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UNIFIED LAGRANGIAN DISPLACEMENT FORMULATION OF THE NON-LINEAR THEORY OF THIN SHELLS

FORMULAÇÃO LAGRANGIANA UNIFICADA BASEADA EM DESLOCAMENTOS DA TEORIA NÃO-LINEAR DE CASCAS FINAS

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ABSTRACT

Deformation of a thin shell is described entirely by three displacements of its reference surface. No restrictions are imposed on magnitudes of the displacements, rotations, strains and/or changes of curvatures of the surface. Explicit form of Lagrangian incremental shell equations is derived for arbitrary configuration-dependent external static surface and boundary loads as well as for arbitrary work-conjugate static and geometric boundary conditions. The most general form of the Lagrangian buckling equations for thin shells is presented.

Keywords: Shells ▪ Non-Linear Theory ▪ Incremental Formulations ▪ Buckling Equations

RESUMO

Neste trabalho o campo de deformações de uma casca fina é inteiramente descrito por três deslocamentos definidos em sua superfície de referência. Não há restrições impostas sobre as magnitudes dos deslocamentos, rotações, deformações e/ou variações de curvatura da superfície. Uma forma explicita das equações incrementais lagrangianas de cascas é obtida para carregamentos externos estáticos arbitrários, de superfície e de fronteira, dependentes da configuração, bem como para condições de contorno arbitrárias, geométricas e estáticas de trabalho-conjugado. A forma mais geral das equações lagrangianas de flambagem para cascas finas é apresentada.

Palavras-chave: Cascas ▪ Teoria Não-Linear ▪ Formulações Incrementais ▪ Equações de Flambagem

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INTRODUCTION

Within the field of thin shell theory hundreds of specialized versions of shell equations were proposed in the literature, each of them having a limited range of applications. The specialized versions are usually derived assuming different constraints on deformation or stress state in the shell space, restricting the magnitudes of strains, displacements or rotations, discussing only special material behaviour, shell geometry or external loads, using particular sets of independent field variables in the resulting boundary value problem etc., [14÷16].

The rapid development of computer hardware and software based on the finite element method makes it possible to solve more and more complex shell problems with sufficient accuracy. However, the shell finite elements available in the literature are usually based on some particular simplified versions of shell theory, and their applicability is restricted to the limited range of applicability of the shell theory itself. Any change in underlying version of shell theory results in the need of developing a new shell finite element, what makes the shell analysis so complex and time consuming.

The aim of this paper is to present a unified formulation of a wide class of non-linear theories of thin shells. In our development we apply only one apparent assumption: the deformation of the shell as a three-dimensional body is determined entirely by deformation of its reference surface. No restrictions are imposed here on magnitudes of the displacements, rotations, strains and/or changes of curvature of the reference surface. For different material behaviour the reduction from three-dimensional solid mechanics to the two-dimensional shell theory may have different analytic representation, which is treated here as part of constitutive relations of the shell.

Let us note that members of that class of shell theories are various versions of the classical linear and geometrically non-linear theory of thin isotropic elastic shells based on the Kirchhoff-Love type constraints [16]. Another example of member of that class is the bending theory of rubber-like shells developed in [22], where the three-dimensional shell deformation was expressed through deformation of its reference surface applying a relaxed normality hypothesis and incompressibility condition. Still another members of that class are some simple

versions of inelastic shell theory expressed entirely in terms of the reference surface deformation as well, as discussed in [20], for example.

A common feature of that class of shell theories is that their equilibrium conditions have the same form following from the principle of virtual displacements written for the reference surface. The geometry of undeformed reference surface is usually the only one which is known in advance, and an arbitrary deformation of the surface can always be described by components of the displacement vector $\mathbf{u}$ relative to the undeformed surface geometry. Therefore, the resulting boundary value problem of the shell can always be expressed in the Lagrangian description in terms of displacements as the only independent field variables.

The unified formulation of Lagrangian non-linear shell equations presented here is based on generalization of our results given in [19,15] for the geometrically non-linear theory of elastic shells, which were extended in [22] into the large-strain bending theory of rubber-like shells. Our Lagrangian shell equations (8), (9) are two-dimensionally exact for the shell reference surface. They are valid for arbitrary configuration-dependent external surface and boundary forces and moments, as well as for arbitrary work-conjugate set of static and geometric boundary conditions. In order to allow correct numerical implementation, the Lagrangian shell equations are presented in Section 4 in the consistent incremental form applying the general Newton-Kantorovich method [6] to the functional (6) of principle of virtual displacements. In particular, we take into account that, in general, the successive approximations to the unknown equilibrium state may not belong to the equilibrium path. This results in some unballanced forces (14) appearing explicitly at each iteration step. We managed to calculate explicitly Gateaux derivatives of all corresponding fields, and to derive the explicit form of the general Lagrangian incremental shell equations (24)÷(27). As a particular case of the incremental shell equations follows the explicit form of the most general buckling equations (28) for thin shells.

GEOMETRIC RELATIONS

In this report we apply the system of notation used by Pietraszkiewicz [14÷16].

Let the reference surface M of undeformed shell be defined by the position vector $\mathbf{r}(\theta^\alpha)$, where θ^α, $\alpha = 1, 2$, are surface curvilinear coordinates. On M we have the natural base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r}_{,\alpha}$, the (covariant components

of the surface) metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with determinant $a = |a_{\alpha\beta}|$, the unit normal vector $\mathbf{n} = a^{-1/2}\mathbf{a}_1 \times \mathbf{a}_2$, the curvature tensor $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ and the permutation tensor $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta)\cdot\mathbf{n}$. The contravariant base vectors $\mathbf{a}^\alpha$ are defined through $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$, and $a^{\alpha\beta} = \mathbf{a}^\beta \cdot \mathbf{a}^\beta$ are used to raise indices on M.

The boundary contour C of M consists of the finite set of piecewise smooth curves $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along C. With each regular point $M \in C$ we associate the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds \equiv \mathbf{r}' = t^\alpha \mathbf{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r}_{,\nu} = \mathbf{t} \times \mathbf{n} = \nu^\alpha \mathbf{a}_\alpha$, $\nu^\alpha = \varepsilon^{\alpha\beta} t_\beta$, where $(\,)_{,\nu}$ denotes the outward normal derivative at C.

