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ON THE VECTOR OF CHANGE OF BOUNDARY CURVATURE IN THE NON-LINEAR T-R TYPE THEORY OF SHELLS [1]

(Paper dedicated to Prof. E. I. Mikhailovskii on his 60th birthday)

W. PIETRASZKIEWICZ
Polish Academy of Sciences, Institute of Fluid-Flow Machines
ul. Fiszera 14, 80-952 Gdańsk, Poland

A non-linear deformation of the shell lateral boundary surface is analysed. Two definitions of the vector of change of boundary curvature are discussed. The first definition of the vector was used in the author's earlier papers. The second definition of the vector is applied here, from which an alternative set of deformational boundary quantities is derived. In the case of small strains, the quantities are consistently simplified and expressed in terms of physical components of shell strain measures at the boundary contour.

1. Introduction. Within the linear theory of shells with account of transverse shear and normal strains, which is also called the Timoshenko-Reissner (T-R) type theory, deformational boundary quantities were introduced by Shamina [1]. Pietraszkiewicz [2-4] derived such quantities for the non-linear shell deformation compatible with the linear distribution of displacements across the shell thickness. These five deformational quantities are: the elongation γ_{tt}, the transverse shear γ_{t3} and three components of the vector $\mathbf{k}_t$ of change of curvature of the shell boundary contour. The components of $\mathbf{k}_t$ derived in [2-4] are complex functions of shell strain measures. In the case of small strains, these complex expressions were consistently simplified in [3,4] and presented in an easily readable form through physical components of the strain measures at the shell boundary contour.

Deformational boundary quantities, appropriate for the geometrically non-linear T-R type shell theory, were also proposed by Mikhailovskii in [5]. I noticed that the quantities derived in [5], which are analogous to the components of $\mathbf{k}_t$, do not agree with the corresponding quantities following from [3,4] and, when linearized, they do not reduce themselves to the results of [1]. After a correspondence with the author [6] I came to the conclusion that the deformational boundary quantities introduced in [2-4] and [5] differ from each other simply by definition.

In this report I analyse an exact non-linear deformation of the shell lateral boundary surface. I show that the total rotation of the shell lateral boundary element can

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be defined in two alternative, non-equivalent ways depending upon the choice of an orthonormal triad of vectors describing the geometry of the lateral boundary element of the deformed shell. In the triad used in [2-4], one of the unit vectors was taken to be tangential to the boundary contour of the deformed shell. In the alternative triad introduced here I take one of the unit vectors to be colinear with a vector, into which a unit vector normal to a reference surface of undeformed shell is transfomed. For such an alternative choice of the triad, exact expressions for the deformational boundary quantities are derived and their relation to those given in [3,4] is discussed. A consistent reduction of these expressions is carried out for the geometrically non-linear and linear T-R type theories of shells. In the case of geometric non-linearity, the relations derived here are equivalent to those of Mikhailovskii [5] within an error permitted in the shell theory.

2. Deformation of the shell boundary. A non-linear deformation of the shell and its lateral boundary surface was decribed in detail in [3,4]. We use notations and some basic relations derived in those papers.

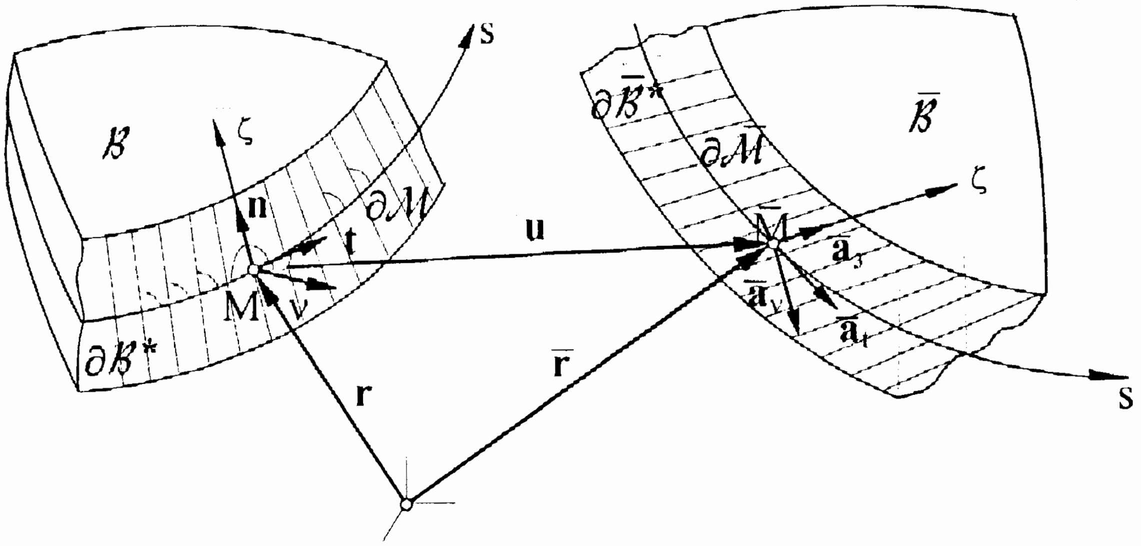

Fig. 1

Let the lateral boundary surface $\partial\mathcal{B}^*$ of the undeformed shell $\mathcal{B}$ be defined by the position vector $\mathbf{p}(s,\zeta) = \mathbf{r}(s) + \zeta\mathbf{n}(s)$, see Fig. 1, where $\mathbf{r}$ is the position vector of the boundary contour $\partial\mathcal{M}$ of the shell reference surface $\mathcal{M}$, $\mathbf{n}$ is the unit vector normal to $\mathcal{M}$, s is the arc length along $\partial\mathcal{M}$, $-h^- \le \zeta \le h^+$ is the distance from $\mathcal{M}$, and $h = h^- + h^+$ is the shell thickness. The surface $\partial\mathcal{B}^*$ is rectilinear and orthogonal to $\mathcal{M}$ along $\partial\mathcal{M}$.

