Arch. Mech., 50, 3, pp. 483–495, Warszawa 1998
FIFTY YEARS OF THE ARCHIVES OF MECHANICS

On the general form of jump conditions for thin irregular shells

Dedicated to Prof. Henryk Zorski
on the occasion of his 70-th birthday

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THE PAPER deals with the nonlinear theory of thin shell structures in the presence of irregularities in geometry, deformation, material properties and loading. The irregular shell is modelled by a reference network being a union of piecewise smooth surfaces and space curves, with various fields satisfying relaxed smoothness, differentiability, and regularity requirements. Transforming the virtual work principle postulated for the entire reference network, the corresponding local field equations and side conditions (boundary and jump conditions) are derived. It is shown that no more than four static and work-conjugate kinematic jump conditions can correctly be formulated whenever the shell deformation is assumed to be entirely determined by deformation of the reference network capable of resisting to stretching and bending. This assumption includes various special formulations of the Kirchhoff–Love type theory of elastic shells, as well as their substantial generalizations accounting for finite strains and inelastic deformations.

1. Introduction

THE NONLINEAR theory of thin shells is well developed for an idealized shell structure defined through a reference surface consisting of a single, smooth and regular surface element (see [1] and references given there).

Real shell structures usually contain folds, stiffeners, branches, self-intersections and additional design elements which make some fields non-smooth or discontinuous along specified curves on the reference surface. The shell deformation itself may not be smooth along some curves, and stiffeners, branching regions or technological connections may possess their own mechanical properties. All such problems of thin shells are referred to in this paper as irregular ones. In order to model and analyze such irregular shell problems it is not enough to formulate the local equilibrium equations, kinematic relations and boundary conditions supplemented by appropriate constitutive equations. In addition, we have to provide some jump conditions along the singular curves.

In an engineering approach to the problem [2–5] the irregular shell structure is first divided into regular parts, and each part is modelled separately by various known analytical or numerical methods. Then the regular parts are assembled

back into the whole shell structure by adjusting boundary conditions of the adjacent regular shell parts along the junction, with account of possibly different mechanical properties of the junction itself, if necessary. Such an approach replaces the problem of modelling and analysis of the whole irregular shell structure by another problem of an assemblage of its regular parts analyzed separately. In this alternative approach any particular assembling technique (which may depend upon the solution method applied) should be regarded as an additional mechanical postulate and can lead to different forms of the jump conditions. It is not apparent under which assumptions, or whether at all, both the problems are mechanically equivalent. Take the very popular finite element method as an example. Various shell finite elements can be used to model regular parts of the shell structure, and several techniques can be applied to assemble the regular parts modelled by the finite elements into the whole irregular shell structure. A critical review of various assembling techniques [6, 7] suggests that each of them is element-dependent and applicable to a limited class of shell problems.

The aim of this paper is to reveal the general structure of the jump conditions appropriate for the nonlinear theory of thin irregular shells. The irregular shell structure is modelled here by a material surface-like continuum, called a shell reference network, capable of resisting the stretching and bending. Such a model includes the classical Kirchhoff–Love type linear and geometrically nonlinear theories of thin elastic shells [1], finite strain theories of rubber-like shells [8, 9] and the theory of elastic-plastic shells undergoing finite strains [10]. The network is composed of piecewise smooth surfaces and space curves. Each space curve in the network may represent a singular surface curve, but also a one-dimensional continuum endowed with its own mechanical properties. Then the principle of virtual work is postulated in the entire reference network, with various fields in principle satisfying the relaxed smoothness, differentiability and regularity assumptions. The non-standard transformation of the principle leads to the local Lagrangian equilibrium equations and boundary conditions known from the theory of thin regular shells, [1]. Additionally, along the singular curves we obtain the general form of four jump conditions appropriate for the theory of thin irregular shells. The jump conditions do not depend on the solution methods or assembling techniques applied, and are valid for unrestricted displacements, rotations, strains and/or bendings of the reference network.

2. Basic postulates

A consistent formulation of field equations and side conditions (boundary and jump conditions) for thin irregular shell structures can be based on the following two postulates:

I. *The deformation of the entire thin irregular shell structure is determined, within a sufficient accuracy, by the deformation of a distinguished material surface-like continuum, called the shell reference network.*

II. *The equlibrium conditions of the entire thin irregular shell structure are determined, to within a sufficient accuracy, by the principle of virtual work for the network capable of resisting the stretching and bending.*

The first of these postulates is kinematic in nature and can be regarded as definition of the thin irregular shell structure. The second one should be regarded as the basic dynamic postulate of the theory. Both postulates are independent of specific constitutive equations needed to model particular classes of materials.

Let us denote by M an undeformed configuration of the reference network referred to in the Postulate I. Then, formally, the principle of virtual work for the irregular shell structure modelled by M can be written in the form

$$\mathcal{G} \equiv \mathcal{G}_{\text{int}} - \mathcal{G}_{\text{ext}} - \mathcal{G}_{\Gamma} = 0, \tag{2.1}$$

where $\mathcal{G}_{\text{int}}$ stands for the internal virtual work, $\mathcal{G}_{\text{ext}}$ denotes the external virtual work, while the term $\mathcal{G}_{\Gamma}$ has been included in (2.1) to account for an additional virtual work due to generalized forces and moments at the shell junctions, multiple-shell intersections and other singular curves denoted jointly by Γ.