Let $\bar{M}$ and $\bar{C}$ be deformed configurations of M and C defined by the position vectors $\bar{\mathbf{r}}(\theta^\alpha) = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha)$ and $\mathbf{r}[\theta^\alpha(s)] = \mathbf{r}(s) + \mathbf{u}(s)$, respectively, where $\mathbf{u} = u_\alpha \mathbf{a}^\alpha + w\mathbf{n}$ is the displacement vector while θ^α and s are convected coordinates. With $\bar{M}$ and $\bar{C}$ we can associate analogously defined quantities, only now marked by an overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\mathbf{a}}^\beta$, $\bar{a}^{\alpha\beta}$, $\bar{\mathbf{t}}, \bar{\boldsymbol{\nu}}$ etc. All the quantities can be expressed through the geometry of M or C and the displacement field $\mathbf{u}$ by relations presented in more detail in [15,16]. In particular, on $\bar{M}$ we have

$$\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha} = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}\,, \quad \bar{\mathbf{n}} = \frac{1}{2} j^{-1} \varepsilon^{\alpha\beta} \bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta\,,$$

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta}, \quad \bar{b}_{\alpha\beta} = b_{\alpha\beta} - \kappa_{\alpha\beta},$$

$$\gamma_{\alpha\beta} = \frac{1}{2}(\bar{\mathbf{r}}_{,\alpha} \cdot \bar{\mathbf{r}}_{,\beta} - a_{\alpha\beta}), \quad \kappa_{\alpha\beta} = \bar{\mathbf{r}}_{,\alpha} \cdot \bar{\mathbf{n}}_{,\beta} + b_{\alpha\beta}, \tag{1}$$

$$j^2 = \frac{\bar{a}}{a} = \frac{1}{2}\,\varepsilon^{\alpha\lambda}\varepsilon^{\beta\kappa}\bar{a}_{\alpha\beta}\bar{a}_{\lambda\kappa}\,,$$

$$\bar{\mathbf{a}}^\alpha = \bar{a}^{\alpha\beta}\bar{\mathbf{a}}_\beta, \qquad \bar{a}^{\alpha\beta} = j^{-2}[(1 + 2\gamma^\kappa_\kappa)\, a^{\alpha\beta} - 2\gamma^{\alpha\beta}]\,,$$

where $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are the Lagrangian symmetric surface strain measures.

Along the deformed shell boundary contour $\bar{C}$ we have [16,8]

$$\bar{\mathbf{r}}' = \mathbf{t} + \mathbf{u}' = \bar{a}_t \bar{\mathbf{t}}\,, \quad \bar{\mathbf{n}} = j^{-1}\bar{\mathbf{r}}_{,\nu} \times \bar{\mathbf{r}}'\,,$$
$$\bar{\mathbf{r}}_{,\nu} = \boldsymbol{\nu} + \mathbf{u}_{,\nu} = \bar{a}_t^{-1}(j\bar{\boldsymbol{\nu}} + 2\gamma_{\nu t}\bar{\mathbf{t}})\,,$$

$$\bar{a}_t = |\bar{\mathbf{r}}'| \,, \qquad 2\gamma_{\nu t} = \bar{\mathbf{r}}_{,\nu} \cdot \bar{\mathbf{r}}' \,, \tag{2}$$
$$j^2 = \frac{\bar{a}}{a} = |\bar{\mathbf{r}}_{,\nu}|^2 |\bar{\mathbf{r}}'|^2 - (\bar{\mathbf{r}}_{,\nu} \cdot \bar{\mathbf{r}}')^2 \,,$$
$$\bar{\mathbf{a}}^\beta = j^{-1}(\bar{a}_t \nu^\beta - 2\gamma_{\nu t} \bar{a}_t^{-1} t^\beta)\bar{\boldsymbol{\nu}} + \bar{a}_t^{-1} t^\beta \bar{\mathbf{t}} \,.$$

All the vectors in (1) and (2) are understood to be expressed through components with respect to the known bases $\mathbf{a}_\alpha$, $\mathbf{n}$ or $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ of M or C, respectively.

DISPLACEMENT SHELL EQUATIONS

Within the class of non-linear theories of thin shells discussed here the deformation of the shell as a three-dimensional body is assumed to be determined entirely by deformation of its reference surface. Therefore, the equilibrium conditions of the shell should follow from the Lagrangian principle of virtual displacements for the reference surface [15,8,17]

$$G[\mathbf{u}; \delta\mathbf{u}] = \int_M \int (N^{\alpha\beta} \delta\gamma_{\alpha\beta} + M^{\alpha\beta} \delta\kappa_{\alpha\beta}) dA -$$
$$- \int_M \int (\mathbf{p} \cdot \delta\mathbf{u} + \mathbf{h} \cdot \delta\bar{\mathbf{n}}) dA - \int_{C_f} (\mathbf{T} \cdot \delta\mathbf{u} + \mathbf{H} \cdot \delta\bar{\mathbf{n}}) ds = 0 \,, \tag{3}$$

which is valid for all kinematically admissible virtual displacements $\delta\mathbf{u}$. In (3) $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are the internal 2nd Piola-Kirchhoff type stress and couple resultants, $\mathbf{p}(\mathbf{u})$ and $\mathbf{h}(\mathbf{u})$ are the external surface force and moment vectors, per unit area of M, $\mathbf{T}(\mathbf{u})$ and $\mathbf{H}(\mathbf{u})$ are the external boundary force and moment vectors, per unit length of C, while δ is the symbol of variation.