Within the non-linear theory of shells taking into account of transverse shear and normal strains, it is assumed that during the deformation process the surface $\partial\mathcal{B}^*$ moves into the lateral boundary surface $\partial\overline{\mathcal{B}}^*$ of the deformed shell described by the position vector

$$\overline{\mathbf{p}}(s,\zeta) = \overline{\mathbf{r}}(s) + \zeta\overline{\mathbf{a}}_3(s),$$
$$\overline{\mathbf{r}} = \mathbf{r} + \mathbf{u}, \quad \overline{\mathbf{a}}_3 = \mathbf{n} + \boldsymbol{\beta} \tag{1}$$

where $\bar{\mathbf{r}}$ is the position vector of the deformed boundary contour $\partial\overline{\mathcal{M}}$, $\mathbf{u}$ is the displacement field, $\boldsymbol{\beta}$ is the difference vector, and (s, ζ) are convected surface coordinates of $\partial\overline{\mathcal{B}}^*$. According to (1), the surface $\partial\overline{\mathcal{B}}^*$ is again rectilinear, although not orthogonal to the deformed reference surface $\overline{\mathcal{M}}$ along $\partial\overline{\mathcal{M}}$.

During the deformation process an orthonormal triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ associated with $\partial\mathcal{B}^*$ moves into a skew triad of non-unit vectors $\bar{\mathbf{a}}_\nu, \bar{\mathbf{a}}_t, \bar{\mathbf{a}}_3$ (see Fig. 1) such that

$$\begin{aligned} &\bar{\mathbf{a}}_\nu = \bar{\mathbf{r}}_{,\alpha}\nu^\alpha, \quad \bar{\mathbf{a}}_t = \bar{\mathbf{r}}_{,\alpha}t^\alpha = a_t\bar{\mathbf{t}}, \quad \bar{\mathbf{a}}_3 = a_3\mathbf{d}, \\ &a_t = |\bar{\mathbf{a}}_t| = \sqrt{1+2\gamma_{tt}}, \quad a_3 = |\bar{\mathbf{a}}_3| = \sqrt{1+2\gamma_{33}}, \\ &\gamma_{tt} = \gamma_{\alpha\beta}t^\alpha t^\beta, \quad \gamma_{\nu t} = \gamma_{\alpha\beta}\nu^\alpha t^\beta, \ \text{etc.}, \quad (\ )_{,\alpha} \equiv \frac{\partial}{\partial\theta^\alpha}(\ ). \end{aligned} \tag{2}$$

Here θ^α, $\alpha = 1, 2$, are convected curvilinear surface coordinates on $\mathcal{M}$ and $\overline{\mathcal{M}}$, $\mathbf{a}_\alpha = \mathbf{r}_{,\alpha}$ are the natural base vectors of $\mathcal{M}$, $\mathbf{t} = \mathbf{a}_\alpha t^\alpha = d\mathbf{r}/ds \equiv \mathbf{r}'$ is the unit tangent vector and $\boldsymbol{\nu} = \mathbf{a}_\alpha\nu^\alpha \equiv \mathbf{r}_{,\nu}$ is the outward normal vector of $\partial\mathcal{M}$ such that $\boldsymbol{\nu} = \mathbf{t}\times\mathbf{n}$.

The spatial Green strain tensor E_{ab}, $a = 1, 2, 3$, is expressed in the neighbourhood of $\mathcal{M}$ through the shell strain measures $\gamma_{ab}, \pi_{\alpha b}$ according to

$$E_{\alpha\beta} = \gamma_{\alpha\beta} + \zeta\pi_{(\alpha\beta)}, \quad E_{\alpha 3} = \gamma_{\alpha 3} + \frac{1}{2}\zeta\pi_{\alpha 3}, \quad E_{33} = \gamma_{33}, \tag{3}$$

where $\pi_{(\alpha\beta)} = \frac{1}{2}(\pi_{\alpha\beta} + \pi_{\beta\alpha})$, and $\gamma_{ab}, \pi_{\alpha b}, a = 1, 2, 3$, are quadratic functions of $\mathbf{u}, \boldsymbol{\beta}$ and their surface gradients given in [3,4].

While in convected coordinates (s, ζ) the vectors $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ constitute a natural surface basis of $\partial\overline{\mathcal{B}}^*$, the vector $\bar{\mathbf{a}}_\nu$ is not normal to $\partial\overline{\mathcal{B}}^*$. For further discussion it is convenient to introduce three other vectors

$$\begin{aligned} &\mathbf{a}_\mu = \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 = a_\mu\boldsymbol{\mu}, \quad a_\mu = |\mathbf{a}_\mu| = \sqrt{(1+2\gamma_{tt})(1+2\gamma_{33}) - 4\gamma_{t3}^2}, \\ &\mathbf{a}_m = \mathbf{a}_\mu \times \bar{\mathbf{a}}_t = a_m\mathbf{m}, \quad a_m = |\mathbf{a}_m| = a_\mu a_t, \quad \gamma_{\nu 3} = \gamma_{\alpha 3}\nu^\alpha, \\ &\mathbf{a}_\tau = \bar{\mathbf{a}}_3 \times \mathbf{a}_\mu = a_\tau\boldsymbol{\tau}, \quad a_\tau = |\mathbf{a}_\tau| = a_\mu a_3, \quad \gamma_{t3} = \gamma_{\alpha 3}t^\alpha. \end{aligned} \tag{4}$$