Within the classical theory of smooth shells, the undeformed configuration $M \subset \mathcal{E}$ ($\mathcal{E}$ denotes the three-dimensional Euclidean point space, i.e. the physical space, and E denotes its translation space) of the shell reference surface is assumed to be a connected and oriented regular surface of class C^2 or higher, with a piecewise smooth boundary ∂M. A typical point of M is then identified by the position vector relative to the origin of a fixed Cartesian coordinate system in the space. For simplicity, we shall use the same symbol $\mathbf{Y}$ for points on M and their position vectors. A deformation of M is described by a map $\boldsymbol{\chi} : M \to \mathcal{E}$, which carries each surface point $\mathbf{Y} \in M$ into its spatial place

$$\mathbf{y} = \boldsymbol{\chi}(\mathbf{Y}) = \mathbf{Y} + \mathbf{u}(\mathbf{Y}) \tag{2.2}$$

in the deformed configuration $m = \boldsymbol{\chi}(M)$ of M, where the spatial vector field $\mathbf{u} : M \to E$ denotes the associated displacement field. In the classical theory of thin smooth shells one also assumes that the deformation map $\boldsymbol{\chi}$ is globally invertible (on the codomain $m = \boldsymbol{\chi}(M)$), is of class C^2 or higher and admits the extension of the same class to the boundary ∂M. Thus, the surface deformation gradient $\mathbf{F} = \mathbf{P}\,\text{Grad}_s\boldsymbol{\chi}$, where $\mathbf{P}$ is a perpendicular projection [11] on the tangent plane $T_{\mathbf{y}}m$, is well defined at each point of M including the boundary, is continuous at all interior points of M and has continuous extensions to the boundary.

In this paper all concepts, including surface differential operators relevant to the smooth surfaces, are understood in the sense defined in [11, 12]. Additionally, we assume that M can be described locally in the parametric form $\mathbf{Y} = \mathbf{Y}(\xi^{\beta})$, where $(\xi^{\beta}, \beta = 1, 2)$ are surface coordinates chosen in any convenient way. Then the natural base vectors $\mathbf{A}_{\alpha}$ and the unit normal vector $\mathbf{A}_n$ at each interior point $\mathbf{Y} \in \text{int}\, M$, the metric tensor $\mathbf{A}$ and the curvature tensor $\mathbf{B}$ are defined in the usual manner [1].

The regularity assumptions stated above for the smooth surface M are too restrictive if various geometric, material and kinematic irregularities are to be admitted. These regularity assumptions may be relaxed in many ways. For a wide class of shell structures it can be assumed that the undeformed reference surface M is only piecewise smooth of a suitable class (the relevant mathematical definition can be found in [13]). Parallelly, we may also admit the deformation $\chi : M \to \mathcal{E}$ of the reference surface to be only piecewise smooth of a corresponding class. Therefore, the theory of irregular shells developed here relies on concepts of piecewise smooth surfaces (and some generalizations thereof) and of piecewise smooth surface fields largely based on our paper [12].

3. Geometry and deformation

Various irregularities encountered in the analysis of irregular shell problems can be grouped into three broad classes, [12]: 1. The undeformed configuration of the shell reference surface is not smooth (thus it may contain folds) or it is not a surface in the classical sense (for example, two smooth intersecting surfaces do not form a surface as the whole). Such irregularities may be called geometric. 2. Deformation of the shell reference surface (smooth or not in the undeformed configuration) fails to be smooth. We may refer to this kind of irregularities as kinematic, since they are associated with the deformation. 3. The shell structure cannot be considered as a single shell (smooth or non-smooth and undergoing smooth or non-smooth deformation) but rather as a union of some number of single shells interconnected along junctions. This type of irregularities may be called mechanical, because the junctions may have their own mechanical properties possibly quite different from the properties of the adjacent shells. The basic assumption made in the three cases is that all irregularities are restricted to distinct curves and points (i.e. to sets of zero area measure) on the shell reference surface-like continuum. Under this assumption, all the three classes of irregularities can be considered at once as follows.

In the most general case, the undeformed configuration of the reference surface-like continuum can be defined to be a network $M \subset \mathcal{E}$ consisting of a finite number of surface elements $M^{(k)}$, $k = 1, 2, ..., K$, with the following properties:

1. Each $M^{(k)}$ is a bounded, oriented, connected and smooth surface of class C^n, $n \geq 2$, whose boundary $\partial M^{(k)}$ consists of a finite number of closed Jordan curves oriented consistently with $M^{(k)}$ that do not meet in cusps.

2. No two distinct surface elements $M^{(k)}$ have common interior points.

3. Two or more distinct surface elements $M^{(k)}$ may have a spatial curve $\Gamma^{(a)}$ as a common part of the boundaries. Such a curve is defined by

$$\Gamma^{(a)} = \partial M^{(k_1)} \cap \partial M^{(k_2)} \cap \ldots \cap \partial M^{(k_m)} \quad \text{if} \quad k_1 \neq k_2 \neq \ldots k_m \,. \tag{3.1}$$

4. Two or more distinct curves $\Gamma^{(a)}$ may have in common only single isolated points.

We then define the network M as the union of all closed surface elements $\overline{M}^{(k)} = M^{(k)} \cup \partial M^{(k)}$, and by Γ we denote the union of all curves $\Gamma^{(a)}$. It is clear that $\Gamma \subset M$. Moreover, the boundary ∂M of M, defined now by

$$\partial M = \left(\bigcup_{k=1}^{K} \partial M^{(k)} \right) \setminus \Gamma, \tag{3.2}$$

consists of a finite number of Jordan curves (not necessarily closed).