Within M variations of $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$ and $\bar{\mathbf{n}}$ are expressed through $\mathbf{u}$ and $\delta\mathbf{u}$ by

$$\delta\gamma_{a\beta} = \frac{1}{2}(\delta\mathbf{u}_{,\alpha} \cdot \bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha \cdot \delta\mathbf{u}_{,\beta}) \,,$$
$$\delta\kappa_{\alpha\beta} = \frac{1}{2}(\bar{\mathbf{n}}_{,\alpha} \cdot \delta\mathbf{u}_{,\beta} + \bar{\mathbf{n}}_{,\beta} \cdot \delta\mathbf{u}_{,\alpha} + \bar{\mathbf{a}}_\alpha \cdot \delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_\beta \cdot \delta\bar{\mathbf{n}}_{,\alpha}) \,, \tag{4}$$
$$\delta\bar{\mathbf{n}} = -\bar{\mathbf{a}}^\beta (\bar{\mathbf{n}} \cdot \delta\mathbf{u}_{,\beta}) \,.$$

At the boundary contour $\bar{C}$ the vector $\bar{\mathbf{n}} = \bar{\mathbf{n}}(s)$ should satisfy the constraints $\bar{\mathbf{r}}' \cdot \bar{\mathbf{n}} = 0$ and $\bar{\mathbf{n}} \cdot \bar{\mathbf{n}} = 1$. Therefore, $\bar{\mathbf{n}}$ on $\bar{C}$ should be expressible

through the geometry of C, three translations $\mathbf{u}(s)$ and one scalar function $\varphi(s) = \varphi[\mathbf{u}_{,\nu}(s),\ \mathbf{u}'(s)]$ describing the rotational deformation of the shell lateral boundary surface.

The general structure of the function φ, and corresponding four work-conjugate static and geometric boundary conditions compatible with the principle of virtual displacement (3), were discussed by Makowski and Pietraszkiewicz [8]. In particular, three physically reasonable special cases of φ were noted in [8]: 1) $n_\nu = \bar{\mathbf{n}} \cdot \boldsymbol{\nu} = j^{-1}(\mathbf{u}' \times \boldsymbol{\nu} - \mathbf{n}) \cdot \mathbf{u}_{,\nu}$ introduced in [19], 2) $v_\nu = \bar{a}_t^{-2}(\bar{\mathbf{n}} - \mathbf{n}) \cdot (\bar{\mathbf{r}}' \times \bar{\mathbf{n}})$ introduced in [13], and 3) ω_t, the angle of total rotation of the boundary, defined in [14] through displacements by $2\cos\omega_t = \bar{\boldsymbol{\nu}} \cdot \boldsymbol{\nu} + \bar{\mathbf{t}} \cdot \mathbf{t} + \bar{\mathbf{n}} \cdot \mathbf{n} - 1$. In what follows all transformations leading to displacement shell equations are performed applying n_ν as the fourth parameter of boundary deformation. Corresponding results for v_ν and ω_t taken as the fourth parameters of boundary deformation are given in [8,18], respectively.

Thus, in terms of $\mathbf{u}$ and n_ν the variation of $\bar{\mathbf{n}}$ of $\bar{C}$ takes the form [15]

$$\delta\mathbf{n} = a_\nu^{-1}[(\boldsymbol{\nu} \times \bar{\mathbf{n}})\bar{\mathbf{n}} \cdot \delta\mathbf{u}' + (\bar{\mathbf{r}}' \times \bar{\mathbf{n}})\delta n_\nu] , \qquad a_\nu = (\bar{\mathbf{r}}' \times \bar{\mathbf{n}}) \cdot \boldsymbol{\nu} , \tag{5}$$

where $\delta n_\nu = \delta n_\nu[\mathbf{u}_{,\nu}, \mathbf{u}'; \delta\mathbf{u}_{,\nu}, \delta\mathbf{u}']$ is non-linear in $\mathbf{u}_{,\nu}$, $\mathbf{u}'$ but is linear in $\delta\mathbf{u}_{,\nu}$, $\delta\mathbf{u}'$.

Introducing (4) and (5) into (3), applying the Stokes' theorem to the surface integrals, then applying integration by parts to the line integrals we can transform (3) into [15]

$$\begin{aligned} G[\mathbf{u}; \delta\mathbf{u}] = &- \int_M \int \{\mathbf{T}^\beta|_\beta + \mathbf{p} + [(\mathbf{h} \cdot \bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}]_{|\beta}\} \cdot \delta\mathbf{u} dA + \\ &+ \int_{C_f} \{[\mathbf{T}^\beta \nu_\beta + \mathbf{F}' - \mathbf{T} - \mathbf{F}^{*\prime} + (\mathbf{h} \cdot \bar{\mathbf{a}}^\beta \nu_\beta)\bar{\mathbf{n}}] \cdot \delta\mathbf{u} + (M - M^*)\delta n_\nu\} ds + \\ &+ \sum_n (\mathbf{F}_n - \mathbf{F}_n^*) \cdot \delta\mathbf{u}_n = 0 , \end{aligned} \tag{6}$$

where

$$\begin{aligned} \mathbf{T}^\beta &= N^{\alpha\beta}\bar{\mathbf{a}}_\alpha + M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha} + \{[M^{\kappa\rho}\bar{\mathbf{a}}_\kappa)_{|\rho} \cdot \bar{\mathbf{a}}^\beta\}\ \bar{\mathbf{n}} , \\ \mathbf{F} &= -a_\nu^{-1}[(\bar{\mathbf{n}} \times \bar{\mathbf{a}}_\alpha) \cdot \boldsymbol{\nu}] M^{\alpha\beta}\nu_\beta \bar{\mathbf{n}} , \qquad \mathbf{F}^* = -a_\nu^{-1}[(\bar{\mathbf{n}} \times \mathbf{H}) \cdot \boldsymbol{\nu}]\bar{\mathbf{n}} , \\ M &= a_\nu^{-1}(\bar{\mathbf{n}} \times \bar{\mathbf{a}}_\alpha) \cdot \bar{\mathbf{r}}' M^{\alpha\beta}\nu_\beta , \qquad M^* = a_\nu^{-1}(\bar{\mathbf{n}} \times \mathbf{H}) \cdot \bar{\mathbf{r}}' , \\ \mathbf{F}_n &= \mathbf{F}(s_n + 0) - \mathbf{F}(s_n - 0) , \qquad \mathbf{u}_n = \mathbf{u}(s_n) . \end{aligned} \tag{7}$$