The unit vector $\boldsymbol{\mu}$ is normal to $\partial\overline{\mathcal{B}}^*$ although not tangential to $\overline{\mathcal{M}}$, the unit vector $\mathbf{m}$ is orthogonal to $\partial\overline{\mathcal{M}}$ but not normal to $\overline{\mathcal{M}}$, and the unit vector $\boldsymbol{\tau}$ is not tangential to $\partial\overline{\mathcal{M}}$ although orthogonal to $\mathbf{d}$ and $\boldsymbol{\mu}$ at $\overline{M} \in \partial\overline{\mathcal{M}}$.

Derivatives of $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ with regard to s can be expressed entirely through the shell strain measures (cf. [3], Section 4.4):

$$\begin{aligned} &\bar{\mathbf{a}}_t' = \bar{\mathbf{a}}_{\alpha,\beta}t^\alpha t^\beta + \bar{\mathbf{a}}_\alpha t^\alpha{}_{,\beta}t^\beta = (t^\alpha|_\beta\,\bar{\mathbf{a}}_\alpha + t^\alpha b_{\alpha\beta}\bar{\mathbf{a}}_3 + \tilde{a}^{dc}\gamma_{c\alpha\beta}t^\alpha\bar{\mathbf{a}}_d)t^\beta, \\ &\bar{\mathbf{a}}_3' = \bar{\mathbf{a}}_{3,\beta}t^\beta = (-b_\beta^\lambda\bar{\mathbf{a}}_\lambda + \gamma_{c3\beta}\tilde{\mathbf{a}}^c)t^\beta, \end{aligned} \tag{5}$$

where

$$\begin{aligned} &\gamma_{cab} = \gamma_{ca;b} + \gamma_{cb;a} - \gamma_{ab;c}, \quad \tilde{a}^{ab}\overline{a}_{ac} = \delta_c^b, \\ &\tilde{a}^{ad} = \frac{1}{2}\frac{a}{\tilde{a}}\epsilon^{abc}\epsilon^{def}(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}), \\ &\frac{\tilde{a}}{a} = \frac{1}{6}\epsilon^{abc}\epsilon^{def}(a_{ad} + 2\gamma_{ad})(a_{be} + 2\gamma_{be})(a_{cf} + 2\gamma_{cf}), \\ &\tilde{a} = |a_{ab}|, \quad \epsilon_{abc} = (\mathbf{a}_a \times \mathbf{a}_b)\cdot\mathbf{a}_c, \quad \tilde{\mathbf{a}}^c = \tilde{a}^{cd}\overline{\mathbf{a}}_d. \end{aligned} \tag{6}$$

In (5) and (6) we have explicitly used spatial bases $\mathbf{a}_a = (\mathbf{a}_\alpha, \mathbf{n})$ at $\mathcal{M}$ and $\overline{\mathbf{a}}_a = (\overline{\mathbf{a}}_\alpha, \overline{\mathbf{a}}_3)$ at $\overline{\mathcal{M}}$. By $(\ )_{;a}$ we have denoted the spatial covariant derivative in the metric $a_{ab} = \mathbf{a}_a \cdot \mathbf{a}_b$ and by $(\ )_{|\alpha}$ the surface covariant derivative in the metric $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$. Spatial indices in the deformed configuration are raised with the help of $\tilde{a}^{ab}$, while surface indices with the help of $\overline{a}^{\alpha\beta}$, where $\tilde{a}^{\alpha\beta} \neq \overline{a}^{\alpha\beta}$, in general.

It was shown in [3] that the rectilinear boundary surface $\partial\overline{\mathcal{B}}^*$ can be defined, with an accuracy up to a rigid-body deformation in space, by three strains $\gamma_{tt}, \gamma_{t3}, \gamma_{33}$ and three components of the vector $\mathbf{k}_t$ of change of curvature of the shell boundary contour. For any particular elastic material behaviour, γ_{33} is expressible entirely through other shell strain measures and surface forces applied to upper and lover shell faces.

An explicit expression for $\mathbf{k}_t$ in terms of $\gamma_{ab}, \pi_{(\alpha\beta)}$ depends upon the choice of an orthonormal triad describing the surface $\partial\overline{\mathcal{B}}^*$ along $\partial\overline{\mathcal{M}}$, which would constitute an analogue of the triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ describing $\partial\mathcal{B}^*$ along $\partial\mathcal{M}$. The unit vector $\boldsymbol{\mu}$ defined in (4) is a unique analogue on $\partial\overline{\mathcal{B}}^*$ of the unit vector $\boldsymbol{\nu}$ on $\partial\mathcal{B}^*$. Two other unit vectors $\overline{\mathbf{t}}$ and $\mathbf{d}$ tangential to $\partial\overline{\mathcal{B}}^*$, which result during the deformation process from the respective unit vectors $\mathbf{t}$ and $\mathbf{n}$ tangential to $\partial\mathcal{B}^*$, are not mutually orthogonal. Therefore, with $\partial\overline{\mathcal{B}}^*$ we can associate either the triad $\boldsymbol{\mu}, \overline{\mathbf{t}}, \mathbf{m}$ or the triad $\boldsymbol{\mu}, \boldsymbol{\tau}, \mathbf{d}$. Each of the triads can be regarded as a different analogue on $\partial\overline{\mathcal{B}}^*$ of the triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ given on $\partial\mathcal{B}^*$.

