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EXPLICIT LAGRANGIAN INCREMENTAL AND BUCKLING EQUATIONS FOR THE NON-LINEAR THEORY OF THIN SHELLS

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Abstract—Non-linear problems of thin elastic shells may conveniently be formulated in the Lagrangian description in terms of three displacement components of the reference surface. Assuming that an approximation to an equilibrium state of the shell is known, the explicit form of incremental equations for the correcting increment of displacements, which allows one to calculate the next approximation, is derived for arbitrary configuration-dependent external static loads, and for arbitrary work-conjugate boundary conditions. The derivation is performed applying the Newton–Kantorovich method which is based essentially on successive approximations to the exact solution of some linearized shell problem. As a special case, the ultimate Lagrangian buckling equations for thin shells are constructed.

1. INTRODUCTION

The Lagrangian non-linear bending theory of thin shells was developed by Pietraszkiewicz and Szwabowicz [1] and Pietraszkiewicz [2, 3]. Several equivalent-to- [2, 3] versions of non-linear shell equations were discussed by Schmidt [4], Iura and Hirashima [5], Stumpf [6], Basar and Krätzig [7], and with modified definitions of the tensor of change of curvature, also in several other papers [8, 9]. In [1–3] it is assumed from the outset that the deformation of a thin shell as a three-dimensional solid is determined entirely by deformation of its reference, usually the middle, surface. The resulting non-linear equilibrium equations and work-conjugate boundary conditions, following from the two-dimensional principle of virtual displacements, are referred to the known geometry of the undeformed reference surface. They are two-dimensionally exact for the reference surface, since no further restrictions are imposed on magnitudes of the surface displacements, rotations, strains and/or changes of curvatures. When appropriate constitutive equations of elastic shells are used, the resulting boundary value problem is formulated entirely in terms of three components of displacement vector **u** of the reference surface as the only independent field variables.

For many years, the Lagrangian displacement shell equations, or their simplified versions, have been used to solve geometrically non-linear problems of thin elastic shells undergoing small strains (e.g. [10–13]) primarily because the simple constitutive equations of such shell theory, to within the consistent first approximation, were well established in the literature [9, 14–16]. As a result, several attempts to derive various Lagrangian equations for superposition of deformations within small-strain elastic shell problems were reported.

In the pioneering paper by Galimov [17] the Lagrangian equilibrium equations and static boundary conditions for superposed deformations were derived in terms of non-symmetric surface stress measures, while Pietraszkiewicz [18] derived the perturbed Lagrangian shell equations in terms of symmetric surface stress measures. However, the effective force and couple resultants used in [17, 18] were not work-conjugate to the geometric parameters of boundary deformation (see also ref. [39]). In the Lagrangian incremental equations derived in [2, 19] the increments associated with boundary rotation were approximated by linear terms and the perturbed equations of Budiansky [20] were given for the membrane fundamental state. In the ones derived by Stein *et al.* [21] there

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were still six incremental boundary conditions, while in the variational formulation of incremental shell equations proposed by Başar and Ding [13] some integrals associated with the increments of external loads were omitted. Lagrangian stability equations for the version of shell theory based on a modified tensor of change of curvature were derived by Nolte [10, 22], Schmidt and Stumpf [23] and Stumpf [24], while the ones based on the shell theory of [2] were given by Stumpf [6]. The final stability equations of [6, 22, 24] were presented in an operator form, where the operators had still to be calculated as Gateaux derivatives of some other operators associated with the fundamental equilibrium state. There are also other forms of incremental or buckling shell equations available in the literature which are referred either to the basis of deformed shell [2, 25–28] or to the intermediate rotated basis [29–32], but we do not discuss such forms here.

Irrespective of the names used for the resulting Lagrangian shell equations in the papers referred to above (incremental, superposed, perturbed, stability, buckling, etc.), the equations were usually derived as some kind of linearization of the field equations about an equilibrium state of the shell. However, when deriving incremental shell equations for the current iteration step, the successive approximations to the unknown equilibrium state may not belong to the equilibrium path. As a result, some unbalanced force vector should always appear explicitly in correct incremental shell equations. Note that only papers [2, 13, 19] introduce explicitly unbalanced force vectors, although some self-correcting numerical procedures certainly had to be used implicitly in [10, 12] in order to obtain the correct final results.

In a recent report by Schieck *et al.* [33] we constructed the consistently simplified two-dimensional strain energy function and the corresponding constitutive equations for shells made of incompressible rubber-like materials undergoing large elastic strains, which generalized and put on a more rational basis earlier results given in [34–37]. This opened the possibility to apply the Lagrangian shell equations to large-strain problems of elastic shells as well. We also proposed in [33] some incremental procedures which allowed us to perform numerical analysis of highly non-linear problems of rubber-like plates and shells using the finite element method. However, the derivation of the explicit Lagrangian incremental shell equations was not included in [33], because of the limited volume of that paper, and because we analysed test problems of rubber-like plates and shells with clamped boundaries only. But still open theoretical and numerical problems remain, however, when non-linear problems of shells with arbitrary boundary conditions are discussed, particularly problems with unrestricted rotational deformation of the shell boundary. In our opinion, the correct derivation of explicit incremental Lagrangian equations for the most general case of shell deformation, external loadings and boundary conditions has enough novelty for itself in order to be published separately, independently of their various possible computer implementations and various possible particular two-dimensional forms of constitutive equations applied.

In this report we present a detailed and explicit derivation of the incremental equations which allows one to analyse the highly non-linear shell problems on the basis of the Lagrangian non-linear bending theory of thin elastic shells as developed in [2, 33]. As a special case, we also construct the explicit form of corresponding buckling shell equations.

After recalling in Sections 2 and 3 some of the geometric formulae and basic relations of the Lagrangian non-linear shell theory, we discuss briefly in Section 4 the general scheme for analysing parametrized non-linear operator equations with the help of the Newton–Kantorovich method. Assuming that an approximation to an equilibrium state of the shell is known, we present as equation (12) the consistently linearized principle of virtual displacements at this approximation, which itself may not be an equilibrium configuration. In Section 5 the directional Gateaux derivatives of various fields, defined inside the reference surface and on its boundary, are explicitly calculated and presented in a readable vector form. This allows one to transform all integrals in the linearized functional of the principle of virtual displacements and present it in the final form (35), from which follow immediately the explicit incremental equations (36) and (37) for the Lagrangian non-linear theory of shells. Lagrangian buckling shell equations (39) and (40) are then derived from the incremental shell equations as their particular case, which is discussed in Section 6.

2. NOTATION AND GEOMETRIC RELATIONS

In this report we apply the system of notation used by Pietraszkiewicz [2, 9].

