Journal of Elasticity **54:** 1–26, 1999.

Jump Conditions in the Non-linear Theory of Thin Irregular Shells

J. MAKOWSKI[1], W. PIETRASZKIEWICZ[2] and H. STUMPF[1]
[1]*Lehrstuhl für Allgemeine Mechanik, Ruhr-Universität Bochum, Universitätsstr. 150 IA3/126, D-44801 Bochum, Germany*
[2]*Institute of Fluid-Flow Machinery of the Polish Academy of Sciences, ul. Fiszera 14, 80-952 Gdańsk, Poland*

Received 14 January 1998; in revised form 14 June 1999

Abstract. The non-linear theory of thin shell structures with irregularities of geometry, material properties, loading and deformation is developed. The irregular shell is modelled by a reference network being a union of piecewise smooth surfaces and curves, with various fields satisfying relaxed regularity requirements. By transforming the virtual work principle postulated for the entire reference network, we derive the corresponding local field equations and side conditions. Particular attention is paid to formulate the general form of static and kinematic jump conditions at singular geometric and physical curves. Several special kinds of irregularities are considered and some particular forms of the jump conditions are discussed.

Key words: shell, jump condition, irregularity.

1. Introduction

Theory and analysis of shell structures are presented in a considerable number of papers and summarized in almost one thousand books and review articles compiled in [1, 2]. It is somehow surprising to note, however, that practically all the papers and books concern regular shells with reference surfaces consisting in fact of a single, smooth and regular surface element. It is also assumed, implicitly rather than explicitly, that:

(a) the reference surface admits a global, regular parametrization;
(b) material properties and the shell thickness vary smoothly over the shell;
(c) the surface deformation can be described by a globally invertible and smoothly differentiable map;
(d) all static and kinematic fields are smoothly differentiable as many times as required over the reference surface.

Real shell structures may contain folds, stiffeners, branches, self-intersections and additional design elements such as technological connections, stepwise thickness changes, parts made of different materials, etc., which cause some fields to be discontinuous along specified curves on the shell reference surface. Imperfections of the real shell shape caused by technological inaccuracies (e.g. when assembling convex or concave metal shells from initially flat sheets), influence additionally the real shell geometry. In many cases stiffeners, branching regions, junctions or technological connections may possess their own mechanical properties different from those of the adjacent shell elements. Furthermore, the shell deformation itself may not be smooth along some lines on the reference surface, e.g. in a technological hinge allowing some rotation about its axis. The non-smoothness of deformation may also appear during the deformation process itself, e.g. when a plastic hinge develops at some level of bending. Finally, the external forces or constraints may be concentrated at some curves or points of the shell space or at its boundary, and this introduces additional irregularities into the shell theory. All such shell problems are regarded in this paper as irregular ones.

The existing methods of analysis of the irregular shells are based on a division of the structure into regular parts, each having a smooth and regular surface as its reference surface. The regular parts are modelled separately, and all the parts are then assembled into the whole structure by adjusting boundary conditions of the adjacent shell elements along the junctions with account of a possibly different mechanical behaviour of the junction itself, if necessary. Such a methodology requires some jump conditions to be applied explicitly at each of the junctions in a way that should be consistent with the particular variant of shell theory employed. Note that the jump conditions are not provided by the shell theory used for the regular parts. As a result, in engineering calculations of irregular shells the jump conditions used in the assembling process are taken intuitively in a form suggested rather by the solution method applied than by the shell theory itself. It should be realized, however, that such an approach replaces the problem of theory and analysis of the whole irregular shell structure by another, supposedly equivalent problem of an assemblage of its regular parts analyzed separately, where any particular assembling technique employed should be regarded as an additional mechanical postulate (see e.g. [3–8]). It is not apparent under which assumptions, or whether at all, both problems are equivalent. A critical review of existing assembling techniques presented in [9, 10] for 5-parameter shell theory suggests that each of them is applicable only to a limited class of shell problems.

The general six-field theory of irregular shells was developed in [9, 11], where the corresponding static and kinematic jump conditions were derived from basic laws of continuum mechanics by their direct specification for shell-like shapes of the body. We refer the reader to those papers for an extensive characterization of various irregular shell problems and a review of the relevant mathematical and shell literature.

