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On deformational boundary quantities in the nonlinear theory of shear-deformable shells

Deformational boundary quantities in the non-linear theory of shells with account of transverse shear and normal strains were derived by Pietraszkiewicz [1,2]. These five quantities to be prescribed along the shell boundary contour are: the elongation γ_{tt}, the transverse shear γ_{t3}, and three components of the vector $\mathbf{k}_t$ of change of curvature of the boundary contour. The components of $\mathbf{k}_t$ are themselves complex functions of the shell strain measures, and follow from differentiation along the boundary contour of the total rotation of the shell lateral boundary surface.

In this report we show that the total rotation can be defined in two alternative, non-equivalent ways. Its first definition leads to the results given in [1,2]. Its second definition leads to the alternative results reported briefly here.

Let the lateral boundary surface $\partial\mathcal{B}^*$ of the undeformed shell $\mathcal{B}$ be defined by the position vector $\mathbf{p}(s,\zeta) = \mathbf{r}(s) + \zeta\mathbf{n}(s)$, where $\mathbf{r}$ is the position vector of the boundary contour $\partial\mathcal{M}$ of the shell reference surface $\mathcal{M}$, $\mathbf{n}$ is the unit vector normal to $\mathcal{M}$, s is the arc length along $\partial\mathcal{M}$, and ζ is the distance from $\mathcal{M}$. The surface $\partial\mathcal{B}^*$ is rectilinear and orthogonal to $\mathcal{M}$ along $\partial\mathcal{M}$. During the shell deformation described in detail in [1,2], it is assumed that $\partial\mathcal{B}^*$ moves into the surface $\partial\bar{\mathcal{B}}^*$ defined by the position vector $\bar{\mathbf{p}}(s,\zeta) = \bar{\mathbf{r}}(s)+\zeta\bar{\mathbf{a}}_3(s)$. Now $\bar{\mathbf{r}} = \mathbf{r}+\mathbf{u}$ is the position vector of the boundary contour $\partial\bar{\mathcal{M}}$ of the deformed reference surface $\bar{\mathcal{M}}$, $\mathbf{u}$ is the displacement field, $\bar{\mathbf{a}}_3 = \mathbf{n}+\boldsymbol{\beta}$, $\boldsymbol{\beta}$ is the difference vector, and s,ζ are convected coordinates. The surface $\partial\bar{\mathcal{B}}^*$ is again rectilinear, although not orthogonal to $\bar{\mathcal{M}}$ along $\partial\bar{\mathcal{M}}$. Within such an assumption, the spatial Green strain tensor E_{ab}, $a = 1,2,3$, in the neighbourhood of $\mathcal{M}$ is expressed through the shell strain measures $\gamma_{ab}, \pi_{\alpha b}$ according to: $E_{\alpha\beta} = \gamma_{\alpha\beta} + \zeta\pi_{(\alpha\beta)}$, $E_{\alpha 3} = \gamma_{\alpha 3} + \frac{1}{2}\zeta\pi_{\alpha 3}$, $E_{33} = \gamma_{33}$, $\alpha = 1,2$, where $\pi_{(\alpha\beta)} = \frac{1}{2}(\pi_{\alpha\beta} + \pi_{\beta\alpha})$, and $\gamma_{ab}, \pi_{\alpha b}$ are quadratic functions of $\mathbf{u}, \boldsymbol{\beta}$ and their surface derivatives.

