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DEFORMATIONAL BOUNDARY QUANTITIES IN THE NONLINEAR THEORY OF SHELLS WITH TRANSVERSE SHEARS†

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Abstract—A thorough analysis is presented of the non linear deformation of the shell lateral boundary surface. The deformation is compatible with the linear distribution of displacements across the shell thickness. It is found that the total rotation of the boundary element can be defined in two ways, by means of two alternative orthonormal triads associated with the deformed shell lateral boundary surface. For both definitions of the rotation, exact expressions for three components of the vector of change of curvature of the boundary contour are derived in terms of shell strain measures. These expressions are then consistently reduced for several particular shell theories.

1. INTRODUCTION

Nonlinear problems of shells with account of transverse shears are usually formulated and solved in terms of displacements and rotations as basic independent field variables. The most general statically and geometrically exact theory of shells was proposed by Simmonds (1984) and developed by Makowski and Stumpf (1990, 1994), with an extensive numerical finite element analysis of representative test examples presented in Chróścielewski *et al.* (1992, 1997), where many references to earlier papers are given.

However, *some* shell problems can be solved in a more convenient way if the corresponding boundary-value problem is formulated in terms of strain and bending measures, associated with the shell reference surface, as basic independent field variables. Such intrinsic shell equations, first proposed by Chien (1944) for thin elastic shells, require boundary conditions to be expressed through the same variables as well. Appropriate combinations of the measures to be assumed at the shell boundary are usually called deformational boundary quantities.

Deformational boundary conditions, expressed in terms of four deformational boundary quantities, were first introduced by Chernykh (1957) into the classical linear theory of thin elastic shells. Under additional requirements discussed by Mikhailovskii and Chernykh (1985), the deformational boundary conditions assure the uniqueness of the corresponding boundary-value problem. The deformational boundary quantities proved very helpful in analyzing linear problems of thin elastic shells, such as thermostatic stresses, shell stiffening, the reinforcement of a shell boundary with a beam, the optimum reinforcement of holes in shells and the connection conditions of two or more shells [see, for example Novozhilov *et al.* (1991)]. The deformational boundary quantities for a nonlinear shell deformation, subjected to the Kirchhoff–Love constraints, were derived by Novozhilov and Shamina (1975) and for a large strain theory of rubber-like shells by Chernykh (1986), where several nonlinear shell problems were also solved. Pietraszkiewicz (1989) derived a set of four deformational boundary conditions for the geometrically nonlinear refined intrinsic shell equations developed by Danielson (1970), Koiter and Simmonds (1973) and Pietraszkiewicz (1977, 1980a).

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Within the linear theory of shells, taking into account transverse shear and normal strains, six deformational boundary quantities were introduced by Shamina (1970). Pietraszkiewicz (1979, 1980b) generalized these quantities to the fully nonlinear range of shell deformation. In latter papers, the linear distribution of displacement field across the shell thickness was assumed. For a linearly elastic material the transverse normal strains were expressed explicitly in terms of other shell strain measures, thus reducing the number of independent deformational boundary quantities to five for this type of shell theory. These quantities to be given along the boundary contour are: the elongation γ_{tt}, the transverse shear γ_{t3} and the three components of the vector $\mathbf{k}_t$ of change of curvature of the shell boundary contour, themselves being complex functions of shell strain measures. The exact expressions for the components of $\mathbf{k}_t$ were then consistently reduced for the geometrically nonlinear shell theory and the results were presented in an easily readable form through physical components of shell strain measures at the boundary.

Deformational boundary quantities, appropriate for the geometrically nonlinear theory of shells with transverse shear and normal strains, have also been proposed by definition in a different form by Mikhailovskii (1995), and then used to derive conditions for the connection of a shell with a beam at a common junction. In that paper, the corresponding vector of change of boundary curvature was calculated from a total rotation tensor, which itself had been assumed as a superposition of two rotations: a finite rotation associated with a Kirchhoff–Love type shell deformation and a small rotation corresponding to small transverse shear strains. Taking into account other definition differences of deformational quantities, it was noticed that components of the vector of change of curvature of Mikhailovskii (1995) do not coincide with the corresponding components derived for the same theory by Pietraszkiewicz (1980b), and when linearized they do not agree with those derived by Shamina (1970). The noted differences are proportional to the transverse shear strains γ_{t3}.

In this report a thorough analysis is presented of deformation of the shell lateral boundary surface, compatible with the linear distribution of displacements across the shell thickness. It is shown that the total rotation of the shell lateral boundary element can be defined in two alternative, non-equivalent ways, depending upon the choice of an orthonormal triad describing the geometry of the deformed boundary element. The first choice of such a triad was discussed in detail by Pietraszkiewicz (1979, 1980b), some of the results of which have been given in Sections 3 and 4, for comparison. Then, in Section 5, consequences of the second choice of such a triad are discussed, exact corresponding expressions for deformational boundary quantities are derived and transformational rules for the recalculation of these quantities from those given by Pietraszkiewicz (1980b) are indicated. By a consistent reduction of the exact relations, several formulae for deformational boundary quantities are obtained in Section 6 for the geometrically nonlinear and linear shell theories with transverse shear and normal strains, for the Kirchhoff–Love type nonlinear theory of shells and for the first-approximation geometrically nonlinear theory of thin isotropic elastic shells. It is also proved that, in the case of geometric nonlinearity, the relations derived here agree with those of Mikhailovskii (1995), to within an error of such a shell theory.

2. NOTATION AND BASIC RELATIONS

Let the undeformed shell $\mathscr{B}$ be parametrized by a normal system of curvilinear coordinates (θ^α, ζ), $\alpha = 1, 2$, such that the position vector $\mathbf{p}(\theta^\alpha, \zeta)$ of any point $P \in \mathscr{B}$ be $\mathbf{p} = \mathbf{r} + \zeta\mathbf{n}$. Here $\mathbf{r} = \mathbf{r}(\theta^\alpha)$ describes a point M on the reference surface $\mathscr{M}$ of $\mathscr{B}$ and ζ is the distance from $\mathscr{M}$. In what follows, $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r}_{,\alpha}$ are the natural base vectors of $\mathscr{M}$, $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ is the covariant (components of the surface) metric tensor with determinant $a = |a_{\alpha\beta}|$, $\mathbf{a}_3 \equiv \mathbf{n} = a^{-1/2}\mathbf{a}_1 \times \mathbf{a}_2$ is the unit normal vector of $\mathscr{M}$, $b_{\alpha\beta} = -\mathbf{n}_{,\alpha} \cdot \mathbf{a}_\beta$ is the covariant curvature tensor of $\mathscr{M}$, and by $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ we denote the surface alternation tensor.

The boundary contour $\partial\mathscr{M}$ of $\mathscr{M}$ consists of piecewise smooth curves described by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along $\partial\mathscr{M}$. At each regular point, $M \in \partial\mathscr{M}$, we have the unit tangent vector $\mathbf{t} = \mathbf{a}_\alpha t^\alpha = \mathrm{d}\mathbf{r}/\mathrm{d}s \equiv \mathbf{r}'$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r}_{,\alpha}\nu^\alpha \equiv \mathbf{r}_{,\nu}$ such that $\boldsymbol{\nu} = \mathbf{t} \times \mathbf{n}$. Therefore, the undeformed shell lateral boundary surface

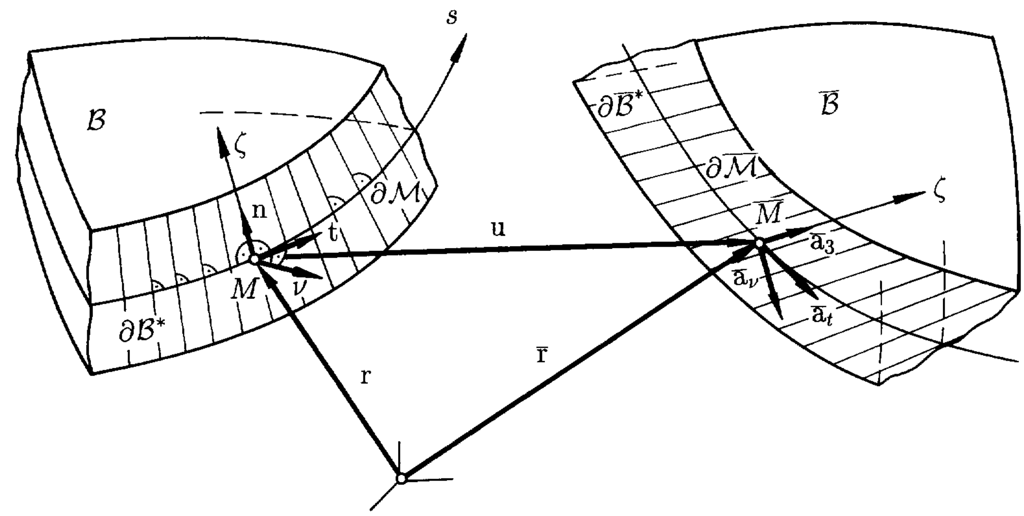

Fig. 1. Geometry of the shell lateral boundary surface in undeformed and deformed configurations.

