_\alpha t^\alpha$ and the outward unit normal vector by $\boldsymbol{\nu} = \mathbf{r},_\nu = \mathbf{a}_\alpha \nu^\alpha = \mathbf{t} \times \mathbf{n}$, $\nu^\alpha = \epsilon^{\alpha\beta} t_\beta$. Here, $(.)'$ indicates differentiation with respect to the arc length s and $(.),_\nu$ denotes the outward normal derivative at C.

Consider now an arbitrary smooth deformation $M - \bar{M}$ of the shell reference surface and let $\mathbf{u}(\Theta^\alpha) = u^\lambda \mathbf{a}_\lambda + w\mathbf{n}$ be the associated displacement field such that $\bar{\mathbf{r}} = \mathbf{r} + \mathbf{u}$. To distinguish all geometric quantities defined on $\bar{M}$ and on its boundary $\bar{C}$ from those on M and C we use an overbar, e.g., $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$, $\bar{a}_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$, $\bar{\boldsymbol{\nu}}$, $\bar{\mathbf{t}}$, etc. The deformation of the shell reference surface may then be expressed in terms of the geometry of M and the displacement field $\mathbf{u}$. In particular, we obtain (cf., Pietraszkiewicz, 1980, 1984a)

$$\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}},_\alpha = \mathbf{a}_\alpha + \mathbf{u},_\alpha, \qquad \bar{\mathbf{n}} = \frac{1}{2} j^{-1} \epsilon^{\alpha\beta} \bar{\mathbf{r}},_\alpha \times \bar{\mathbf{r}},_\beta, \tag{1a}$$

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}) = \frac{1}{2}(\bar{\mathbf{r}},_\alpha \cdot \bar{\mathbf{r}},_\beta - a_{\alpha\beta}), \tag{1b}$$

$$\kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}) = \bar{\mathbf{r}},_\alpha \cdot \bar{\mathbf{n}},_\beta + b_{\alpha\beta}, \tag{1c}$$

$$j^2 = \frac{\bar{a}}{a} = \frac{1}{2} \epsilon^{\alpha\lambda} \epsilon^{\beta\kappa} (\bar{\mathbf{r}},_\alpha \cdot \bar{\mathbf{r}},_\beta)(\bar{\mathbf{r}},_\lambda \cdot \bar{\mathbf{r}},_\kappa). \tag{1d}$$

Similarly, along the boundary we have

$$\bar{\mathbf{r}}' = \bar{\mathbf{a}}_\beta t^\beta = \mathbf{t} + \mathbf{u}' = c_\nu \boldsymbol{\nu} + c_t \mathbf{t} + c\mathbf{n}, \tag{2a}$$

$$\bar{\mathbf{r}},_\nu = \bar{\mathbf{a}}_\beta \nu^\beta = \boldsymbol{\nu} + \mathbf{u},_\nu, \tag{2b}$$

$$\bar{\mathbf{n}} = j^{-1} \bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}}' = n_\nu \boldsymbol{\nu} + n_t \mathbf{t} + n\mathbf{n}, \tag{2c}$$

$$j = |\bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}}'|, \; j^2 = |\bar{\mathbf{r}},_\nu|^2 |\bar{\mathbf{r}}'|^2 - (\bar{\mathbf{r}},_\nu \cdot \bar{\mathbf{r}}')^2. \tag{2d}$$

For future reference we also note the following relationships

$$\bar{\mathbf{a}}_t \equiv \bar{\mathbf{r}}' = \bar{a}_t \bar{\mathbf{t}}, \; \bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_t \times \bar{\mathbf{n}} = \bar{a}_t \bar{\boldsymbol{\nu}}, \tag{3a}$$

$$\bar{a}_t = |\bar{\mathbf{r}}'| = \sqrt{1 + 2\gamma_{tt}}, \; a_\nu = \boldsymbol{\nu} \cdot \bar{\mathbf{a}}_\nu, \tag{3b}$$

$$\bar{\mathbf{r}},_\nu = \bar{a}_t^{-1}(j\bar{\boldsymbol{\nu}} + 2\gamma_{\nu t} \bar{\mathbf{t}}), \tag{3c}$$

$$\bar{\mathbf{a}}^\beta = j^{-1}(\bar{a}_t \nu^\beta - 2\gamma_{\nu t} \bar{a}_t^{-1} t^\beta)\bar{\boldsymbol{\nu}} + \bar{a}_t^{-1} t^\beta \bar{\mathbf{t}}, \tag{3d}$$

$$2\gamma_{\nu t} = 2\gamma_{\alpha\beta}\nu^\alpha t^\beta = \bar{\mathbf{r}},_\nu \cdot \bar{\mathbf{r}}', \; 2\gamma_{tt} = 2\gamma_{\alpha\beta} t^\alpha t^\beta = |\bar{\mathbf{r}}'|^2 - 1. \tag{3e}$$

Statement of the Problem

We are concerned here with the class of nonlinear theories of thin shells for which the deformation of the shell as a three-dimensional body is completely determined by the stretching and bending of its reference surface. A common feature of various shell theories within this class is that their equilibrium conditions may be expressed by the following principle of virtual displacements

$$\iint_M (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})dA = \iint_M (\mathbf{p} \cdot \delta\bar{\mathbf{r}} + \mathbf{h} \cdot \delta\bar{\mathbf{n}})dA + \int_{C_f} (\mathbf{T} \cdot \delta\bar{\mathbf{r}} + \mathbf{H} \cdot \delta\bar{\mathbf{n}})ds. \tag{4}$$

In (4) all quantities are defined with respect to undeformed reference surface M (Lagrangian description) and C_f is the part of C where the external boundary force and moment resultants $\mathbf{T}$ and $\mathbf{H}$ are prescribed. The Lagrangian surface strain measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are defined by (1*b*, *c*) while $\bar{\mathbf{n}}$ on C is given by (2*c*, *d*). The mechanical variables $N^{\alpha\beta}$ and $M^{\alpha\beta}$ in (4) are two-dimensional symmetric second Piola-Kirchhoff type stress resultant and stress couple tensors while $\mathbf{p}$ and $\mathbf{h}$ are the external surface force and moment resultants on M. Explicit expressions for $N^{\alpha\beta}$, $M^{\alpha\beta}$ and $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$, $\mathbf{H}$ in terms of three-dimensional surface and body forces and of the reference surface deformation depend on the particular type of nonlinear shell theory employed.

In view of (1*b*, *c*), the only independent variable undergoing variations in M is the position vector $\bar{\mathbf{r}}$ (or, equivalently, the displacement vector $\mathbf{u}$). Therefore, applying the standard variational procedure, the principle (4) may also be rewritten in the form

$$-\iint_M (\mathbf{T}^\beta{}_{|\beta}+\mathbf{p})\cdot\delta\bar{\mathbf{r}}dA+\int_{C_u}[(\mathbf{T}^\beta\nu_\beta)\cdot\delta\bar{\mathbf{r}}+(M^{\alpha\beta}\bar{\mathbf{a}}_\alpha\nu_\beta)\cdot\delta\bar{\mathbf{n}}]ds$$

$$+\int_{C_f}[(\mathbf{T}^\beta\nu_\beta-\mathbf{T})\cdot\delta\bar{\mathbf{r}}+(M^{\alpha\beta}\bar{\mathbf{a}}_\alpha\nu_\beta-\mathbf{H})\cdot\delta\bar{\mathbf{n}}]ds=0, \quad (5)$$

where

$$\mathbf{T}^\beta=N^{\alpha\beta}\bar{\mathbf{a}}_\alpha+M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha}+\{[(M^{\lambda\kappa}\bar{\mathbf{a}}_\lambda)|_\kappa+\mathbf{h}]\cdot\bar{\mathbf{a}}^\beta\}\bar{\mathbf{n}}, \quad (6)$$