In my ealier papers [2-4] the triad $\boldsymbol{\mu}, \overline{\mathbf{t}}, \mathbf{m}$ was used. According to (1), the surface $\partial\overline{\mathcal{B}}^*$ is constructed along $\partial\overline{\mathcal{M}}$ by attaching at each $\overline{M} \in \partial\overline{\mathcal{M}}$ the rectilinear element $\zeta\overline{\mathbf{a}}_3$. It seemed to me more natural to include the vector $\overline{\mathbf{t}}$ tangential to $\partial\overline{\mathcal{M}}$, rather than $\mathbf{d}$, into the triad describing the bending properties of $\partial\overline{\mathcal{B}}^*$ along $\partial\overline{\mathcal{M}}$. With the choice of the triad $\boldsymbol{\mu}, \overline{\mathbf{t}}, \mathbf{m}$ along $\partial\overline{\mathcal{M}}$, the total rotation tensor $\mathbf{R}_t$, as well as corresponding vectors of change of curvature $\mathbf{k}_t$ and $\mathbf{l}_t$ of the shell boundary contour, can be defined concisely using [7] by the following relations:

$$
\begin{aligned}
&\mathbf{R}_t = \boldsymbol{\mu} \otimes \boldsymbol{\nu} + \overline{\mathbf{t}} \otimes \mathbf{t} + \mathbf{m} \otimes \mathbf{n},\\
&\mathbf{R}_t^{\mathsf{T}} \mathbf{R}_t' = \mathbf{k}_t \times \mathbf{1}, \quad \mathbf{R}_t' \mathbf{R}_t^{\mathsf{T}} = \mathbf{l}_t \times \tilde{\mathbf{1}},\\
&\mathbf{k}_t = -k_{tt}\boldsymbol{\nu} + k_{t\nu}\mathbf{t} - k_{tn}\mathbf{n}, \quad \mathbf{l}_t = \mathbf{R}_t \mathbf{k}_t,\\
&\mathbf{1} = \boldsymbol{\nu} \otimes \boldsymbol{\nu} + \mathbf{t} \otimes \mathbf{t} + \mathbf{n} \otimes \mathbf{n}, \quad \tilde{\mathbf{1}} = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \overline{\mathbf{t}} \otimes \overline{\mathbf{t}} + \mathbf{m} \otimes \mathbf{m}.
\end{aligned} \tag{7}
$$

Here $\otimes$ denotes the tensor product, $(\ )^{\mathsf{T}}$ is the transposition of the tensor and $\mathbf{1}, \tilde{\mathbf{1}}$ are different representations of the metric tensor of the 3D Euclidean vector space.

Exact expressions for the components $k_{tt}, k_{t\nu}, k_{tn}$ of $\mathbf{k}_t$ in terms of the shell strain measures $\gamma_{ab}, \pi_{(\alpha\beta)}$ were derived in [2-4]. In the case of geometric non-linearity, these expressions were consistently simplified in [3,4] and presented in a readable form through physical components of the shell strain measures at the boundary contour $\partial\mathcal{M}$.

3. Alternative vector of change of boundary curvature. With the lateral boundary surface $\partial\overline{\mathcal{B}}^*$ of the deformed shell, we can associate the alternative orthonormal triad $\boldsymbol{\mu}, \boldsymbol{\tau}, \mathbf{d}$ which also fully describes the geometry of $\partial\overline{\mathcal{B}}^*$ along $\partial\overline{\mathcal{M}}$. The choice of such a triad allows one to define an alternative total rotation tensor $\mathbf{Q}_t$ of $\partial\overline{\mathcal{B}}^*$, as well as corresponding vectors of change of curvature $\boldsymbol{\kappa}_t$ and $\boldsymbol{\lambda}_t$, by expressions analogous to (7):

$$
\begin{aligned}
&\mathbf{Q}_t = \boldsymbol{\mu} \otimes \boldsymbol{\nu} + \boldsymbol{\tau} \otimes \mathbf{t} + \mathbf{d} \otimes \mathbf{n}, \\
&\mathbf{Q}_t^{\mathsf{T}} \mathbf{Q}_t' = \boldsymbol{\kappa}_t \times \mathbf{1}, \quad \mathbf{Q}_t' \mathbf{Q}_t^{\mathsf{T}} = \boldsymbol{\lambda}_t \times \hat{\mathbf{1}}, \\
&\boldsymbol{\kappa}_t = -\kappa_{tt} \boldsymbol{\nu} + \kappa_{t\nu} \mathbf{t} - \kappa_{tn} \mathbf{n}, \quad \boldsymbol{\lambda}_t = \mathbf{Q}_t \boldsymbol{\kappa}_t, \\
&\hat{\mathbf{1}} = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{d} \otimes \mathbf{d}.
\end{aligned}
\tag{8}
$$