Each $M^{(k)}$ can be regarded as the reference surface of a regular shell part. Each $\Gamma^{(a)}$ can be regarded as representing a geometric surface curve (e.g. a fold or a curve in M across which some fields fail to be smooth), but also a reference axis of one-dimensional continuum (e.g. an axis of a rod-like element) representing a multiple shell intersection, a technological junction, etc., whenever they are assumed to have mechanical properties of their own. In the former case we call $\Gamma^{(a)}$ a geometric singular curve, and in the latter case – a physical singular curve. Because geometric curves are just a special case of physical curves, no distinction will be made in the general considerations and the two cases will be considered at the same time. The geometry of each smooth surface element $M^{(k)}$ and its boundary $\partial M^{(k)}$, hence the geometry of the curves $\Gamma^{(a)}$, can now be described by the methods of classical differential geometry.

In general, any deformation of the reference network M is described by two maps $\boldsymbol{\chi} : M \setminus \Gamma \to \mathcal{E}$ and $\boldsymbol{\chi}_\Gamma : \Gamma \to \mathcal{E}$, since the singular curve Γ may be admitted to follow its own deformation. We shall assume that the deformation $\boldsymbol{\chi}$ is a continuous function over each smooth surface element $M^{(k)}$ and differentiable of class C^2 in the interior of $M^{(k)}$ (in the relative topology). Under these assumptions the surface gradient $\mathbf{F}$ exists at every point $\mathbf{Y} \in \operatorname{int} M^{(k)}$. However, we shall not assume *a priori* that the deformation $\boldsymbol{\chi}$ is continuous across the singular curve Γ or some parts thereof. Accordingly, we regard $\boldsymbol{\chi}$ as being defined for all points of M except possibly for points belonging to Γ. We shall then assume that the deformation map $\boldsymbol{\chi}$ has a finite limit at every point $\mathbf{Y} \in \Gamma$,

$$\mathbf{y}^{(k)} = \boldsymbol{\chi}^{(k)}(\mathbf{Y}) = \lim_{\mathbf{Z} \to \mathbf{Y}} \boldsymbol{\chi}(\mathbf{Z}) = \mathbf{Y} + \lim_{\mathbf{Z} \to \mathbf{Y}} \mathbf{u}(\mathbf{Z}), \qquad \mathbf{Z} \in M^{(k)}, \tag{3.3}$$

whenever Γ is a part of the boundary $\partial M^{(k)}$. However, in many cases the function $\boldsymbol{\chi}$ may be defined on an entire M and the deformation $\boldsymbol{\chi}_\Gamma$ is then the restriction of $\boldsymbol{\chi}$ to Γ, i.e. $\boldsymbol{\chi}_\Gamma = \boldsymbol{\chi}_{|\Gamma}$.

4. Virtual work expressions

Within a formal (axiomatic) approach, the internal virtual work $\mathcal{G}_{\text{int}}$ is assumed to be a real-valued, additive set function over measurable subsets of the reference network and an absolutely continuous function with respect to the area

measure (line and area measures are understood in the sense of Hausdorff measure). Under these assumptions the internal virtual work $\mathcal{G}_{\text{int}}$ can be written as the sum of virtual work expressions over all mutually disjoint smooth surface elements $M^{(k)}$:

$$\mathcal{G}_{\text{int}} = \sum_k \mathcal{G}_{\text{int}}^{(k)} = \sum_k \iint_{M^{(k)}} W_i \, dA \,, \tag{4.1}$$

where W_i denotes the internal virtual work density per unit area of the undeformed reference network M.

Similarly, the external virtual work $\mathcal{G}_{\text{ext}}$ can naturally be written as the sum of two parts, which account for the virtual work of the external surface forces and moments and the virtual work of the external boundary forces and moments, respectively:

$$\mathcal{G}_{\text{ext}} = \sum_k \iint_{M^{(k)}} W_e \, dA \,, + \int_{\partial M_f} w_e \, dS. \tag{4.2}$$

Here W_e denotes the virtual work density of external surface forces and moments measured per unit area of the undeformed reference network M, while w_e denotes the virtual work density of external forces and moments applied to a part ∂M_f of the boundary ∂M.