and C_u, $C=C_u\cup C_f$, denotes the part of C where the geometric boundary conditions are prescribed. Also, $(.)_{|\beta}$ indicates covariant differentiation in the metric of M. From (5) we directly obtain the familiar equilibrium equations

$$\mathbf{T}^\beta{}_{|\beta}+\mathbf{p}=\mathbf{0} \text{ in } M. \quad (7)$$

The derivation of static and geometric boundary conditions which are consistent with (4) and (5) is, unfortunately, not straightforward and unique and has up until now never been performed in complete generality. Note that $\bar{\mathbf{n}}$ is not an independent variable on C, since by virtue of (2c, d) it is the function of $\bar{\mathbf{r}}_{,\nu}$ and $\bar{\mathbf{r}}'$. Rather, the independent variables undergoing the variation on C are the position vector $\bar{\mathbf{r}}$ (and, hence, $\bar{\mathbf{r}}'$) and its outward normal derivative $\bar{\mathbf{r}}_{,\nu}$. Those variables, however, have to satisfy two identities

$$\bar{\mathbf{r}}'\cdot\bar{\mathbf{n}}=0, \quad \bar{\mathbf{n}}\cdot\bar{\mathbf{n}}=1 \quad \text{along } C. \quad (8)$$

These identities imply that three components of $\bar{\mathbf{r}}$ (or $\mathbf{u}$) and one additional scalar function of position (or displacement) derivatives, say $\varphi(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$, are necessary and sufficient to describe the shell deformation along its boundary uniquely. Consequently, the number of corresponding static boundary conditions can also be reduced to four.

The static boundary and corner conditions may be obtained from (4) as the natural boundary conditions. Indeed, performing the variation of (2c) directly or varying the identities (8), $\delta\bar{\mathbf{n}}$ may be written in the form

$$\delta\bar{\mathbf{n}}=-\nu_\beta\bar{\mathbf{a}}^\beta\{\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}\}-t_\beta\bar{\mathbf{a}}^\beta(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}'). \quad (9a)$$

The expression (9a) may now be substituted into the second line integral of (5) and, subsequently, all terms containing $\delta\bar{\mathbf{r}}'$ may be eliminated by integration by parts. This leads to a reduced form of the line integral along C_f and some additional terms at each corner point $M_n\in C_f$, $n=1,2,\ldots,N$. For arbitrary $\delta\bar{\mathbf{r}}$ and $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ along C_f and $\delta\bar{\mathbf{r}}_n$ at each $M_n\in C_f$ their multipliers should vanish identically, which then gives four natural static boundary conditions along C_f and three natural static conditions at each corner $M_n\in C_f$.

The derivation of static boundary and corner conditions in the way just outlined is not unique, however, since using (8), $\delta\bar{\mathbf{n}}$ may also be expressed in several other, though essentially equivalent, forms such as, for example,

$$\delta\bar{\mathbf{n}}=\bar{\boldsymbol{\nu}}\{\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}\}-\frac{1}{\bar{a}_t^2}\bar{\mathbf{r}}'(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}'), \quad (9b)$$

$$=\frac{1}{a_\nu}[\bar{\mathbf{a}}_\nu\{\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}\}+\boldsymbol{\nu}\times\bar{\mathbf{n}}(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}')]. \quad (9c)$$

Still other forms will be discussed subsequently. Each of the possible forms of $\delta\bar{\mathbf{n}}$ may be used for the derivation of a different set of natural static boundary and corner conditions, and each set of conditions will be consistent with the principle of virtual displacements (4). In particular, the corresponding bending couple resultant, which in each of the cases of (9) is defined by a different expression, performs virtual work on the respective variational expression $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$, $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ or $\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}$.

In the variational principle (4) all virtual displacements are assumed to be kinematically admissible, so that the first line integral over C_u in (5) must vanish identically. It will be shown in Chapter 5 that for any *given* set of the four geometric parameters $\bar{\mathbf{r}}$, φ, which describe an arbitrary deformation of the shell boundary, the kinematically admissible virtual displacement field indeed satisfies the kinematic constraints $\delta\bar{\mathbf{r}}=\mathbf{0}$ and $\delta\bar{\mathbf{n}}=\mathbf{0}$ along C_u. In view of expressions (9), the vanishing of the line integral over C_u in (5) is also assured by the fulfillment of only four kinematic constraints, that is by $\delta\bar{\mathbf{r}}=\mathbf{0}$ and $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}=0$, $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}=0$ or $\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}=0$, respectively, along C_u, and $\delta\bar{\mathbf{r}}_m=\mathbf{0}$ at each corner $M_m\in C_u$, $m=1,2,\ldots,M$. It is apparent that the constraint $\delta\bar{\mathbf{r}}=\mathbf{0}$ is equivalent to the geometric boundary conditions $\bar{\mathbf{r}}=\bar{\mathbf{r}}^*$ along C_u, and that $\delta\bar{\mathbf{r}}_m=\mathbf{0}$ corresponds to the geometric corner condition $\bar{\mathbf{r}}_m=\bar{\mathbf{r}}_m^*$ at each $M_m\in C_u$, where $(.)^*$ denotes the prescribed value. It is not immediately obvious, however, what kind of scalar parameter should be prescribed on C_u in order to satisfy the fourth kinematic constraint $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}=0$, $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}=0$ or $\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}=0$, respectively. The question thus arises whether there exists a scalar function $\varphi(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$ such that its variation will coincide with $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$, $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ or $\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}$, possibly multiplied by some other nonvanishing scalar function $\mu(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$. When such a φ does exist, the fourth geometric boundary condition takes the form $\varphi=\varphi^*$ along C_u. Only in such a case are the four natural static boundary conditions generated from (5) and (9) work-conjugate to the geometric ones.

In the particular case of a pure rotation of the shell boundary, i.e., when three translations are prescribed, Zubov (1982) showed that such a scalar parameter $\varphi(\bar{\mathbf{r}}_{,\nu})$ is the solution of an integrable Pfaffian equation. Later particular definitions of the parameter φ were discussed by Zubov (1984). In this paper we develop an alternative and entirely general approach to the problem of existence and of the form of the parameter φ. This approach is valid for an arbitrary deformation of the shell boundary.

Integrability Conditions

Let ω denotes any variational expression of the type enclosed in braces in (9). Its general form is

$$\omega=\mathbf{A}(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')\cdot\delta\bar{\mathbf{r}}_{,\nu}+\mathbf{B}(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')\cdot\delta\bar{\mathbf{r}}', \quad (10)$$

where vector-valued functions $\mathbf{A}$ and $\mathbf{B}$ must be specified for each particular case.