If the shell is thin in some sense, it can be modelled within a reasonable accuracy by a representative material surface offering resistance to stretching and bending. Such a model includes the classical Kirchhoff–Love type geometrically non-linear theory of elastic shells (see [13, 14] and references cited therein), some finite-strain theories of rubber-like shells [15–17], simple theories of elastic–plastic shells [18] and the theory of elastic–plastic shells undergoing finite strains [19]. Jump conditions appropriate for such a three-field theory of thin irregular shells could also be derived from those formulated in [9, 11] applying respective simplifying hypotheses corresponding to the particular version of thin shell theory. However, it seems to us more reasonable to derive the jump conditions from the onset for the whole class of theories of thin irregular shells, independently of the general results of [9, 11].

In this paper the undeformed geometry of a thin irregular shell structure is described in Chapter 3 as a union of piecewise smooth surfaces and space curves forming together a complex reference network. Each space curve in this network may represent a singular curve on a surface, but also a one-dimensional continuum endowed with its own kinematic and/or physical properties. It is assumed that the deformation of the entire irregular shell structure is determined by the deformation of its reference network, but no restrictions are introduced on the magnitude of the displacements, stretchings and bendings of the network. Then, the principle of virtual work is postulated on the entire reference network, with various fields satisfying relaxed regularity requirements. The non-standard transformations of the principle presented in Chapter 4 lead to the local equilibrium equations and boundary conditions known from the theory of thin smooth shells. Additionally, we obtain in Chapter 5 the general form of jump conditions appropriate for the theory of thin irregular shells. As examples, we discuss in more detail the geometric irregularities (folds, intersections, rigid junctions) and kinematic irregularities (elastic and inelastic junctions), for which particular forms of the jump conditions are derived.

2. Basic notation and preliminary relations for smooth surfaces

List of basic notation:

$\mathcal{E}$	–	three-dimensional Euclidean point space,
E	–	translation (three-dimensional vector) space of $\mathcal{E}$,
M	–	undeformed shell reference surface,
∂M	–	boundary of M,
∂M_f	–	part of the boundary ∂M along which external loads are prescribed,
∂M_d	–	complementary part of the boundary ∂M along which displacement and rotation are prescribed,
$M^{(k)}$	–	smooth surface elements ($k = 1, 2, \ldots, K$),
$\partial M^{(k)}$	–	boundary of $M^{(k)}$,
$\Gamma^{(a)}$	–	smooth curves which are common parts of two or more smooth surface elements,
Γ	–	union of all curves $\Gamma^{(a)}$,
$T_{\boldsymbol{x}}M$	–	tangent space at a regular point $\boldsymbol{x}$ of M,
N	–	deformed shell reference surface,

$\boldsymbol{m}, \boldsymbol{n}$	–	unit normal vectors to the undeformed and deformed reference surfaces,
$\boldsymbol{P}, \boldsymbol{I}$	–	projection and inclusion operators on M,
$\boldsymbol{\chi}$	–	deformation mapping of M,
$\boldsymbol{\chi}_{\Gamma}$	–	deformation mapping of Γ,
$\boldsymbol{u}$	–	displacement vector of the shell reference surface,
ϕ	–	scalar function describing the rotational deformation of the shell lateral boundary surface,
$\mathbb{V} \equiv (\boldsymbol{v}, \boldsymbol{w})$	–	generalized virtual displacements,
$\boldsymbol{v}$	–	virtual displacement vector of the shell reference surface,
$\boldsymbol{w}$	–	virtual change of the unit normal vector $\boldsymbol{n}$,
φ	–	virtual change of ϕ,
$\boldsymbol{F}$	–	surface deformation gradient,
$\boldsymbol{G}$	–	tangential surface deformation gradient,
$\boldsymbol{A}, \boldsymbol{B}$	–	metric tensor and curvature tensor of the shell reference surface,
$\boldsymbol{L}$	–	surface gradient of the virtual displacement field,
$\boldsymbol{D}$	–	appropriate measure of the virtual surface strain,
$\boldsymbol{E}, \boldsymbol{K}$	–	Lagrangian surface strain tensors of stretching and bending,
$\dot{\boldsymbol{E}}, \dot{\boldsymbol{K}}$	–	virtual change of the Lagrangian surface strain measures,
$\boldsymbol{E}_e, \boldsymbol{K}_e$	–	Eulerian surface strain measures,
$\overset{\triangledown}{\boldsymbol{E}}_e, \overset{\triangledown}{\boldsymbol{K}}_e$	–	convective virtual change of the Eulerian surface strain measures,
$\boldsymbol{N}, \boldsymbol{M}$	–	internal surface stress and couple stress tensors of second Piola–Kirchhoff type,
$\boldsymbol{N}_K, \boldsymbol{M}_K$	–	internal surface stress and couple stress tensors of Kirchhoff type,
$\boldsymbol{p}, \boldsymbol{h}$	–	external surface force and moment vectors,
$\boldsymbol{t}^*, \boldsymbol{h}^*$	–	external boundary force and moment vectors,
G_i	–	internal virtual work,
G_e	–	external virtual work,
G_{Γ}	–	additional virtual work of generalized forces acting along Γ,
W_i	–	internal virtual work density,
W_e	–	virtual work density of the external surface loads,
w_e	–	virtual work density of the external boundary loads,
σ_{Γ}	–	virtual work density along regular parts of Γ,
σ_i	–	virtual work at singular points of Γ.