Consistently with the postulate (I), each regular part of the undeformed irregular shell structure is modelled by a smooth surface element $M^{(k)}$, the deformed configuration of which $m^{(k)} = \chi(M^{(k)})$ is determined by the position vector $\mathbf{y} = \chi(\mathbf{Y})$ and the field $\mathbf{n}(\chi(\mathbf{Y}))$ of unit normal vectors specifying its orientation. Moreover, by virtue of the classical theorem of differential geometry, two symmetric surface tensors [1, 8–10]

$$\begin{aligned} \boldsymbol{\Gamma} &= \frac{1}{2}\left(\mathbf{F}^T\mathbf{a}\,\mathbf{F} - \mathbf{A}\right) = \Gamma_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta, \qquad \Gamma_{\alpha\beta} = \frac{1}{2}(a_{\alpha\beta} - A_{\alpha\beta}), \\ \mathbf{K} &= -\left(\mathbf{F}^T\mathbf{b}\,\mathbf{F} - \mathbf{B}\right) = K_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta, \qquad K_{\alpha\beta} = -(b_{\alpha\beta} - B_{\alpha\beta}) \end{aligned} \tag{4.3}$$

provide the Green-type measures of the local strains (stretching and bending) of the reference surface element $M^{(k)}$. Here $\mathbf{a}$ and $\mathbf{b}$ denote the metric and curvature tensors of the deformed surface element $m^{(k)}$, respectively.

When dealing with virtual deformations of the shell reference network, we may consider a one-parameter family of deformations $\mathbf{y} = \chi(\mathbf{Y}, t)$, where t is a scalar (time-like) parameter. We then denote by $\mathbf{v} \equiv \dot{\mathbf{y}} = \dot{\mathbf{u}}$ and $\mathbf{w} \equiv \dot{\mathbf{n}}$ the virtual displacement of the reference surface and the virtual change of the unit normal, respectively. The virtual changes of the local strain measures (4.3) are $\dot{\boldsymbol{\Gamma}} = \dot{\Gamma}_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta$ and $\dot{\mathbf{K}} = \dot{K}_{\alpha\beta}\mathbf{A}^\alpha \otimes \mathbf{A}^\beta$. It is now obvious that the simplest expression for the internal virtual work density W_i must be of the form

$$W_i = \mathbf{N}\cdot\dot{\boldsymbol{\Gamma}} + \mathbf{M}\cdot\dot{\mathbf{K}} = N^{\alpha\beta}\dot{\Gamma}_{\alpha\beta} + M^{\alpha\beta}\dot{K}_{\alpha\beta}\,, \tag{4.4}$$

where $\mathbf{N} = N^{\alpha\beta}\mathbf{A}_\alpha \otimes \mathbf{A}_\beta$ and $\mathbf{M} = M^{\alpha\beta}\mathbf{A} \otimes \mathbf{A}_\beta$ are the symmetric stress resultant and couple surface tensors of the 2-nd Piola–Kirchhoff type.

The virtual work density of the external surface and boundary loads can be represented by invariant expressions as in the classical theories of thin smooth Kirchhoff–Love type shells [1]:

$$W_e = \mathbf{p}\cdot\mathbf{v} + \mathbf{h}\cdot\mathbf{w}\,, \qquad w_e = \mathbf{T}^*\cdot\mathbf{v} + \mathbf{H}^*\cdot\mathbf{w}\,. \tag{4.5}$$

Here $\mathbf{p}$ and $\mathbf{h}$ are the external surface force and moment resultant vectors referred to the undeformed surface element $M^{(k)}$, while $\mathbf{T}^*$ and $\mathbf{H}^*$ are the external boundary force and moment resultants referred to the undeformed boundary ∂M_f.

5. Equilibrium equations and boundary conditions

Derivation of the local equilibrium equations and boundary conditions for thin regular shells may be found in many papers (see e.g. [1, 8–10]). The same procedure, under suitable regularity assumptions, can essentially be applied to each smooth surface element $M^{(k)}$. Accordingly, we present below only the main steps of this derivation.

The virtual strains can be obtained by varying the components of the strain tensors (4.3). At each internal point of $M^{(k)}$, the field of unit normal vectors $\mathbf{n}$ can be expressed in terms of first derivatives of the displacement field, $\mathbf{n} = \mathbf{n}(\mathbf{u}_{,\beta})$, since it must satisfy the three geometric constraints, $\mathbf{a}_\beta\cdot\mathbf{n} = 0$ and $\mathbf{n}\cdot\mathbf{n} = 1$, where $\mathbf{a}_\beta$ are the natural base vectors of $m^{(k)} = \chi(M^{(k)})$. Variations of these constraints lead to $\mathbf{w} = -(\mathbf{n}\cdot\mathbf{v}_{,\beta})\mathbf{a}^\beta$. Then, it is not difficult to show that the internal virtual work density (4.4) can be written in the form [1]

$$W_i = \mathbf{T}^\beta\cdot\mathbf{v}_{,\beta} + (\mathbf{H}^\beta\cdot\mathbf{w})_{|\beta}\,, \tag{5.1}$$

where the generalized stress resultants $\mathbf{T}^\beta$ and couple resultants $\mathbf{H}^\beta$ are defined by

$$\mathbf{T}^\beta = N^{\alpha\beta}\mathbf{a}_\alpha + M^{\alpha\beta}\mathbf{n}_{,\alpha} + \left[(M^{\lambda\alpha}\mathbf{a}_\lambda)_{|\alpha}\cdot\mathbf{a}^\beta\right]\mathbf{n}, \qquad \mathbf{H}^\beta = M^{\alpha\beta}\mathbf{a}_\alpha\,. \tag{5.2}$$

Here $(\cdot)_{|\alpha}$ denotes covariant differentiation in the undeformed metric $A_{\alpha\beta}$.