The ω defined by (10) may be considered as a differential 1-form on the infinite-dimensional space consisting of the ordered pairs $(\bar{\mathbf{r}}_{,\nu}(s), \bar{\mathbf{r}}'(s))$ of vector-valued functions defined along C (cf., Cartan, 1970). However, for our present purposes it is sufficient to consider ω at an arbitrary fixed point $M\in C$. Then ω, as defined by (10), may be regarded as a differential 1-form on the six-dimensional manifold X with local coordinates ξ_i, $i=1,2,\ldots,6$ (in a neighborhood of $x_o\in X$), which may be identified with components of $(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$ in the orthonormal base $\{\boldsymbol{\nu},\mathbf{t},\mathbf{n}\}$, i.e.,

$$(\xi_i)=(\boldsymbol{\nu}\cdot\bar{\mathbf{r}}_{,\nu},\ \mathbf{t}\cdot\bar{\mathbf{r}}_{,\nu},\ \mathbf{n}\cdot\bar{\mathbf{r}}_{,\nu},\ \boldsymbol{\nu}\cdot\bar{\mathbf{r}}',\ \mathbf{t}\cdot\bar{\mathbf{r}}',\ \mathbf{n}\cdot\bar{\mathbf{r}}'). \quad (11)$$

Here x_o with local coordinates (1, 0, 0, 0, 1, 0) signifies the undeformed state of the shell boundary. Thus, the 1-form ω may be rewritten as

$$\omega=\sum_{i=1}^{6}A_i(\xi_j)\,\delta\xi_i, \quad (12)$$

where $\delta\xi_i$ are understood to be differentials in the usual sense and A_i, $i=1,2,\ldots,6$, are defined as components of $(\mathbf{A},\mathbf{B})$ in the base $\{\boldsymbol{\nu},\mathbf{t},\mathbf{n}\}$, that is

$$(A_i)=(\boldsymbol{\nu}\cdot\mathbf{A},\ \mathbf{t}\cdot\mathbf{A},\ \mathbf{n}\cdot\mathbf{A},\ \boldsymbol{\nu}\cdot\mathbf{B},\ \mathbf{t}\cdot\mathbf{B},\ \mathbf{n}\cdot\mathbf{B}). \quad (13)$$

The interpretation of ω as a differential 1-form makes it possible to apply some basic definitions and theorems of the theory of differential forms (cf., Cartan, 1970, Westenholz, 1981). For convenience they are briefly summarized below in our notation.

The 1-form (10) is said to be *exact* on X if there exists a scalar-valued function $\varphi(\bar{\mathbf{r}},_\nu,\bar{\mathbf{r}}')$, called the *primitive* of ω, such that $\omega = \delta\varphi$, i.e., $\partial\varphi/\partial\bar{\mathbf{r}},_\nu = \mathbf{A}$ and $\partial\varphi/\partial\bar{\mathbf{r}}' = \mathbf{B}$. According to the lemma of Poincaré, the necessary condition for ω to be exact is $d\omega = 0$, where $d\omega$ denotes the exterior derivative of ω. In the notation of (12), the condition $d\omega = 0$ reads

$$A_{i,j} - A_{j,i} = 0 \text{ for } i,j = 1,2,\ldots,6, \tag{14}$$

which implies that the matrix $\partial A_i/\partial\xi_j \equiv A_{i,j}$ has to be symmetric. In a sufficiently small neighborhood of $x_o \in X$ the conditions (14) are also sufficient for ω to be exact.

The 1-form (10) is said to be *integrable* on X if there exist scalar-valued functions $\mu(\bar{\mathbf{r}},_\nu,\bar{\mathbf{r}}')$, called the *integrating factor*, and $\varphi(\bar{\mathbf{r}},_\nu,\bar{\mathbf{r}}')$ such that $\mu\omega = \delta\varphi$, i.e., $\mu^{-1}(\partial\varphi/\partial\bar{\mathbf{r}},_\nu) = \mathbf{A}$ and $\mu^{-1}(\partial\varphi/\partial\bar{\mathbf{r}}') = \mathbf{B}$. According to the theorem of Frobenius, the necessary condition for ω to be integrable is $\omega\wedge d\omega = 0$, where $\wedge$ denotes the exterior product. In a sufficiently small neighborhood of $x_o \in X$ this condition is also sufficient for ω to be integrable. In the notation of (12) the integrability condition $\omega\wedge d\omega = 0$ takes the form (cf., Ince, 1956)

$$A_i(A_{k,j} - A_{j,k}) + A_j(A_{i,k} - A_{k,i}) + A_k(A_{j,i} - A_{i,j}) = 0 \tag{15}$$

for $i, j, k = 1, 2, \ldots, 6$. There are twenty such equations of which only ten are independent. It is obvious that the exact 1-form is integrable and that $\mu\omega$ is exact if ω is integrable.

Now we are in a position to discuss the problem of existence of the fourth geometric boundary condition corresponding to various variational expressions enclosed in braces in (9). Consider the case (9*b*) for which the natural static boundary conditions were given first by Galimov (1951) and were rederived in different but equivalent forms in many subsequent papers. In this case $\omega = \bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$, and the corresponding vector-valued functions $\mathbf{A}$ and $\mathbf{B}$, calculated with the help of (9*a*) and (3*d*), take the form

$$\mathbf{A} = -\bar{a}_t j^{-1}\bar{\mathbf{n}}, \quad \mathbf{B} = 2\bar{a}_t{}^{-1} j^{-1}\gamma_{\nu t}\bar{\mathbf{n}}, \tag{16}$$

where all quantities on the right-hand sides are functions of $\bar{\mathbf{r}},_\nu$ and $\bar{\mathbf{r}}'$ given by (2*c*,*d*) and (3*e*). Differentiation of (16), with respect to $\bar{\mathbf{r}},_\nu$ and $\bar{\mathbf{r}}'$, gives

$$\frac{\partial\mathbf{A}}{\partial\bar{\mathbf{r}},_\nu} = \bar{a}_t^2 j^{-2}\,(\bar{\boldsymbol{\nu}}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\boldsymbol{\nu}}), \tag{17a}$$

$$\frac{\partial\mathbf{A}}{\partial\bar{\mathbf{r}}'} = j^{-1}\,\bar{\mathbf{t}}\otimes\bar{\mathbf{n}} - 2j^{-2}\gamma_{\nu t}\,(\bar{\boldsymbol{\nu}}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\boldsymbol{\nu}}) = \left(\frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}},_\nu}\right)^T, \tag{17b}$$

$$\frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}}'} = \bar{a}_t{}^{-2}\,[\bar{\mathbf{n}}\otimes\bar{\boldsymbol{\nu}} + 4j^{-2}\gamma_{\nu t}^2\,(\bar{\boldsymbol{\nu}}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\boldsymbol{\nu}}) - 2j^{-1}\gamma_{\nu t}\,(\bar{\mathbf{t}}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\mathbf{t}})]. \tag{17c}$$

Since (17*a*) is symmetric and (17*b*) holds, (17*c*) leads to the only nonvanishing expression

$$\frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}}'} - \left(\frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}}'}\right)^T = \bar{a}_t{}^{-2}\,(\bar{\mathbf{n}}\otimes\bar{\boldsymbol{\nu}} - \bar{\boldsymbol{\nu}}\otimes\bar{\mathbf{n}}). \tag{18}$$

Therefore, conditions (14) are identically satisfied for any combination of $i, j \in (1, 2, \ldots, 6)$ except for $(i, j) = (4, 5)$, $(4, 6)$, and $(5, 6)$. As a result, the 1-form $\omega = \bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ is not exact on X.

If components of (17) are introduced into (15), the integrability conditions are satisfied identically for any combination of $i, j, k \in (1, 2, \ldots, 6)$, except for such combinations in which any two of three indices i, j, k assume the values $(4, 5)$, $(4, 6)$ or $(5, 6)$ while the remaining third index assumes the value 1, 2 or 3. For example, it is easy to see that for $(i, j, k) = (1, 4, 5)$ the left-hand side of (15) is $\bar{a}_t{}^{-1}j^{-1}\,(\boldsymbol{\nu}\cdot\bar{\mathbf{n}})(\mathbf{n}\cdot\bar{\mathbf{t}})$, which does not vanish identically. As a result, the 1-form $\omega = \bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ is not integrable on X.

The variational expression $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ may itself be represented in several different but equivalent forms. If $\boldsymbol{\beta} = \bar{\mathbf{n}} - \mathbf{n}$ is the difference vector, then $\delta\bar{\mathbf{n}} = \delta\boldsymbol{\beta}$ and $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}} = \bar{\boldsymbol{\nu}}\cdot\delta\boldsymbol{\beta} \equiv \delta\bar{\beta}_\nu$, which was used by Pietraszkiewicz (1977). However, in the expression $\delta\bar{\beta}_\nu$ the symbol δ should not be understood as the symbol of variation, since so defined $\delta\bar{\beta}_\nu \neq \delta(\bar{\boldsymbol{\nu}}\cdot\boldsymbol{\beta})$.