$\partial\mathscr{B}^*$ defined by $\mathbf{p}(s,\zeta) = \mathbf{r}(s)+\zeta\mathbf{n}(s)$, $-h^- \leqslant \zeta \leqslant +h^+$, where $h = h^- + h^+$ denotes the shell thickness, is rectilinear and orthogonal to $\mathscr{M}$ along $\partial\mathscr{M}$ (Fig. 1).

Let $\bar{\mathscr{M}}$ and $\partial\bar{\mathscr{M}}$ be deformed configurations of $\mathscr{M}$ and $\partial\mathscr{M}$ defined by the position vectors $\bar{\mathbf{r}}(\theta^\alpha) = \mathbf{r}(\theta^\alpha)+\mathbf{u}(\theta^\alpha)$ and $\bar{\mathbf{r}}[\theta^\alpha(s)] = \mathbf{r}(s)+\mathbf{u}(s)$, respectively, where $\mathbf{u}$ is the displacement field of the reference surface, while θ^α and s are convected coordinates. Then with $\bar{\mathscr{M}}$ and $\partial\bar{\mathscr{M}}$ we can associate analogously defined geometric quantities, only now marked by an overbar: $\bar{\mathbf{a}}_\alpha = \bar{\mathbf{r}}_{,\alpha}$, $\bar{a}_{\alpha\beta} = \bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_\beta$, $\bar{a} = |\,\bar{a}_{\alpha\beta}\,|$, $\bar{\mathbf{n}} = \bar{a}^{-1/2}\bar{\mathbf{a}}_1 \times \bar{\mathbf{a}}_2$, $\bar{b}_{\alpha\beta} = -\bar{\mathbf{n}}_{,\alpha}\cdot\bar{\mathbf{a}}_\beta$, $\bar{\varepsilon}_{\alpha\beta} = (\bar{\mathbf{a}}_\alpha \times \bar{\mathbf{a}}_\beta)\cdot\bar{\mathbf{n}}$, $\bar{\mathbf{t}} = \bar{\mathbf{r}}'/|\,\bar{\mathbf{r}}'\,|$, $\bar{\boldsymbol{\nu}} = \bar{\mathbf{t}}\times\bar{\mathbf{n}}$, $\bar{\mathbf{a}}_\nu = \bar{\mathbf{r}}_{,\alpha}\nu^\alpha$, etc. Note that here $\bar{\mathbf{a}}_\nu$ is not colinear with $\bar{\boldsymbol{\nu}}$, due to the shear distortion of $\bar{\mathscr{M}}$. All the barred surface quantities can now be expressed, if necessary, through the geometry of $\mathscr{M}$ and $\partial\mathscr{M}$ and the displacement field $\mathbf{u}$ [see Pietraszkiewicz (1980a, 1989)].

Within the nonlinear theory of shells, in which transverse shear and normal strains are taken into account, the deformation field in the neighborhood of $\mathscr{M}$ is usually approximated by its linear part (Pietraszkiewicz, 1979). Therefore, any material fibre which is initially normal to $\mathscr{M}$, $\mathbf{p} = \mathbf{r}+\zeta\mathbf{n}$, after deformation, takes the position

$$\bar{\mathbf{p}}(\theta^\alpha,\zeta) = \bar{\mathbf{r}}(\theta^\alpha)+\zeta\bar{\mathbf{a}}_3(\theta^\alpha), \tag{1}$$

where $\bar{\mathbf{a}}_3 = \mathbf{n}+\boldsymbol{\beta}$, with the difference vector, $\boldsymbol{\beta}$, as an additional independent field variable. Corresponding components of the spatial Green strain tensor $\mathrm{E}(\theta^\alpha,\zeta)$ are then approximated in the neighborhood of $\mathscr{M}$ by

$$E_{\alpha\beta} = \gamma_{\alpha\beta}+\zeta\pi_{(\alpha\beta)}, \quad E_{\alpha 3} = \gamma_{\alpha 3}+\tfrac{1}{2}\zeta\pi_{\alpha 3}, \quad E_{33} = \gamma_{33}, \tag{2}$$

where $\pi_{(\alpha\beta)} = 1/2(\pi_{\alpha\beta}+\pi_{\beta\alpha})$ and γ_{ab}, $\pi_{\alpha\mathrm{b}}$, $a = 1, 2, 3$, are quadratic functions of $\mathbf{u}$, $\boldsymbol{\beta}$ and their surface derivatives.

The vector $\bar{\mathbf{a}}_3$ in eqn (1) is neither unit nor normal to $\bar{\mathscr{M}}$, in general, and can be represented through the unit vector $\mathbf{d}$ according to

$$\bar{\mathbf{a}}_3 = a_3\mathbf{d}, \quad a_3 = |\,\bar{\mathbf{a}}_3\,| = \sqrt{1+2\gamma_{33}}. \tag{3}$$

For a detailed description of the shell deformation compatible with eqn (1) we refer to Pietraszkiewicz (1979, 1980b).

3. DEFORMATION OF THE SHELL BOUNDARY

During the deformation compatible with eqn (1), the shell undeformed lateral boundary surface $\partial\mathscr{B}^*$ moves into the deformed lateral boundary surface $\partial\bar{\mathscr{B}}^*$ defined by

$$\bar{\mathbf{p}}(s,\zeta) = \bar{\mathbf{r}}(s) + \zeta\bar{\mathbf{a}}_3(s), \tag{4}$$

which is again rectilinear, although no longer orthogonal to $\bar{\mathscr{M}}$ along $\partial\bar{\mathscr{M}}$ (Fig. 1). The orthonormal triad $\mathbf{v}$, $\mathbf{t}$, $\mathbf{n}$, associated with $\partial\mathscr{B}^*$, constitutes a spatial basis at $M \in \partial\mathscr{M}$. During the deformation process, this triad moves into a skew triad of non-unit vectors $\bar{\mathbf{a}}_v$, $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ such that

$$\begin{aligned} \bar{\mathbf{a}}_t &= \bar{\mathbf{r}}_{,\alpha}t^\alpha = a_t\bar{\mathbf{t}}, \quad a_t = |\,\bar{\mathbf{a}}_t\,| = \sqrt{1+2\gamma_{tt}}, \quad \gamma_{tt} = \gamma_{\alpha\beta}t^\alpha t^\beta, \\ \bar{\mathbf{a}}_v &= \bar{\mathbf{r}}_{,\alpha}v^\alpha, \quad a_v = |\,\bar{\mathbf{a}}_v\,| = \sqrt{1+2\gamma_{vv}}, \quad \gamma_{vv} = \gamma_{\alpha\beta}v^\alpha v^\beta. \end{aligned} \tag{5}$$

Note that while in convected coordinates (s, ζ), $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ constitute a surface basis for $\partial\bar{\mathscr{B}}^*$, the vector $\bar{\mathbf{a}}_v$ is not normal to $\partial\bar{\mathscr{B}}^*$ at $\bar{M} \in \partial\bar{\mathscr{M}}$. Therefore, it is convenient to introduce three other vectors

$$\begin{aligned} \mathbf{a}_\mu &= \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 = a_\mu\boldsymbol{\mu}, \quad a_\mu = |\,\mathbf{a}_\mu\,| = \sqrt{(1+2\gamma_{tt})(1+2\gamma_{33}) - 4\gamma_{t3}^2}, \\ \mathbf{a}_m &= \mathbf{a}_\mu \times \bar{\mathbf{a}}_t = a_m\mathbf{m}, \quad a_m = |\,\bar{\mathbf{a}}_m\,| = a_\mu a_t, \\ \mathbf{a}_\tau &= \bar{\mathbf{a}}_3 \times \mathbf{a}_\mu = a_\tau\boldsymbol{\tau}, \quad a_\tau = |\mathbf{a}_\tau = a_\mu a_3, \\ \gamma_{t3} &= \gamma_{\alpha 3}t^\alpha, \quad \gamma_{v3} = \gamma_{v3}v^\alpha, \quad \gamma_{vt} = \gamma_{\alpha\beta}v^\alpha t^\beta. \end{aligned} \tag{6}$$

The unit vector $\boldsymbol{\mu}$ is now normal to $\partial\bar{\mathscr{B}}^*$ although not tangential to $\bar{\mathscr{M}}$, the unit vector $\mathbf{m}$ is orthogonal to $\partial\bar{\mathscr{M}}$, but not normal to $\bar{\mathscr{M}}$, and the unit vector $\boldsymbol{\tau}$ is not tangential to $\partial\bar{\mathscr{M}}$ at $\bar{M} \in \bar{\mathscr{M}}$, although orthogonal to $\mathbf{d}$ and $\boldsymbol{\mu}$.