The tensors $\mathbf{T} = \mathbf{T}^\beta \otimes \mathbf{A}_\beta$ and $\mathbf{H} = \mathbf{H}^\beta \otimes \mathbf{A}_\beta$ are assumed to be of class C^1 in the interior of each smooth surface element $M^{(k)}$ and to have extensions of the same class to the boundary $\partial M^{(k)}$, with finite limits $\mathbf{T}^{(k)}(\mathbf{Y})$ and $\mathbf{H}^{(k)}(\mathbf{Y})$ at each point $\mathbf{Y} \in \partial M^{(k)}$. Then the surface divergence theorem can be applied on each smooth surface element $M^{(k)}$ to give

$$\iint\limits_{M^{(k)}} W_i\, dA = -\iint\limits_{M^{(k)}} (\mathrm{Div}_s\mathbf{T})\cdot\mathbf{v}\, dA + \int\limits_{\partial M^{(k)}} \left(\mathbf{T}_\nu^{(k)}\cdot\mathbf{v}^{(k)} + \mathbf{H}_\nu^{(k)}\cdot\mathbf{w}^{(k)}\right) dS, \tag{5.3}$$

where $\mathrm{Div}_s\mathbf{T} = \mathbf{T}^{\beta}{}_{|\beta}$ at every interior point of $M^{(k)}$, $\mathbf{T}_{\nu}^{(k)} \equiv \mathbf{T}^{(k)}\boldsymbol{\nu}^{(k)}$ and $\mathbf{H}_{\nu}^{(k)} \equiv \mathbf{H}^{(k)}\boldsymbol{\nu}^{(k)}$ at the boundary point $\mathbf{Y} \in \partial M^{(k)}$, with $\boldsymbol{\nu}^{(k)}$ denoting the unit outward normal vector along $\partial M^{(k)}$.

In the same manner, the external surface virtual work density $(4.5)_1$ can be rewritten in the form

$$W_e = \mathbf{l}\cdot\mathbf{v} - [(\mathbf{h}\cdot\mathbf{a}^{\beta})\mathbf{n}\cdot\mathbf{v}]_{|\beta}\,, \qquad \mathbf{l} \equiv \mathbf{p} + [(\mathbf{h}\cdot\mathbf{a}^{\beta})\mathbf{n}]_{|\beta} \tag{5.4}$$

and

$$\iint\limits_{M^{(k)}} (\mathbf{p}\cdot\mathbf{v} + \mathbf{h}\cdot\mathbf{w})\,dA = \iint\limits_{M^{(k)}} \mathbf{l}\cdot\mathbf{v}\,dA - \int\limits_{\partial M^{(k)}} \left((\mathbf{h}^{(k)}\cdot\boldsymbol{\mu}^{(k)})\mathbf{n}^{(k)}\cdot\mathbf{v}^{(k)}\right) dS\,, \tag{5.5}$$

where $\boldsymbol{\mu} = \mathbf{a}^{\beta}\nu_{\beta}$.

Along the boundary $\partial M^{(k)}$ of each smooth surface element $M^{(k)}$, the surface gradient $\mathrm{Grad}_s\mathbf{u}$ of the displacement field can be decomposed into tangential and normal derivatives (under the assumption that $\mathrm{Grad}_s\mathbf{u}$ admits a continuous extension to the boundary $\partial M^{(k)}$):

$$\begin{gathered} \mathrm{Grad}_s\mathbf{u} = \mathbf{u}_{,\nu} \otimes \boldsymbol{\nu} + \mathbf{u}' \otimes \boldsymbol{\tau}, \\ \mathbf{u}_{,\nu} \equiv (\mathrm{Grad}_s\mathbf{u})\boldsymbol{\nu} = \mathbf{u}_{,\beta}\nu^{\beta}, \qquad \mathbf{u}' \equiv (\mathrm{Grad}_s\mathbf{u})\boldsymbol{\tau} = \mathbf{u}_{,\beta}\tau^{\beta}, \end{gathered} \tag{5.6}$$

where $\boldsymbol{\tau}$ is the unit tangent vector of $\partial M^{(k)}$. From (5.6) it follows that the field of unit normal vectors $\mathbf{n}$ along $\partial m^{(k)} = \boldsymbol{\chi}(\partial M^{(k)})$ can be regarded as a function of $\mathbf{u}_{,\nu}$ and $\mathbf{u}'$, i.e. $\mathbf{n} = \mathbf{n}(\mathbf{u}_{,\nu}, \mathbf{u}')$, subjected to only two independent constraints

$$\mathbf{y}'\cdot\mathbf{n} = (\boldsymbol{\tau} + \mathbf{u}')\cdot\mathbf{n} = 0, \qquad \mathbf{n}\cdot\mathbf{n} = 1. \tag{5.7}$$

As a result, $\mathbf{n} = \mathbf{n}(\mathbf{u}_{,\nu}, \mathbf{u}')$ along $\partial M^{(k)}$ is expressible through $\mathbf{u}'$ and a scalar function $\phi = \phi(\mathbf{u}_{,\nu}, \mathbf{u}')$ describing the rotational deformation of the shell lateral boundary surface. The structure of the function $\phi(\mathbf{u}_{,\nu}, \mathbf{u}')$ was discussed in [14, 15], where the general expression for $\mathbf{w} \equiv \dot{\mathbf{n}}$ in terms of $\varphi \equiv \dot{\phi}$ and $\mathbf{v}'$ was derived in the form

$$\mathbf{w} = \mathbf{q}\varphi + \mathbf{L}\,\mathbf{v}', \qquad \mathbf{q}(\phi, \mathbf{u}') \equiv \partial_{\phi}\mathbf{n}, \qquad \mathbf{L}(\phi, \mathbf{u}') \equiv \partial_{\mathbf{u}'}\mathbf{n}. \tag{5.8}$$