With $\partial\mathcal{B}^*$ we can associate the natural orthonormal triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ along $\partial\mathcal{M}$, such that $\mathbf{t} = d\mathbf{r}/ds \equiv \mathbf{r}' = \mathbf{r}_{,\alpha}t^\alpha$, $\boldsymbol{\nu} = \mathbf{r}_{,\alpha}\nu^\alpha \equiv \mathbf{r}_{,\nu} = \mathbf{t}\times\mathbf{n}$, where $(\ )_{,\alpha} \equiv \partial/\partial\theta^\alpha(\ )$ and θ^α are surface curvilinear coordinates on $\mathcal{M}$. After the shell deformation the triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ is transformed into the skew triad of non-unit vectors $\bar{\mathbf{a}}_\nu, \bar{\mathbf{a}}_t, \bar{\mathbf{a}}_3$, such that $\bar{\mathbf{a}}_\nu = \bar{\mathbf{r}}_{,\alpha}\nu^\alpha$, $\bar{\mathbf{a}}_t = \bar{\mathbf{r}}_{,\alpha}t^\alpha = a_t\bar{\mathbf{t}}$, $\bar{\mathbf{a}}_3 = a_3\mathbf{d}$, where $a_t = |\bar{\mathbf{a}}_t|$, $a_3 = |\bar{\mathbf{a}}_3|$ and $\bar{\mathbf{t}}, \mathbf{d}$ are the unit vectors.

In order to define the total rotation of $\partial\bar{\mathcal{B}}^*$ relative to $\partial\mathcal{B}^*$, we should associate also with $\partial\bar{\mathcal{B}}^*$ some orthonormal triad. Let us introduce the vector $\mathbf{a}_\mu = \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 = a_\mu\boldsymbol{\mu}$, where $a_\mu = |\bar{\mathbf{a}}_\mu|$ and $\boldsymbol{\mu}$ is the unit vector normal to $\partial\bar{\mathcal{B}}^*$ along $\partial\bar{\mathcal{M}}$. Since $\bar{\mathbf{t}}$ and $\mathbf{d}$ are not mutually orthogonal, it is apparent that we can associate with $\partial\bar{\mathcal{B}}^*$ two different triads of unit vectors: $\boldsymbol{\mu}, \bar{\mathbf{t}}, \mathbf{m}$ and $\boldsymbol{\mu}, \boldsymbol{\tau}, \mathbf{d}$. Here $\mathbf{a}_m = \mathbf{a}_\mu \times \bar{\mathbf{a}}_t = a_m\mathbf{m}$, $a_m = |\mathbf{a}_m| = a_\mu a_t$ and $\mathbf{a}_\tau = \bar{\mathbf{a}}_3 \times \mathbf{a}_\mu = a_\tau\boldsymbol{\tau}$, $a_\tau = |\mathbf{a}_\tau| = a_\mu a_3$. Each of the two triads can be regarded as a different analogue on $\partial\bar{\mathcal{B}}^*$ of the triad $\boldsymbol{\nu}, \mathbf{t}, \mathbf{n}$ given on $\partial\mathcal{B}^*$.

In the papers [1,2] the total rotation of $\partial\bar{\mathcal{B}}^*$ relative to $\partial\mathcal{B}^*$ was described in terms of the triad $\boldsymbol{\mu}, \bar{\mathbf{t}}, \mathbf{m}$. We introduced the total rotation tensor $\mathbf{R}_t = \boldsymbol{\mu}\otimes\boldsymbol{\nu} + \bar{\mathbf{t}}\otimes\mathbf{t} + \mathbf{m}\otimes\mathbf{n}$, which differentiated with respect to s leads to $\mathbf{R}_t^T\mathbf{R}_t' = \mathbf{k}_t \times \mathbf{1}$, where $\mathbf{1} = \boldsymbol{\nu}\otimes\boldsymbol{\nu} + \mathbf{t}\otimes\mathbf{t} + \mathbf{n}\otimes\mathbf{n}$ and $\mathbf{k}_t = -k_{tt}\boldsymbol{\nu} + k_{t\nu}\mathbf{t} - k_{tn}\mathbf{n}$. Exact expressions for $k_{tt}, k_{t\nu}, k_{tn}$ in terms of the shell strain measures, and several approximate expressions appropriate for particular shell theories, were derived in [1,2].