The total rotation of the shell boundary is described by the total rotation tensor $\mathbf{R}_t = \bar{\boldsymbol{\nu}}\otimes\boldsymbol{\nu} + \bar{\mathbf{t}}\otimes\mathbf{t} + \bar{\mathbf{n}}\otimes\mathbf{n}$ such that $\bar{\mathbf{n}} = \mathbf{R}_t\mathbf{n}$ (cf., Pietraszkiewicz, 1979). The skew-symmetric tensors $\delta\mathbf{R}_t\mathbf{R}_t^T$ and $\mathbf{R}_t^T\delta\mathbf{R}_t$ can be associated with the respective axial vectors of virtual rotation $\delta\boldsymbol{\omega}_t$ and $\delta\mathbf{w}_t$ such that

$$\delta\mathbf{R}_t\mathbf{R}_t^T = \delta\boldsymbol{\omega}_t\times\mathbf{1},\ \mathbf{R}_t^T\delta\mathbf{R}_t = \delta\mathbf{w}_t\times\mathbf{1}, \tag{19a}$$

$$\delta\boldsymbol{\omega}_t = \mathbf{R}_t\delta\mathbf{w}_t, \tag{19b}$$

where $\mathbf{1}$ is the metric tensor of the three-dimensional Euclidean space. Since $\delta\bar{\mathbf{n}} = \delta\boldsymbol{\omega}_t\times\bar{\mathbf{n}} = \mathbf{R}_t(\delta\mathbf{w}_t\times\mathbf{n})$, it follows that $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}} = \delta\boldsymbol{\omega}_t\cdot\bar{\mathbf{t}} = \delta\mathbf{w}_t\cdot\mathbf{t}$. An expression analogous to $\delta\boldsymbol{\omega}_t\cdot\bar{\mathbf{t}}$ was used in a number of papers, for example, by Wempner (1981), Sakurai et al. (1983), and Axelrad (1987) while $\delta\mathbf{w}_t\cdot\mathbf{t}$ may be found in the recent paper of Szwabowicz (1986). Likewise, in the definitions of $\delta\boldsymbol{\omega}_t$ and $\delta\mathbf{w}_t$ the symbol δ should not be understood as the symbol of variation, since the symbols $\boldsymbol{\omega}_t$ or $\mathbf{w}_t$ alone have no geometric meaning.

The variational expression $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$ may also be transformed as follows:

$$\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}} = -\bar{\mathbf{n}}\cdot\delta\bar{\boldsymbol{\nu}} = -\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}},_{\bar{\nu}} = \tag{20a}$$

$$= -(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}),_{\bar{\nu}} + \bar{b}_\alpha^\beta\bar{\nu}^\alpha\bar{\mathbf{a}}_\beta\cdot\delta\bar{\mathbf{r}}. \tag{20b}$$

The expression $(\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}),_{\bar{\nu}}$ was used by Koiter (1966), Danielson (1970), and Zubov (1982).

From the discussion just presented it is seen that, as far as their representations in terms of derivatives $\bar{\mathbf{r}},_\nu$ and $\bar{\mathbf{r}}'$ are concerned, all differential 1-forms $\delta\bar{\beta}_\nu$, $\delta\boldsymbol{\omega}_t\cdot\bar{\mathbf{t}}$, $\delta\mathbf{w}_t\cdot\mathbf{t}$, $-\bar{\mathbf{n}}\cdot\delta\bar{\boldsymbol{\nu}}$ and $-\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}},_{\bar{\nu}}$ are equivalent to the 1-form $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}}$, i.e., all 1-forms are defined by the same expression (10) with (16). As a result, neither of those 1-forms is exact or integrable as well. In all cases there exists no function φ such that $\mu\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}} = \delta\varphi$, and the natural bending couple generated by (5) with (9*b*) does not possess a work-conjugate geometric counterpart. In the language of analytic mechanics, this means that all kinematic constraints which are equivalent to $\bar{\boldsymbol{\nu}}\cdot\delta\bar{\mathbf{n}} = 0$ are nonholonomic constraints. As a result, all those versions of the nonlinear theory of thin shells, in which the natural static boundary conditions are constructed with the help of (5) and (9*b*), can not be presented in a variational form which requires a functional to be stationary. In particular, it immediately follows from this discussion that several variational principles, which have been proposed in the literature for such versions of geometrically nonlinear first-approximation theories of elastic shells, must be incorrect.

The discussion of the exactness and integrability of two other differential 1-forms appearing in braces in (9*a*) and (9*c*) is given in the Appendix. There is proved that $\omega = \bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}},_\nu$ also is not integrable on X, while at the same time it is confirmed that $\omega = \boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}$ is indeed exact on X.

It is quite obvious that $\boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}}$ is exact because $\boldsymbol{\nu}$ is not varied and, therefore, $\omega = \boldsymbol{\nu}\cdot\delta\bar{\mathbf{n}} = \delta(\boldsymbol{\nu}\cdot\bar{\mathbf{n}}) \equiv \delta n_\nu$. As a result, the specification of $\bar{\mathbf{r}}$ and n_ν along C_u establishes those geometric boundary conditions which are work-conjugate to the corresponding static ones following from (5) and (9*c*). Such a complete set of the work-conjugate boundary conditions was originally derived by Pietraszkiewicz and Szwabowicz (1981) with the help of a modified tensor of change of curvature and rederived by Pietraszkiewicz (1984a,b) using $\kappa_{\alpha\beta}$ defined by (1*c*). Within the geometrically nonlinear first-approximation theory of elastic shells, this led to a number of results on variational principles, consistently approximated relations for shells undergoing restricted rotations, stability equations, and superposed deformations, which have been summarized by Szwabowicz (1982), Pietraszkiewicz (1984a), Schmidt (1985), and Stumpf (1986) where further references are given.

General Form of Boundary Conditions

The differential 1-forms previously discussed are only examples of the variety of 1-forms which may appear in the boundary line integral of the principle of virtual displacements. Each particular 1-form generates a different set of the natural static boundary and corner conditions on C_f. Indeed, each of the 1-forms enclosed in braces in (9) may be multiplied by a nonvanishing scalar function $\eta(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$. This leads to a modification of the corresponding natural boundary condition for the bending couple resultant which essentially consists of a division by η. Furthermore, an additional term of the type $\mathbf{c}(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')\cdot\delta\bar{\mathbf{r}}'$ may also be added to each of the 1-forms. If, then, the same term is substracted from the 1-form one obtains, upon substituting (9) into (5) and integrating by parts, appropriate modifications of the corresponding force boundary and corner conditions. Among the variety of the 1-forms, which may be obtained by such transformations, the most important are the exact, or even only integrable 1-forms, since only those 1-forms generate the proper natural static boundary and corner conditions which are work-conjugate to the geometric ones.

Let the expression (9*a*) be rewritten as

$$\delta\bar{\mathbf{n}} = -\nu_\beta j^{-1}\bar{\mathbf{a}}^\beta\ \{\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu}\} - t_\beta\bar{\mathbf{a}}^\beta\ (\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}'), \tag{21}$$

$$\mathbf{d} = \bar{\mathbf{r}}_{,\nu}\times\bar{\mathbf{r}}'. \tag{22}$$

The simple variational expression appearing in (21),

$$\vartheta = \mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu} = A_1\delta\xi_1 + A_2\delta\xi_2 + A_3\delta\xi_3, \tag{23}$$

$$A_1 = \xi_2\xi_6 - \xi_3\xi_5,\ A_2 = \xi_3\xi_4 - \xi_1\xi_6,\ A_3 = \xi_1\xi_5 - \xi_2\xi_4, \tag{24}$$

may also be regarded as a differential 1-form of the type (10) on the six-dimensional manifold X, only in this case $\mathbf{B} \equiv \mathbf{0}$. It is easy to verify that the 1-form (23) is not integrable on X, for the conditions (15) are not identically satisfied when, for example, $(i, j, k) = (1, 2, 4)$. Our aim is to transform the expression (23) in such a way as to represent it in terms of an exact 1-form.