Since (6) should be satisfied identically for all kinematically admissible $\delta\mathbf{u}$, from (6) follow the Lagrangian equilibrium equations and static boundary and corner conditions

$$\begin{aligned}
&\mathbf{T}^{\beta}|_{\beta} + \mathbf{p} + [(\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}}]_{|\beta} = \mathbf{0} \qquad \text{in } M\ , \\
&\mathbf{T}^{\beta}\nu_{\beta} + \mathbf{F}' = \mathbf{T} + \mathbf{F}^{*\prime} - (\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta}\nu_{\beta})\bar{\mathbf{n}}\ , \qquad M = M^{*} \qquad \text{on } C_f\ , \qquad (8)\\
&\mathbf{F}_n = \mathbf{F}_n^{*} \qquad \text{at each corner } M_n \in C_f\ .
\end{aligned}$$

Corresponding work-conjugate geometric boundary conditions are

$$\mathbf{u}(s) = \mathbf{u}^{*}(s)\ , \qquad n_{\nu}(s) = n_{\nu}^{*}(s) \qquad \text{on } C_u\ . \qquad (9)$$

All the vectors appearing in (8), (9) are understood to be expressed through components with respect to known bases $\mathbf{a}_{\alpha}$, $\mathbf{n}$ or $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ of undeformed M or C, respectively.

In the case of an elastic material the constitutive equations for $N^{\alpha\beta}$, $M^{\alpha\beta}$ compatible with (3) are

$$N^{\alpha\beta} = \frac{\partial\Sigma}{\partial\gamma_{\alpha\beta}}\ , \qquad M^{\alpha\beta} = \frac{\partial\Sigma}{\partial\kappa_{\alpha\beta}}\ , \qquad (10)$$

where $\Sigma = \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ is a two-dimensional strain energy function defined over M. In the particular case of isotropic elastic material undergoing small strains (but unrestricted rotations) the strain energy Σ is, to the first approximation, a quadratic function of the surface strain measures, [5,14,16]. In the particular case of large-strain bending theory of shells made of isotropic elastic incompressible rubber-like materials the structure of Σ, to the first approximation, is given in [22]. Therefore, for each particular elastic material the constitutive equations can explicitly be expressed in the form $N^{\alpha\beta} = N^{\alpha\beta}[\gamma_{\alpha\beta}(\mathbf{u}), \kappa_{\alpha\beta}(\mathbf{u})]$, $M^{\alpha\beta} = M^{\alpha\beta}[\gamma_{\alpha\beta}(\mathbf{u}), \kappa_{\alpha\beta}(\mathbf{u})]$ for any $\mathbf{u}$. As a result, the boundary value problem (8), (9) of the Lagrangian non-linear theory of thin elastic shells is expressed entirely in terms of displacements $\mathbf{u}$ as the only independent field variables.

It should be pointed out that the underlying principle of virtual displacements (3) is an incremental principle, which itself does not reguire $N^{\alpha\beta}$ and $M^{\alpha\beta}$ to be derivable from the strain energy function. Therefore, our resulting

shell relations (8), (9) are valid also for inelastic shells. However, in the case of inelastic material behaviour $N^{\alpha\beta}$ and $M^{\alpha\beta}$ at successive equilibrium configurations $\mathbf{u}$ should be calculated in an incremental-iterative way applying appropriate two-dimensional incremental form of constitutive equations for $N^{\alpha\beta}$ and $M^{\alpha\beta}$.

INCREMENTAL SHELL EQUATIONS

The highly non-linear boundary value problem (8), (9) can effectively be solved only by incremental-iterative procedures applying computerized numerical methods, for which the shell equations (8), (9) should be presented in a consistent incremental form.

In general, the external loads $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$ may be specified arbitrarily, or through several independent dimensionless parameters $(\lambda_1, \lambda_2, \ldots, \lambda_p) \in \Lambda \subset R^p$. In the latter case any information concerning the principal features of the solution manifold can be obtained analysing the set of solution submanifolds corresponding to a smoothly varying single parameter. Therefore, in the following considerations we restrict ourselves to the case when the external loads are specified by a single parameter $\lambda \in \Lambda \subset R$.

For smoothly varying λ the regular solutions of (8), (9) form an equilibrium path $\mathbf{u}(\lambda)$ for which $G[\mathbf{u}(\lambda); \delta\mathbf{u}] = 0$ for all kinematically admissible virtual displacements $\delta\mathbf{u}$. For tracing $\mathbf{u}(\lambda)$ it is convenient to apply the Newton-Kantorovich method [6].

Let $\mathbf{u}_m = \mathbf{u}(\lambda_m)$ be an equilibrium state associated with some $\lambda = \lambda_m$, and let $\mathbf{u}_m^{(i)}$ be a known i-th approximation to $\mathbf{u}_m$, which in general may not belong to the equilibrium path $\mathbf{u}(\lambda)$. In order to calculate the correction $\Delta\mathbf{u}_m^{(i+1)}$ such that $\mathbf{u}_m^{(i+1)} = \mathbf{u}_m^{(i)} + \Delta\mathbf{u}_m^{(i+1)}$ is the next approximation to $\mathbf{u}_m$ we linearize $G[\mathbf{u}; \delta\mathbf{u}]$ at $\mathbf{u}_m^{(i)}$ in the direction $\Delta\mathbf{u}_m^{(i+1)}$, what leads to the functional equation [6]

$$\mathbf{G}[\mathbf{u}_m^{(i)}; \delta\mathbf{u}] + \Delta G[\mathbf{u}_m^{(i)}; \delta\mathbf{u}, \Delta\mathbf{u}_m^{(i+1)}] = 0 \,, \tag{11}$$

where ΔG is the Gateaux derivative of (3) taken at $\mathbf{u}_m^{(i)}$ in the kinematically admissible direction $\Delta\mathbf{u}_m^{(i+1)}$. When $\mathbf{u}_m^{(i)}$ does not belong to the equilibrium path the first term of (11) allows to calculate the unballanced force vector. The

second term of (11) is linear in the unknown $\Delta\mathbf{u}_m^{(i+1)}$ and allows to calculate the tangent stiffness matrix at $\mathbf{u}_m^{(i)}$ of the problem.