Explicit expressions for the vector-valued function $\mathbf{q} = \mathbf{q}(\mathbf{u}_{,\nu}, \mathbf{u}')$ and the tensor-valued function $\mathbf{L} = \mathbf{L}(\mathbf{u}_{,\nu}, \mathbf{u}')$ depend on the particular definition of the scalar-valued function $\phi = \phi(\mathbf{u}_{,\nu}, \mathbf{u}')$ employed.

With the help of (5.8), the second term in the line integral of (5.3) can be transformed further to

$$\int\limits_{\partial M^{(k)}} \mathbf{H}_{\nu}^{(k)}\cdot\mathbf{w}^{(k)}\,dS = \int\limits_{\partial M^{(k)}} \left(-\mathbf{f}^{(k)}\cdot(\mathbf{v}^{(k)})' + H^{(k)}\varphi^{(k)}\right) dS, \tag{5.9}$$

where

$$\mathbf{f}^{(k)} = -\mathbf{L}^T \mathbf{H}_\nu^{(k)}, \qquad H^{(k)} = \mathbf{q}\cdot\mathbf{H}_\nu^{(k)}. \tag{5.10}$$

Along each $\partial M^{(k)}$ there may be singular points (e.g. corner points) P_a, $a = 1, ..., A$, described by $S = S_a$, at which the field $\mathbf{f}^{(k)}\cdot\mathbf{v}^{(k)}$ is not differentiable. At such points we assume the existence of finite limits of $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$ defined by

$$\mathbf{f}_a^{(k)\pm} = \lim_{h\to 0} \mathbf{f}^{(k)}(S_a \pm h), \qquad \mathbf{v}_a^{(k)\pm} = \lim_{h\to 0} \mathbf{v}^{(k)}(S_a \pm h). \tag{5.11}$$

Then the line integral (5.9) can be transformed by applying the integration by parts leading to

$$\iint\limits_{M^{(k)}} W_i\, dA = -\iint\limits_{M^{(k)}} (\mathrm{Div}_s \mathbf{T})\cdot\mathbf{v}\, dA + \int\limits_{\partial M^{(k)}} (\mathbf{P}_\nu^{(k)}\cdot\mathbf{v}^{(k)} + H^{(k)}\varphi^{(k)})\, dS + \sum_{P_a\in\partial M^{(k)}} \left(\mathbf{f}_a^{(k)+}\cdot\mathbf{v}_a^{(k)+} - \mathbf{f}_a^{(k)-}\cdot\mathbf{v}_a^{(k)-}\right), \tag{5.12}$$

where

$$\mathbf{P}_\nu^{(k)} = \mathbf{T}_\nu^{(k)} + (\mathbf{f}^{(k)})'. \tag{5.13}$$

By virtue of (5.12) and (4.1), the internal virtual work for the entire reference network M can be written in the form

$$\mathcal{G}_{\mathrm{int}} = -\iint\limits_{M} (\mathrm{Div}_s \mathbf{T})\cdot\mathbf{v}\, dA + \int\limits_{\partial M} (\mathbf{P}_\nu\cdot\mathbf{v} + H\varphi)\, dS + \int\limits_{\Gamma} (\llbracket \mathbf{P}_\nu\cdot\mathbf{v}\rrbracket + \llbracket H\varphi\rrbracket)\, dS + \sum_{P_i\in\Gamma} [\mathbf{f}\cdot\mathbf{v}]_i + \sum_{P_b\in\partial M_f} [\mathbf{f}\cdot\mathbf{v}]_b\,. \tag{5.14}$$

Here the jump $\llbracket \mathbf{P}_\nu\cdot\mathbf{v}\rrbracket$ at each regular point $\mathbf{Y}\in\Gamma^{(a)} \equiv \Gamma^{(1,2,\ldots,n)}$ of the common curve for $n \geq 2$ adjacent surface elements $M^{(k)}$ is defined by

$$\llbracket \mathbf{P}\cdot\mathbf{v}\rrbracket = \pm\mathbf{P}_\nu^{(1)}\cdot\mathbf{v}^{(1)} \pm \mathbf{P}_\nu^{(2)}\cdot\mathbf{v}^{(2)} \pm \ldots \pm \mathbf{P}_\nu^{(n)}\cdot\mathbf{v}^{(n)}, \tag{5.15}$$

with the jump $\llbracket H\varphi\rrbracket$ being defined in the same way. The signs in the definition (5.15) must be chosen consistently with a fixed orientation of the curve $\Gamma^{(a)}$. If the orientation of $\Gamma^{(a)}$ coincides with the orientation of the boundary curve $\partial M^{(k)}$, then the sign "$-$" must be chosen for the corresponding term and the "$+$" sign otherwise. If we denote by $\boldsymbol{\tau}_\Gamma$ the unit tangent vector specifying the orientation of the curve $\Gamma^{(a)}$, then $\boldsymbol{\nu}^{(k)} = \pm\boldsymbol{\tau}_\Gamma\times\mathbf{A}_n^{(k)}$, and the sign must be chosen in such a way that the boundary $\partial M^{(k)}$ be consistently oriented with $M^{(k)}$.

