Bernoulli Numbers and Rotational Kinematics[1]

W. Pietraszkiewicz.[2] The kinematic differential Eqs. (21) and (22) of Pfister (1998) are expressed through unconventional tensor power series coe $(\cdot)$ and Ber $(\cdot)$, and the proposed proof is indirect and confusing indeed. Perhaps it may be of some interest to readers of this Journal to note that equivalent equations follow directly as a result of simple transformations, which are analogous to those used by Pietraszkiewicz and Badur (1983) in discussion of spatial changes of the rotation field in continuum mechanics.

With notation used by Pfister (1998), the standard representation of the rotation tensor $\mathbf{R} \in SO(3)$ given already by Gibbs (1884) reads

$$\mathbf{R} = \cos\Psi\mathbf{1} + \sin\Psi\mathbf{n}\times\mathbf{1} + (1-\cos\Psi)\mathbf{n}\otimes\mathbf{n}, \quad (1)$$

where $\otimes$ denotes the tensor product. For a time-dependent $\mathbf{R} = \mathbf{R}(t)$ it follows from (1) that

$$\dot{\mathbf{R}} = (-\sin\Psi\mathbf{1} + \cos\Psi\mathbf{n}\times\mathbf{1} + \sin\Psi\mathbf{n}\otimes\mathbf{n})\dot{\Psi} + \sin\Psi\dot{\mathbf{n}}\times\mathbf{1} + (1-\cos\Psi)(\dot{\mathbf{n}}\otimes\mathbf{n} + \mathbf{n}\otimes\dot{\mathbf{n}}). \quad (2)$$

With $\mathbf{R}^T$ denoting the transposed rotation tensor ($= \mathbf{R}^{-1}$), the angular velocity vector $\boldsymbol{\Omega}$ is an axial vector of the skew-symmetric tensor $\dot{\mathbf{R}}\mathbf{R}^T$, that is $\dot{\mathbf{R}}\mathbf{R}^T = \boldsymbol{\Omega}\times\mathbf{1}$. Introducing (1) and (2) into the left-hand side of this relation, after some algebra we obtain

$$\boldsymbol{\Omega} = \sin\Psi\dot{\mathbf{n}} + (1-\cos\Psi)\mathbf{n}\times\dot{\mathbf{n}} + \dot{\Psi}\mathbf{n}. \quad (3)$$

The Eq. (3) can be solved for $\dot{\mathbf{n}}$ leading to

$$\dot{\mathbf{n}} = \frac{1}{2\,\mathrm{tg}\,\Psi/2}(\boldsymbol{\Omega} - \dot{\Psi}\mathbf{n}) - \frac{1}{2}\mathbf{n}\times\boldsymbol{\Omega}. \quad (4)$$

The simple vector Eqs. (3) and (4) are just the canonical forms of the kinematic differential equations equivalent to (21) and (22) of Pfister (1998). Indeed, introducing the finite rotation vector $\boldsymbol{\Psi} = \Psi\mathbf{n}$ and taking into account that

$$\dot{\boldsymbol{\Psi}} = \dot{\Psi}\mathbf{n} + \Psi\dot{\mathbf{n}}, \quad \boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}} = \boldsymbol{\Psi}\cdot\boldsymbol{\Omega} = \Psi\dot{\Psi},$$

$$\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\dot{\boldsymbol{\Psi}}) = (\boldsymbol{\Psi}\cdot\dot{\boldsymbol{\Psi}})\boldsymbol{\Psi} - \Psi^2\dot{\boldsymbol{\Psi}}, \quad (5)$$

the relations (3) and (4) can be transformed into the forms

$$\boldsymbol{\Omega} = \mathbf{A}\cdot\dot{\boldsymbol{\Psi}}, \quad \dot{\boldsymbol{\Psi}} = \mathbf{A}^{-1}\cdot\boldsymbol{\Omega}, \quad (6)$$

where

$$\mathbf{A} = \mathbf{1} + \frac{1-\cos\Psi}{\Psi^2}\boldsymbol{\Psi}\times\mathbf{1} + \frac{\Psi-\sin\Psi}{\Psi^3}\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\mathbf{1}),$$

$$\mathbf{A}^{-1} = \frac{1}{\Psi^2}(\boldsymbol{\Psi}\otimes\boldsymbol{\Psi}) - \frac{1}{2}\boldsymbol{\Psi}\times\mathbf{1} - \frac{1}{2\Psi\,\mathrm{tg}\,\Psi/2}\boldsymbol{\Psi}\times(\boldsymbol{\Psi}\times\mathbf{1}). \quad (7)$$

The Eqs. (6) with (7) coincide with (21) and (22) of Pfister (1998), if trigonometric functions in (7) are replaced by Gibbs' functions $\mathrm{gib}_i(\cdot)$. However, in the Eq. (22) of Pfister (1998) two obvious misprints should be corrected: 1 should be replaced by **1**, and the closing parenthesis should be inserted after Ber $(-\hat{\boldsymbol{\Psi}})$.

For other definitions of the finite rotation vector used in the literature, such as tg $\Psi/2\mathbf{n}$, 2 tg $\Psi/2\mathbf{n}$, sin Ψ $\mathbf{n}$, or tg $\Psi/4\mathbf{n}$, for example, appropriate equivalent forms of the kinematic differential equations can easily be derived by analogous transformations of (3) and (4).

References

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