Let $\mathbf{r}(\theta^\alpha)$ be the position vector of the reference surface $\mathscr{M}$ of undeformed shell, where θ^α, $\alpha = 1, 2$, are surface curvilinear coordinates. With each point $M \in \mathscr{M}$ we associate the natural base vectors $\mathbf{a}_\alpha = \partial \mathbf{r}/\partial \theta^\alpha \equiv \mathbf{r}_{,\alpha}$, the covariant (components of the surface) metric tensor $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ with determinant $a = |a_{\alpha\beta}|$, the unit normal vector $\mathbf{n} = a^{-1/2}\mathbf{a}_1 \times \mathbf{a}_2$, the curvature tensor $b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n}_{,\beta}$ and the permutation tensor $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta)\cdot \mathbf{n}$. The contravariant base vectors $\mathbf{a}^\alpha$ are defined through $\mathbf{a}^\alpha \cdot \mathbf{a}_\beta = \delta^\alpha_\beta$, and the contravariant metric tensor $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ is used to raise indices of tensor components on $\mathscr{M}$.

The boundary contour $\mathscr{C}$ of $\mathscr{M}$ is assumed to consist of the finite set of piecewise smooth curves defined by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along $\mathscr{C}$. With each regular point $M \in \mathscr{C}$ we associate the unit tangent vector $\mathbf{t} = d\mathbf{r}/ds \equiv \mathbf{r}' = t^\alpha \mathbf{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r}_{,\nu} = \nu^\alpha \mathbf{a}_\alpha = \mathbf{t} \times \mathbf{n}$, $\nu^\alpha = \varepsilon^{\alpha\beta} t_\beta$, where $(\ )_{,\nu}$ denotes the outward normal derivative at $\mathscr{C}$.

Let $\bar{\mathscr{M}}$ be the deformed configuration of $\mathscr{M}$ defined by $\bar{\mathbf{r}}(\theta^\alpha) = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha)$, where $\mathbf{u}: \mathscr{M} \to E^3$ is the displacement field and θ^α are the same surface convected coordinates. With each point $\bar{M} \in \bar{\mathscr{M}}$ we now associate geometric quantities which are defined analogously to those defined on $\mathscr{M}$, only now marked by an overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\mathbf{a}}^\alpha$, $\bar{a}^{\alpha\beta}$, $\bar{\mathbf{t}}$, $\bar{\boldsymbol{\nu}}$, etc. The quantities defined on $\bar{\mathscr{M}}$ and the deformation of $\mathscr{M}$ are given by the relations [2, 9]

$$
\begin{aligned}
&\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha} = \mathbf{a}_\alpha + \mathbf{u}_{,\alpha}, \qquad \bar{\mathbf{n}} = \tfrac{1}{2} j^{-1} \varepsilon^{\alpha\beta} \bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta,\\
&\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \qquad \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}),\\
&j^2 = \frac{\bar{a}}{a} = \frac{1}{2}\varepsilon^{\alpha\lambda}\varepsilon^{\beta\kappa}\bar{a}_{\alpha\beta}\bar{a}_{\lambda\kappa},\\
&\bar{\mathbf{a}}^\alpha = \bar{a}^{\alpha\beta}\bar{\mathbf{a}}_\beta, \quad \bar{a}^{\alpha\beta} = j^{-2}[(1 + 2\gamma^\kappa_\kappa)a^{\alpha\beta} - 2\gamma^{\alpha\beta}],
\end{aligned} \tag{1}
$$

where $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are the Lagrangian surface strain measures.

Along the deformed shell boundary contour $\bar{\mathscr{C}}$ we have the following relationships [2, 9]:

$$
\begin{aligned}
\bar{\mathbf{r}}' &= \bar{\mathbf{a}}_\alpha t^\alpha = \mathbf{t} + \mathbf{u}' = \bar{a}_t \bar{\mathbf{t}}, \quad \bar{a}_t = |\bar{\mathbf{r}}'|,\\
\bar{\mathbf{n}} &= j^{-1}\bar{\mathbf{r}}_{,\nu} \times \bar{\mathbf{r}}', \qquad \bar{\mathbf{r}}' \times \bar{\mathbf{n}} = \bar{a}_t \bar{\boldsymbol{\nu}},\\
j^2 &= |\bar{\mathbf{r}}_{,\nu}|^2 |\bar{\mathbf{r}}'|^2 - (\bar{\mathbf{r}}_{,\nu}\cdot\bar{\mathbf{r}}')^2,\\
2\gamma_{\nu t} &= 2\gamma_{\alpha\beta}\nu^\alpha t^\beta = \bar{\mathbf{r}}_{,\nu}\cdot\bar{\mathbf{r}}',\\
\bar{\mathbf{a}}^\beta &= j^{-1}(\bar{a}_t \nu^\beta - 2\gamma_{\nu t}\bar{a}_t^{-1} t^\beta)\bar{\boldsymbol{\nu}} + \bar{a}_t^{-1} t^\beta \bar{\mathbf{t}}.
\end{aligned} \tag{2}
$$

It is apparent from (1) and (2) that the geometry of $\bar{\mathscr{M}}$ is entirely determined by the geometry of $\mathscr{M}$ and $\mathbf{u}_{,\alpha}$, while the geometry of $\bar{\mathscr{C}}$ is entirely determined by the one of $\mathscr{C}$ and $\mathbf{u}_{,\nu}$, $\mathbf{u}'$.

3. LAGRANGIAN NON-LINEAR DISPLACEMENT SHELL EQUATIONS

In order to make the paper self-contained, let us recall here some basic results on Lagrangian non-linear shell equations [2].

Let $\bar{\mathscr{M}}$ be the reference surface of a deformed shell in an equilibrium state, under the configuration-dependent static surface force $\mathbf{p}$ and the surface moment $\mathbf{h}$, both defined per unit area of undeformed surface $\mathscr{M}$, as well as under boundary force $\mathbf{T}$ and the boundary moment $\mathbf{H}$, both defined per unit length of undeformed boundary $\mathscr{C}$. Then for an additional kinematically admissible virtual displacement field $\delta\mathbf{u}: \mathscr{M} \to E^3$ the internal virtual work, performed by the internal second Piola–Kirchhoff-type stress and couple resultants $N^{\alpha\beta}$ and $M^{\alpha\beta}$ on respective variations of the strain measures $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$, should be equal to the external virtual work, performed by the external surface and boundary loads on respective variations of displacemental parameters or, equivalently,

$$G[\mathbf{u};\delta\mathbf{u}] = G_i[\mathbf{u};\delta\mathbf{u}] - G_e[\mathbf{u};\delta\mathbf{u}] = 0,$$

$$G_i[\mathbf{u};\delta\mathbf{u}] = \iint_{\mathcal{M}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})\,\mathrm{d}A,$$

$$G_e[\mathbf{u};\delta\mathbf{u}] = \iint_{\mathcal{M}} (\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{h}\cdot\delta\bar{\mathbf{n}})\,\mathrm{d}A + \int_{\mathscr{C}_f} (\mathbf{T}\cdot\delta\mathbf{u} + \mathbf{H}\cdot\delta\bar{\mathbf{n}})\,\mathrm{d}s. \tag{3}$$

Here $\mathscr{C}_f$ denotes that part of the shell boundary $\mathscr{C}$ where static boundary conditions are prescribed, and all the functionals G depend non-linearly on $\mathbf{u}$ but are linear in $\delta\mathbf{u}$.