The total rotation of $\partial\bar{\mathcal{B}}^*$ relative to $\partial\mathcal{B}^*$ can alternatively be described in terms of the triad $\boldsymbol{\mu}, \boldsymbol{\tau}, \mathbf{d}$. Let us introduce the alternative total rotation tensor $\mathbf{Q}_t = \boldsymbol{\mu}\otimes\boldsymbol{\nu} + \boldsymbol{\tau}\otimes\mathbf{t} + \mathbf{d}\otimes\mathbf{n}$, then differentiate it with respect to s to obtain $\mathbf{Q}_t^T\mathbf{Q}_t' = \boldsymbol{\kappa}_t \times \mathbf{1}$, where now $\boldsymbol{\kappa}_t = -\kappa_{tt}\boldsymbol{\nu} + \kappa_{t\nu}\mathbf{t} - \kappa_{tn}\mathbf{n}$ is the alternative vector of change of curvature of $\partial\bar{\mathcal{B}}^*$ along $\partial\bar{\mathcal{M}}$. Each of the components of $\boldsymbol{\kappa}_t$ can now be expressed entirely in terms of the shell strain measures $\gamma_{ab}, \pi_{(\alpha\beta)}$ in the way presented in detail in [3]. These expressions are:

$$-\kappa_{tt} = \frac{1}{a_\tau a_3}[a_3^2(a_t^2(a_t^2\sigma_t - 2\tau_t\gamma_{\nu t} - \gamma_{\alpha 3\beta}t^\alpha t^\beta) + 2\gamma_{t3}(2\tau_t\gamma_{\nu 3} - 2\sigma_t\gamma_{t3} + \gamma_{33\beta}t^\beta)] - \sigma_t,$$

$$\kappa_{t\nu} = \frac{1}{a_\mu a_3}\sqrt{\frac{\bar{a}}{a}}(\tau_t + \nu_\rho\bar{a}^{\rho c}\gamma_{c3\beta}t^\beta) - \tau_t, \tag{1}$$

$$-\kappa_{tn} = \frac{1}{a_\mu a_\tau}\sqrt{\frac{\bar{a}}{a}}[a_3^2(\kappa_t - \nu_\rho\bar{a}^{\rho c}\gamma_{c\alpha\beta}t^\alpha t^\beta) + 2\gamma_{t3}(\tau_t + \nu_\rho\bar{a}^{\rho c}\gamma_{c3\beta}t^\beta)] - \kappa_t.$$

Here σ_t, τ_t and κ_t are the normal curvature, the geodesic torsion and the geodesic curvature of $\partial\mathcal{M}$, respectively, $a_\mu, a_\tau, a_3, \sqrt{\bar{a}/a}$, $\bar{a}^{\rho c}, \gamma_{cab}$ are known functions of the shell strain measures $\gamma_{ab}, \pi_{(\alpha\beta)}$, while $\gamma_{\nu t} = \gamma_{\alpha\beta}\nu^\alpha t^\beta$, $\gamma_{\nu 3} =$

$\gamma_{\alpha 3}\nu^{\alpha}$ etc. are physical components of the shell strain tensor at $\partial\mathcal{M}$.

The expressions (1) for the deformational boundary quantities $\kappa_{tt}, \kappa_{t\nu}, \kappa_{tn}$ are exact within the assumed linear distribution of displacement field across the shell thickness, i.e. they are valid for unrestricted values of $\gamma_{ab}, \pi_{(\alpha\beta)}$. In the definition of γ_{cab} given in [1,2], only the first derivative of $E_{\alpha\beta}$ in the normal direction parametrised by ζ is present, with the value taken at $\zeta = 0$. As a result, any more complex distribution of deformation across the shell thickness, such as used in [4,5] for rubber-like shells for example, cannot change the expressions (1).