Derivatives of $\bar{\mathbf{a}}_t$ and $\bar{\mathbf{a}}_3$ with regard to s can be expressed entirely through the shell strain measures [Pietraszkiewicz (1979) Section 4.4]:

$$\begin{aligned} \bar{\mathbf{a}}_t' &= \bar{\mathbf{a}}_{\alpha,\beta}t^\alpha t^\beta + \bar{\mathbf{a}}_\alpha t^\alpha{}_{,\beta}t^\beta = (t^\alpha{}_{|\beta}\bar{\mathbf{a}}_\alpha + t^\alpha b_{\alpha\beta}\bar{\mathbf{a}}_3 + \bar{a}^{dc}\gamma_{c\alpha\beta}t^\alpha\bar{\mathbf{a}}_d)t^\beta, \\ \bar{\mathbf{a}}_3' &= \bar{\mathbf{a}}_{3,\beta}t^\beta = (-b_\beta^\lambda\bar{\mathbf{a}}_\lambda + \gamma_{c3\beta}\bar{\mathbf{a}}^c)t^\beta, \end{aligned} \tag{7}$$

where

$$\begin{aligned} \gamma_{cab} &= \gamma_{ca;b} + \gamma_{cb;a} - \gamma_{ab;c}, \quad \bar{a}^{ab}\bar{a}_{ac} = \delta_c^b, \\ \bar{a}^{ad} &= \frac{1}{2}\frac{a}{\bar{a}}\varepsilon^{abc}\varepsilon^{def}(a_{be}+2\gamma_{be})(a_{cf}+2\gamma_{cf}), \\ \frac{\bar{a}}{a} &= \frac{1}{6}\varepsilon^{abc}\varepsilon^{def}(a_{ad}+2\gamma_{ad})(a_{be}+2\gamma_{be})(a_{cf}+2\gamma_{cf}), \\ \bar{a} &= |\,a_{ab}\,|, \quad \varepsilon_{abc} = (\mathbf{a}_a \times \mathbf{a}_b)\cdot\mathbf{a}_c, \quad \bar{\mathbf{a}}^c = \bar{a}^{cd}\bar{\mathbf{a}}_d. \end{aligned} \tag{8}$$

In eqns (7) and (8) we have explicitly used spatial bases $\mathbf{a}_a = (\mathbf{a}_\alpha, \mathbf{a}_3 \equiv \mathbf{n})$ at $\mathscr{M}$ and $\bar{\mathbf{a}}_a = (\bar{\mathbf{a}}_\alpha, \bar{\mathbf{a}}_3)$ at $\bar{\mathscr{M}}$ respectively, as well as denoting, by $(\;)_{;a}$ the spatial covariant derivative in the metric $a_{ab} = \mathbf{a}_a \cdot \mathbf{a}_b$ and by $(\;)_{|\alpha}$ the surface covariant derivative in the metric $a_{\alpha\beta}$. Spatial indices in the deformed configuration are raised with the help of $\bar{a}^{ab}$, where $\bar{a}^{\alpha\beta} \neq \bar{a}^{\alpha\beta}$, in general.

4. DEFORMATIONAL BOUNDARY QUANTITIES

It was shown by Pietraszkiewicz (1979, 1980b) that the rectilinear boundary surface $\partial\bar{\mathscr{B}}^*$ can be defined, with an accuracy up to a rigid-body deformation in space, by three strains γ_{tt}, γ_{t3} and γ_{33}, and the vector $\mathbf{k}_t$ of change of curvature of the shell boundary contour. Within the constraint (1) and for any particular elastic material behavior, the transverse normal strain γ_{33} is expressible through other shell strain measures. Therefore, only five independent scalar deformational boundary quantities, γ_{tt}, γ_{t3}, and three components of $\mathbf{k}_t$, define the surface $\partial\bar{\mathscr{B}}^*$.

An explicit expression for $\mathbf{k}_t$ in terms of γ_{ab}, $\pi_{(\alpha\beta)}$ depends upon the choice of the orthonormal triad describing the surface $\partial\bar{\mathscr{B}}^*$ along $\partial\bar{\mathscr{M}}$, which would constitute an analogue of the triad $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ describing $\partial\mathscr{B}^*$ along $\partial\mathscr{M}$. The unit vector $\boldsymbol{\mu}$ defined in eqn (6) is a unique analogue on $\partial\bar{\mathscr{B}}^*$ of the unit vector $\boldsymbol{\nu}$ on $\partial\mathscr{B}^*$. Two other unit vectors $\bar{\mathbf{t}}$ and $\mathbf{d}$ tangential to $\partial\bar{\mathscr{B}}^*$, which result during the deformation process from the respective $\mathbf{t}$ and $\mathbf{n}$ tangential to $\partial\mathscr{B}^*$, are not mutually orthogonal. We are free to include only one of them into the triad and a third unit vector is then uniquely defined by a vector product with $\boldsymbol{\mu}$. It follows, from this discussion, that with $\partial\bar{\mathscr{B}}^*$ we can associate two different orthonormal triads: $\boldsymbol{\mu}$, $\bar{\mathbf{t}}$, $\mathbf{m}$ or $\boldsymbol{\mu}$, $\boldsymbol{\tau}$, $\mathbf{d}$. Each of the triads can be regarded as an analogue on $\partial\bar{\mathscr{B}}^*$ of the triad $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ on $\partial\mathscr{B}^*$.

In the papers of Pietraszkiewicz (1979, 1980b), the description of $\partial\bar{\mathscr{B}}^*$ was performed in terms of the triad $\boldsymbol{\mu}$, $\bar{\mathbf{t}}$, $\mathbf{m}$. It seemed more natural to include $\bar{\mathbf{t}}$, the unit tangent to $\partial\bar{\mathscr{M}}$, rather than $\mathbf{d}$, into the triad describing $\partial\bar{\mathscr{B}}^*$ along $\partial\bar{\mathscr{M}}$. Let us recall some results associated with such a choice for a further comparison.

The total rotation tensor $\mathbf{R}_t$ associated with the triad $\boldsymbol{\mu}$, $\bar{\mathbf{t}}$, $\mathbf{m}$, as well as corresponding vectors $\mathbf{k}_t$ and $\mathbf{l}_t$ of change of curvature, can concisely be defined according to Pietraszkiewicz and Badur (1983) by

$$
\begin{aligned}
\mathbf{R}_t &= \boldsymbol{\mu}\otimes\boldsymbol{\nu}+\bar{\mathbf{t}}\otimes\mathbf{t}+\mathbf{m}\otimes\mathbf{n},\\
\mathbf{R}_t^{\mathrm{T}}\mathbf{R}_t' &= \mathbf{k}_t\times\mathbf{1},\quad \mathbf{R}_t'\mathbf{R}_t^{\mathrm{T}} = \mathbf{l}_t\times\bar{\mathbf{1}},\\
\mathbf{k}_t &= +k_{tt}\boldsymbol{\nu}+k_{t\nu}\mathbf{t}-k_{tn}\mathbf{n},\quad \mathbf{l}_t = \mathbf{R}_t\mathbf{k}_t,\\
\mathbf{1} &= \boldsymbol{\nu}\otimes\boldsymbol{\nu}+\mathbf{t}\otimes\mathbf{t}+\mathbf{n}\otimes\mathbf{n},\quad \bar{\mathbf{1}} = \boldsymbol{\mu}\otimes\boldsymbol{\mu}+\bar{\mathbf{t}}\otimes\bar{\mathbf{t}}+\mathbf{m}\otimes\mathbf{m}.
\end{aligned}\tag{9}
$$

Here $\otimes$ denotes the tensor product, $(\ )^{\mathrm{T}}$ the transposition of the tensor and $\mathbf{1}$, $\bar{\mathbf{1}}$ are different representations of the metric tensor of the three-dimensional (3D) Euclidean vector space.