Variations of the surface strain measures appearing in $(3)_2$ are given through $\mathbf{u}$ and $\delta\mathbf{u}$ by

$$\delta\gamma_{\alpha\beta} = \tfrac{1}{2}(\delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}_\beta + \bar{\mathbf{a}}_\alpha\cdot\delta\mathbf{u}_{,\beta}),$$

$$\delta\kappa_{\alpha\beta} = \tfrac{1}{2}(\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \bar{\mathbf{n}}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha} + \bar{\mathbf{a}}_\alpha\cdot\delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_\beta\cdot\delta\bar{\mathbf{n}}_{,\alpha}),$$

$$\delta\bar{\mathbf{n}} = -\,\bar{\mathbf{a}}^\beta(\bar{\mathbf{n}}\cdot\delta\mathbf{u}_{,\beta}). \tag{4}$$

At the boundary contour $\mathscr{C}$ the vector $\bar{\mathbf{n}} = \bar{\mathbf{n}}(s)$ should satisfy the constraints $\bar{\mathbf{r}}'\cdot\bar{\mathbf{n}} = 0$ and $\bar{\mathbf{n}}\cdot\bar{\mathbf{n}} = 1$. As a result, three components of translation $\mathbf{u}(s)$ and one scalar function $\varphi(s) = \varphi[\mathbf{u}_{,\nu}(s), \mathbf{u}'(s)]$ describing the rotational deformation are necessary and sufficient in order to describe the deformation of the shell boundary uniquely.

The general structure of the function φ and associated with φ work-conjugate boundary conditions compatible with (3) were discussed by Makowski and Pietraszkiewicz [39]. Three physically reasonable particular cases of φ are known in the literature: (1) $n_\nu = \bar{\mathbf{n}}\cdot\boldsymbol{\nu} = j^{-1}(\mathbf{u}'\times\boldsymbol{\nu} - \mathbf{n})\cdot\mathbf{u}_{,\nu}$ introduced in [1]; (2) $\vartheta_\nu = \bar{a}_t^{-2}(\bar{\mathbf{n}} - \mathbf{n})\cdot(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})$ introduced in [38]; and (3) ω_t, the angle of total rotation of the boundary, defined in [41] through displacements by $2\cos\omega_t + 1 = \bar{\boldsymbol{\nu}}\cdot\boldsymbol{\nu} + \bar{\mathbf{t}}\cdot\mathbf{t} + \bar{\mathbf{n}}\cdot\mathbf{n}$. Since almost all general results available in the literature have been obtained using the function n_ν, in this report we also apply n_ν, in terms of which the variation of $\bar{\mathbf{n}}$ of $\bar{\mathscr{C}}$ takes the form

$$\delta\bar{\mathbf{n}} = a_\nu^{-1}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\delta\mathbf{u}' + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\delta n_\nu],$$

$$a_\nu = (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\cdot\boldsymbol{\nu}, \tag{5}$$

where $\delta n_\nu = \delta n_\nu[\mathbf{u}_{,\nu}, \mathbf{u}'; \delta\mathbf{u}_{,\nu}, \delta\mathbf{u}']$ is non-linear in $\mathbf{u}_{,\nu}, \mathbf{u}'$ but is linear in $\delta\mathbf{u}_{,\nu}, \delta\mathbf{u}'$.

Introducing (4) and (5) into (3), applying Stokes' theorem to the surface integral and then applying integration by parts to the line integral, the principle of virtual displacements (3) can be transformed into

$$G[\mathbf{u};\delta\mathbf{u}] = -\iint_{\mathcal{M}} \{\mathbf{T}^\beta|_\beta + \mathbf{p} + [(\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}]|_\beta\}\cdot\delta\mathbf{u}\,\mathrm{d}A$$

$$+ \int_{\mathscr{C}_f} [(\mathbf{P} - \mathbf{P}^*)\cdot\delta\mathbf{u} + (M - M^*)\delta n_\nu]\,\mathrm{d}s + \sum_n (\mathbf{F}_n - \mathbf{F}_n^*)\cdot\delta\mathbf{u}_n = 0, \tag{6}$$

where

$$\mathbf{T}^\beta = N^{\alpha\beta}\bar{\mathbf{a}}_\alpha + M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha} + \{[(M^{\kappa\rho}\bar{\mathbf{a}}_\kappa)|_\rho]\cdot\bar{\mathbf{a}}^\beta\}\bar{\mathbf{n}},$$

$$\mathbf{P} = \mathbf{T}^\beta\nu_\beta + \mathbf{F}', \qquad \mathbf{P}^* = \mathbf{T} - (\mathbf{h}\cdot\bar{\mathbf{a}}^\beta\nu_\beta)\bar{\mathbf{n}} + \mathbf{F}^{*\prime},$$

$$\mathbf{F} = -\,a_\nu^{-1}[(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\boldsymbol{\nu}]M^{\alpha\beta}\nu_\beta\bar{\mathbf{n}}, \qquad \mathbf{F}^* = -\,a_\nu^{-1}[(\bar{\mathbf{n}}\times\mathbf{H})\cdot\boldsymbol{\nu}]\bar{\mathbf{n}},$$

$$M = a_\nu^{-1}(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\alpha)\cdot\bar{\mathbf{r}}'M^{\alpha\beta}\nu_\beta, \qquad M^* = a_\nu^{-1}(\bar{\mathbf{n}}\times\mathbf{H})\cdot\bar{\mathbf{r}}',$$

$$\mathbf{F}_n = \mathbf{F}(s_n + 0) - \mathbf{F}(s_n - 0), \qquad \mathbf{u}_n = \mathbf{u}(s_n). \tag{7}$$

Alternative to $(7)_{3,4}$ definitions associated with ϑ_ν and ω_t as the fourth parameters of boundary deformation are given in [39, 40], respectively.

Since (6) should be satisfied identically for all kinematically admissible $\delta\mathbf{u}$, from (6) follow the Lagrangian equilibrium equations and static boundary conditions

$$\mathbf{T}^\beta|_\beta + \mathbf{p} + [(\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}]|_\beta = \mathbf{0} \quad \text{in } \mathcal{M},$$

$$\mathbf{P}(s) = \mathbf{P}^*(s), \qquad M(s) = M^*(s) \quad \text{on } \mathscr{C}_f,$$
$$\mathbf{F}_n = \mathbf{F}_n^* \quad \text{at each corner } M_n \in \mathscr{C}_f. \tag{8}$$

Corresponding geometric boundary conditions, which are work-conjugate to the static ones $(8)_{2,3}$, have the form

$$\mathbf{u}(s) = \mathbf{u}^*(s), \quad n_v(s) = n_v^*(s) \qquad \text{on } \mathscr{C}_u, \tag{9}$$

In the case of an elastic material, to which we confine ourselves in this report, the constitutive equations compatible with (3) are

$$N^{\alpha\beta} = \frac{\partial \Sigma}{\partial \gamma_{\alpha\beta}}, \qquad M^{\alpha\beta} = \frac{\partial \Sigma}{\partial \kappa_{\alpha\beta}}, \tag{10}$$

where $\Sigma = \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ is the two-dimensional strain energy function of the shell.

