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CLOSED-FORM FORCE-ELONGATION RELATIONS FOR THE UNIAXIAL VISCOELASTIC BEHAVIOR OF BIOLOGICAL SOFT TISSUES

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Introduction

Within the Quasi-linear Viscoelasticity, a closed-form solution to the hereditary integral in the case of uniaxial deformation is given. Loading with a constant rate, a logarithmic relaxation function and a polynomial representation of the elastic response is assumed. The resulting force-elongation relation is enhanced to comply with experimental limitations.
Fung (1972, 1981) proposed to use a Quasi-linear Viscoelasticity (QLV) for the description of the nonlinear behavior of soft biological tissues under uniaxial deformation. The QLV combines both elastic and time-dependent components of a tissue's mechanical response using a hereditary integral formulation. A vast variety of soft tissue structures were modeled using this theory (see e.g. Haut and Little, 1972, Woo et al., 1980, Myers et al., 1991, Visarius, 1994, and references given there).
In QLV theory the hereditary integral is given in terms of the reduced relaxation function and the elastic response, which must be determined experimentally. As a result, the hereditary integral is usually evaluated by direct numerical integration of the experimental data.
The purpose of this note is to provide a closed-form analytical expression to the hereditary integral for loading with a constant rate, and enhance this expression to comply with experimental constraints which are usually associated with one-dimensional simple extension tests.

Hereditary Integral

According to QLV proposed by Fung (1972, 1981), the Lagrangian stress and strain measures are related through the hereditary integral. In one-dimensional simple extension tests it is more convenient to use alternative global work-conjugate variables - the tensile force F and the elongation u - which are related through the analogous hereditary integral

$$F(t) = \int_0^t G(t-\tau) \cdot \frac{\partial F^e(u)}{\partial u} \cdot \frac{\partial u(\tau)}{\partial \tau} d\tau \quad . \tag{1}$$

In (1) t denotes time, with loading and motion starting at t=0, $F^e(u)$ is the time-dependent elastic response (which can be a non-linear function of u) and $G(t)$ is the reduced relaxation function, which can be defined in terms of a relaxation spectrum $S(t)$ by

$$G(t) = \frac{1}{1 + \int_0^\infty S(\tau) d\tau} \cdot \left[1 + \int_0^\infty S(\tau) \cdot e^{-\frac{t}{\tau}} d\tau \right] \quad . \tag{2}$$

In order to account for the weakly frequency dependent behavior of soft biological tissues, Fung (1981) proposed $S(\tau)$ of the form

$$\begin{aligned} S(\tau) &= \frac{c}{\tau} && \text{for } \tau_1 \leq \tau \leq \tau_2 \\ &= 0 && \text{for } \tau < \tau_1, \tau > \tau_2 \end{aligned} \quad . \tag{3}$$

Performing time integration in (2) with (3) we obtain the well-known expression

$$G(t) = \frac{1 + c \cdot \left[E_1\left(\frac{t}{\tau_2}\right) - E_1\left(\frac{t}{\tau_1}\right) \right]}{1 + c \cdot \ln\left(\frac{\tau_2}{\tau_1}\right)} \quad , \tag{4}$$

where $E_1(x) = \int_0^\infty e^{\frac{-y}{y}} dy$ is the exponential integral function.

Experimental values of c, τ_1 and τ_2 for various soft tissues are available, see Visarius (1994, Table 4.3.1). It is known, however, that the values of c, τ_1 and τ_2 are sensitive to small changes of experimental data, cf. Sauren and Rousseau (1983), Dortmans et al. (1984). Besides, the form (4) itself is inconvenient to use in parametric studies or for the derivation of analytical solutions.

Many experimental results indicate that $G(t)$ is almost logarithmic in a certain time domain. This allows us to use here the simple formula

$$G(t) = A \cdot \ln(t) + B \quad , \tag{5}$$

where the constants A and B are either expressed through c, τ_1 and τ_2, as in Tanaka and Fung (1974), or determined directly from experimental data.