Suppose, for a moment, that $\bar{\mathbf{r}}$ is prescribed along C. Then so is $\bar{\mathbf{r}}'$ and, hence, the coordinates ξ_4, ξ_5, ξ_6 are not varied but rather play the role of parameters in (23). Thus, (23) may now be regarded as a differential 1-form on the three-dimensional submanifold $Y\subset X$ with local coordinates ξ_1, ξ_2, ξ_3. It then follows from (23) that for different $i, j \in (1, 2, 3)$ $A_{i,j} \neq A_{j,i}$, and the 1-form ϑ is not exact on Y. There is only one integrability condition (15) for $(i, j, k) = (1, 2, 3)$. Using (24), it is easy to verify that this condition is identically satisfied. Therefore, the 1-form ϑ is integrable on Y. In order to find its integrating factor and its primitive on Y, we follow the method of integration of total differential equations (see Ince, 1956, Section 2.8).

Let, for a moment, one of the coordinates ξ_i be constant. Since in the undeformed configuration $A_3 = 1$, $\xi_5 = 1$, it is convenient to assume this to be ξ_2. Then the 1-form $\vartheta = A_1(\xi_3)d\xi_1 + A_3(\xi_1)d\xi_3$, given on the two-dimensional submanifold $Z\subset Y$ with local coordinates ξ_1, ξ_3, is always integrable on Z. Its integrating factor then is $\lambda = -\xi_5/A_1A_3$ and its primitive is given by $\kappa = \ln|A_1/A_3|$. If now ξ_2 is again allowed to vary, that is, the 1-form ϑ is again supposed to be given on Y, then the functions $\lambda(\xi_i)$ and $\kappa(\xi_i)$, $i = 1, 2, 3$, previously calculated allow one to evaluate the function $S(\chi, \xi_2) = \lambda A_2 - \kappa_{,2}$. In the present case it vanishes, so that $S = 0$. As a result, $\lambda(\xi_i)$ is the respective integrating factor and $\kappa(\xi_i)$ is the primitive of the 1-form ϑ on Y, such that $\lambda\vartheta = \delta\kappa$ holds. This can easily be confirmed by a direct analysis. Moreover, $\beta(\kappa)$ is also the primitive of ϑ for any differentiable scalar function β so that, as a result, the general form of the primitive of ϑ on Y is given by $\beta\ \{\ln[\ |\ (A_1/A_3)]\} \equiv h(A_1/A_3)$, where h is an arbitrary differentiable function (cf., Zubov, 1984).

If one follows the same procedure keeping ξ_1 or ξ_3 temporarily constant, one finds that $k(A_2/A_3)$ or $l(A_1/A_2)$ are also primitives of ϑ for arbitrary differentiable functions k and l. But from the identities (8), the fact that $\bar{\mathbf{r}}$ is prescribed along C and the arbitrariness of h, it is seen that primitives k and l are different but equivalent forms of the primitive h.

Now we remove the initial constraint that $\bar{\mathbf{r}}$ is prescribed along C and allow it to vary again. Thus, we return to the 1-form ϑ given on X, according to (23) and (24). Let $\varphi(\bar{\mathbf{r}}',\alpha)$, $\alpha = A_1/A_3 = n_\nu/n$ be an arbitrary differentiable scalar-valued function of its arguments. The variation of α then leads to

$$\delta\alpha = -A_3^{-2}\ (\xi_5\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu} - \xi_2\mathbf{d}\cdot\delta\bar{\mathbf{r}}'), \tag{25}$$

which allows one to derive the expression for the variation of φ in the form

$$\delta\varphi = \eta\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu} + \mathbf{c}\cdot\delta\bar{\mathbf{r}}' \tag{26}$$

$$\eta = -A_3^{-2}\xi_5\chi,\quad \mathbf{c} = \boldsymbol{\lambda} + A_3^{-2}\xi_2\chi\mathbf{d}, \tag{27a}$$

$$\boldsymbol{\lambda} = \frac{\partial\varphi}{\partial\bar{\mathbf{r}}'},\quad \chi = \frac{\partial\varphi}{\partial\alpha}. \tag{27b}$$

It follows from (26) that we have, in fact, constructed the scalar-valued function $\eta(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$ and the vector-valued function $\mathbf{c}(\bar{\mathbf{r}}_{,\nu},\bar{\mathbf{r}}')$ which have allowed us to transform the nonintegrable 1-form (23) into the exact 1-form ψ defined by the right-hand side of (26), such that $\psi = \delta\varphi$. If now (26) is solved for $\mathbf{d}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ and the result is introduced into (21) we obtain

$$\delta\bar{\mathbf{n}} = \bar{\mathbf{a}}^\alpha\nu_\alpha f\ \{\delta\varphi\} - \bar{\mathbf{a}}^\alpha\ ([\nu_\alpha f\boldsymbol{\lambda} + (\nu_\alpha g + t_\alpha)\bar{\mathbf{n}}]\cdot\delta\bar{\mathbf{r}}'), \tag{28}$$

$$f = \frac{jn^2}{c_t\chi},\quad g = \frac{\xi_2}{\xi_5} = \frac{1}{\bar{a}_t^2}\left(\frac{c_\nu n - cn_\nu}{jc_t} + 2\gamma_{\nu t}\right). \tag{29}$$

This is yet another expression for $\delta\bar{\mathbf{n}}$. It differs qualitatively from (9*a*, *b*) and (21), since it is given directly in terms of the exact 1-form $\delta\varphi$.

If now (28) is substituted into the boundary integral of (5) and the term containing $\delta\bar{\mathbf{r}}'$ is eliminated by integration by parts, the integral takes the final form

$$\int_{C_f} [(\mathbf{P}-\mathbf{P}^*)\cdot\delta\bar{\mathbf{r}} + (M-M^*)\delta\varphi]ds + \sum_n\ (\mathbf{F}_n - \mathbf{F}_n^*)\cdot\delta\bar{\mathbf{r}}_n, \tag{30}$$

where the effective force resultants and the bending couple resultant are defined by

$$\mathbf{P} = \mathbf{T}^\beta\nu_\beta + \mathbf{F}',\quad \mathbf{F} = fM_{\nu\nu}\boldsymbol{\lambda} + (gM_{\nu\nu} + M_{\nu t})\bar{\mathbf{n}}, \tag{31a}$$

$$\mathbf{P}^* = \mathbf{T} + \mathbf{F}^{*\prime},\ \mathbf{F}^* = (\mathbf{H}\cdot\bar{\mathbf{a}}^\beta)\ [\nu_\beta f\boldsymbol{\lambda} + (\nu_\beta g + t_\beta)\bar{\mathbf{n}}], \tag{31b}$$

$$M = fM_{\nu\nu},\ M^* = f(\mathbf{H}\cdot\bar{\mathbf{a}}^\beta\nu_\beta), \tag{31c}$$

$$\mathbf{F}_n = \mathbf{F}(s_n+0) - \mathbf{F}(s_n-0),\ \mathbf{F}_n^* = \mathbf{F}^*(s_n+0) - \mathbf{F}^*(s_n-0), \tag{31d}$$

$$\bar{\mathbf{r}}_n = \bar{\mathbf{r}}(s_n), \tag{31e}$$

$$M_{\nu\nu} = M^{\alpha\beta}\nu_\alpha\nu_\beta,\quad M_{\nu t} = M^{\alpha\beta}\nu_\alpha t_\beta. \tag{31f}$$