Let us recall that the function Σ is usually constructed as some kind of approximation to the corresponding three-dimensional elastic strain energy function of the shell, and is approximate virtually by definition. Its explicit form depends on the mechanical properties of the material the shell is made of. In the particular case of an isotropic elastic material and small-strain (but unrestricted rotation) shell theory the Σ is, to the first approximation, a quadratic function of the strain measures $\gamma_{\alpha\beta}$, $\kappa_{\alpha\beta}$, cf. [9, 14–16, 41]. In the case of large-strain theory of shells made of isotropic elastic incompressible rubber-like materials the consistently simplified structure of Σ, within the consistent first approximation and the simplest approximation, and corresponding constitutive equations of the type (10) are given in [33]. Still other forms of Σ compatible with (3) were proposed in [34–37, 42, 43] for large-strain problems of elastic shells.

In each particular case of $\Sigma = \Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$ the constitutive equations are of the form $N^{\alpha\beta} = N^{\alpha\beta}(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$, $M^{\alpha\beta} = M^{\alpha\beta}(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$, where $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\mathbf{u})$, $\kappa_{\alpha\beta} = \kappa_{\alpha\beta}(\mathbf{u})$ are known functions of $\mathbf{u}$ following from (1). Since all other geometric quantities in (7) associated with $\bar{\mathscr{M}}$ and $\bar{\mathscr{C}}$ are understood to be expressed through $\mathbf{u}$ as well, the boundary value problem (8) and (9) or its weak formulation (3) of the Lagrangian non-linear theory of thin elastic shells is expressed entirely in terms of displacements $\mathbf{u}$ as the only independent field variables.

4. SOLUTION OF PARAMETRIZED NON-LINEAR OPERATOR EQUATIONS

The highly non-linear boundary value problem (8) and (9) can be effectively solved only by numerical methods, provided the correct incremental-iterative procedure in the total Lagrangian description is developed.

It has been pointed out in Section 1 that many incremental forms of Lagrangian shell equations available in the literature do not take into account that successive approximations to the unknown equilibrium state follow a path which may not be the equilibrium path. Therefore, in what follows, we briefly recall some basic facts about the incremental-iterative procedures which are used for solving highly non-linear problems; cf. [44, 45].

In general, the configuration-dependent external loads applied to the shell may be specified by several independent dimensionless parameters $(\lambda_1, \lambda_2, \ldots, \lambda_p) = \lambda \in \Lambda \subset R^p$, and the boundary value problem (8) and (9) can be presented in the form

$$F(\mathbf{u}, \lambda) = 0, \tag{11}$$

where the non-linear continuously differentiable operator F is defined on the product space $C(\mathscr{M}, E^3) \times R^p$ with values in the Banach space, and $C(\mathscr{M}, E^3)$ is a set of all configurations of the shell reference surface in the three-dimensional Euclidean space, i.e. infinite-dimensional configuration space. Under rather general conditions (see Chapter 4 of [44]) the solution manifold consisting of all points $(\mathbf{u}, \lambda)$ satisfying (11) has the structure of a p-dimensional differential manifold in $C(\mathscr{M}, E^3)$. However, any information concerning the principal features of this manifold can be obtained by analysing the set of one-dimensional submanifolds corresponding to the smoothly varying single parameters. Therefore, we also restrict our further discussion to the case when the external loads are specified by a single parameter $\lambda \in \Lambda \subset R$.

For smoothly varying λ the regular solutions of (11) or (8) and (9) form a one-dimensional submanifold $\mathbf{u}(\lambda)$ in $C(\mathcal{M}, E^3)$ which is usually called the equilibrium path. According to the principle of virtual displacements (3), $\mathbf{u}(\lambda)$ is a weak solution of the boundary value problem (8) and (9) if $G[\mathbf{u}(\lambda); \delta\mathbf{u}] = 0$ for all kinematically admissible virtual displacements $\delta\mathbf{u}$. For tracing the equilibrium path $\mathbf{u}(\lambda)$ it is convenient to apply the Newton–Kantorovich method (cf. [45]), which essentially is based on successive approximationsto the exact solution of some linearized problem.

Let the equilibrium path $\mathbf{u}(\lambda)$ be divided into the finite number of equilibrium states corresponding to some values $\lambda_0, \lambda_1, \ldots, \lambda_m, \lambda_{m+1}, \ldots, \lambda_M \in \Lambda$ of the load parameter λ. Let $\mathbf{u}_m^{(i)}$ denote the known ith approximation to the equilibrium state $\mathbf{u}_m = \mathbf{u}(\lambda_m)$. It is apparent that, in general, the approximation $\mathbf{u}_m^{(i)}$ may not belong to the equilibrium path $\mathbf{u}(\lambda)$. In order to calculate the correction $\Delta\mathbf{u}_m^{(i+1)}$ which would allow us to determine the next approximation $\mathbf{u}_m^{(i+1)} = \mathbf{u}_m^{(i)} + \Delta\mathbf{u}_m^{(i+1)}$ to $\mathbf{u}_m$, it is convenient to linearize $G[\mathbf{u}; \delta\mathbf{u}]$ at the approximation $\mathbf{u}_m^{(i)}$, what leads to the following equation for $\Delta\mathbf{u}_m^{(i+1)}$ (cf. [45, Chapter 3]):

$$G[\mathbf{u}_m^{(i)}; \delta\mathbf{u}] + \Delta G[\mathbf{u}_m^{(i)}; \Delta\mathbf{u}_m^{(i+1)}, \delta\mathbf{u}] = 0. \tag{12}$$

The first term of (12) denotes the value of the functional $(3)_1$ evaluated at the approximation $\mathbf{u}_m^{(i)}$. Since $\mathbf{u}_m^{(i)}$ may not belong to the equilibrium path, this term does not vanish, in general, and allows one to calculate the unbalanced force vector. The second term of (12) denotes the directional Gateaux derivative of the functional $(3)_1$ taken at the approximation $\mathbf{u}_m^{(i)}$ in the kinematically admissible direction $\Delta\mathbf{u}_m^{(i+1)}$. This term is linear in the unknown $\Delta\mathbf{u}_m^{(i+1)}$ and allows one to calculate the tangent stiffness matrix at $\mathbf{u}_m^{(i)}$ of the non-linear shell problem.

5. INCREMENTAL LAGRANGIAN SHELL EQUATIONS

In order to simplify the notation, in this Section 5 we set $\mathbf{u}_m^{(i)} \equiv \mathbf{u}$ and $\Delta\mathbf{u}_m^{(i+1)} \equiv \Delta\mathbf{u}$, while the values at $\mathbf{u}_m^{(i)}$ of the corresponding external surface forces and moments as well as the boundary forces and moments approximating $\mathbf{p}_m = \mathbf{p}(\lambda_m)$, $\mathbf{h}_m = \mathbf{h}(\lambda_m)$, $\mathbf{T}_m = \mathbf{T}(\lambda_m)$ and $\mathbf{H}_m = \mathbf{H}(\lambda_m)$ we denote in short by $\mathbf{p}, \mathbf{h}, \mathbf{T}$ and $\mathbf{H}$, respectively.