We can now calculate $\mathbf{Q}_t^{\mathsf{T}}$ and $\mathbf{Q}_t'$ from $(8)_1$, and evaluate $\boldsymbol{\kappa}_t$ by inverting $(8)_2$ and applying some geometric identities. This leads to two equivalent definitions of the components of $\boldsymbol{\kappa}_t$:

$$
\begin{aligned}
-\kappa_{tt} &= \frac{1}{a_\tau a_3} \mathbf{a}_\tau' \cdot \bar{\mathbf{a}}_3 - \sigma_t = -\frac{1}{a_\tau a_3} \mathbf{a}_\tau \cdot \bar{\mathbf{a}}_3' - \sigma_t, \\
\kappa_{t\nu} &= \frac{1}{a_3 a_\mu} \bar{\mathbf{a}}_3' \cdot \mathbf{a}_\mu - \tau_t = -\frac{1}{a_3 a_\mu} \bar{\mathbf{a}}_3 \cdot \mathbf{a}_\mu' - \tau_t, \\
-\kappa_{tn} &= \frac{1}{a_\mu a_\tau} \mathbf{a}_\mu' \cdot \mathbf{a}_\tau - \kappa_t = -\frac{1}{a_\mu a_\tau} \mathbf{a}_\mu \cdot \mathbf{a}_\tau' - \kappa_t.
\end{aligned}
\tag{9}
$$

where σ_t, τ_t and κ_t are the normal curvature, the geodesic torsion and the geodesic curvature of $\partial\mathcal{M}$, respectively. From the relations (4) we can also find that

$$
\begin{aligned}
\mathbf{a}_\tau &= a_3^2 \bar{\mathbf{a}}_t - 2\gamma_{t3} \bar{\mathbf{a}}_3, \\
\mathbf{a}_\tau' &= 2\gamma_{33}' \bar{\mathbf{a}}_t + a_3^2 \bar{\mathbf{a}}_t' - 2\gamma_{t3}' \bar{\mathbf{a}}_3 - 2\gamma_{t3} \bar{\mathbf{a}}_3', \\
\mathbf{a}_\mu' &= \bar{\mathbf{a}}_t' \times \bar{\mathbf{a}}_3 + \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3'.
\end{aligned}
\tag{10}
$$

It is now apparent that all three components of $\boldsymbol{\kappa}_t$ can be calculated from only two differential expressions (5). Therefore, introducting (5) and (10) into (9) we obtain

$$
\begin{aligned}
-\kappa_{tt} &= \frac{1}{a_\tau a_3} [a_3^2 (a_t^2 \sigma_t - 2\tau_t \gamma_{\nu t} - \gamma_{\alpha 3\beta} t^\alpha t^\beta) + 2\gamma_{t3} (2\tau_t \gamma_{\nu 3} - 2\sigma_t \gamma_{t3} + \gamma_{33\beta} t^\beta)] - \sigma_t, \\
\kappa_{t\nu} &= \frac{1}{a_\mu a_3} \sqrt{\frac{\tilde{a}}{a}} \, (\tau_t + \nu_\rho \tilde{a}^{\rho c} \gamma_{c3\beta} t^\beta) - \tau_t, \\
-\kappa_{tn} &= \frac{1}{a_\mu a_\tau} \sqrt{\frac{\tilde{a}}{a}} \, [a_3^2 (\kappa_t - \nu_\rho \tilde{a}^{\rho c} \gamma_{c\alpha\beta} t^\alpha t^\beta) + 2\gamma_{t3} (\tau_t + \nu_\rho \tilde{a}^{\rho c} \gamma_{c3\beta} t^\beta)] - \kappa_t.
\end{aligned}
\tag{11}
$$

The expressions (11) for three deformational boundary quantities are exact within the assumed linear approximation (1) of the displacement field across the shell thickness, i.e. they are valid for unrestricted values of γ_{ab} and $\pi_{(\alpha\beta)}$. In definitions $(6)_1$ of γ_{cab} included into (11) only first derivatives in the normal direction of the spatial Green strains $E_{\alpha\beta}$ are present. This means that any more complex distribution of the shell deformation across the shell thickness, such as used in [5] for example, cannot change the expressions (11).

The two different vectors $\boldsymbol{\kappa}_t$ and $\mathbf{k}_t$ of change of boundary curvature can be related through the proper orthogonal tensor $\mathbf{P}_t$ rotating $\boldsymbol{\mu}, \bar{\mathbf{t}}, \mathbf{m}$ into $\boldsymbol{\mu}, \boldsymbol{\tau}, \mathbf{d}$:

$$
\begin{aligned}
&\mathbf{P}_t = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \boldsymbol{\tau} \otimes \bar{\mathbf{t}} + \mathbf{d} \otimes \mathbf{m}, \quad \mathbf{Q}_t = \mathbf{P}_t \mathbf{R}_t, \\
&\mathbf{P}_t^{\mathsf{T}} \mathbf{P}_t' = \boldsymbol{\omega}_t \times \tilde{\mathbf{1}}, \quad \boldsymbol{\omega}_t = -\omega_{tt} \boldsymbol{\mu} + \omega_{t\nu} \bar{\mathbf{t}} - \omega_{tn} \mathbf{m}
\end{aligned}
\tag{12}
$$

$$-\omega_{tt} = \frac{1}{a_\tau a_3}\mathbf{a}'_\tau \cdot \bar{\mathbf{a}}_3 - \frac{1}{a_t a_m}\bar{\mathbf{a}}'_t \cdot \mathbf{a}_m,$$

$$\omega_{t\nu} = \frac{1}{a_3 a_\mu}\bar{\mathbf{a}}'_3 \cdot \mathbf{a}_\mu - \frac{1}{a_m a_\mu}\mathbf{a}'_m \cdot \mathbf{a}_\mu, \tag{13}$$

$$-\omega_{tn} = \frac{1}{a_\mu a_\tau}\mathbf{a}'_\mu \cdot \mathbf{a}_\tau - \frac{1}{a_\mu a_t}\mathbf{a}'_\mu \cdot \bar{\mathbf{a}}_t.$$

Explicit expressions for the components of $\boldsymbol{\omega}_t$ in terms of $\gamma_{ab}, \pi_{(\alpha\beta)}$ can now be derived with the help of (10) and (2)-(6).