From (5) and (30) it follows that the static boundary and corner conditions take the form

$$\mathbf{P}(s) = \mathbf{P}^*(s),\ M(s) = M^*(s)\ \text{on}\ C_f, \tag{32a}$$

$$\mathbf{F}_n = \mathbf{F}_n^*\ \text{at each corner}\ M_n \in C_f. \tag{32b}$$

Furthermore, it is seen from (5) and (30) that the geometric boundary conditions which are work-conjugate to the static ones (32) are given by

$$\bar{\mathbf{r}}(s) = \bar{\mathbf{r}}^*(s),\ \varphi[\bar{\mathbf{r}}'(s),\ \alpha(s)] = \varphi^*(s)\ \text{on}\ C_u, \tag{33}$$

where, by definition, $\alpha(s) = \alpha\ [\bar{\mathbf{r}}_{,\nu}(s),\ \bar{\mathbf{r}}'(s)]$. It also follows from (30) and (28) that $\bar{\mathbf{r}}$ is kinematically admissible if $\delta\bar{\mathbf{r}} = \mathbf{0}$ and $\delta\varphi = 0$ on C_u. Hence, $\delta\bar{\mathbf{n}} = \mathbf{0}$ on C_u as well.

The general forms (32) and (33) of four work-conjugate static and geometric boundary conditions are derived here in terms of the function $\varphi(\bar{\mathbf{r}}',\alpha)$ in which $\alpha = n_\nu/n$ is an intermediate variable. The analysis clearly indicates that the choice of α by no means is the only possible choice of such an intermediate variable. Other scalar functions of $\bar{\mathbf{r}},_\nu$, $\bar{\mathbf{r}}'$ may also be chosen instead of α. However, since φ is an arbitrary function of its arguments, this would lead to formally different, though essentially equivalent representations of the boundary conditions.

It follows from the analysis that $\varphi(\bar{\mathbf{r}}', \alpha)$ should be differentiable with respect to both arguments. The definition of f in (29) indicates also that $\partial\varphi/\partial\alpha$ should not vanish identically in some neighborhood of the undeformed shell boundary. However, in order to be physically meaningful, φ has to satisfy a number of additional requirements based on mathematical and mechanical considerations. This allows one to arrive at a more restricted class of admissible scalar functions for the description of rotational deformation of the shell boundary. In particular, φ must vanish in the undeformed state, that is $\varphi(\mathbf{t}, 0) = 0$, and upon linearization it should coincide with the linearized rotation of the shell boundary $\varphi_\nu = \mathbf{n}\cdot\mathbf{u},_\nu$ which is used in the classical linear theory of shells. It is also reasonable to require φ to be a monotonous function of α, at least in some neighborhood of the undeformed state.

Some Special Cases

We close our considerations with a brief discussion of some particular definitions of φ which may be used in the nonlinear theory of thin shells. For any choice of φ, the corresponding work-conjugate static boundary conditions may be derived directly from (32), (31), (29), and (27*b*).

It follows from (2*a*) and (2*b*) that the identities (8) imply that

$$\frac{n_t}{n} = -c_t^{-1}\,(c_\nu\alpha + c), \tag{34}$$

$$n^2 = [1 + \alpha^2 + c_t^{-2}\,(c_\nu\alpha + c)^2]^{-1}. \tag{35}$$

Therefore, $\bar{\mathbf{n}}$ may be expressed as the function of $\bar{\mathbf{r}}'$ and α,

$$\bar{\mathbf{n}}(\bar{\mathbf{r}}',\alpha) = n[\alpha\boldsymbol{\nu} - c_t^{-1}\,(c_\nu\alpha + c)\mathbf{t} + \mathbf{n}], \tag{36}$$

where n is to be determined from (35). For an arbitrary deformation of the boundary the sign of n, following from (35), is not unique. However, it follows from (2*c*) and (2*d*) that in some neighborhood of the undeformed state, n must be positive.

It immediately follows from (36) that $n_\nu = \boldsymbol{\nu}\cdot\bar{\mathbf{n}}$ is a particular case of φ indeed. The corresponding work-conjugate static boundary and corner conditions were given by Pietraszkiewicz (1984a, b).

Novozhilov and Shamina (1975) used the fourth geometric boundary parameter ϑ_ν defined by

$$\vartheta_\nu = \bar{a}_t^{-2}\,(\bar{\mathbf{n}} - \mathbf{n})\cdot\bar{\mathbf{a}}_\nu. \tag{37}$$

From (3*a*) and (36) it is seen that

$$\bar{\mathbf{a}}_\nu = n[(c_t - c\frac{n_t}{n})\boldsymbol{\nu} + (c\alpha - c_\nu)\mathbf{t} + (c_\nu\frac{n_t}{n} - c_t\alpha)\mathbf{n}], \tag{38}$$

and, according to (36) and (37),

$$\vartheta_\nu = n\bar{a}_t^{-2}\,[c_t\alpha + c_t^{-1}c_\nu(c_\nu\alpha + c)]. \tag{39}$$

Therefore, $\vartheta_\nu = \vartheta_\nu\,(\bar{\mathbf{r}}', \alpha)$ is also a particular case of φ. Let us derive the corresponding work-conjugate static boundary conditions.

Taking the variation of (37), and introducing it into the expression (9*a*) multiplied by $\bar{\mathbf{a}}_t \times \bar{\mathbf{n}}$, we obtain after some transformations

$$\delta\bar{\mathbf{n}} = n^{-1}\bar{a}_t\bar{\boldsymbol{\nu}}\delta\vartheta_\nu + \{n^{-1}\,[2\vartheta_\nu\bar{\boldsymbol{\nu}}\otimes\bar{\mathbf{t}} + \bar{a}_t^{-1}\bar{\boldsymbol{\nu}}\otimes(\bar{\mathbf{n}}\times\mathbf{n})] - \bar{a}_t^{-1}\bar{\mathbf{t}}\otimes\bar{\mathbf{n}}\}\cdot\delta\bar{\mathbf{r}}'. \tag{40}$$

Note that the expression (40) has the same structure as the general expression (28). Introducing (40) into the line integral of (5), we finally obtain the following definitions for the natural static parameters on the boundary which are work-conjugate to the geometric parameters $\bar{\mathbf{r}}$ and ϑ_ν,

$$\mathbf{F} = (M_{\nu t} + 2\bar{a}_t^{-2}\gamma_{\nu t}M_{\nu\nu})\bar{\mathbf{n}} - jn^{-1}\bar{a}_t^{-1}M_{\nu\nu}\,(2\vartheta_\nu\bar{\mathbf{t}} + \bar{a}_t^{-1}\bar{\mathbf{n}}\times\mathbf{n}), \tag{41a}$$

$$\mathbf{F}^* = \bar{a}_t^{-1}\,(\mathbf{H}\cdot\bar{\mathbf{t}})\bar{\mathbf{n}} - n^{-1}\,(\mathbf{H}\cdot\bar{\boldsymbol{\nu}})(2\vartheta_\nu\bar{\mathbf{t}} + \bar{a}_t^{-1}\bar{\mathbf{n}}\times\mathbf{n}), \tag{41b}$$

$$M = jn^{-1}M_{\nu\nu},\; M^* = n^{-1}\bar{a}_t(\mathbf{H}\cdot\bar{\boldsymbol{\nu}}). \tag{41c}$$

Thus, the set of work-conjugate boundary conditions takes the general form (32) and (33), except that $\vartheta_\nu\,[\bar{\mathbf{r}},_\nu(s), \bar{\mathbf{r}}'(s)]$ stands for φ in (33), and definitions (41) are used in (32).