In order to simplify notation, in the following part of this Section we set $\mathbf{u}_m^{(i)} \equiv \mathbf{u}$, $\Delta\mathbf{u}_m^{(i+1)} \equiv \Delta\mathbf{u}$, while the values at $\mathbf{u}_m^{(i)}$ of corresponding external loadings we denote shortly by $\mathbf{p}$, $\mathbf{h}$, $\mathbf{T}$ and $\mathbf{H}$.

Let us consider a curve $\mathbf{u}(\eta)$ through the i-th approximation $\mathbf{u}$ to $\mathbf{u}_m$, which in the neighbourhood of $\mathbf{u}$ takes the form $\mathbf{u}(\eta) = \mathbf{u} + \eta\Delta\mathbf{u}$. The directional Gateaux derivative of the functional G, taken at the i-th approximation $\mathbf{u}$ to $\mathbf{u}_m$ in the direction $\Delta\mathbf{u}$, is given by

$$\Delta G[\mathbf{u}; \delta\mathbf{u}, \Delta\mathbf{u}] = \frac{d}{d\eta}\, G[\mathbf{u}(\eta); \delta\mathbf{u}]_{|\eta=0} \,, \tag{12}$$

where $G[\mathbf{u}(\eta); \delta\mathbf{u}]$ is defined analogously as the functional $G[\mathbf{u}; \delta\mathbf{u}]$, only now $\mathbf{u}(\eta)$ appears in place of $\mathbf{u}$.

Along the curve $\mathbf{u}(\eta)$ the external loads are denoted by $\mathbf{p}(\eta)$, $\mathbf{h}(\eta)$, $\mathbf{T}(\eta)$ and $\mathbf{H}(\eta)$, while the internal stress and couple resultants by $N^{\alpha\beta}(\eta)$ and $M^{\alpha\beta}(\eta)$, respectively. The corresponding Gateaux derivatives of those fields are defined according to

$$\begin{aligned} \Delta\mathbf{p} &= \frac{d}{d\eta}\, \mathbf{p}(\eta)_{|\eta=0} \,, \qquad \Delta\mathbf{T} = \frac{d}{d\eta}\, \mathbf{T}(\eta)_{|\eta=0} \,, \\ \Delta N^{\alpha\beta} &= \frac{d}{d\eta}\, N^{\alpha\beta}(\eta)_{|\eta=0} \,, \qquad etc. \end{aligned} \tag{13}$$

Let us apply the linearization (11) to the already transformed functional (6). Since our $\mathbf{u} \equiv \mathbf{u}_m^{(i)}$ may not belong to the equilibrium path, let us introduce the unballanced residual surface and boundary forces and couples

$$\begin{aligned} \mathbf{p}_R &= \mathbf{T}^{\beta}|_{\beta} + \mathbf{p} + [(\mathbf{h} \cdot \bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}}]|_{\beta} \,, \\ \mathbf{P}_R &= \mathbf{T}^{\beta}\nu_{\beta} + \mathbf{F}' - \mathbf{T} - \mathbf{F}^{*\prime} + (\mathbf{h} \cdot \bar{\mathbf{a}}^{\beta}\nu_{\beta})\bar{\mathbf{n}} \,, \\ M_R &= M - M^{*}, \ \mathbf{F}_{nR} = \mathbf{F}_n - \mathbf{F}_n^{*} \,. \end{aligned} \tag{14}$$

The quantities (14) allow to evaluate the first term in (11) in the form

$$G[\mathbf{u};\delta\mathbf{u}] = -\int_M \int \mathbf{p}_R\cdot\delta\mathbf{u}dA+\int_{C_f}(\mathbf{P}_R\cdot\delta\mathbf{u}+M_R\delta n_\nu)\,ds+\sum_n \mathbf{F}_{nR}\cdot\delta\mathbf{u}_n\ . \quad (15)$$

In order to calculate the second term of (11), let us remind that along the curve $\mathbf{u}(\eta)$ the shell geometry is defined by $\bar{\mathbf{a}}_\alpha(\eta)$, $\bar{\mathbf{a}}^\beta(\eta)$ and $\bar{\mathbf{n}}(\eta)$. Therefore, taking Gateaux derivatives of the identities $\bar{\mathbf{a}}^\beta(\eta)\cdot\bar{\mathbf{a}}_\alpha(\eta)=\delta^\alpha_\beta$, $\bar{\mathbf{a}}^\beta(\eta)\cdot\bar{\mathbf{n}}(\eta)=0$ we have in M

$$\begin{aligned}&\Delta\bar{\mathbf{a}}_\alpha = \Delta\mathbf{u}_{,\alpha}\,, \qquad \Delta\bar{\mathbf{n}} = -\bar{\mathbf{a}}^\kappa(\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\kappa})\,,\\ &\Delta\bar{\mathbf{a}}^\beta = -(\bar{\mathbf{a}}^\beta\cdot\Delta\mathbf{u}_{,\kappa})\,\bar{\mathbf{a}}^\kappa + \bar{a}^{\beta\kappa}(\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\kappa})\,\bar{\mathbf{n}}\ .\end{aligned} \quad (16)$$

$$\begin{aligned}&\Delta(\delta\bar{\mathbf{n}}) = \mathbf{B}^\beta\cdot\delta\mathbf{u}_{,\beta}\,,\\ &\mathbf{B}^\beta = [(\bar{\mathbf{a}}^\beta\cdot\Delta\mathbf{u}_{,\kappa})\,\bar{\mathbf{a}}^\kappa - a^{\beta\kappa}(\mathbf{n}\cdot\Delta\mathbf{u}_{,\kappa})\mathbf{n}]\otimes\mathbf{n}+(\mathbf{n}\cdot\Delta\mathbf{u}_{,\kappa})\,\bar{\mathbf{a}}^\beta\otimes\bar{\mathbf{a}}^\kappa\ .\end{aligned} \quad (17)$$