If $\mathbf{Q}_t$ following from $(12)_1$ is introduced into $(8)_2$, then using $(7)_2$ we obtain

$$\boldsymbol{\kappa}_t = \mathbf{k}_t + \mathbf{R}_t^\top \boldsymbol{\omega}_t,$$
$$\kappa_{tt} = k_{tt} + \omega_{tt}, \quad \kappa_{t\nu} = k_{t\nu} + \omega_{t\nu}, \quad \kappa_{tn} = k_{tn} + \omega_{tn}, \tag{14}$$

which provide simple transformation rules between $\boldsymbol{\kappa}_t$ and $\mathbf{k}_t$.

In order to present (11) in a more readable form, let us express all the spatial tensors $\gamma_{c\alpha\beta}$ through physical components of γ_{ab} and $\pi_{(\alpha\beta)}$ at $\partial\mathcal{M}$. For this reason, we should express all the spatial covariant derivatives $\gamma_{ab;c}$ in terms of the surface covariant derivatives $\gamma_{\alpha\beta|\lambda}, \gamma_{\alpha3|\lambda}$ as well as $\gamma_{33}, \pi_{(\alpha\beta)}$ and $b_{\alpha\beta}$, the curvature tensor of $\mathcal{M}$, keeping in mind that

$$\Gamma^3_{\alpha\beta} = b_{\alpha\beta}, \quad \Gamma^\rho_{3\beta} = -b^\rho_\beta, \quad \Gamma^3_{3\alpha} = \Gamma^3_{33} = \Gamma^\alpha_{33} = 0,$$
$$\gamma_{\alpha\beta,3} = \frac{d}{d\zeta}E_{\alpha\beta}(\zeta)\,|_{\zeta=0} = \pi_{(\alpha\beta)}. \tag{15}$$

Taking into account geometric identities given in [3], after transformations we obtain

$$\gamma_{\nu tt} = \gamma_{\lambda\alpha\beta}\nu^\lambda t^\alpha t^\beta = 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu\gamma_{\nu t} + 2\kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) - 2\sigma_t\gamma_{\nu 3},$$

$$\gamma_{ttt} = \gamma_{\lambda\alpha\beta}t^\lambda t^\alpha t^\beta = \gamma'_{tt} + 2\kappa_t\gamma_{\nu t} - 2\sigma_t\gamma_{t3},$$

$$\gamma_{3tt} = \gamma_{3\alpha\beta}t^\alpha t^\beta = -\pi_{(tt)} - 2\sigma_t\gamma_{33} + 2\kappa_\nu\gamma_{\nu 3} + 2\gamma'_{t3},$$

$$\gamma_{\nu 3t} = \gamma_{\alpha 3\beta}t^\alpha t^\beta = \pi_{(\nu t)} + 2\sigma_t\gamma_{\nu t} - 2\tau_t\gamma_{\nu\nu} + \gamma'_{\nu 3} + \kappa_\nu\gamma_{\nu 3} - \gamma_{t3,\nu} - \kappa_t\gamma_{t3}, \tag{16}$$

$$\gamma_{t3t} = \gamma_{\alpha 3\beta}t^\alpha t^\beta = \pi_{(tt)} + 2\sigma_t\gamma_{tt} - 2\tau_t\gamma_{\nu t},$$

$$\gamma_{33t} = \gamma_{33\beta}t^\beta = \gamma'_{33} - 2\tau_t\gamma_{\nu 3} + 2\sigma_t\gamma_{t3}.$$

It is now apparent that using (8) and identifying the spatial alternation tensor $\epsilon^{\alpha\beta 3}$ with the surface alternation tensor $\epsilon^{\alpha\beta}$, we are also able to express $\sqrt{\tilde{a}/a}$, $\nu_\alpha\tilde{a}^{\alpha\beta}\nu_\beta$, $\nu_\alpha\tilde{a}^{\alpha\beta}t_\beta$, $\nu_\alpha\tilde{a}^{\alpha 3}$ in terms of physical components of $\gamma_{ab}, \pi_{(\alpha\beta)}$ at $\partial\mathcal{M}$. As a result, the deformational boundary quantities (11) can be expressed explicitly in terms of the physical components.

4. Geometrically non-linear T-R type theory of shells. In the case of geometric non-linearity, $\gamma_{ab} \sim h\pi_{(\alpha\beta)} \sim O(\eta)$, where $\eta \ll 1$. Then all the relations can consistently be approximated by linear terms in each of the strain measures. In particular, we have

$$
\begin{aligned}
&a_\mu \simeq 1+\gamma_{tt}+\gamma_{33}, \quad a_\tau \simeq 1+\gamma_{tt}+2\gamma_{33}, \quad a_3 \simeq 1+\gamma_{33},\\
&\frac{1}{a_\mu} \simeq 1-\gamma_{tt}-\gamma_{33}, \quad \frac{1}{a_\tau} \simeq 1-\gamma_{tt}-2\gamma_{33}, \quad \frac{1}{a_3} \simeq 1-\gamma_{33},\\
&\sqrt{\frac{\tilde{a}}{a}} \simeq 1+\gamma_{\nu\nu}+\gamma_{tt}+\gamma_{33}, \quad \sqrt{\frac{a}{\tilde{a}}} \simeq 1-\gamma_{\nu\nu}-\gamma_{tt}-\gamma_{33},\\
&\nu_\alpha \tilde{a}^{\alpha\beta}\nu_\beta \simeq 1-2\gamma_{\nu\nu}, \quad \nu_\alpha \tilde{a}^{\alpha\beta}t_\beta \simeq -2\gamma_{\nu t}, \quad \nu_\alpha \tilde{a}^{\alpha 3} \simeq -2\gamma_{\nu 3}.
\end{aligned}
\tag{17}
$$