Finally, the rotational deformation of the shell boundary may also be described by the total rotation tensor $\mathbf{R}_t$. Noting (3*a*), (2*a*), and (36) it is seen that the tensor $\mathbf{R}_t$ referred to the undeformed base vectors takes the form

$$\mathbf{R}_t = \bar{a}_t^{-1}n\,(\{[c_t + c_t^{-1}c(c_\nu\alpha + c)]\boldsymbol{\nu} + (c\alpha - c_\nu)\mathbf{t} - [c_\nu c_t^{-1}\,(c_\nu\alpha + c) + c_t\alpha]\mathbf{n}\}\otimes\boldsymbol{\nu} + n^{-1}(c_\nu\boldsymbol{\nu} + c_t\mathbf{t} + c\mathbf{n})\otimes\mathbf{t} + [\alpha\boldsymbol{\nu} - c_t^{-1}\,(c_\nu\alpha + c)\mathbf{t} + \mathbf{n}]\otimes\mathbf{n}). \tag{42}$$

It follows from (42) and (2*a*) that $\mathbf{R}_t = \mathbf{R}_t(\bar{\mathbf{r}}',\alpha)$ and the parameter φ may be defined as some scalar function of $\mathbf{R}_t$, that is $\varphi = \varphi(\mathbf{R}_t)$. In particular, the angle of total rotation ω_t corresponding to $\mathbf{R}_t$ is given by $\omega_t = \arccos\,(1/2\; tr\mathbf{R}_t - 1/2)$, where it follows from (42) and (34) that

$$tr\mathbf{R}_t = n^2\,[1 + \bar{a}_t^2\,(c_\nu c\alpha - c_\nu^2)]. \tag{43}$$

Therefore, the angle of total rotation ω_t may also be chosen as the fourth geometric parameter of the boundary deformation. This choice has been found by Simmonds (1985b) to be the most natural one in the displacement form of nonlinear equations which govern an axisymmetric deformation of shells of revolution.

The work-conjugate static boundary conditions corresponding to the particular cases of φ discussed above, are obviously quite complex. More suitable particular forms of φ may be obtained under additional, more restrictive, mathematical and mechanical requirements.

Concluding Remarks

In this paper an entirely general approach to the derivation of the work-conjugate static and geometric boundary conditions has been developed for a class of nonlinear theories of thin shells. In this approach, basic theorems of the theory of differential forms have been applied to various variational expressions which may appear in the boundary line integral of the principle of virtual displacements. It has been shown that the majority of static boundary conditions, which have been proposed in the literature, do not possess work-conjugate geometric counterparts, because the corresponding differential forms are not integrable. Such static boundary conditions are, however, hardly acceptable in the consistent formulation of the nonlinear theory of shells.

The general forms of the four geometric boundary conditions and of the corresponding work-conjugate static boundary conditions have been derived for the first time in the literature. They have been expressed in terms of an arbitrary scalar function φ of displacement derivatives which describes the rotational deformation of the shell boundary. Since φ is arbitrary, one has a wide range of possibilities to choose the form of boundary conditions to be used in the nonlinear theory of thin shells. This freedom of choice enables one to select φ in such a way that it best suits the particular version of the shell theory or the particular shell problem at hand. As an example, three particular definitions of φ have been discussed.

In the analysis it has been assumed that the deformation of the shell as a three-dimensional body is entirely determined by the stretching and bending of its reference surface. No restrictions have, however, been imposed on the magnitudes of the displacements, rotations, strains and/or changes of curvatures of the reference surface. There is not even a need to specify the material behavior of the shell, since the principle (4) itself does not require $N^{\alpha\beta}$ and $M^{\alpha\beta}$ to be derivable from a strain energy function. Therefore, the boundary conditions derived here are valid for a large class of three-dimensionally different (even inelastic) shell theories which have the same two-dimensional mathematical structure implied by the principle of virtual displacements (4). This generalizes considerably the results available in the literature for some simple versions of nonlinear theory of thin shells.

For any shell theory it is necessary to specify on M and C_f how the fields $N^{\alpha\beta}$, $M^{\alpha\beta}$, $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$, $\mathbf{H}$ are related to the corresponding three-dimensional external surface and body forces and to the deformation of the reference surface. For the geometrically nonlinear first-approximation theory of thin elastic shells such definitions have been given, for example, by Pietraszkiewicz and Szwabowicz (1981) and Pietraszkiewicz (1984b). In simple versions of the finite-strain bending theory of elastic rubberlike shells developed by Chernykh (1980) and Simmonds (1985a), the corresponding definitions should also explicitly take into account the appropriate approximate form of the shell deformation in the transverse normal direction. As was noted by Stumpf and Makowski (1986) and Makowski and Stumpf (1986), the finite strain theory of elastic shells may have a richer mathematical structure than the one discussed here, if the transverse normal strains are fully accounted for. However, in the majority of cases it is usually sufficient to express the transverse normal strains in terms of the stretching and bending of the reference surface.

The work-conjugate boundary conditions derived here allow for a thin shell to formulate properly the nonlinear boundary value problem in terms of displacements as basic independent field variables. Such displacement form of nonlinear shell equations is used most often to analyze problems of flexible shells. In the case of conservative loads the work-conjugate boundary conditions allow to construct various functionals, whose stationarity conditions are equivalent to the proper field equations and boundary conditions.

References

Axelrad, E. L., 1987, *Theory of Flexible Shells*, North-Holland Pub. Co., Amsterdam, Chapter 1, Section 7.

Cartan, H., 1970, *Differential Forms*, Kershaw Publishing Company Ltd., Chapter 1, Section 2.1.

Chernykh, K. F., 1980, "Nonlinear Theory of Isotropically Elastic Shells," *Mechanics of Solids*, (Izvestiia AN SSSR, MTT), Vol. 15, pp. 118-127.

Galimov, K. Z., 1950, "General Theory of Elastic Shells with Finite Displacements," *Izvestiia Kazanskogo Filiala AN SSSR, Seriia Fiz.-Mat. i Tekhn. Nauk*, (in Russian), No. 2, pp. 3-38.

Galimov, K. Z., 1951, "On the General Theory of Plates and Shells with Finite Displacements and Deformations," *Prikladnaia Matematika i Mekhanika*, (in Russian), Vol. 15, pp. 723-742.

Ince, E. L., 1956, *Ordinary Differential Equations*, Dover Publications, Inc., Chapter 2.

Koiter, W. T., 1964, "On the Dynamic Boundary Conditions in the Theory of Thin Shells," *Proceedings, Koninkl. Nederl. Akademie van Wetenschappen-Amsterdam*, Series B, Vol. 67, pp. 117-126.

Koiter, W. T., 1966, "On the Non-Linear Theory of Thin Elastic Shells," *Proceedings, Koninkl. Nederl. Akademie van Wetenschappen – Amsterdam*, Series B, Vol. 69, pp. 1-54.

Love, A. E. H., 1927, *A Treatise on the Mathematical Theory of Elasticity*, 4th Ed., Dover Publications, Chapter 24, Articles 329 and 332.

Makowski, J., and Stumpf, H., 1986, "Finite Strains and Rotations in Shells," *Finite Rotations in Structural Mechanics*, W. Pietraszkiewicz, ed., Springer-Verlag, Berlin, pp. 175-194.

Novozhilov, V. V., and Shamina, V. A., 1975, "On Kinematic Boundary Conditions in Non-Linear Elasticity Problems," *Mechanics of Solids*, (Izvestiia AN SSSR, MTT), Vol. 5, pp. 63-74.

Pietraszkiewicz, W., 1977, "Introduction to the Non-Linear Theory of Shells," *Ruhr-Universität Bochum, Mitteilungen aus dem Institut für Mechanik*, No. 10, pp. 1-154.

Pietraszkiewicz, W., 1979, *Finite Rotations and Lagrangian Description in the Non-Linear Theory of Shells*, Polish Scientific Publishers, Warszawa, Poznań.