Introducing eqns (5)–(7) into eqn (9), after some transformations, we can derive exact formulae for the components of $\mathbf{k}_t$ expressed entirely in terms of $\gamma_{\alpha\beta}$, $\pi_{(\alpha\beta)}$ [see eqn (62) of Pietraszkiewicz (1980b)]:

$$
\begin{aligned}
-k_{tt} &= \frac{1}{a_t a_m}[a_t^2(2\gamma_{t3}'+\sigma_t-\pi_{(tt)})-2\gamma_{t3}\gamma_{tt}']-\sigma_t,\\
k_{t\nu} &= \frac{1}{a_\mu a_m}\sqrt{\frac{\tilde{a}}{a}}[a_t^2(\tau_t+\nu_\lambda\tilde{a}^{\lambda c}\gamma_{c3\beta}t^\beta)+2\gamma_{t3}(\kappa_t-\nu_\lambda\tilde{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha t^\beta)]-\tau_t,\\
-k_{tn} &= \frac{1}{a_\mu a_t}\sqrt{\frac{\tilde{a}}{a}}(\kappa_t-\nu_\lambda\tilde{a}^{\lambda c}\gamma_{c\alpha\beta}t^\alpha t^\beta)-\kappa_t,
\end{aligned}\tag{10}
$$

where σ_t, τ_t, κ_t are the normal curvature, the geodesic torsion and the geodesic curvature of $\partial\mathscr{M}$, respectively, and $\pi_{(tt)} = \pi_{(\alpha\beta)}t^\alpha t^\beta$.

In the case of geometric nonlinearity, i.e. when $\gamma_{ab}\sim h\pi_{(ab)}\sim O(\eta)$, $\eta\ll 1$, the expressions (10), written in terms of physical components of the shell strain measures, can consistently be reduced to [somewhat refining eqn (92) of Pietraszkiewicz (1980b)]:

$$
\begin{aligned}
k_{tt} &= \pi_{(tt)}+(\sigma_t-\underset{\sim}{\pi_{(tt)}})(\gamma_{tt}+\gamma_{33})-\underline{2\gamma_{t3}'},\\
k_{t\nu} &= \pi_{(\nu t)}+2(\sigma_t-\underset{\sim}{\pi_{(tt)}})\gamma_{\nu t}-(\tau_t+\underset{\sim}{\pi_{(\nu t)}})(\gamma_{\nu\nu}+\gamma_{33})\\
&\quad +\underline{\gamma_{\nu t}'+\kappa_\nu\gamma_{\nu 3}-\gamma_{t3,\nu}+\kappa_t\gamma_{t3}},
\end{aligned}
$$

$$k_{tn} = 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu \gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}) - \underset{\sim\sim\sim\sim\sim\sim}{\underline{2(\sigma_t - \pi_{(tt)})\gamma_{\nu 3}}}. \tag{11}$$

In the relations (11), terms underlined by a wavy line are responsible for a geometric nonlinearity (apart from the nonlinearity of strain–displacement relations), and those underlined by a solid line take account of transverse shears. Indeed, if terms underlined by a wavy line are omitted, and in expressions for all the remaining shell strain measures only linear terms of $\mathbf{u}$, $\boldsymbol{\beta}$ are taken into account, the relations (11) reduce themselves to those given by Shamina (1970) for the linear theory of shells with transverse shear and normal strains. However, in order to make such a comparison, formulae (4.17) of Shamina (1970) should still be expressed explicitly through physical components along $\partial\mathcal{M}$.

5. ALTERNATIVE DEFORMATIONAL BOUNDARY QUANTITIES

With the lateral boundary surface $\partial\bar{\mathscr{B}}^*$ of the deformed shell, we can associate an alternative orthonormal triad $\boldsymbol{\mu}$, $\boldsymbol{\tau}$, $\mathbf{d}$, which also fully describes the geometry of $\partial\bar{\mathscr{B}}^*$ along $\partial\bar{\mathcal{M}}$. This triad allows one to define an alternative total rotation tensor $\mathbf{Q}_t$ of $\partial\bar{\mathscr{B}}^*$, as well as corresponding vectors of change of curvature $\boldsymbol{\kappa}_t$ and $\boldsymbol{\lambda}_t$, by expressions analogous to eqn (9):

$$\begin{aligned}
\mathbf{Q}_t &= \boldsymbol{\mu} \otimes \boldsymbol{\nu} + \boldsymbol{\tau} \otimes \mathbf{t} + \mathbf{d} \otimes \mathbf{n},\\
\mathbf{Q}_t^T\mathbf{Q}'_t &= \boldsymbol{\kappa}_t \times \mathbf{1}, \quad \mathbf{Q}'_t\mathbf{Q}_t^T = \boldsymbol{\lambda}_t \times \bar{\mathbf{1}},\\
\boldsymbol{\kappa}_t &= +\kappa_{tt}\boldsymbol{\nu} + \kappa_{t\nu}\mathbf{t} - \kappa_{tn}\mathbf{n}, \quad \boldsymbol{\lambda}_t = \mathbf{Q}_t\boldsymbol{\kappa}_t,\\
\bar{\mathbf{1}} &= \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \boldsymbol{\tau} \otimes \boldsymbol{\tau} + \mathbf{d} \otimes \mathbf{d}.
\end{aligned} \tag{12}$$

An explicit relation for $\boldsymbol{\kappa}_t$ follows now from the inverse of eqn $(12)_2$ to be

$$\boldsymbol{\kappa}_t = \tfrac{1}{2}(\boldsymbol{\nu} \times \mathbf{Q}_t^T\mathbf{Q}'_t\boldsymbol{\nu} + \mathbf{t} \times \mathbf{Q}_t^T\mathbf{Q}'_t\mathbf{t} + \mathbf{n} \times \mathbf{Q}_t^T\mathbf{Q}'_t\mathbf{n}). \tag{13}$$

The rules for differentiating the triad $\boldsymbol{\nu}$, $\mathbf{t}$, $\mathbf{n}$ along $\partial\mathcal{M}$ are known:

$$\boldsymbol{\nu}' = \kappa_t\mathbf{t} - \tau_t\mathbf{n}, \quad \mathbf{t}' = \sigma_t\mathbf{n} - \kappa_t\boldsymbol{\nu}, \quad \mathbf{n}' = \tau_t\boldsymbol{\nu} - \sigma_t\mathbf{t}. \tag{14}$$

We can now calculate $\mathbf{Q}'_t$ from eqn $(12)_1$, introduce the result into eqn (13), and use eqns (6) and (14), which leads to two equivalent definitions of the components of $\boldsymbol{\kappa}_t$:

$$\begin{aligned}
-\kappa_{tt} &= \frac{1}{a_\tau a_3}\mathbf{a}'_\tau \cdot \bar{\mathbf{a}}_3 - \sigma_t = -\frac{1}{a_\tau a_3}\mathbf{a}_\tau \cdot \bar{\mathbf{a}}'_3 - \sigma_t,\\
\kappa_{t\nu} &= \frac{1}{a_3 a_\mu}\bar{\mathbf{a}}'_3 \cdot \mathbf{a}_\mu - \tau_t = -\frac{1}{a_3 a_\mu}\bar{\mathbf{a}}_3 \cdot \mathbf{a}'_\mu - \tau_t,\\
-\kappa_{tn} &= \frac{1}{a_\mu a_\tau}\mathbf{a}'_\mu \cdot \mathbf{a}_\tau - \kappa_t = -\frac{1}{a_\mu a_\tau}\mathbf{a}_\mu \cdot \mathbf{a}'_\tau - \kappa_t.
\end{aligned} \tag{15}$$

From the relations (6) we can show that

$$\begin{aligned}
\mathbf{a}_\tau &= a_3^2\bar{\mathbf{a}}_t - 2\gamma_{t3}\bar{\mathbf{a}}_3,\\
\mathbf{a}'_\tau &= 2\gamma'_{33}\bar{\mathbf{a}}_t + a_3^2\bar{\mathbf{a}}'_t - 2\gamma'_{t3}\bar{\mathbf{a}}_3 - 2\gamma_{t3}\bar{\mathbf{a}}'_3,\\
\mathbf{a}'_\mu &= \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}_3 + \bar{\mathbf{a}}_t \times \bar{\mathbf{a}}'_3.
\end{aligned} \tag{16}$$