When (17) together with (16) are introduced into (11), and only linear terms in each of $\gamma_{ab}, \pi_{(\alpha\beta)}$ are taken into account, we obtain

$$
\begin{aligned}
\kappa_{tt} &\simeq \pi_{(tt)} + (\sigma_t - \underset{\smile}{\pi_{(tt)}})\,(\gamma_{tt}+\gamma_{33}) - \underline{2\gamma'_{t3}} + \underline{2\gamma'_{t3}},\\
\kappa_{t\nu} &\simeq \pi_{(\nu t)} + 2(\sigma_t - \underset{\smile}{\pi_{(tt)}})\,\gamma_{\nu t} - (\tau_t + \underset{\smile}{\pi_{(\nu t)}})\,(\gamma_{\nu\nu}+\gamma_{33})\\
&\qquad + \underline{\gamma'_{\nu 3} + \kappa_\nu\gamma_{\nu 3} - \gamma_{t3,\nu} + \kappa_t\gamma_{t3}} - \underline{2\kappa_t\gamma_{t3}}\,,\\
\kappa_{tn} &\simeq 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu\gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu}-\gamma_{tt}) - \underline{2(\sigma_t - \pi_{(tt)})\gamma_{\nu 3}} - \underline{2(\tau_t + \pi_{(\nu t)})\gamma_{t3}}.
\end{aligned}
\tag{18}
$$

The relations (18) provide consistently approximated explicit expressions for components of the vector $\boldsymbol{\kappa}_t$ of change of boundary curvature, which are appropriate for the geometrically non-linear theory of shells with account of transverse shear and normal strains.

When (18) are compared with corresponding simplified expressions for components of $\mathbf{k}_t$ given in (92) of [4], it is seen that in (18) the transverse shears are present not only through terms underlined by a solid line, as in (92) of [4]. Here we have additional terms proportional to γ_{t3} which are underlined by a dashed line. This difference is a result of the use of the alternative vector $\boldsymbol{\kappa}_t$ here, as compared with the vector $\mathbf{k}_t$ used in [4]. Indeed, under small strains components of $\boldsymbol{\omega}_t$ defined in (13) can consistently be approximated by

$$
\omega_{tt} \simeq 2\gamma'_{t3}, \quad \omega_{t\nu} \simeq -2\kappa_t\gamma_{t3}, \quad \omega_{tn} \simeq -2(\tau_t + \underset{\smile}{\pi_{(\nu t)}})\gamma_{t3}. \tag{19}
$$

It is now apparent that the components of $\boldsymbol{\kappa}_t$ given in (18) can be recalculated from the components of $\mathbf{k}_t$ given in (92) of [4] by applying transformation rules (14) with (19).

In some applications it may be convenient to have the components of $\boldsymbol{\kappa}_t$ expressed through the surface tensor $\beta_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})$ of change of curvature of the reference surface. In the case of geometric non-linearity, from kinematic relations for $\pi_{(\alpha\beta)}$ given in [3,4] we obtain

$$\pi_{(\alpha\beta)} = \tfrac{1}{2}(\overline{\mathbf{a}}_\alpha \cdot \overline{\mathbf{a}}_{3,\beta} + \overline{\mathbf{a}}_\beta \cdot \overline{\mathbf{a}}_{3,\alpha}) + b_{\alpha\beta}$$

$$\simeq \beta_{\alpha\beta} - (b_{\alpha\beta} - \underset{\sim}{\beta_{\alpha\beta}})\gamma_{33} + \gamma_{\alpha3|\beta} + \gamma_{\beta3|\alpha},$$

$$\pi_{(tt)} \simeq \beta_{tt} - (\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{33} + 2\kappa_t\gamma_{\nu3} - 2\gamma'_{t3}, \tag{20}$$

$$\pi_{(\nu t)} \simeq \beta_{\nu t} + (\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{33} + \gamma'_{\nu3} - \kappa_\nu\gamma_{\nu3} + \gamma_{t3,\nu} - \kappa_t\gamma_{t3},$$

which introduced into (18) allows one to find equivalent expressions for the deformational boundary quantities:

$$\kappa_{tt} \simeq \beta_{tt} + (\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{tt} + \underline{2\kappa_t\gamma_{\nu3}} + 2\gamma'_{t3},$$

$$\kappa_{t\nu} \simeq \beta_{\nu t} + 2(\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{\nu t} - (\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{\nu\nu} + \underline{2\gamma'_{\nu3}} - \underline{2\kappa_t\gamma_{t3}}, \tag{21}$$

$$\kappa_{tn} \simeq 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu\gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) - \underline{2(\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{\nu3}} - \underline{2(\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{t3}}.$$

It is interesting to note that the relations (21) do not depend explicitly upon the normal strain γ_{33}.

In the relations (18)-(21) terms underlined by a wavy line are responsible for the geometric non-linearity, together with the non-linearity of strain-displacement relations. Therefore, if terms underlined by a wavy line are omitted, and the remaining strain measures are defined only by linear expressions in $\mathbf{u}, \boldsymbol{\beta}$, the relations (18)-(21) reduce themselves to the form appropriate for the linear T-R type theory of shells. These linear expressions differ from those given by Shamina [1] only by terms underlined by a dashed line.