In this paper we shall use the coordinate-free approach to the description of surface geometry and to the analysis on surfaces developed in [11, 20, 29], where all necessary definitions, theorems and relations can be found. We recall below some of those relations in order to fix notation and make the paper self-contained.

Let $\mathcal{E}$ denote the three-dimensional point space (the physical space) and E its translation (three-dimensional vector) space. Within the classical theory of smooth shells the undeformed configuration $M \subset \mathcal{E}$ of the shell reference surface is assumed to be a connected, oriented and regular surface of class C^n, $n \geqslant 2$, with a

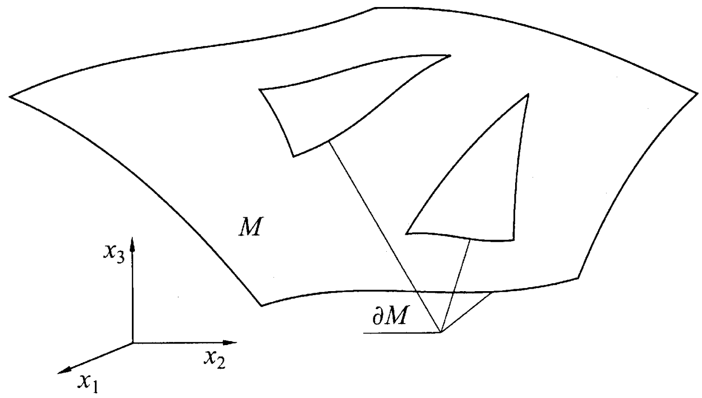

Figure 1.

boundary ∂M consisting of a finite number of closed, piecewise smooth curves that do not meet in cusps, Figure 1.

At each point $\boldsymbol{x} \in M$ we can define a two-dimensional subspace $T_{\boldsymbol{x}}M$ of E, called a tangent space, which allows to decompose E additively into $E = T_{\boldsymbol{x}}M \oplus T_{\boldsymbol{x}}M^{\perp}$, where $T_{\boldsymbol{x}}M^{\perp}$ is an orthogonal complement of $T_{\boldsymbol{x}}M$. The surface M is endowed with local structure by $T_{\boldsymbol{x}}M$ and a locally unique map $\pi_{\boldsymbol{x}}\colon T_{\boldsymbol{x}}M \to \mathcal{E}$ with range $Rg(\pi_{\boldsymbol{x}}) \subset M$ in an open neighbourhood of the zero vector $\mathbf{0} \in T_{\boldsymbol{x}}M$. Then at $\boldsymbol{x} \in M$ we can introduce two operators: an inclusion $\boldsymbol{I}(\boldsymbol{x})\colon T_{\boldsymbol{x}}M \to E$, defined by $\boldsymbol{I}(\boldsymbol{x}) = \nabla\pi_{\boldsymbol{x}}(\mathbf{0})$, where $\nabla\pi_{\boldsymbol{x}}(\mathbf{0})\colon T_{\boldsymbol{x}}M \to E$ is a spatial gradient of $\pi_{\boldsymbol{x}}$, and a perpendicular projection $\boldsymbol{P}(\boldsymbol{x})\colon E \to T_{\boldsymbol{x}}M$ defined by $\boldsymbol{P}(\boldsymbol{x}) = \boldsymbol{I}(\boldsymbol{x})^T$; $\boldsymbol{I}(\boldsymbol{x})$ maps any vector $\boldsymbol{t} \in T_{\boldsymbol{x}}M$ into itself but considered as an element of E, and $\boldsymbol{P}(\boldsymbol{x})$ assigns to any vector $\boldsymbol{u} \in E$ its tangential component on $T_{\boldsymbol{x}}M$. These operators satisfy the relations