This means that we are able to calculate all three components of $\boldsymbol{\kappa}_t$ from only two differential expressions (7) for $\bar{\mathbf{a}}_t'$ and $\bar{\mathbf{a}}_3'$. Therefore, introducing eqns (7) and (16) into eqn (15) after transformations we obtain

$$-\kappa_{tt} = \frac{1}{a_\tau a_3}[a_3^2(a_t^2\sigma_t - 2\tau_t\gamma_{vt} - \gamma_{\alpha 3\beta}t^\alpha t^\beta) + 2\gamma_{t3}(2\tau_t\gamma_{v3} - 2\sigma_t\gamma_{t3} + \gamma_{33\beta}t^\beta)] - \sigma_t,$$

$$\kappa_{tv} = \frac{1}{a_\mu a_3}\sqrt{\frac{\tilde{a}}{a}}(\tau_t + v_\rho \tilde{a}^{\rho c}\gamma_{c3\beta}t^\beta) - \tau_t,$$

$$-\kappa_{tn} = \frac{1}{a_\mu a_\tau}\sqrt{\frac{\tilde{a}}{a}}[a_3^2(\kappa_t - v_\rho \tilde{a}^{\rho c}\gamma_{c\alpha\beta}t^\alpha t^\beta) + 2\gamma_{t3}(\tau_t + v_\rho \tilde{a}^{\rho c}\gamma_{c3\beta}t^\beta)] - \kappa_t. \quad (17)$$

The expressions (17) for the deformational boundary quantities are exact within the linear approximation of shell deformation (1), (2) and (4), i.e. they are valid for unrestricted values of γ_{ab} and $\pi_{(\alpha\beta)}$. Note, however, that a more complex distribution of shell deformation in the normal direction [as was used in modeling the behavior of rubber-like shells by Stumpf and Makowski (1986); Schieck *et al.* (1992); Kabric and Chernykh (1996), for example] cannot influence the expressions (17), since only the first derivatives in the normal direction of the spatial Green strains $E_{\alpha\beta}$ are used in the derivation of eqn (17).

Let us introduce the proper orthogonal tensor

$$\mathbf{P}_t = \boldsymbol{\mu} \otimes \boldsymbol{\mu} + \boldsymbol{\tau} \otimes \bar{\mathbf{t}} + \mathbf{d} \otimes \mathbf{m}, \quad \mathbf{Q}_t = \mathbf{P}_t\mathbf{R}_t. \quad (18)$$

The tensor $\mathbf{P}_t$ rotates the orthonormal triad $\boldsymbol{\mu}$, $\bar{\mathbf{t}}$, $\mathbf{m}$ (used in Section 4) into the triad $\boldsymbol{\mu}$, $\boldsymbol{\tau}$, $\mathbf{d}$ used in this section. The axial vector, $\boldsymbol{\omega}_t$, corresponding to the skew-symmetric tensor $\mathbf{P}_t^T\mathbf{P}_t'$ is given by

$$\mathbf{P}_t^T\mathbf{P}_t' = \boldsymbol{\omega}_t \times \mathbf{1}, \quad \boldsymbol{\omega}_t = -\omega_{tt}\boldsymbol{\mu} + \omega_{tv}\bar{\mathbf{t}} - \omega_{tn}\mathbf{m},$$

$$-\omega_{tt} = \frac{1}{a_\tau a_3}\mathbf{a}_\tau' \cdot \bar{\mathbf{a}}_3 - \frac{1}{a_t a_m}\bar{\mathbf{a}}_t' \cdot \mathbf{a}_m,$$

$$\omega_{tv} = \frac{1}{a_3 a_\mu}\bar{\mathbf{a}}_3' \cdot \mathbf{a}_\mu - \frac{1}{a_m a_\mu}\mathbf{a}_m' \cdot \mathbf{a}_\mu,$$

$$-\omega_{tn} = \frac{1}{a_\mu a_\tau}\mathbf{a}_\mu' \cdot \mathbf{a}_\tau - \frac{1}{a_\mu a_t}\mathbf{a}_\mu' \cdot \bar{\mathbf{a}}_t. \quad (19)$$

Explicit expressions for the components of $\boldsymbol{\omega}_t$, in terms of shell strain measures, follow now using eqns (16) and (3)–(8).

If eqn $(18)_2$ is introduced into eqn $(12)_2$, then using eqn $(9)_2$ after transformations we obtain

$$\boldsymbol{\kappa}_t = \mathbf{k}_t + \mathbf{R}_t^T\boldsymbol{\omega}_t.$$

$$\kappa_{tt} = k_{tt} + \omega_{tt}, \quad \kappa_{tv} = k_{tv} + \omega_{tv}, \quad \kappa_{tn} = k_{tn} + \omega_{tn}. \quad (20)$$

This provides simple transformation rules between $\boldsymbol{\kappa}_t$ and $\mathbf{k}_t$. These rules explicitly indicate that the alternative deformational boundary quantities (17) are functionally dependent upon those given in eqn (10).

In order to allow a further discussion of possible simplifications of eqn (17), it is convenient to have all the spatial tensors $\gamma_{c\alpha\beta}$ expressed through physical components of γ_{ab} and $\pi_{(\alpha\beta)}$ along $\partial\mathcal{M}$. For this reason, we should express all spatial covariant derivatives $\gamma_{ab;c}$ in terms of surface covariant derivatives $\gamma_{\alpha\beta|\lambda}$, $\gamma_{\alpha 3|\lambda}$ and $\pi_{(\alpha\beta)}$, γ_{33}, $b_{\alpha\beta}$ keeping in mind that

$$\Gamma^3_{\alpha\beta} = b_{\alpha\beta},\quad \Gamma^\rho_{3\beta} = -b^\rho_\beta,\quad \Gamma^3_{3\alpha} = \Gamma^3_{33} = \Gamma^\alpha_{33} = 0,$$

$$\gamma_{\alpha\beta,3} = \frac{\mathrm{d}}{\mathrm{d}\zeta}E_{\alpha\beta}(\zeta)\,|_{\zeta=0} = \pi_{(\alpha\beta)}. \tag{21}$$

Such transformations for $\gamma_{\alpha3\beta}$, for example, can be performed as follows:

$$\begin{aligned}\gamma_{\alpha3\beta} &= \gamma_{\alpha3;\beta}+\gamma_{\alpha\beta;3}-\gamma_{3\beta;\alpha}\\ &= \gamma_{\alpha3,\beta}-\Gamma^\rho_{\alpha\beta}\gamma_{\rho3}-\Gamma^3_{\alpha\beta}\gamma_{33}-\Gamma^\rho_{3\beta}\gamma_{\alpha\rho}-\Gamma^3_{3\beta}\gamma_{\alpha3}\\ &\quad+\gamma_{\alpha\beta,3}-\Gamma^\rho_{\alpha3}\gamma_{\rho\beta}-\Gamma^3_{\alpha3}\gamma_{3\beta}-\Gamma^\rho_{\beta3}\gamma_{\alpha\rho}-\Gamma^3_{\beta3}\gamma_{\alpha3}\\ &\quad-\gamma_{3\beta,\alpha}+\Gamma^\rho_{3\alpha}\gamma_{\rho\beta}-\Gamma^3_{3\alpha}\gamma_{3\beta}+\Gamma^\rho_{\beta\alpha}\gamma_{3\rho}+\Gamma^3_{\beta\alpha}\gamma_{33}\\ &= \pi_{(\alpha\beta)}+\gamma_{\alpha3|\beta}-\gamma_{\beta3|\alpha}+2b^\rho_\beta\gamma_{\alpha\rho}.\end{aligned} \tag{22}$$