Let us consider a curve $\mathbf{u}(\eta)$ through the ith approximation $\mathbf{u}$ to $\mathbf{u}_m$ such that in the neighbourhood of $\mathbf{u}$

$$\mathbf{u}(\eta) = \mathbf{u} + \eta\Delta\mathbf{u}. \tag{13}$$

The directional Gateaux derivative of $G[\mathbf{u}; \delta\mathbf{u}]$ taken at the ith approximation $\mathbf{u}$ in the kinematically admissible direction $\Delta\mathbf{u}$ is given by

$$\Delta G[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = \frac{\mathrm{d}}{\mathrm{d}\eta} G[\mathbf{u}(\eta); \delta\mathbf{u}]|_{\eta=0}, \tag{14}$$

where $G[\mathbf{u}(\eta); \delta\mathbf{u}]$ is defined analogously as the functional $G[\mathbf{u}; \delta\mathbf{u}]$ in $(3)_1$, only now $\mathbf{u}(\eta)$ appears in place of $\mathbf{u}$.

The components of the Lagrangian surface stress and strain measures associated with the curve $\mathbf{u}(\eta)$ are denoted by $N^{\alpha\beta}(\eta)$, $M^{\alpha\beta}(\eta)$, $\gamma_{\alpha\beta}(\eta)$ and $\kappa_{\alpha\beta}(\eta)$, while the corresponding external surface loads by $\mathbf{p}(\eta)$, $\mathbf{h}(\eta)$, $\mathbf{T}(\eta)$ and $\mathbf{H}(\eta)$. In deriving Gateaux derivatives at $\mathbf{u}$ in the kinematically admissible direction $\Delta\mathbf{u}$ of all the fields appearing in the shell theory, we follow our paper [33].

From (14) and (3) it follows that

$$\Delta G[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = \Delta G_i[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] - \Delta G_e[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}], \tag{15}$$

$$\Delta G_i[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = \iint_{\mathcal{M}} (\Delta\sigma_M + \Delta\sigma_G)\,\mathrm{d}A, \tag{16}$$

$$\begin{aligned}\Delta\sigma_M &= \Delta N^{\alpha\beta}\delta\gamma_{\alpha\beta} + \Delta M^{\alpha\beta}\delta\kappa_{\alpha\beta},\\ \Delta\sigma_G &= N^{\alpha\beta}\Delta(\delta\gamma_{\alpha\beta}) + M^{\alpha\beta}\Delta(\delta\kappa_{\alpha\beta}),\end{aligned} \tag{17}$$

$$\begin{aligned}\Delta G_e[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = &\iint_{\mathcal{M}} [\Delta\mathbf{p}\cdot\delta\mathbf{u} + \Delta\mathbf{h}\cdot\delta\bar{\mathbf{n}} + \mathbf{h}\cdot\Delta(\delta\bar{\mathbf{n}})]\,\mathrm{d}A\\ &+ \int_{\mathscr{C}_f} [\Delta\mathbf{T}\cdot\delta\mathbf{u} + \Delta\mathbf{H}\cdot\delta\bar{\mathbf{n}} + \mathbf{H}\cdot\Delta(\delta\bar{\mathbf{n}})]\,\mathrm{d}s,\end{aligned} \tag{18}$$

where $\Delta\mathbf{p}$, $\Delta\mathbf{h}$, $\Delta\mathbf{T}$ and $\Delta\mathbf{H}$ are the Gateaux derivatives at $\mathbf{u}$ of the respective external loads.

Within the reference surface $\mathcal{M}$ the Gateaux derivatives of various fields are given by

$$\Delta\bar{\mathbf{n}} = -\bar{\mathbf{a}}^{\beta}(\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\beta}), \qquad \Delta(\delta\bar{\mathbf{n}}) = \mathbf{B}^{\beta}\cdot\delta\mathbf{u}_{,\beta}, \tag{19}$$

$$\mathbf{B}^{\beta} = [(\bar{\mathbf{a}}^{\beta}\cdot\Delta\mathbf{u}_{,\kappa})\bar{\mathbf{a}}^{\kappa} - \bar{a}^{\beta\kappa}(\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\kappa})\bar{\mathbf{n}}]\otimes\bar{\mathbf{n}} + (\bar{\mathbf{n}}\cdot\Delta\mathbf{u}_{,\kappa})\bar{\mathbf{a}}^{\beta}\otimes\bar{\mathbf{a}}^{\kappa}, \tag{20}$$

$$\Delta\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{\mathbf{a}}_{\alpha}\cdot\Delta\mathbf{u}_{,\beta} + \bar{\mathbf{a}}_{\beta}\cdot\Delta\mathbf{u}_{,\alpha}),$$

$$\Delta\kappa_{\alpha\beta} = \tfrac{1}{2}(\bar{\mathbf{n}}_{,\alpha}\cdot\Delta\mathbf{u}_{,\beta} + \bar{\mathbf{n}}_{,\beta}\cdot\Delta\mathbf{u}_{,\alpha} + \bar{\mathbf{a}}_{\alpha}\cdot\Delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_{\beta}\cdot\Delta\bar{\mathbf{n}}_{,\alpha}), \tag{21}$$

$$\Delta(\delta\gamma_{\alpha\beta}) = \tfrac{1}{2}(\Delta\mathbf{u}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\mathbf{u}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha}),$$

$$\begin{aligned}\Delta(\delta\kappa_{\alpha\beta}) = \tfrac{1}{2}\{&\Delta\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\bar{\mathbf{n}}_{,\beta}\cdot\delta\mathbf{u}_{,\alpha} + \Delta\mathbf{u}_{,\alpha}\cdot\delta\bar{\mathbf{n}}_{,\beta} + \Delta\mathbf{u}_{,\beta}\cdot\delta\bar{\mathbf{n}}_{,\alpha}\\ &+ \bar{\mathbf{a}}_{\alpha}\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\beta} + \bar{\mathbf{a}}_{\beta}\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\alpha}\},\end{aligned} \tag{22}$$

$$\begin{aligned}\Delta N^{\alpha\beta} &= C_1^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_2^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu},\\ \Delta M^{\alpha\beta} &= C_3^{\alpha\beta\lambda\mu}\Delta\gamma_{\lambda\mu} + C_4^{\alpha\beta\lambda\mu}\Delta\kappa_{\lambda\mu},\end{aligned} \tag{23}$$

where $C_k^{\alpha\beta\lambda\mu}$, $k = 1, 2, 3, 4$, are the tangent elasticities at $\mathbf{u}$, defined as the second partial derivatives of the strain energy function $\Sigma(\gamma_{\alpha\beta}, \kappa_{\alpha\beta})$. For each particular form of Σ the elasticities $C_k^{\alpha\beta\lambda\mu}$ are known functions of $\mathbf{u}$.