Similarly, let us introduce $\bar{\mathbf{n}}(\eta)$ and $\delta\bar{\mathbf{n}}(\eta)$ on $\bar{C}$, defined by respective formulae $(2)_1$ and (5), where now $\mathbf{u}(\eta)$ stands for $\mathbf{u}$. This allows to calculate Gateaux derivatives of $\bar{\mathbf{n}}$ and $\delta\bar{\mathbf{n}}$ on C in the form

$$\begin{aligned}&\Delta\bar{\mathbf{n}} = a_\nu^{-1}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})\,\bar{\mathbf{n}}\cdot\Delta\mathbf{u}'+(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\Delta n_\nu]\,,\\ &\Delta(\delta\bar{\mathbf{n}}) = \mathbf{A}\cdot\delta\mathbf{u}'+\mathbf{B}\delta n_\nu + a_\nu^{-1}(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\,\Delta(\delta n_\nu)\,,\end{aligned} \quad (18)$$

$$\begin{aligned}&\mathbf{A} = -a_\nu^{-2}[\boldsymbol{\nu}\cdot(\Delta\mathbf{u}'\times\bar{\mathbf{n}}+\bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})](\boldsymbol{\nu}\times\bar{\mathbf{n}})\otimes\bar{\mathbf{n}}+a_\nu^{-1}[\boldsymbol{\nu}\times(\Delta\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}+\bar{\mathbf{n}}\otimes\Delta\bar{\mathbf{n}}]\,,\\ &\mathbf{B} = -a_\nu^{-2}[\boldsymbol{\nu}\cdot(\Delta\mathbf{u}'\times\bar{\mathbf{n}}+\bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})](\bar{\mathbf{r}}'\times\bar{\mathbf{n}})+a_\nu^{-1}(\bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}}+\Delta\mathbf{u}'\times\bar{\mathbf{n}})\,.\end{aligned} \quad (19)$$

With the help of (16)÷(19), (12), (13), (6), (7) and some transformations, the second term of (11) can be presented in the form

$$\Delta G[\mathbf{u};\delta\mathbf{u},\Delta\mathbf{u}] = -\int_M \int \{[(\Delta\mathbf{T}^\beta + \mathbf{S}^\beta)_{|\beta} + \Delta\mathbf{p} + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\,\bar{\mathbf{n}} -$$
$$-\mathbf{h}\cdot\mathbf{B}^\beta]_{|\beta}\}\cdot\delta\mathbf{u}dA + \tag{20}$$
$$+\int_{C_f}\Big(\Big\{(\Delta\mathbf{T}^\beta + \mathbf{S}^\beta)\,\nu_\beta + (\Delta\mathbf{F}+\mathbf{C})' - \Delta\mathbf{T} - (\Delta\mathbf{F}^* + \mathbf{C}^*)' + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\,\bar{\mathbf{n}} -$$
$$-\mathbf{h}\cdot\mathbf{B}^\beta]\nu_\beta\Big\}\cdot\delta\mathbf{u} + (\Delta M + K - \Delta M^* - K^*)\,\delta n_\nu + M_R\Delta(\delta n_\nu)\Big)ds +$$
$$+\sum_n(\Delta\mathbf{F}_n + \mathbf{C}_n - \Delta\mathbf{F}_n^* - \mathbf{C}_n^*)\cdot\delta\mathbf{u}_n\ ,$$

where

$$\Delta\mathbf{T}^\beta = \Delta N^{\alpha\beta}\bar{\mathbf{a}}_\alpha + \Delta M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha} + [(\Delta M^{\kappa\rho}\bar{\mathbf{a}}_\kappa)_{|\rho}\cdot\bar{\mathbf{a}}^\beta]\bar{\mathbf{n}}\ ,$$
$$\mathbf{S}^\beta = N^{\alpha\beta}\Delta\mathbf{u}_{,\alpha} + M^{\alpha\beta}\Delta\bar{\mathbf{n}}_{,\alpha} + [(M^{\kappa\rho}\Delta\mathbf{u}_{,\kappa})_{|\rho}\cdot\bar{\mathbf{a}}^\beta]\,\bar{\mathbf{n}} - (M^{\kappa\rho}\bar{\mathbf{a}}_\kappa)_{|\rho}\cdot\mathbf{B}^\beta\ , \tag{21}$$

$$\Delta\mathbf{F} = -a_\nu^{-1}[(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\boldsymbol{\nu}]\Delta M^{\alpha\beta}\nu_\beta\bar{\mathbf{n}}\ ,$$
$$\mathbf{C} = -a_\nu^{-1}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha}]M^{\alpha\beta}\nu_\beta\bar{\mathbf{n}} - M^{\alpha\beta}\bar{\mathbf{a}}_\alpha\nu_\beta\cdot\mathbf{A}\ , \tag{22}$$
$$\Delta M = a_\nu^{-1}(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\bar{\mathbf{r}}'\Delta M^{\alpha\beta}\nu_\beta\ ,$$
$$K = a_\nu^{-1}[(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha}]\,M^{\alpha\beta}\nu_\beta + M^{\alpha\beta}\bar{\mathbf{a}}_\alpha\nu_\beta\cdot\mathbf{B}\ ,$$

$$\Delta\mathbf{F}^* = -a_\nu^{-1}[(\bar{\mathbf{n}}\times\Delta\mathbf{H})\cdot\boldsymbol{\nu}]\bar{\mathbf{n}}\ ,\qquad \mathbf{C}^* = -\mathbf{H}\cdot\mathbf{A}\ , \tag{23}$$
$$\Delta M^* = a_\nu^{-1}(\bar{\mathbf{n}}\times\Delta\mathbf{H})\cdot\bar{\mathbf{r}}'\ ,\qquad K^* = \mathbf{H}\cdot\mathbf{B}\ ,$$

The fields $\Delta\mathbf{T}^{\beta}$, $\Delta\mathbf{F}$, ΔM, $\Delta\mathbf{F}_n$ are linear in $\Delta\mathbf{u}$ and represent the material part of changes of $\mathbf{u}$, while the fields $\mathbf{S}^{\beta}$, $\mathbf{C}$, K, $\mathbf{C}_n$ are also linear in $\Delta\mathbf{u}$ but represent the geometric part of changes of $\mathbf{u}$ at the incremental step.