Corresponding expressions for $\gamma_{\lambda\alpha\beta}$, $\gamma_{3\alpha\beta}$ and $\gamma_{33\beta}$ can be obtained in the same way. Then using the geometric relations

$$\begin{gathered}\sigma_t = b_{\alpha\beta}t^\alpha t^\beta,\quad \tau_t = -b_{\alpha\beta}v^\alpha t^\beta,\quad \kappa_t = -v_\alpha t^\alpha|_\beta t^\beta = t_\alpha v^\alpha|_\beta t^\beta,\\ \sigma_v = b_{\alpha\beta}v^\alpha v^\beta,\quad \tau_v = -b_{\alpha\beta}t^\alpha v^\beta = \tau_t,\quad \kappa_v = -t_\alpha v^\alpha|_\beta v^\beta = v_\alpha t^\alpha|_\beta v^\beta,\\ t_\alpha t^\alpha|_\beta t^\beta = 0,\quad v_\alpha v^\alpha|_\beta t^\beta = 0,\quad \varepsilon^{\alpha\beta} = v^\alpha t^\beta - t^\alpha v^\beta,\end{gathered} \tag{23}$$

we are able to calculate the following expression:

$$\begin{aligned}\gamma_{vtt} &= \gamma_{\lambda\alpha\beta}v^\lambda t^\alpha t^\beta = 2\gamma'_{vt}-\gamma_{tt,v}+2\kappa_v\gamma_{vt}+2\kappa_t(\gamma_{vv}-\gamma_{tt})-2\sigma_t\gamma_{v3},\\ \gamma_{ttt} &= \gamma_{\lambda\alpha\beta}t^\lambda t^\alpha t^\beta = \gamma'_{tt}+2\kappa_t\gamma_{vt}-2\sigma_t\gamma_{t3},\\ \gamma_{3tt} &= \gamma_{3\alpha\beta}t^\alpha t^\beta = -\pi_{(tt)}-2\sigma_t\gamma_{33}+2\kappa_v\gamma_{v3}+2\gamma'_{t3},\\ \gamma_{v3t} &= \gamma_{\alpha3\beta}t^\alpha t^\beta = \pi_{(vt)}+2\sigma_t\gamma_{vt}-2\tau_t\gamma_{vv}+\gamma'_{v3}+\kappa_v\gamma_{v3}-\gamma_{t3,v}-\kappa_t\gamma_{t3},\\ \gamma_{t3t} &= \gamma_{\alpha3\beta}t^\alpha t^\beta = \pi_{(tt)}+2\sigma_t\gamma_{tt}-2\tau_t\gamma_{vt},\\ \gamma_{33t} &= \gamma_{33\beta}t^\beta = \gamma'_{33}-2\tau_t\gamma_{v3}+2\sigma_t\gamma_{t3}.\end{aligned} \tag{24}$$

It is now apparent that using eqn (8) and identifying the spatial $\varepsilon^{\alpha\beta3}$ with the surface $\varepsilon^{\alpha\beta}$, we are also able to express $\sqrt{\bar{a}/a}$, $v_\alpha\bar{a}^{\alpha\beta}v_\beta$, $v_\alpha\bar{a}^{\alpha\beta}t_\beta$, $v_\alpha\bar{a}^{\alpha3}$ in terms of physical components of γ_{ab}, $\pi_{(\alpha\beta)}$ along $\partial\mathcal{M}$. As a result, the deformational boundary quantities, eqn (17), can be expressed explicitly in terms of the physical components. However, these exact expressions are quite complex and are not presented here.

6. DEFORMATIONAL BOUNDARY QUANTITIES FOR PARTICULAR SHELL THEORIES

From exact expressions (17), consistently reduced formulae can be derived for several special versions of shell theory. In the case of geometric nonlinearity, i.e. when $\gamma_{ab} \sim h\pi_{(\alpha\beta)} \sim O(\eta)$, $\eta \ll 1$, all the relations should consistently be approximated by linear terms in each of the strain measures. In particular, in this case we can use the following consistent approximations:

$$\begin{gathered}a_\mu \simeq 1+\gamma_{tt}+\gamma_{33},\quad a_\tau \simeq 1+\gamma_{tt}+2\gamma_{33},\quad a_3 \simeq 1+\gamma_{33},\\ \frac{1}{a_\mu} \simeq 1-\gamma_{tt}-\gamma_{33},\quad \frac{1}{a_\tau} \simeq 1-\gamma_{tt}-2\gamma_{33},\quad \frac{1}{a_3} \simeq 1-\gamma_{33},\end{gathered}$$

$$\sqrt{\frac{\tilde{a}}{a}} \simeq 1+\gamma_{vv}+\gamma_{tt}+\gamma_{33}, \quad \sqrt{\frac{a}{\tilde{a}}} \simeq 1-\gamma_{vv}-\gamma_{tt}-\gamma_{33},$$

$$v_\alpha \tilde{a}^{\alpha\beta} v_\beta \simeq 1-2\gamma_{vv}, \quad v_\alpha \tilde{a}^{\alpha\beta} t_\beta \simeq -2\gamma_{vt}, \quad v_\alpha \tilde{a}^{\alpha 3} \simeq -2\gamma_{v3}. \tag{25}$$

When eqn (25) together with eqn (24) is introduced into eqn (17) and only linear terms in each of γ_{ab} and $\pi_{(\alpha\beta)}$ are taken into account, we obtain

$$\kappa_{tt} \simeq \pi_{(tt)}+(\sigma_t-\pi_{(tt)})(\gamma_{tt}+\gamma_{33})-\underline{2\gamma'_{t3}}+2\gamma'_{t3},$$

$$\kappa_{tv} \simeq \pi_{(vt)}+2(\sigma_t-\pi_{(tt)})\gamma_{vt}-(\tau_t+\pi_{(vt)})(\gamma_{vv}+\gamma_{33})$$
$$+\underline{\gamma'_{v3}+\kappa_v\gamma_{v3}-\gamma_{t3,v}+\kappa_t\gamma_{t3}}-2\kappa_t\gamma_{t3},$$

$$\kappa_{tn} \simeq 2\gamma'_{vt}-\gamma_{tt,v}+2\kappa_v\gamma_{vt}+\kappa_t(\gamma_{vv}-\gamma_{tt})-\underline{2(\sigma_t-\pi_{(tt)})\gamma_{v3}}-2(\tau_t+\pi_{(vt)})\gamma_{t3}. \tag{26}$$

The relations (26) provide explicit expressions of deformational boundary quantities for the geometrically nonlinear theory of shells with transverse shear and normal strains.

When eqn (26) is compared with eqn (11), it is seen that the transverse shears are present in eqn (26) not only through terms underlined by a solid line as in eqn (11), but also through additional terms proportional to γ_{t3}, which are underlined by a dashed line. This is the result of the alternative vector $\boldsymbol{\kappa}_t$ used here, as compared with $\mathbf{k}_t$ used to derive eqn (11). Indeed, under the small strains, components of ω_t defined in eqn (19) can consistently be approximated using eqns (6), (16), (23), $(24)_4$ and (25), which yields

$$\omega_{tt} \simeq 2\gamma'_{t3}, \quad \omega_{tv} \simeq -2\kappa_t\gamma_{t3}, \quad \omega_{tn} \simeq -2(\tau_t+\pi_{(vt)})\gamma_{t3}. \tag{27}$$

It is now apparent that the components of $\boldsymbol{\kappa}_t$ given by eqn (26) can be recalculated from components of $\mathbf{k}_t$ given in eqn (11) according to transformation rules (20), where the approximate relations (27) should be applied.

In the relations (26) terms underlined by a wavy line are responsible for the geometric non-linearity, apart from the nonlinearity of strain–displacement relations. Therefore, if these terms are omitted, and in the definitions of all the remaining shell strain measures only linear terms of $\mathbf{u}$, $\boldsymbol{\beta}$ are taken into account, the relations (26) reduce themselves to the following form appropriate to the linear theory of shells with transverse shear and normal strains:

$$\kappa_{tt} \simeq \pi_{(tt)}+\sigma_t(\gamma_{tt}+\gamma_{33})-2\gamma'_{t3}+2\gamma'_{t3},$$

$$\kappa_{vt} \simeq \pi_{(vt)}+2\sigma_t\gamma_{vt}-\tau_t(\gamma_{vv}+\gamma_{33})+\gamma'_{v3}+\kappa_v\gamma_{v3}-\gamma_{t3,v}+\kappa_t\gamma_{t3}-2\kappa_t\gamma_{t3},$$

$$\kappa_{tn} \simeq 2\gamma'_{vt}-\gamma_{tt,v}+2\kappa_v\gamma_{vt}+\kappa_t(\gamma_{vv}-\gamma_{tt})-2\sigma_t\gamma_{v3}-2\tau_t\gamma_{t3}. \tag{28}$$

These expressions differ from those following from Shamina (1970) by terms underlined by a dashed line. This difference results again from the use of the vector $\boldsymbol{\kappa}_t$ here, while Shamina (1970) derived her deformational quantities from a vector which was equivalent to our $\mathbf{k}_t$.