The jumps at all singular points of M have been divided in (5.14) into the jumps $[\mathbf{f}\cdot\mathbf{v}]_i$ at the internal points $P_i\in\Gamma$, $i = 1, ..., I$, and the jumps $[\mathbf{f}\cdot\mathbf{v}]_b$ at the

boundary points $P_b \in \partial M$. At each internal point P_i being the common point of $m \geq 2$ adjacent branches $\Gamma^{(a)}$ of Γ, and at each boundary point P_b being the common point of $t \geq 2$ adjacent parts $\partial M^{(t)}$ of ∂M and adjacent branches $\Gamma^{(q)}$ of Γ approaching P_b from inside of M, these jumps are defined by

$$
\begin{aligned}
[\mathbf{f}\cdot\mathbf{v}]_i &= \pm\mathbf{f}_i^{(1)\pm}\cdot\mathbf{v}_i^{(1)\pm} \pm \mathbf{f}_i^{(2)\pm}\cdot\mathbf{v}_i^{(2)\pm} \pm \ldots \pm \mathbf{f}_i^{(m)\pm}\cdot\mathbf{v}_i^{(m)\pm}, \\
[\mathbf{f}\cdot\mathbf{v}]_b &= \pm\mathbf{f}_b^{(1)\pm}\cdot\mathbf{v}_b^{(1)\pm} \pm \mathbf{f}_b^{(2)\pm}\cdot\mathbf{v}_b^{(2)\pm} \pm \ldots \pm \mathbf{f}_b^{(t)\pm}\cdot\mathbf{v}_b^{(t)\pm} \\
&\quad \pm\mathbf{f}_i^{(1)\pm}\cdot\mathbf{v}_b^{(1)\pm} \pm \mathbf{f}_i^{(2)\pm}\cdot\mathbf{v}_b^{(2)\pm} \pm \ldots \pm \mathbf{f}_i^{(q)\pm}\cdot\mathbf{v}_b^{(q)\pm}.
\end{aligned} \tag{5.16}
$$

Similar transformations can be applied to the external virtual work, which gives

$$
\begin{aligned}
\mathcal{G}_{\text{ext}} = \iint_{M\setminus\Gamma} \mathbf{l}\cdot\mathbf{v}\,dA - \int_{\Gamma} [\![(\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\cdot\mathbf{v}]\!]\,dS + \int_{\partial M_f} (\{\mathbf{P}^* - (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v} + H^*\varphi)\,dS \\
- \int_{\partial M_d} (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\cdot\mathbf{v}\,dS + \sum_{P_b\in\partial M_f} [\mathbf{f}^*\cdot\mathbf{v}]_b,
\end{aligned} \tag{5.17}
$$

where $\partial M_d = \partial M\setminus\partial M_f$ is the complementary part of ∂M, and

$$
\mathbf{f}^* = -\mathbf{L}^T\mathbf{H}^*, \qquad H^* = \mathbf{q}\cdot\mathbf{H}^*, \qquad \mathbf{P}^* = \mathbf{T}^* + (\mathbf{f}^*)'. \tag{5.18}
$$

The jumps $[\![(\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\cdot\mathbf{v}]\!]$ along the common curve $\Gamma^{(a)}$ for $n \geq 2$ adjacent surface elements are defined analogously to (5.15). However, the jumps $[\mathbf{f}^*\cdot\mathbf{v}]_b$ in (5.17) take into account only those limiting values which are obtained by approaching P_b along branches of the boundary ∂M_f. If the boundary point P_b is a common point of $t \geq 2$ adjacent parts $\partial M^{(t)}$ of ∂M and q adjacent branches of $\Gamma^{(q)}$ of Γ, then

$$
[\mathbf{f}^*\cdot\mathbf{v}]_b = \pm\mathbf{f}_b^{*(1)\pm}\cdot\mathbf{v}_b^{(1)\pm} \pm \mathbf{f}_b^{*(2)\pm}\cdot\mathbf{v}_b^{(2)\pm} \pm \ldots \pm \mathbf{f}_b^{*(t)\pm}\cdot\mathbf{v}_b^{(t)\pm}. \tag{5.19}
$$

Introducing (5.14) and (5.17) into the principle of virtual work (2.1) we obtain

$$
\begin{aligned}
&-\iint_{M\setminus\Gamma} (\mathrm{Div}_s\mathbf{T} + \mathbf{l})\cdot\mathbf{v}\,dA \\
&+ \int_{\partial M_f} (\{\mathbf{P}_\nu - \mathbf{P}^* + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v} + (H - H^*)\varphi)\,dS + \sum_{P_b\in\partial M_f} [(\mathbf{f} - \mathbf{f}^*)\cdot\mathbf{v}]_b \\
&+ \int_{\partial M_d} (\{\mathbf{P}_\nu + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v} + H\varphi)\,dS + \sum_{P_b\in\partial M_d} [\mathbf{f}\cdot\mathbf{v}]_b \\
&+ \int_{\Gamma} ([\![\{\mathbf{P}_\nu + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v}]\!] + [\![H\varphi]\!])\,dS + \sum_{P_i\in\Gamma} [\mathbf{f}\cdot\mathbf{v}]_i - \mathcal{G}_\Gamma = 0.
\end{aligned} \tag{5.20}
$$