Mikhailovskii [5] proposed deformational boundary quantities for the geometrically nonlinear T-R type theory of shells by introducing a total rotation tensor $\check{\mathbf{Q}}_t$. The tensor $\check{\mathbf{Q}}_t$ was assumed to be a superposition of two rotations: a finite rotation associated with a Kirchhoff-Love type geometrically non-linear theory of shells, which rotates the triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ into an intermediate orthonormal triad $\overline{\boldsymbol{\nu}}, \overline{\mathbf{t}}, \overline{\mathbf{n}}$, and a small additional rotation corresponding to a linear approximation in small shears $\gamma_{\nu3}$ and γ_{t3}:

$$\check{\mathbf{Q}}_t \simeq [\overline{\mathbf{1}} + 2\gamma_{\nu3}(\overline{\boldsymbol{\nu}} \otimes \overline{\mathbf{n}} - \overline{\mathbf{n}} \otimes \overline{\boldsymbol{\nu}}) + 2\gamma_{t3}(\overline{\mathbf{t}} \otimes \overline{\mathbf{n}} - \overline{\mathbf{n}} \otimes \overline{\mathbf{t}})](\overline{\boldsymbol{\nu}} \otimes \boldsymbol{\nu} + \overline{\mathbf{t}} \otimes \mathbf{t} + \overline{\mathbf{n}} \otimes \mathbf{n}),$$

$$\overline{\mathbf{1}} = \overline{\boldsymbol{\nu}} \otimes \overline{\boldsymbol{\nu}} + \overline{\mathbf{t}} \otimes \overline{\mathbf{t}} + \overline{\mathbf{n}} \otimes \overline{\mathbf{n}}. \tag{22}$$

Differentiating $\check{\mathbf{Q}}_t$ with regard to $\overline{s}$, the length parameter along the deformed boundary contour $\partial\overline{\mathcal{M}}$, it was found in [5] that with accuracy to small terms $\beta_{\alpha\beta}\gamma_{\lambda3}$,

$$\frac{d\check{\mathbf{Q}}_t}{d\overline{s}}(\check{\mathbf{Q}}_t)^T \simeq \check{\boldsymbol{\lambda}}_t \times \overline{\mathbf{1}}, \quad \check{\boldsymbol{\lambda}}_t = -\check{\kappa}_{tt}\overline{\boldsymbol{\nu}} + \check{\kappa}_{t\nu}\overline{\mathbf{t}} - \check{\kappa}_{tn}\overline{\mathbf{n}}. \tag{23}$$

$$-\check{\kappa_{tt}} \simeq \overline{\sigma}_t - \frac{1}{a_t}\sigma_t - 2\overline{\kappa}_t\gamma_{\nu 3} - 2\frac{d\gamma_{t3}}{d\overline{s}},$$

$$\check{\kappa_{t\nu}} \simeq \overline{\tau}_t - \frac{1}{a_t}\tau_t + 2\frac{d\gamma_{\nu 3}}{d\overline{s}} - 2\overline{\kappa}_t\gamma_{t3}, \quad (24)$$

$$-\check{\kappa_{tn}} \simeq \overline{\kappa}_t - \frac{1}{a_t}\kappa_t + 2\overline{\sigma}_t\gamma_{\nu 3} + 2\overline{\tau}_t\gamma_{t3}.$$

Let us show that, to within an error of geometric non-linearity, the relations (24) are compatible with (18) and (21). Indeed, to within small strains

$$\overline{\sigma}_t \simeq \sigma_t - \beta_{tt}, \quad \overline{\tau}_t \simeq \tau_t + \beta_{\nu t}, \quad \overline{\kappa}_t \simeq \kappa_t, \quad (25)$$

and changing differentiation in (8) from d/ds into $d/d\overline{s} = a_t{}^{-1}d/ds$ we obtain

$$\frac{d\mathbf{Q}_t}{d\overline{s}}\mathbf{Q}_t^{\mathrm{T}} = \overline{\boldsymbol{\lambda}}_t \times \hat{\mathbf{i}}, \quad \overline{\boldsymbol{\lambda}}_t = \frac{1}{a_t}\boldsymbol{\lambda}_t = \frac{1}{a_t}(-\kappa_{tt}\boldsymbol{\mu} + \kappa_{t\nu}\boldsymbol{\tau} - \kappa_{tn}\mathbf{d}). \quad (26)$$

But for small shears $\gamma_{\nu 3}$ and γ_{t3},

$$\boldsymbol{\mu} \simeq \overline{\boldsymbol{\nu}} - 2\gamma_{\nu 3}\overline{\mathbf{n}}, \quad \boldsymbol{\tau} \simeq \overline{\mathbf{t}} - 2\gamma_{t3}\overline{\mathbf{n}},$$
$$\mathbf{d} \simeq \overline{\mathbf{n}} + 2\gamma_{\nu 3}\overline{\boldsymbol{\nu}} + 2\gamma_{t3}\overline{\mathbf{t}}. \quad (27)$$

If (27) is introduced into (26) then, with the help of (25) and (21), and with accuracy to small terms $\beta_{\alpha\beta}\gamma_{\lambda 3}$, we can approximately represent $\overline{\boldsymbol{\lambda}}_t$ in the intermediate basis $\overline{\boldsymbol{\nu}}, \overline{\mathbf{t}}, \overline{\mathbf{n}}$ with components (24) proposed by Mikhailovskii [5].

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