At the undeformed boundary contour $\mathscr{C}$ of the reference shell surface the Gateaux derivatives of $\bar{\mathbf{n}}$ and $\delta\bar{\mathbf{n}}$ are

$$\begin{aligned}\Delta\bar{\mathbf{n}} &= a_{\nu}^{-1}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})\bar{\mathbf{n}}\cdot\Delta\mathbf{u}' + (\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\Delta n_{\nu}],\\ \Delta(\delta\bar{\mathbf{n}}) &= \mathbf{A}\cdot\delta\mathbf{u}' + \mathbf{B}\delta n_{\nu} + \mathbf{D}\Delta(\delta n_{\nu}),\end{aligned} \tag{24}$$

$$\begin{aligned}\mathbf{A} &= -a_{\nu}^{-2}[\boldsymbol{\nu}\cdot(\Delta\mathbf{u}'\times\bar{\mathbf{n}} + \bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})](\boldsymbol{\nu}\times\bar{\mathbf{n}})\otimes\bar{\mathbf{n}} + a_{\nu}^{-1}[\boldsymbol{\nu}\times(\Delta\bar{\mathbf{n}}\otimes\bar{\mathbf{n}} + \bar{\mathbf{n}}\otimes\Delta\bar{\mathbf{n}})],\\ \mathbf{B} &= -a_{\nu}^{-2}[\boldsymbol{\nu}\cdot(\Delta\mathbf{u}'\times\bar{\mathbf{n}} + \bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}})](\bar{\mathbf{r}}'\times\bar{\mathbf{n}}) + a_{\nu}^{-1}(\Delta\mathbf{u}'\times\bar{\mathbf{n}} + \bar{\mathbf{r}}'\times\Delta\bar{\mathbf{n}}),\\ \mathbf{D} &= a_{\nu}^{-1}\bar{\mathbf{r}}'\times\bar{\mathbf{n}}.\end{aligned} \tag{25}$$

Now we are in a position to derive the incremental equilibrium equations with corresponding boundary and corner conditions which should be satisfied by the incremental stress measures $\Delta N^{\alpha\beta}[\mathbf{u};\Delta\mathbf{u}]$ and $\Delta M^{\alpha\beta}[\mathbf{u};\Delta\mathbf{u}]$.

Let us introduce (21), (22) and $(19)_1$ into the first term of (16) and apply the Stokes' theorem, what after transformations leads to

$$\iint_{\mathcal{M}}\Delta\sigma_M\,\mathrm{d}A = -\iint_{\mathcal{M}}(\Delta\mathbf{T}^{\beta})|_{\beta}\cdot\delta\mathbf{u}\,\mathrm{d}A + \int_{\mathscr{C}_f}(\Delta\mathbf{P}\cdot\delta\mathbf{u} + \Delta M\delta n_{\nu})\,\mathrm{d}s + \sum_n\Delta\mathbf{F}_n\cdot\delta\mathbf{u}_n, \tag{26}$$

where

$$\begin{aligned}\Delta\mathbf{T}^{\beta} &= \Delta N^{\alpha\beta}\bar{\mathbf{a}}_{\alpha} + \Delta M^{\alpha\beta}\bar{\mathbf{n}}_{,\alpha} + \{[(\Delta M^{\kappa\rho}\bar{\mathbf{a}}_{\kappa})|_{\rho}]\cdot\bar{\mathbf{a}}^{\beta}\}\bar{\mathbf{n}},\\ \Delta\mathbf{P} &= \Delta\mathbf{T}^{\beta}\nu_{\beta} + \Delta\mathbf{F}', \quad \Delta\mathbf{F} = -a_{\nu}^{-1}[(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_{\alpha})\cdot\boldsymbol{\nu}]\Delta M^{\alpha\beta}\nu_{\beta}\bar{\mathbf{n}},\\ \Delta M &= a_{\nu}^{-1}(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_{\alpha})\cdot\bar{\mathbf{r}}'\Delta M^{\alpha\beta}\nu_{\beta},\\ \Delta\mathbf{F}_n &= \Delta\mathbf{F}(s_n + 0) - \Delta\mathbf{F}(s_n - 0).\end{aligned} \tag{27}$$

Upon introducing (22), $(19)_1$ and $(4)_3$ into the second term of (16) and again applying the Stokes' theorem, we obtain

$$\begin{aligned}\iint_{\mathcal{M}}\Delta\sigma_G\mathrm{d}A &= \iint_{\mathcal{M}}(N^{\alpha\beta}\Delta\mathbf{u}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + M^{\alpha\beta}\{\Delta\bar{\mathbf{n}}_{,\alpha}\cdot\delta\mathbf{u}_{,\beta} + \Delta\mathbf{u}_{,\alpha}\cdot\delta\bar{\mathbf{n}}_{,\beta} + \bar{\mathbf{a}}_{\alpha}\cdot[\Delta(\delta\bar{\mathbf{n}})]_{,\beta}\})\mathrm{d}A\\ &= -\iint_{\mathcal{M}}\mathbf{S}^{\beta}|_{\beta}\cdot\delta\mathbf{u}\,\mathrm{d}A + \int_{\mathscr{C}_f}[\mathbf{Q}\cdot\delta\mathbf{u} + K\delta n_{\nu} + M\Delta(\delta n_{\nu})]\,\mathrm{d}s + \sum_n\mathbf{C}_n\cdot\delta\mathbf{u}_n,\end{aligned} \tag{28}$$

where

$$\begin{aligned}\mathbf{Q} &= \mathbf{S}^{\beta}\nu_{\beta} + \mathbf{C}'\\ \mathbf{S}^{\beta} &= N^{\alpha\beta}\Delta\mathbf{u}_{,\alpha} + M^{\alpha\beta}\Delta\bar{\mathbf{n}}_{,\alpha} + [(M^{\kappa\rho}\Delta\mathbf{u}_{,\kappa})|_{\rho}\cdot\bar{\mathbf{a}}^{\beta}]\bar{\mathbf{n}} - (M^{\kappa\rho}\bar{\mathbf{a}}_{\kappa})|_{\rho}\cdot\mathbf{B}^{\beta},\\ \mathbf{C} &= -a_{\nu}^{-1}[(\boldsymbol{\nu}\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha}]M^{\alpha\beta}\nu_{\beta}\bar{\mathbf{n}} - M^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\nu_{\beta}\cdot\mathbf{A},\\ K &= a_{\nu}^{-1}[(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha}]M^{\alpha\beta}\nu_{\beta} + M^{\alpha\beta}\bar{\mathbf{a}}_{\alpha}\nu_{\beta}\cdot\mathbf{B},\end{aligned}$$

$$M = M^{\alpha\beta}\bar{\mathbf{a}}_\alpha \nu_\beta \cdot \mathbf{D}, \qquad \mathbf{C}_n = \mathbf{C}(s_n + 0) - \mathbf{C}(s_n - 0). \tag{29}$$

Taking into account the identity

$$\Delta\mathbf{u}_{,\alpha} = \bar{\mathbf{a}}_\kappa(\Delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}^\kappa) + \bar{\mathbf{n}}(\Delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{n}}), \tag{30}$$

we also have

$$(\boldsymbol{\nu}\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha} = [(\bar{\mathbf{n}}\times\bar{\mathbf{a}}_\kappa)\cdot\boldsymbol{\nu}](\Delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}^\kappa),$$

$$(\bar{\mathbf{r}}'\times\bar{\mathbf{n}})\cdot\Delta\mathbf{u}_{,\alpha} = [(\mathbf{n}\times\bar{\mathbf{a}}_\kappa)\cdot\bar{\mathbf{r}}'](\Delta\mathbf{u}_{,\alpha}\cdot\bar{\mathbf{a}}^\kappa). \tag{31}$$

This allows one to note that with (31) the first terms of $\mathbf{C}$ and K in (29) have the structure similar to $\Delta\mathbf{F}$ and ΔM in (27).