Since (11) with (15) and (20) should vanish for any kinematically admissible $\delta\mathbf{u}$, from (11) we obtain the following incremental equilibrium equations as well as the incremental static boundary and corner conditions for the Lagrangian non-linear theory of thin shells

$$(\Delta\mathbf{T}^{\beta} + \mathbf{S}^{\beta})_{|\beta} + \Delta\mathbf{p} + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]_{|\beta} + \mathbf{p}_R = \mathbf{0} \qquad \text{in } M\ , \tag{24}$$

$$\left.\begin{aligned} &(\Delta\mathbf{T}^{\beta} + \mathbf{S}^{\beta})\,\nu_{\beta} + (\Delta\mathbf{F} + \mathbf{C})' = \Delta\mathbf{T} + (\Delta\mathbf{F}^{*} + \mathbf{C}^{*})' - \\ &- [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\,\dot{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]\nu_{\beta} - \mathbf{P}_R \\ &\Delta M + K = \Delta M^{*} + K^{*}\ , \qquad M_R = 0 \end{aligned}\right\} \text{ on } C_f\ , \tag{25}$$

$$\Delta\mathbf{F}_n + \mathbf{C}_n = \Delta\mathbf{F}_n^{*} + \mathbf{C}_n^{*} - \mathbf{F}_{nR} \qquad \text{at each corner } M_n \in C_f\ .$$

The corresponding work-conjugate geometric boundary conditions to be satisfied at each incremental step are

$$\Delta\mathbf{u} = \mathbf{0}\ , \qquad \Delta n_{\nu} = 0 \qquad \text{on } C_u \tag{26}$$

All the vectors in (24)÷(26) are given through components in the respective undeformed bases $\mathbf{a}_{\alpha}$, $\mathbf{n}$, and $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$.

The incremental shell equations (24)÷(26) constitute the linearized boundary value problem for the increment $\Delta\mathbf{u} \equiv \Delta\mathbf{u}_m^{(i+1)}$ which allows from known $\mathbf{u}_m^{(i)}$ to calculate the next approximation $\mathbf{u}_m^{(i+1)}$ to the equilibrium state $\mathbf{u}_m$.

In the case of an elastic material $\Delta N^{\alpha\beta}$ and $\Delta M^{\alpha\beta}$ follow directly from the constitutive equations (10)

$$\begin{aligned} \Delta N^{\alpha\beta} &= C_1^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_2^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu}\ , \\ \Delta M^{\alpha\beta} &= C_3^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_4^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu}\ , \end{aligned} \tag{27}$$

where $C_k^{\alpha\beta\lambda\mu}$, $k = 1,\ldots,4$ are the tangent elasticities at $\mathbf{u}$, defined as second partial derivatives of Σ with respect to $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$ (see (85) of [22]), while

$\Delta\gamma_{\alpha\beta}$, $\Delta\kappa_{\alpha\beta}$ are Gateaux derivatives at $\mathbf{u}$ of the surface strain measures (see (82) of [22]). Therefore, for each particular form of Σ, [5,16,22], the tangent elasticities can be explicitly calculated as known functions of $\mathbf{u}$. In the case of inelastic material behaviour the tangent elasticities at each $\mathbf{u}$ should be calculated by some independent incremental-iterative procedure.

The set of incremental shell equations (24)÷(27) with (14), (16), (17), (19) and (21)÷(23) derived here generalizes considerably the previous incremental formulations [7,15,1] which were valid for small strains, linear elastic behaviour and restricted class of external loads or boundary conditions.

LAGRANGIAN BUCKLING SHELL EQUATIONS

Buckling shell equations are usually derived through linearization of the boundary value problem about an equilibrium state of the shell [23].

Let $\mathbf{u}$ be an equilibrium state whose stability properties are analysed. Since at $\mathbf{u}$ we have $G[\mathbf{u};\delta\mathbf{u}] = 0$, according to (3), linearization of G at $\mathbf{u}$ in a kinematically admissible direction $\Delta\mathbf{u}$ (note that now $\Delta\mathbf{u}$ has different meaning from $\Delta\mathbf{u} \equiv \Delta\mathbf{u}_m^{(i+1)}$ used in the previous Section) leads according to (11) to the functional equation $\Delta G[\mathbf{u};\delta\mathbf{u},\Delta\mathbf{u}] = 0$. Here ΔG can be explicitly calculated from (6), and the calculation procedure is exactly the same as the one performed in the previous Section, where ΔG has been calculated at an approximation $\mathbf{u}_m^{(i)}$ to an equilibrium $\mathbf{u}_m$ in a kinematically admissible direction $\Delta\mathbf{u}_m^{(i+1)}$. Therefore, it is apparent that now ΔG at $\mathbf{u}$ in the direction $\Delta\mathbf{u}$ takes formally exactly the form (20) with $M_R \equiv 0$. From vanishing of ΔG for any $\delta\mathbf{u}$ we immediately arrive at the following explicit form of Lagrangian buckling equations for thin shells

$$
\begin{aligned}
&(\Delta\mathbf{T}^{\beta} + \mathbf{S}^{\beta})_{|\beta} + \Delta\mathbf{p} + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]_{|\beta} = \mathbf{0} \qquad \text{in } M\ ,\\
&\left.\begin{aligned}
&(\Delta\mathbf{T}^{\beta} + \mathbf{S}^{\beta})\,\nu_{\beta} + (\Delta\mathbf{F} + \mathbf{C})' = \Delta\mathbf{T} + (\Delta\mathbf{F}^{*} + \mathbf{C}^{*})' -\\
&- [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\,\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]\nu_{\beta}\\
&\Delta M + K = \Delta M^{*} + K^{*}
\end{aligned}\right\} \text{on } C_f\ , \qquad (28)\\
&\Delta\mathbf{F}_n + \mathbf{C}_n = \Delta\mathbf{F}_n^{*} + \mathbf{C}_n^{*} \qquad \text{at each corner } M_n \in C_f\ ,\\
&\Delta\mathbf{u} = \mathbf{0}\ , \qquad \Delta n_{\nu} = 0 \qquad \text{on } C_u\ ,
\end{aligned}
$$

where all the quantities are defined through $\mathbf{u}$ and $\Delta\mathbf{u}$ by exactly the same formulae as analogously denoted respective quantities of the previous Section have been defined through $\mathbf{u}_m^{(i)} \equiv \mathbf{u}$ and $\Delta\mathbf{u}_m^{(i+1)} \equiv \Delta\mathbf{u}$. All the vectors in (28) are understood again to be expressed through components with respect to known bases $\mathbf{a}_\alpha, \mathbf{n}$ and $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ of M and C, respectively.