In some applications it may be convenient to have the components of $\boldsymbol{\kappa}_t$ expressed through the tensor $\beta_{\alpha\beta} = -(\bar{b}_{\alpha\beta}-b_{\alpha\beta})$ of change of curvature of the reference surface rather than through $\pi_{(\alpha\beta)}$. In the case of geometric nonlinearity, the kinematic relation for $\pi_{(\alpha\beta)}$ [cf. Pietraszkiewicz (1979)] gives

$$\pi_{(\alpha\beta)} = \tfrac{1}{2}(\bar{\mathbf{a}}_\alpha \cdot \bar{\mathbf{a}}_{3,\beta} + \bar{\mathbf{a}}_\beta \cdot \bar{\mathbf{a}}_{3,\alpha}) + b_{\alpha\beta}$$
$$\simeq \beta_{\alpha\beta} - (b_{\alpha\beta} - \underset{\sim}{\beta_{\alpha\beta}})\gamma_{33} + \gamma_{\alpha 3|\beta} + \gamma_{\beta 3|\alpha},$$
$$\pi_{(\mathrm{tt})} \simeq \beta_{\mathrm{tt}} - (\sigma_{\mathrm{t}} - \underset{\sim}{\beta_{\mathrm{tt}}})\gamma_{33} + 2\kappa_{\mathrm{t}}\gamma_{\nu 3} - 2\gamma'_{\mathrm{t}3},$$
$$\pi_{(\nu\mathrm{t})} \simeq \beta_{\nu\mathrm{t}} + (\tau_{\mathrm{t}} + \underset{\sim}{\beta_{\nu\mathrm{t}}})\gamma_{33} + \gamma'_{\nu 3} - \kappa_\nu\gamma_{\nu 3} + \gamma_{\mathrm{t}3,\nu} - \kappa_{\mathrm{t}}\gamma_{\mathrm{t}3}, \tag{29}$$

which introduced into eqn (26) allows one to find equivalent expressions for the deformational boundary quantities (Pietraszkiewicz, 1997):

$$\kappa_{\mathrm{tt}} \simeq \beta_{\mathrm{tt}} + (\sigma_{\mathrm{t}} - \underset{\sim}{\beta_{\mathrm{tt}}})\gamma_{\mathrm{tt}} + \underline{2\kappa_{\mathrm{t}}\gamma_{\nu 3}} + \underline{2\gamma'_{\mathrm{t}3}},$$
$$\kappa_{\mathrm{t}\nu} \simeq \beta_{\nu\mathrm{t}} + 2(\sigma_{\mathrm{t}} - \underset{\sim}{\beta_{\mathrm{tt}}})\gamma_{\nu\mathrm{t}} - (\tau_{\mathrm{t}} + \underset{\sim}{\beta_{\nu\mathrm{t}}})\gamma_{\nu\nu} + \underline{2\gamma'_{\nu 3}} - \underline{2\kappa_{\mathrm{t}}\gamma_{\mathrm{t}3}},$$
$$\kappa_{\mathrm{tn}} \simeq 2\gamma'_{\nu\mathrm{t}} - \gamma_{\mathrm{tt},\nu} + 2\kappa_\nu\gamma_{\nu\mathrm{t}} + \kappa_{\mathrm{t}}(\gamma_{\nu\nu} - \gamma_{\mathrm{tt}}) - \underline{2(\sigma_{\mathrm{t}} - \underset{\sim}{\beta_{\mathrm{tt}}})\gamma_{\nu 3}} - \underline{2(\tau_{\mathrm{t}} + \underset{\sim}{\beta_{\nu\mathrm{t}}})\gamma_{\mathrm{t}3}}. \tag{30}$$

It is interesting to note that, in this representation, the normal strain γ_{33} has disappeared from all the components of $\boldsymbol{\kappa}_{\mathrm{t}}$. If all the quadratic terms underlined by a wave line are omitted, and the remaining shell strain measures are defined only by linear terms in $\mathbf{u}$, $\boldsymbol{\beta}$, the relations (30) reduce themselves to the form equivalent to eqn (28) for the linear theory of shells with transverse shear and normal strains. If, additionally, all the terms with $\gamma_{\nu 3}$ and $\gamma_{\mathrm{t}3}$ are omitted in eqn (30), we obtain deformational boundary quantities of the classical linear theory of thin shells proposed by Chernykh (1957, 1964).

In the classical nonlinear theory of thin shells based on the Kirchhoff–Love constraints, it is assumed from the outset that $\gamma_{\alpha 3} = \gamma_{33} = 0$ and, therefore, $\tilde{a} = \bar{a}$, $\bar{\mathbf{a}}_3 = \bar{\mathbf{n}}$, $\mathbf{a}_\mu = a_{\mathrm{t}}\bar{\mathbf{v}}$, $\mathbf{a}_{\mathrm{m}} = a_{\mathrm{t}}^2\bar{\mathbf{n}}$, $\mathbf{a}_\tau = \bar{\mathbf{a}}_{\mathrm{t}}$. In this case $\pi_{(\alpha\beta)} = \beta_{\alpha\beta}$, $\boldsymbol{\omega}_{\mathrm{t}} = 0$, $\boldsymbol{\kappa}_{\mathrm{t}} = \mathbf{k}_{\mathrm{t}}$, and both the exact relations (17) and (10) reduce to

$$-\kappa_{\mathrm{tt}} = a_{\mathrm{t}}\bar{\sigma}_{\mathrm{t}} - \sigma_{\mathrm{t}} = \frac{1}{a_{\mathrm{t}}}(\sigma_{\mathrm{t}} - \beta_{\mathrm{tt}}) - \sigma_{\mathrm{t}},$$
$$\kappa_{\mathrm{t}\nu} = a_{\mathrm{t}}\bar{\tau}_{\mathrm{t}} - \tau_{\mathrm{t}} = \frac{1}{a_{\mathrm{t}}}\sqrt{\frac{\bar{a}}{a}}[\tau_{\mathrm{t}} + \nu_\rho\bar{a}^{\rho\lambda}(\beta_{\lambda\beta} + 2b^\kappa_\beta\gamma_{\kappa\lambda})t^\beta] - \tau_{\mathrm{t}},$$
$$-\kappa_{\mathrm{tn}} = a_{\mathrm{t}}\bar{\kappa}_{\mathrm{t}} - \kappa_{\mathrm{t}} = \frac{1}{a_{\mathrm{t}}^2}\sqrt{\frac{\bar{a}}{a}}[\kappa_{\mathrm{t}} - \nu_\rho\bar{a}^{\rho\lambda}(2\gamma_{\lambda\alpha|\beta} - \gamma_{\alpha\beta|\lambda})t^\alpha t^\beta] - \kappa_{\mathrm{t}}, \tag{31}$$

where $\bar{\sigma}_{\mathrm{t}}$, $\bar{\tau}_{\mathrm{t}}$ and $\bar{\kappa}_{\mathrm{t}}$ are the normal curvature, the geodesic torsion and the geodesic curvature of $\partial\bar{\mathscr{M}}$, respectively. The formulae for $-\kappa_{\mathrm{tt}}$ and $-\kappa_{\mathrm{tn}}$ are exactly those derived by Novozhilov and Shamina (1975), while that for $\kappa_{\mathrm{t}\nu}$ agrees with the one derived by Pietraszkiewicz (1979). However, using an equivalent second expression for $\kappa_{\mathrm{t}\nu}$ in eqn $(15)_2$ we would obtain an equivalent relation

$$\kappa_{\mathrm{t}\nu} = -\frac{1}{a_{\mathrm{t}}}\sqrt{\frac{\bar{a}}{a}}\nu_\rho\bar{a}^{\rho\lambda}(b_{\lambda\beta} - \beta_{\lambda\beta})t^\beta - \tau_{\mathrm{t}}, \tag{32}$$

which now coincides with the one given by Novozhilov and Shamina (1975). The relations $(31)_{2,3}$ were presented by Pietraszkiewicz (1979) also in terms of physical components of $\gamma_{\alpha\beta}$, $\beta_{\alpha\beta}$ along $\partial\mathscr{M}$.

Within the first-approximation geometrically nonlinear theory of thin isotropic elastic shells, estimates for the strain measures are:

$$\gamma_{\alpha\beta} \sim h\pi_{(\alpha\beta)} \sim \mathrm{O}(\eta), \quad \gamma_{\alpha 3} \sim h\pi_{\alpha 3} \sim \mathrm{O}(\eta\theta), \quad \gamma_{33} \sim \mathrm{O}(\nu\eta), \quad \eta \ll 1, \quad \theta^2 \ll 1, \tag{33}$$

where θ is a small parameter given in Pietraszkiewicz (1989) and ν is the Poisson ratio. Under these estimates, the relations (30) can consistently be reduced to

$$\begin{aligned}
\kappa_{tt} &\simeq \beta_{tt} + (\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{tt}, \\
\kappa_{t\nu} &\simeq \beta_{\nu t} + 2(\sigma_t - \underset{\sim}{\beta_{tt}})\gamma_{\nu t} - (\tau_t + \underset{\sim}{\beta_{\nu t}})\gamma_{\nu\nu}, \\
\kappa_{tn} &\simeq 2\gamma'_{\nu t} - \gamma_{tt,\nu} + 2\kappa_\nu \gamma_{\nu t} + \kappa_t(\gamma_{\nu\nu} - \gamma_{tt}).
\end{aligned} \tag{34}$$

These are exactly expressions for deformational boundary quantities derived in eqn (6.8) of Pietraszkiewicz (1989) for the geometrically nonlinear (so-called refined) intrinsic shell equations. With appropriate modifications of definitions of the surface strain and bending measures, the deformational boundary quantities (34) can also be used together with intrinsic shell equations discussed by Libai and Simmonds (1983), Axelrad and Emmerling (1988), Libai and Bert (1994a,b) and Valid (1995).