The exterior part of (15), given by (18), can also be transformed using $(4)_3$, $(5)_1$, $(19)_2$ and $(24)_2$, applying Stokes' theorem to some integrals over $\mathcal{M}$, and then applying integration by parts to some integrals over $\mathscr{C}_f$, which gives

$$\begin{aligned}\Delta G_e[\mathbf{u};\Delta\mathbf{u},\delta\mathbf{u}] = &\iint_{\mathcal{M}} \{\Delta\mathbf{p} + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}]|_\beta - (\mathbf{h}\cdot\mathbf{B}^\beta)|_\beta\}\cdot\delta\mathbf{u}\,\mathrm{d}A\\ &+\int_{\mathscr{C}_f}\{[\Delta\mathbf{P}^* - (\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta\nu_\beta)\bar{\mathbf{n}} + \mathbf{h}\cdot\mathbf{B}^\beta\nu_\beta]\cdot\delta\mathbf{u}\\ &+(\Delta M^* + K^*)\delta n_\nu + M^*\Delta(\delta n_\nu)\}\,\mathrm{d}s + \sum_n(\Delta\mathbf{F}_n^* + \mathbf{C}_n^*)\cdot\delta\mathbf{u}_n,\end{aligned} \tag{32}$$

where

$$\Delta\mathbf{P}^* = \Delta\mathbf{T} + (\Delta\mathbf{F}^* + \mathbf{C}^*)',$$

$$\Delta\mathbf{F}^* = -a_\nu^{-1}[(\bar{\mathbf{n}}\times\Delta\mathbf{H})\cdot\boldsymbol{\nu}]\bar{\mathbf{n}}, \qquad \Delta M^* = a_\nu^{-1}(\bar{\mathbf{n}}\times\Delta\mathbf{H})\cdot\bar{\mathbf{r}}',$$

$$\mathbf{C}^* = -\mathbf{H}\cdot\mathbf{A}, \qquad K^* = \mathbf{H}\cdot\mathbf{B}, \qquad M^* = \mathbf{H}\cdot\mathbf{D}. \tag{33}$$

Finally, the first term in expression (12) can also be transformed in a similar way as principle $(3)_1$ has been transformed into form (6). For $\mathbf{u}$ belonging to the equilibrium path this term should vanish identically. But, in general, the value $\mathbf{u}$ (recall that $\mathbf{u}$ means $\mathbf{u}_m^{(l)}$ in this Section 5, according to our convention) need not belong to the equilibrium path. Therefore, let us introduce the unbalanced residual force vectors corresponding to $\mathbf{u}$:

$$\mathbf{p}_R = \mathbf{T}^\beta|_\beta + \mathbf{p} + [(\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}}]|_\beta, \quad \mathbf{P}_R = \mathbf{P} - \mathbf{P}^*,$$

$$M_R = M - M^*, \qquad \mathbf{F}_{nR} = \mathbf{F}_n - \mathbf{F}_n^*. \tag{34}$$

Now all the transformed integrals (26), (28), (32) and (6) with (34) can be introduced into (12), which gives

$$\begin{aligned}&-\iint_{\mathcal{M}}\{(\Delta\mathbf{T}^\beta + \mathbf{S}^\beta)|_\beta + \Delta\mathbf{p} + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^\beta]|_\beta + \mathbf{p}_R\}\cdot\delta\mathbf{u}\,\mathrm{d}A\\ &+\int_{\mathscr{C}_f}(\{\Delta\mathbf{P} + \mathbf{Q} - \Delta\mathbf{P}^* + [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^\beta]\nu_\beta + \mathbf{P}_R\}\cdot\delta\mathbf{u}\\ &+(\Delta M + K - \Delta M^* - K^* + M_R)\delta n_\nu + M_R\Delta(\delta n_\nu))\,\mathrm{d}s\\ &+\sum_n(\Delta\mathbf{F}_n + \mathbf{C}_n - \Delta\mathbf{F}_n^* - \mathbf{C}_n^* + \mathbf{F}_{nR})\cdot\delta\mathbf{u}_n = 0.\end{aligned} \tag{35}$$

Since (35) should vanish identically for any kinematically admissible $\delta\mathbf{u}$, from (35) we obtain the following incremental equilibrium equations as well as the incremental static boundary and corner conditions for the Lagrangian non-linear theory of thin shells:

$$[\Delta\mathbf{T}^\beta + \mathbf{S}^\beta + (\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^\beta]|_\beta + \Delta\mathbf{p} + \mathbf{p}_R = \mathbf{0} \quad \text{in } \mathcal{M},$$

$$\left.\begin{aligned}\Delta\mathbf{P} + \mathbf{Q} &= \Delta\mathbf{P}^* - [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^\beta)\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^\beta]\nu_\beta - \mathbf{P}_R\\ \Delta M + K &= \Delta M^* + K^*\end{aligned}\right\} \quad \text{on } \mathscr{C}_f,$$

$$\Delta\mathbf{F}_n + \mathbf{C}_n = \Delta\mathbf{F}_n^* + \mathbf{C}_n^* - \mathbf{F}_{nR} \quad \text{at each corner } M_n \in \mathscr{C}_f. \tag{36}$$

The corresponding work-conjugate geometric boundary conditions to be satisfied at each incremental step are

$$\Delta\mathbf{u} = \mathbf{0}, \qquad \Delta n_\nu = 0 \quad \text{on } \mathscr{C}_u. \tag{37}$$

Equations (36) and (37) represent the required set of linear conditions for the increment $\Delta\mathbf{u}$ to be satisfied at the ith iteration step. When $\Delta\mathbf{u}$ is obtained from (36) and (37) it allows one to determine the $(i+1)$th approximation to the equilibrium state $\mathbf{u}_m$.

Equations (36) and (37) have an easily interpretable structure. The vectors $\Delta\mathbf{T}^\beta$, $\Delta\mathbf{P}$, $\Delta\mathbf{F}_n$ and ΔM, being linear in $\Delta\mathbf{u}$, represent the material part of changes of $\mathbf{u}$ dependent on the constitutive equations, while the vectors $\mathbf{S}^\beta$, $\mathbf{Q}$, $\mathbf{C}_n$ and K, being linear in $\Delta\mathbf{u}$ as well, represent the geometric part of changes of $\mathbf{u}$ caused by the change of shell geometry at the incremental step. Equations (36) and (37) are given for arbitrary configuration-dependent external static loads, and for arbitrary work-conjugate incremental static and geometric boundary conditions associated with n_ν taken as the fourth parameter of boundary deformation. Alternative forms of incremental boundary conditions associated with other definitions of the fourth parameter can easily be derived, if necessary, by taking Gateaux derivatives of respective boundary quantities defined in [39, 40]. The explicit appearance in (36) of the unbalanced force vectors $\mathbf{p}_R$, $\mathbf{P}_R$ and $\mathbf{F}_{nR}$ assures the correct equilibrium balance at each step.