The explicit Lagrangian buckling shell equations (28) extend to the large-strain range of deformation and arbitrary loading the stability equations derived within small-strain theory of elastic shells by Stumpf [23] in operator form and by Nolte [9] in explicit form.

REMARKS ON COMPUTER IMPLEMENTATION

The unified formulation of the non-linear displacement 3-field theory of shells presented here is necessarily quite complex because of its generality and versatility. Note that $\kappa_{\alpha\beta}$ appearing in the underlying principle of virtual displacements (3) is expressible according to $(1)_3$ in terms of $\mathbf{u}_{,\alpha}$ and $\mathbf{u}_{,\alpha\beta}$. As a result, in any consistent finite element approximation of the resulting displacement boundary value problem C^1 interelement continuity is required. Additionally, in order to allow numerical analysis for various material laws, the element geometry and kinematics should be decoupled from the constitutive equations.

These requirements were fulfilled by Harte [4] in the case of thin shells made of linearly-elastic material undergoing small strains and moderate rotations and extended by Schieck [21] to the case of shells made of rubber-like incompressible elastic material undergoing large strains and unrestricted rotations. In those papers a triangular high-precision doubly-curved shell finite element with 54 degrees of freedom proposed already by Cowper [3] is selected. In the element of [21] biquintic polynomials are applied as shape functions for all three displacement components. As 18 degrees of freedom at each node the quantities u_α, w; $u_{\alpha,\beta}, w_{,\beta}$; $u_{\alpha,\beta\gamma}, w_{,\beta\gamma}$ are used, and the Gauss integration is performed in 21 points. The geometry of the element is calculated exactly from the given shell geometry. The shape functions are then condensed in such a way that C^1 interelement continuity of all displacement components is assured. The element is capable to represent only approximately the constant strain modes and the strain-free rigid-body modes. Test show, however, that the approximation error

quickly approaches zero with the mesh refinement. The strain-displacement relations and material laws are disconnected from the finite element kinematics. Therefore, they can easily be changed, if necessary.

The C^1 shell element described above was used with the MESY 3 computer code of structural analysis, and several test examples were run on CDC Cyber 205 vector computer applying special algorithms and programming techniques. In particular, the number of DOF of the element was combined with the number of integration points, what led to the vector length $54 \times 21 = 1134$ and allowed to increase the computation speed by the factor 23 over non-vectorized algorithms. The concise description of the vectorized algorithms developed is given by Nolte and Schieck [10,11] while the modified vectorized subroutine for calculation of the element tangent stiffness matrix and the residual force vector is described in Schieck [21].

With the help of the C^1 triangular shell finite element described above, several numerical results for highly non-linear one- and two-dimensional problems of elastic shells were presented in [21,22]. The application of the element to problems of elasto-plastic shells undergoing large strains is under development. The use of other material laws, and application of other C^1 shell finite elements, within the proposed unified displacement formulation of the non-linear theory of thin shells will hopefully be the subject of research in the future.

The C^1 continuity requirement, and associated complexity of the finite elements, is considered to be a disadvantage of the displacement formulation of thin shell theory as compared with a more complex 6-field theory of shear-deformable shells [14]. In the latter one both displacements and rotations are the independent field variables, and the finite element approximation expressed in terms of those variables requires only C^0 interelement continuity [2]. However, while the 3-field shell theory has been presented here in supposedly ultimate formulation, the 6-field shell theory is still under development and several important questions of the theory itself and its FE approximation are still under discussion. C^0 shell finite elements bring themselves some problems (locking effects, spurious zero-energy modes etc.) which are still waiting for commonly accepted satisfactory solutions. The complexity of 3-field C^1 finite elements may become less important already in the near future when powerful parallel processors of the next generation are installed into computers of PC

class, and then much better approximation quality of C^1 finite elements may become a decicive advantage. One can also expect that 6-field shell models will be absorbed in the future by unified computer codes of 3D analysis of structures, while the special structure of boundary conditions required by the 3-field shell theory does not allow it to be degenerated from the structure of 3D theory. Therefore, one might anticipate that this theory would remain as the only "shell" theory also in the future structural analysis. These are some arguments why the derivation of 2D constitutive laws for various material behaviour, and development of computer codes based on C^1 shell finite elements, seems to be of some importance also for future analysis of shell structures.

CONCLUSIONS

In this report a unified formulation of a wide class of non-linear theories of thin shells has been presented. The analysis has been based on only one assumption: the deformation of the shell as a three-dimensional body is determined entirely by deformation of its reference surface. Basic shell equations, in the global (8), (9) and consistent incremental (24)÷(26) forms, have been explicitly derived in the Lagrangian description in terms of displacements of the reference surface as the only independent field variables. The most general explicit form of Lagrangian buckling shell equations (28) have also been derived. Particular attention has been paid to consistency of work-conjugate boundary conditions, and to precise evaluation of unballanced forces when successive approximations to an equilibrium state do not follow the equilibrium path.

Our formulation of shell equations is valid for an arbitrary geometry of the shell reference surface, for unrestricted displacements, rotations, strains and/or changes of curvatures of the reference surface, for arbitrary configuration-dependent external surface and boundary loadings, and for arbitrary set of four work-conjugate static and geometric boundary conditions. Therefore, our formulation contains many specialized versions of non-linear shell equations available in the literature.

We have explicitly applied here the constitutive equations of elastic shells (10), (27), since for such material bahaviour effective computer FEM programs were developed, and several one- and two-dimensional non-linear problems of shells within small-strain [4,9,12] and large-strain [21,22] range of deformation were

analysed. However, our formulation of the non-linear shell theory is applicable to some problems of inelastic shells as well, provided corresponding incremental constitutive equations for the surface stress measures are available.

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