Pietraszkiewicz, W., 1980, "Finite Rotations in the Non-Linear Theory of Thin Shells," *Thin Shell Theory, New Trends and Applications*, W. Olszak, ed., Springer-Verlag, Wien, pp. 153-208.

Pietraszkiewicz, W., 1984a, "Lagrangian Description and Incremental Formulation in the Non-Linear Theory of Thin Shells," *International Journal of Non-Linear Mechanics*, Vol. 19, pp. 115-140.

Pietraszkiewicz, W., 1984b, "On Entirely Lagrangian Displacement Form of Non-Linear Shell Equations," *Flexible Shells*, E. L. Axelrad and F. A. Emmerling, eds., Springer-Verlag, Berlin, pp. 106-123.

Pietraszkiewicz, W., and Szwabowicz, M. L., 1981, "Entirely Lagrangian Nonlinear Theory of Thin Shells," *Archives of Mechanics*, Vol. 33, pp. 273-288.

Sakurai, T., Hasegawa, A., and Nishino, F., 1983, "A Finite Displacement Formulation of Elastic Shells," *Proceedings of the Japan Society of Civil Engineers*, No. 330, pp. 151-159.

Schmidt, R., and Pietraszkiewicz, W., 1981, "Variational Principles in the Geometrically Non-Linear Theory of Shells Undergoing Moderate Rotations," *Ingenieur-Archiv*, Vol. 50, pp. 187-201.

Schmidt, R., 1985, "A Current Trend in Shell Theory: Constrained Geometrically Non-Linear Kirchhoff - Love Type Theories Based on Polar Decomposition of Strains and Rotations," *Computers and Structures*, Vol. 20, pp. 265-275.

Simmonds, J. G., 1985a, "The Strain - Energy Density of Rubber-Like Shells," *International Journal of Solids and Structures*, Vol. 21, pp. 67-77.

Simmonds, J. G., 1985b, "A New Displacement Form for the Nonlinear Equations of Motion of Shells of Revolution," ASME JOURNAL OF APPLIED MECHANICS, Vol. 52, pp. 507-509.

Stumpf, H., 1986, "General Concept of the Analysis of Thin Elastic Shells," *Zeitschrift für Angewandte Matematik und Mechanik*, Vol. 66, pp. 337-350.

Stumpf, H., and Makowski, J., 1986, "On Large Strain Deformations of Shells," *Acta Mechanica*, Vol. 65, pp. 153-168.

Szwabowicz, M. L., 1982, "Geometrically Non-Linear Theory of Thin Elastic Shells in the Lagrangian Description," Ph.D. Dissertation, (in Polish), Institute of Fluid-Flow Machinery, PASci, Gdańsk, Poland.

Szwabowicz, M. L., 1986, "Variational Formulation in the Geometrically Non-Linear Thin Elastic Shell Theory," *International Journal of Solids and Structures*, Vol. 22, pp. 1161-1175.

Thomson, W., and Tait, P. G., 1883, *Treatise on Natural Philosophy*, 2nd Ed., Vol. 1, Part 2, Cambridge, U.K.

Von Westenholz, C., 1981, *Differential Forms in Mathematical Physics*, 2nd Ed., North-Holland Pub. Co., Amsterdam, Chapter 8.

Wempner, G., 1981, *Mechanics of Solids with Applications to Thin Bodies*, Sijthoff and Noordhoff, Alphen aan den Rijn, Chapter 12.

Zubov, L. M., 1982, *Methods of Non-Linear Elasticity in Shell Theory*, (in Russian), Rostov University Press, Chapter 4, Section 5.

Zubov, L. M., 1984, "On Boundary Condition for Rotation Angle of the Edge in Non-Linear Shell Theory," *Voprosy Dinamiki i Prochnosti*, (in Russian), Vol. 44, pp. 22-28.

APPENDIX

Let us verify the exactness and integrability of the 1-form $\omega=\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ appearing in (9a). In this case $\mathbf{A}=\bar{\mathbf{n}}$, $\mathbf{B}=\mathbf{0}$ so that the differentiation of (2c) gives

$$\frac{\partial\mathbf{A}}{\partial\bar{\mathbf{r}}_{,\nu}}=-\bar{a}_t j^{-1}\bar{\nu}\otimes\bar{\mathbf{n}},\quad \frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}}_{,\nu}}=\frac{\partial\mathbf{B}}{\partial\bar{\mathbf{r}}'}=\mathbf{0},\qquad (A1a)$$

$$\frac{\partial\mathbf{A}}{\partial\bar{\mathbf{r}}'}=2(\bar{a}_t j)^{-1}\gamma_{\nu t}\bar{\nu}\otimes\bar{\mathbf{n}}-\bar{a}_t{}^{-1}\bar{\mathbf{t}}\otimes\bar{\mathbf{n}},\qquad (A1b)$$

what implies that the conditions (14) are not satisfied and the 1-form $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ is not exact on X. If we introduce (A1) into (15), eleven conditions of (15) are identically satisfied while nine are not satisfied. For example, the left-hand side of (15) for (i, j, k) = (1, 3, 4) is $j^{-1}(\nu\cdot\bar{\mathbf{n}})\xi_2$ what does not identically vanish. As a result, the 1-form $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{r}}_{,\nu}$ is also not integrable on X.

In the case of the 1-form $\nu\cdot\delta\bar{\mathbf{n}}$ it follows from (9a) and (3d) that

$$\mathbf{A}=-\bar{a}_t j^{-1}(\nu\cdot\bar{\nu})\bar{\mathbf{n}},\quad \mathbf{B}=\bar{a}_t{}^{-1}(2j^{-1}\gamma_{\nu t}\nu\cdot\bar{\nu}-\nu\cdot\bar{\mathbf{t}})\bar{\mathbf{n}}.\qquad (A2)$$

Differentiation of (A2) with (2) and (3) gives

$$\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{r}}_{,\nu}} = \bar{a}_t^2 j^{-2}\,[(\nu\cdot\bar{\nu})(\bar{\nu}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\nu}) - (\nu\cdot\bar{\mathbf{n}})\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}], \tag{A3a}$$

$$\frac{\partial \mathbf{A}}{\partial \bar{\mathbf{r}}'} = -2j^{-2}\gamma_{\nu t}\,(\nu\cdot\bar{\nu})(\bar{\nu}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\nu}) + j^{-1}(\nu\cdot\mathfrak{t})\bar{\mathbf{n}}\otimes\bar{\nu}$$

$$+ j^{-1}(\nu\cdot\bar{\nu})\mathfrak{t}\otimes\bar{\mathbf{n}} + 2j^{-2}\gamma_{\nu t}(\nu\cdot\bar{\mathbf{n}})\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} = \left(\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{r}}_{,\nu}}\right)^T \tag{A3b}$$

$$\frac{\partial \mathbf{B}}{\partial \bar{\mathbf{r}}'} = 2\bar{a}_t^{-2} j^{-1}\gamma_{\nu t}\,(2j^{-1}\gamma_{\nu t}\nu\cdot\bar{\nu} - \nu\cdot\mathfrak{t})(\bar{\nu}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\bar{\nu}) +$$

$$+ \bar{a}_t^{-2}(\nu\cdot\mathfrak{t} - 2j^{-1}\gamma_{\nu t}\nu\cdot\bar{\nu})(\mathfrak{t}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\mathfrak{t}) - \tag{A3c}$$

$$- \bar{a}_t^{-2}(1 + 4j^{-2}\gamma_{\nu t}^2)(\nu\cdot\bar{\mathbf{n}})\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}.$$

Since ($A3a$) and ($A3c$) are symmetric and ($A3b$) holds, the component matrix A_{ij} is also symmetric and all the conditions (14) are identically satisfied. Therefore, the 1-form $\nu\cdot\delta\bar{\mathbf{n}}$ is exact on X.