For the approximation of the elastic response $F^e(u)$, series expansions in terms of polynomials or exponential functions are primarily used in the literature. In this note we prefer to apply the simplest polynomials, thus representing $F^e(u)$ in the form

$$F^e(u) = \sum_{i=1}^{n} c_i \cdot u^i \ , n \geq 1 \tag{6}$$

with constants c_i to be determined experimentally. For the loading at a constant rate R the following relation is introduced

$$u = R \cdot t \quad . \tag{7}$$

Force-elongation Relation

Introducing (5), (6) and (7) into (1) we obtain

$$F(t) = \int_0^t [A \cdot \ln(t-\tau) + B] \cdot \left[\sum_{i=1}^{n} i \cdot c_i \cdot u^{i-1} \right] \cdot R \, d\tau \quad . \tag{8}$$

The time integration can now be directly performed in (8) with the use of substitution and the rule of de l'Hospital, which yields

$$F(u) = \sum_{i=1}^{n} c_i \cdot u^i \cdot \left[A \cdot \left(\ln\left(\frac{u}{R}\right) + f_i \right) + B \right] \quad , \tag{9}$$

$$\text{where} \qquad f_i = i \cdot \sum_{k=1}^{i} \frac{(i-1)! \cdot (-1)^k}{(k-1)! \cdot (i-k)! \cdot k^2} \quad . \tag{10}$$

The numerical values of the first ten factors f_i calculated from (10) are listed in Table 1.

TABLE 1: Factors f_i for $1 \le i \le 10$

i	1	2	3	4	5	6	7	8	9	10
f_i	-1.00	-1.50	-1.83	-2.08	-2.28	-2.45	-2.59	-2.72	-2.83	-2.93

The closed-form expression (9) provides us the analytical force-elongation relation $F(u)$ with free coefficients A, B and c_i to be determined from the non-linear curve fitting of the experimental relaxation and elastic response test data.

Enhanced Force-elongation Relation

It is impossible to experimentally impose an instantaneous elongation u to the tissue. As a result, errors are introduced into such a modeling process which, in turn, effect the accuracy of experimentally determined coefficients of (5) and (6) as well as predictions of tissue behavior made with (9).

In case of the reduced relaxation function (5), an adjustment of these coefficients may be achieved by extrapolation of the experimental data. Following Myers et al. (1991), we can approximate the reduced relaxation function over the ramp portion of the relaxation test by a linearly decreasing function

$$G(t) = 1 - \alpha \cdot t \quad . \tag{11}$$

The approximately corrected elastic response F^{ec} can be calculated by deconvolution of F^e according to

$$F^e = \int_0^t G(t-\tau) \cdot \frac{\partial F^{ec}(u(\tau))}{\partial u} \cdot \frac{\partial u(\tau)}{\partial \tau} d\tau \quad , \tag{12}$$

where (6) and (5) are used for F^e and G, respectively, and R describing u in (7) has to be replaced by the specific ramp R_e chosen in the experiment.

To solve (12) for F^{ec} the Laplace transform of (12) is taken to give

$$\tilde{F}^e(s) = \tilde{G}(s) \cdot s \cdot \tilde{F}^{ec}(s) \tag{13}$$

$$\text{with} \qquad \tilde{f}(s) = \int_0^\infty f(t) \cdot e^{-st} dt \tag{14}$$

denoting the Laplace transform of a function $f(t)$ (Erdelyi et al., 1954). Applying (14) to (11) and (6) we obtain

$$\widetilde{G}(s) = \frac{s-\alpha}{s^2} \tag{15}$$

$$\widetilde{F}^{e}(s) = \sum_{i=1}^{n} c_i \cdot R_e^i \cdot \frac{i!}{s^{i+1}} \tag{16}$$

which introduced into (13) allows one to solve it for $\widetilde{F}^{ec}(s)$:

$$\widetilde{F}^{ec}(s) = \frac{1}{s-\alpha} \cdot \sum_{i=1}^{n} c_i \cdot R_e^i \cdot \frac{i!}{s^i} \quad . \tag{17}$$