In the case of geometric non-linearity, $\gamma_{ab} \sim h\pi_{(\alpha\beta)} \sim O(\eta)$, where $\eta \ll 1$. Then all the relations (1) can consistently be approximated by linear terms in each of the shell strain measures. Additionally, the shell bending tensor $\pi_{(\alpha\beta)}$ can be expressed through the tensor $\beta_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta})$ of change of curvature of the reference surface by the consistently approximated relation $\pi_{(\alpha\beta)} \simeq \beta_{\alpha\beta} - (b_{\alpha\beta} - \beta_{\alpha\beta})\gamma_{33} + \gamma_{\alpha 3}|_{\beta} + \gamma_{\beta 3}|_{\alpha}$, where $\bar{b}_{\alpha\beta} = -\bar{\mathbf{r}}_{,\alpha} \cdot \bar{\mathbf{n}}_{,\beta}$ and $b_{\alpha\beta} = -\mathbf{r}_{,\alpha} \cdot \mathbf{n}_{,\beta}$ are curvature tensors of $\bar{\mathcal{M}}$ and $\mathcal{M}$, respectively, and $(\ )|_{\alpha}$ is the surface covariant derivative in the undeformed metric $a_{\alpha\beta}$. After all the consistent approximations of (2), performed in terms of physical components of $\gamma_{ab}, \beta_{\alpha\beta}$ at $\partial\mathcal{M}$, we obtain

$$\begin{aligned}
\kappa_{tt} &\simeq \beta_{tt} + (\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{tt} + \underline{2\kappa_t\gamma_{\nu 3}} + \underline{2\gamma'_{t3}}, \\
\kappa_{t\nu} &\simeq \beta_{\nu t} + 2(\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{\nu t} - (\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{\nu\nu} + \underline{2\gamma'_{\nu 3}} - \underline{2\kappa_t\gamma_{t3}}, \\
\kappa_{tn} &\simeq 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu\gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) - \underline{2(\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{\nu 3}} - \underline{2(\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{t3}}.
\end{aligned} \tag{2}$$

where $\beta_{\nu t} = \beta_{\alpha\beta}\nu^{\alpha}t^{\beta}$, $\beta_{tt} = \beta_{\alpha\beta}t^{\alpha}t^{\beta}$.

The relations (2) provide consistently approximated explicit expressions for the deformational boundary quantities $\kappa_{tt}, \kappa_{t\nu}, \kappa_{tn}$ appropriate for the geometrically non-linear theory of shells with account of transverse shear and normal strains. In the relations (2) terms underlined by the wavy line are responsible for the geometric non-linearity, together with the non-linearity of strain-displacement relations, while terms underlined by the solid and dashed lines take account of transverse shears. Note that the relations (2) do not depend explicitly upon the normal strains γ_{33}.

The alternative expressions (2) derived here differ only by terms proportional to γ_{t3}, and underlined by the dashed line, from the corresponding expressions for $k_{tt}, k_{t\nu}, k_{tn}$ given in [2]. This is the result of using the alternative total rotation tensor $\mathbf{Q}_t$ here as compared with $\mathbf{R}_t$ used in [1,2]. Within the error of geometric non-linearity the relations (2) can be shown [3] to be equivalent to those introduced by definition in a different form by Mikhailovskii [6].

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References

1 PIETRASZKIEWICZ, W.: Finite rotations and Lagrangean description in the non-linear theory of shells. Polish Sci. Publ. Warszawa-Poznań, 1979.
2 PIETRASZKIEWICZ, W.: Finite rotations in shells. In: KOITER, W.T.; MIKHAILOV, G.K., (eds.): Theory of Shells. North-Holland, Amsterdam 1980, pp. 445-471.
3 PIETRASZKIEWICZ, W.: Deformational boundary quantities in the non-linear theory of shells with transverse shears. Int. J. Solids Structures (submitted).
4 STUMPF, H; MAKOWSKI, J.: On large strain deformations of shells. Acta Mechanica 65 (1986), 153-168.
5 SCHIECK, B; PIETRASZKIEWICZ, W.; STUMPF, H.: Theory and numerical analysis of shells undergoing large elastic strains. Int. J. Solids Structures 29 (1992), 689-709.
6 MIKHAILOVSKII, E.I.: Boundary conditions for the reinforced edge of rigid-flexible shells in the non-linear Timoshenko-Reissner type theory. Mechanics of Solids (English transl. Izvestiia RAN, MTT)(1995), 109-119.

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