For arbitrary but kinematically admissible virtual deformations, the fields $\mathbf{v}$ and φ vanish identically along ∂M_d. Then, from (5.20) we obtain the Lagrangian local equilibrium equations

$$\text{Div}_s \mathbf{T} + \mathbf{l} = \mathbf{0} \qquad \text{at each regular } \mathbf{Y} \in M, \tag{5.21}$$

and the static boundary and corner conditions

$$\begin{aligned} &\mathbf{P}_\nu - \mathbf{P}^* + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n} = \mathbf{0}, \qquad H - H^* = 0 \quad \text{along regular parts of } \partial M_f, \\ &\mathbf{f}_b - \mathbf{f}_b^* = \mathbf{0} \qquad \text{at each singular point } P_b \in \partial M_f. \end{aligned} \tag{5.22}$$

Correspondingly, the work-conjugate geometric boundary conditions take the form

$$\mathbf{u} = \mathbf{u}^*, \qquad \phi = \phi^* \quad \text{along } \partial M_d. \tag{5.23}$$

As it has been expected, the local equilibrium conditions for thin irregular shells are the same as in the classical nonlinear theory of thin smooth shells [1].

6. General form of jump conditions

The new aspects of the theory of thin irregular shells, as compared with regular ones, lie exclusively in the concept of jump conditions needed, besides the constitutive relations, to obtain the complete formulation of the boundary value problem. The considerations of the previous section have shown that if the equilibrium equations and boundary conditions hold, then the principle of virtual work (5.20) asserts that

$$\int_\Gamma \left([\![\{\mathbf{P}_\nu + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v}]\!] + [\![H\varphi]\!] \right) dS + \sum_{P_i \in \Gamma} [\mathbf{f}\cdot\mathbf{v}]_i - \mathcal{G}_\Gamma = 0. \tag{6.1}$$

Equation (6.1) expresses the most general form of jump conditions in the weak form, and all considerations leading to (6.1) provide an unambiguous interpretation of the final results. No other results or consequences can be obtained when one applies the two hypotheses from Sec. 2, on which the whole theory of thin irregular shells is based. The weak form (6.1) of the jump conditions must hold for all cases, whether the irregularities are of geometric, kinematic or static nature, and (6.1) is entirely independent of the mechanical properties assigned to the curve Γ and the points P_i, P_b. Of course, so far we have not said anything about the possible physical meaning of the curve Γ and the points P_i, P_b, because our aim has been to include as many special cases as possible within this general framework. Moreover, at the beginning of the analysis, no *a priori* information about the form of $\mathcal{G}_\Gamma$ has been at our disposal.

In general, the $\Gamma^{(a)}$ are either geometric or physical curves embedded in the shell reference network M. In the first case, each $\Gamma^{(a)}$ represents simply a geometric curve on M, across which some surface fields fail to be continuous or smooth of the required class. In the second case, each $\Gamma^{(a)}$ is said to be a physical curve in the sense that it can be equipped with specific physical properties possibly quite distinct from the mechanical properties of the regular shell parts. Such a $\Gamma^{(a)}$ can model various physical situations encountered in real shell structures. The same applies to points P_i and P_b. In either case, it becomes obvious that the most general form of the virtual work expression $\mathcal{G}_\Gamma$ allowed within the theory of thin irregular shells is

$$\mathcal{G}_\Gamma = \int_\Gamma \sigma_\Gamma(\mathbf{Y})\, dS + \sum_{P_i \in \Gamma} \sigma_i\,, \tag{6.2}$$

where σ_Γ is the virtual work density along regular parts of the curve Γ, and σ_i is the virtual work at any singular point of Γ. The functions σ_Γ and σ_i must be specified in each particular case of the irregularity.

In view of (6.2), and taking further into account that the equation (6.1) must hold for each curve $\Gamma^{(a)}$ as well as for every part of it, we obtain the corresponding local form of the jump conditions:

$$[\![\{\mathbf{P}_\nu + (\mathbf{h}\cdot\boldsymbol{\mu})\mathbf{n}\}\cdot\mathbf{v}]\!] + [\![H\varphi]\!] - \sigma_\Gamma = 0, \quad \text{at regular points of } \Gamma, \tag{6.3}$$

$$[\mathbf{f}\cdot\mathbf{v}]_i - \sigma_i = 0, \quad \text{at each internal singular point } P_i \in \Gamma. \tag{6.4}$$

The jump conditions (6.3) and (6.4) constitute the additional set of local relations for thin irregular shell structures to be satisfied along the singular curves Γ.

The jump conditions (6.3) and (6.4) are valid for unrestricted displacements, rotations, strains and/or bending of the reference network M. Their general form depends neither on the assembling techniques nor on the solution methods applied. For special types of irregularities, the jump conditions can be considerably simplified and presented in a more explicit uncoupled form. Such particular forms of the jump conditions appropriate for only geometric irregularities (folds, intersections, rigid junctions) and for some simple kinematic irregularities (elastic and visco-elastic junctions) will be discussed in [16].

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Received October 7, 1997.