Using

$$t^n \;\to\; \frac{n!}{s^{n+1}} \quad \text{and} \quad \frac{1}{s^m \cdot (s-\alpha)} \;\to\; \frac{(-1)^m}{(-\alpha)^m} \cdot e^{\alpha \cdot t} + \sum_{j=1}^{m} \frac{t^{(j-1)} \cdot (-1)^{(m-j)}}{(-\alpha)^{(m+1-j)} \cdot (j-1)!} \quad . \tag{18}$$

the back-transformation of (17) takes the form

$$F^{ec}(u) = \sum_{i=1}^{n} c_i \cdot R_e^i \cdot i! \cdot \left[\frac{1}{\alpha^i} \cdot \left(e^{\frac{u\alpha}{R_e}} - 1 \right) - \sum_{j=1}^{i-1} \frac{\left(\frac{u}{R_e} \right)^j}{\alpha^{i-j} \cdot (j-1)!} \right] \quad . \tag{19}$$

The exponential term in (19) can be expanded into series

$$e^{\frac{\alpha \cdot u}{R_e}} = \sum_{k=0}^{\infty} \frac{1}{k!} \cdot \left(\frac{\alpha \cdot u}{R_e} \right)^k \quad , \tag{20}$$

which introduced into (19), after contraction of some terms, leads to

$$F^{ec}(u) = \sum_{k=0}^{\infty} u^k \cdot \frac{1}{k!} \cdot \sum_{i=1}^{n} c_i \cdot i! \cdot \left(\frac{\alpha}{R_e} \right)^{(k-i)} \quad . \tag{21}$$

In the practically important case of truncation of the expansion (20) the following estimate of the error may be given (Madelung 1957)

$$e^{\frac{\alpha \cdot u}{R_e}} = \sum_{k=1}^{l} \frac{1}{k!} \cdot \left(\frac{\alpha \cdot u}{R_e}\right)^k + r_l \quad , \tag{22}$$

$$r_l \le \frac{1}{l!} \cdot \left|\frac{\alpha \cdot u}{R_e}\right|^l \quad \text{for} \quad \left|\frac{\alpha \cdot u}{R_e}\right| \le \frac{l+1}{2} \quad . \tag{23}$$

Upon substitution of (21) into (9) the enhanced form of the force-elongation relation for the one-dimensional simple extension tests is obtained

$$F = \sum_{i=1}^{\max(l,n)} d_i \cdot u^i \cdot \left[A \cdot \left(\ln\left(\frac{u}{R}\right) + f_i \right) + B \right] \quad , \tag{24}$$

where

$$d_i = \sum_{k=1}^{\min(i,n)} \frac{k!}{i!} \cdot \left(\frac{\alpha}{R_e}\right)^{(i-k)} \cdot c_k \quad . \tag{25}$$

The expression (24) is a closed-form force-elongation relation which already accounts for the experimental limitations inherent to biomechanical testing. It incorporates corrected parameters d_i which account for the unwanted relaxation during the elastic test.

Discussion and Conclusion

The systematic characterization of the nonlinear material properties of biological soft tissues is a challenge for bioengineers since decades. This study provides a closed-form solution of the hereditary integral of one of the most popular viscoelastic theories, the quasi-linear theory proposed by Fung (1972).

Several models used in the literature can be found as special cases of the general closed-form solution. From (9) the analogous solution to the form used by Haut and Little (1972) can be derived for i=2. Expressions (17) and (19) can be used with n=4 to directly deduce results obtained by Myers et al. (1991) (note the misprint in eq. 15 of the publication).

The closed-form relation presented allows for studies of isolated parameters (e.g. the loading rate) as well as for implementation as a material law into other models. Accuracy and computation speed are further advantages of the closed-form, since no iterative numerical solution is necessary.

The experimental application of the model with an analysis of its predictive ability and the influence of the loading rate will be published in a separate paper.

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