Let us recall that the Lagrangian non-linear theory of shells, for which (36) and (37) are derived, is valid for unrestricted displacements, rotations, strains and changes of curvature of the reference surface. In this paper we have used explicitly the constitutive equations (10) of an elastic material. We have done this for definiteness of our development, since the corresponding numerical procedure, based on a triangular doubly curved finite element of C^1 type with 54 degrees of freedom, was developed in [46] and successfully applied to solve some highly non-linear one- and two-dimensional problems of elastic shells within the small- and large-strain range of deformation [33, 46]. It should be pointed out again that the incremental equations (36) and (37) follow from direct linearization of the principle of virtual displacements $(3)_1$. However, the principle itself does not require $N^{\alpha\beta}$ and $M^{\alpha\beta}$ to be derivable from the strain energy function. Since $(3)_1$ is an incremental principle, it is applicable both to elastic and inelastic shells. Therefore, our resulting incremental shell equations may be applied to solve problems of inelastic shells as well, provided corresponding incremental constitutive equations are introduced to evaluate $\Delta N^{\alpha\beta}$ and $\Delta M^{\alpha\beta}$ at each iteration step.

The set of incremental equations (36) and (37) may be viewed as direct generalization of the analogous incremental formulation in total Lagrangian description proposed for geometrically non-linear problems of elastic shells in [2, 19]. Note that the incremental shell equations of [2, 19] were derived by an engineering approach, analysing the superposition of two deformations with subsequent linearization of the second deformation about the intermediate configuration. Our equations (36) and (37) are valid for large elastic strains, and the influence of surface moment $\mathbf{h}$ is additionally taken into account. Applying the exact formula (24) for $\Delta(\delta\bar{\mathbf{n}})$ at $\mathscr{C}_f$ we are also able to account exactly the incremental changes of rotational deformation of the shell boundary, while in [2, 19] those changes were approximated by some linearized incremental rotations and the resulting error was included into the residual force vector.

The recent incremental formulation proposed by Basar and Ding [13] for the geometrically non-linear elastic shell theory of [7], which is essentially equivalent to the one given in [2, 3], seems to be more restrictive as compared with the one developed here. Apart from the explicit use of classical constitutive equations for thin elastic shells, which are valid for small strains, and the omission of the less important external surface moment $\mathbf{h}$, the increments $\Delta\mathbf{p}$, $\Delta\mathbf{T}$ and $\Delta\mathbf{H}$ of the external surface and boundary loads at the iteration step $\Delta\mathbf{u}$ are omitted in [13] as well. This unnecessarily restricts the class of admissible loads.

6. LAGRANGIAN BUCKLING SHELL EQUATIONS

Buckling shell equations, called also the equations of critical equilibrium of the shell, can be derived through the linearization of the non-linear boundary value problem about a given finitely deformed equilibrium state of the shell [6, 30].

Let $\mathbf{u}$ be a regular solution of (8) and (9) determining the equilibrium state whose stability properties are analysed. At this configuration $G[\mathbf{u}; \delta\mathbf{u}] = 0$, according to $(3)_1$. Linearization of G at $\mathbf{u}$ in the kinematically admissible direction $\Delta\mathbf{u}$ leads immediately to the linear equation for $\Delta\mathbf{u}$:

$$\Delta G[\mathbf{u}; \Delta\mathbf{u}, \delta\mathbf{u}] = 0. \tag{38}$$

It is apparent that (38) is a particular case of the general linearized equation (12). However, now $\mathbf{u}$ is an equilibrium state, and the first term in (12) vanishes identically.

The Gateaux derivative of G in (38) can now be explicitly calculated, performing exactly the same transformations as in Section 5. This leads immediately to the following explicit form of Lagrangian buckling equations for thin shells:

$$[\Delta\mathbf{T}^{\beta} + \mathbf{S}^{\beta} + (\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]|_{\beta} + \Delta\mathbf{p} = \mathbf{0} \quad \text{in } \mathcal{M}, \tag{39}$$

with corresponding work-conjugate boundary conditions

$$\left.\begin{aligned} \Delta\mathbf{P} + \mathbf{Q} &= \Delta\mathbf{P}^{*} - [(\Delta\mathbf{h}\cdot\bar{\mathbf{a}}^{\beta})\bar{\mathbf{n}} - \mathbf{h}\cdot\mathbf{B}^{\beta}]\nu_{\beta} \\ \Delta M + K &= \Delta M^{*} + K^{*} \end{aligned}\right\} \quad \text{on } \mathscr{C}_f,$$

$$\Delta\mathbf{F}_n + \mathbf{C}_n = \Delta\mathbf{F}_n^{*} + \mathbf{C}_n^{*} \quad \text{at each corner } M_n \in \mathscr{C}_f,$$

$$\Delta\mathbf{u} = \mathbf{0}, \qquad \Delta n_{\nu} = 0 \quad \text{on } \mathscr{C}_u. \tag{40}$$

All the quantities are defined in (39) and (40) through $\mathbf{u}$ and $\Delta\mathbf{u}$ by exactly the same formulae as the analogously denoted respective quantities of Section 5 have been defined through $\mathbf{u} \equiv \mathbf{u}_m^{(i)}$ and $\Delta\mathbf{u} = \Delta\mathbf{u}_m^{(i+1)}$.

The Lagrangian buckling equations (39) and (40) of thin shells are again valid for arbitrary displacements, rotations, strains and changes of curvature of the reference surface, for arbitrary configuration-dependent external static surface and boundary loads as well as for arbitrary work-conjugate static and geometric boundary conditions associated with n_{ν} as the fourth parameter of boundary deformation. Therefore, the explicit Lagrangian buckling equations (39) and (40) represent the ultimate form of such equations within the non-linear theory of thin shells in the Lagrangian description. Their accuracy and the range of applicability in engineering problems depends only on the accuracy of the basic assumption of thin shell theory, that deformation of the shell is determined by deformation of its reference surface, and by the accuracy of the particular constitutive equations used in the analysis.

The buckling shell equations (39) and (40) extend to the large-strain range of deformation and arbitrary loading the explicit stability equations derived by Nolte [10] for the geometrically non-linear theory of thin elastic shells loaded by dead-type external surface force and boundary force and moment.

7. CONCLUSIONS

In this report we have developed the general form of incremental equations, allowing one to solve highly non-linear bending problems of thin shells. Our analysis is based on the Lagrangian non-linear theory of thin elastic shells developed in [2, 3] and on some results given for the large-strain problems of rubber-like shells in [33]. Explicit incremental shell equations are derived using the Newton–Kantorovich method, which essentially consists of successive approximations to the exact solution of some linearized shell problem. Assuming that an approximation to an equilibrium state of the shell is known, the solution of the incremental shell equations with respect to increments of displacements allows one to calculate the next approximation. In our procedure we have fully taken into account that the successive approximations to the unknown equilibrium state may not belong to the equilibrium path. As a result, some unbalanced force vectors appear explicitly in our incremental shell equations. As a special case of the incremental shell equations, the Lagrangian buckling equations for thin shells have been constructed.

We have explicitly applied here the constitutive equations (10) of elastic shells, since for such shells the effective computer programs were developed in [10, 46], and many one- and two-dimensional non-linear problems of elastic shells within small- and large-strain ranges were analysed. However, our incremental and buckling equations are applicable to inelastic problems of thin shells as well, provided some incremental constitutive equations for the shell stress measures are available.

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