Mikhailovskii (1995) proposed deformational boundary quantities for the geometrically nonlinear theory of shells with transverse shear and normal strains, by introducing a total rotation tensor $\mathbf{Q}_t^\vee$ being a superposition of two rotations: a finite rotation associated with a Kirchhoff–Love type geometrically nonlinear theory of shells and a small rotation corresponding to a linear approximation in small shears $\gamma_{\alpha 3}$:

$$\begin{aligned}
\mathbf{Q}_t^\vee &\simeq [\bar{\mathbf{1}} + 2\gamma_{\nu 3}(\bar{\boldsymbol{\nu}} \otimes \bar{\mathbf{n}} - \bar{\mathbf{n}} \otimes \bar{\boldsymbol{\nu}}) + 2\gamma_{t3}(\bar{\mathbf{t}} \otimes \bar{\mathbf{n}} - \bar{\mathbf{n}} \otimes \bar{\mathbf{t}})](\bar{\boldsymbol{\nu}} \otimes \boldsymbol{\nu} + \bar{\mathbf{t}} \otimes \mathbf{t} + \bar{\mathbf{n}} \otimes \mathbf{n}), \\
\bar{\mathbf{1}} &= \bar{\boldsymbol{\nu}} \otimes \bar{\boldsymbol{\nu}} + \bar{\mathbf{t}} \otimes \bar{\mathbf{t}} + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}.
\end{aligned} \tag{35}$$

Differentiating $\mathbf{Q}_t$ with regard to $\bar{s}$, the length parameter along the deformed boundary contour $\partial\bar{\mathscr{M}}$, it was found that with accuracy to small terms $\beta_{\alpha\beta}\gamma_{\lambda 3}$,

$$\frac{\mathrm{d}\mathbf{Q}_t^\vee}{\mathrm{d}\bar{s}} \simeq -\kappa_{tt}^\vee(\bar{\mathbf{n}} \otimes \mathbf{t} - \bar{\mathbf{t}} \otimes \mathbf{n}) + \kappa_{t\nu}^\vee(\bar{\boldsymbol{\nu}} \otimes \mathbf{n} - \bar{\mathbf{n}} \otimes \boldsymbol{\nu}) - \kappa_{tn}^\vee(\bar{\mathbf{t}} \otimes \boldsymbol{\nu} - \bar{\boldsymbol{\nu}} \otimes \mathbf{t}), \tag{36}$$

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{Q}_t^\vee}{\mathrm{d}\bar{s}}(\mathbf{Q}_t^\vee)^{\mathrm{T}} &\simeq \boldsymbol{\lambda}_t^\vee \times \bar{\mathbf{1}}, \quad \boldsymbol{\lambda}_t^\vee = -\kappa_{tt}^\vee \bar{\boldsymbol{\nu}} + \kappa_{t\nu}^\vee \bar{\mathbf{t}} - \kappa_{tn}^\vee \bar{\mathbf{n}}, \\
-\kappa_{tt}^\vee &\simeq \bar{\sigma}_t - \frac{1}{a_t}\sigma_t - 2\bar{\kappa}_t\gamma_{\nu 3} - 2\frac{\mathrm{d}\gamma_{t3}}{\mathrm{d}\bar{s}}, \\
\kappa_{t\nu}^\vee &\simeq \bar{\tau}_t - \frac{1}{a_t}\tau_t + 2\frac{\mathrm{d}\gamma_{\nu 3}}{\mathrm{d}\bar{s}} - 2\bar{\kappa}_t\gamma_{t3}, \\
-\kappa_{tn}^\vee &\simeq \bar{\kappa}_t - \frac{1}{a_t}\kappa_t + 2\bar{\sigma}_t\gamma_{\nu 3} + 2\bar{\tau}_t\gamma_{t3}.
\end{aligned} \tag{37}$$

Let us show that, to within an error of geometric nonlinearity, the relations (36) and (37) are consistent with the relations (30). Indeed, to within small strains

$$\bar{\sigma}_t \simeq \sigma_t - \beta_{tt}, \quad \bar{\tau}_t \simeq \tau_t + \beta_{\nu t}, \quad \bar{\kappa}_t \simeq \kappa_t, \tag{38}$$

and changing differentiation in eqn (12) from $\mathrm{d}/\mathrm{d}s$ to $\mathrm{d}/\mathrm{d}\bar{s} = a_t^{-1}\, d/ds$ we obtain

$$\frac{\mathrm{d}\mathbf{Q}_t}{\mathrm{d}\bar{s}}\mathbf{Q}_t^{\mathrm{T}} = \bar{\boldsymbol{\lambda}}_t \times \hat{\mathbf{1}}, \quad \bar{\boldsymbol{\lambda}}_t = \frac{1}{a_t}\boldsymbol{\lambda}_t = \frac{1}{a_t}(-\kappa_{tt}\boldsymbol{\mu} + \kappa_v \boldsymbol{\tau} - \kappa_{tn}\mathbf{d}), \tag{39}$$

but for small $\gamma_{\alpha 3}$

$$\boldsymbol{\mu} \simeq \bar{\mathbf{v}} - 2\gamma_{v3}\bar{\mathbf{n}}, \quad \boldsymbol{\tau} \simeq \bar{\mathbf{t}} - 2\gamma_{t3}\bar{\mathbf{n}},$$
$$\mathbf{d} \simeq \bar{\mathbf{n}} + 2\gamma_{v3}\bar{\mathbf{v}} + 2\gamma_{t3}\bar{\mathbf{t}}. \tag{40}$$

If eqn (40) is introduced into eqn (39), then with the help of eqns (38), (30) and (31), and with accuracy to small terms $\beta_{\alpha\beta}\gamma_{\lambda 3}$, we can approximately represent $\bar{\boldsymbol{\lambda}}_t$ on the basis $\bar{\mathbf{v}}$, $\bar{\mathbf{t}}$, $\bar{\mathbf{n}}$ with components (37) proposed by Mikhailovskii (1995). This clearly indicates that with accuracy to the small terms, the deformational boundary quantities (37) are equivalent to those of eqn (30). As a result, they are also expressible in terms of the previous quantities (26) by linear transformations eqn $(20)_2$ with eqn (27).

7. CONCLUSIONS

In this report, an entirely general approach to the derivation of deformational boundary quantities has been developed for the nonlinear theory of shells with transverse shear and normal strains. It is only assumed here that the displacement field is distributed linearly across the shell thickness. No kind of restrictions are introduced about the magnitudes of the displacements, rotations, strains and/or bendings of the shell material elements.

It has been noted that the total rotation of the shell lateral boundary element can be defined by two alternative, non-equivalent ways through two alternative orthonormal triads of vectors associated with the lateral boundary surface of the deformed shell. The choice of the first triad was already discussed by Pietraszkiewicz (1979, 1980b). The consequences of the choice of the second triad, for an exact description of deformation of the shell lateral boundary surface, have been discussed in detail in this report.

We have derived exact alternative expressions (17) for components of the vector of change of curvature of the shell boundary contour, as well as their two consistently reduced forms, eqns (26) and (30), appropriate for geometrically nonlinear theory of shells which, when linearized, give deformational boundary quantities for the linear shell theory as well. All the expressions derived here are new in the literature. We have also confirmed our results by reducing them further with the help of additional simplifying assumptions, corresponding to several particular shell theories, for which such deformational boundary quantities were proposed in the literature.

To each of the two alternative sets of deformational boundary quantities discussed here, there corresponds a different set of work-conjugate static boundary quantities, expressed entirely in terms of internal stress and couple resultant tensors, so that the virtual work performed by both sets of corresponding work-conjugate quantities remains the same. The problem of such static boundary quantities should be addressed in a separate paper.

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