$$\begin{aligned} &\boldsymbol{It} = \boldsymbol{t}, \qquad \boldsymbol{t}\cdot(\boldsymbol{Pu}) = (\boldsymbol{It})\cdot\boldsymbol{u},\\ &\boldsymbol{PP}^T = \boldsymbol{A} = \boldsymbol{I}^T\boldsymbol{I}, \qquad \boldsymbol{IP} = \mathbf{1} - \boldsymbol{m}\otimes\boldsymbol{m}, \end{aligned} \tag{2.1}$$

where $\boldsymbol{A}(\boldsymbol{x})\colon T_{\boldsymbol{x}}M \to T_{\boldsymbol{x}}M$ and $\mathbf{1}(\boldsymbol{x})\colon E \to E$ are identity maps (metric tensors) on $T_{\boldsymbol{x}}M$ and E, respectively, and $\boldsymbol{m}(\boldsymbol{x}) \in T_{\boldsymbol{x}}M^{\perp}$ is the unit normal vector assigning the orientation to M at each $\boldsymbol{x} \in M$.

Let $\boldsymbol{\Phi}\colon M \to F$ be a differentiable surface field, where F denotes any finite-dimensional inner-product vector space. Special cases of F (special surface fields $\boldsymbol{\Phi}$) to be used in this paper are: $\mathbb{R}$ (the scalar field ϕ), $T_{\boldsymbol{x}}M$ (the tangential vector field $\boldsymbol{t}$), E (the spatial vector field $\boldsymbol{u}$), $T_{\boldsymbol{x}}M \otimes T_{\boldsymbol{x}}M$ (the tangential tensor field $\boldsymbol{T}$) and $E \otimes T_{\boldsymbol{x}}M$ (the mixed tensor field $\boldsymbol{S}$). The surface gradient of $\boldsymbol{\Phi}$ at $\boldsymbol{x} \in M$ is the unique surface field $\mathrm{Grad}_s\,\boldsymbol{\Phi}(\boldsymbol{x}) \in F \otimes T_{\boldsymbol{x}}M$ defined by $\mathrm{Grad}_s\,\boldsymbol{\Phi}(\boldsymbol{x}) = \nabla(\boldsymbol{\Phi}\circ\pi_{\boldsymbol{x}})(\mathbf{0})$. In particular, the Weingarten map (the surface curvature tensor), $\boldsymbol{B}(\boldsymbol{x}) \in T_{\boldsymbol{x}}M \otimes T_{\boldsymbol{x}}M$, is defined at any $\boldsymbol{x} \in M$ by $\boldsymbol{B} = -\boldsymbol{P}(\mathrm{Grad}_s\,\boldsymbol{m})$.

The surface divergence of a differentiable tangential surface vector field $\boldsymbol{t}(\boldsymbol{x}) \in T_{\boldsymbol{x}}M$ is a surface scalar field $\mathrm{Div}_s\,\boldsymbol{t}(\boldsymbol{x}) \in \mathbb{R}$ defined at each $\boldsymbol{x} \in M$ by $\mathrm{Div}_s\,\boldsymbol{t} =$

$\mathrm{tr}(\boldsymbol{P}\,\mathrm{Grad}_s\,\boldsymbol{t})$. The surface divergence of a differentiable surface field $\boldsymbol{\Psi}(\boldsymbol{x}) \in F \otimes T_{\boldsymbol{x}}M$ is the unique surface field $\mathrm{Div}_s\boldsymbol{\Psi}(\boldsymbol{x}) \in F$ defined at each $\boldsymbol{x} \in M$ by $(\mathrm{Div}_s\boldsymbol{\Psi})\boldsymbol{w} = \mathrm{Div}_s(\boldsymbol{\Psi}^T\boldsymbol{w})$ for any $\boldsymbol{w} \in F$. Then the classical surface divergence theorem takes the form

$$\int_{\partial M} \boldsymbol{\Psi}\boldsymbol{\nu}\,\mathrm{d}s = \iint_M \mathrm{Div}_s\boldsymbol{\Psi}\,\mathrm{d}a, \tag{2.2}$$

where $\boldsymbol{\nu}(\boldsymbol{x}) \in T_{\boldsymbol{x}}M$ is the outward unit normal to ∂M at $\boldsymbol{x} \in \partial M$ while $\mathrm{d}s$ and $\mathrm{d}a$ are differential length and area elements of ∂M and M, respectively. In particular, for a spatial vector $\boldsymbol{u}(\boldsymbol{x}) \in E$ and a mixed tensor $\boldsymbol{S}(\boldsymbol{x}) \in E \otimes T_{\boldsymbol{x}}M$ surface fields we have

$$\int_{\partial M} \boldsymbol{u} \cdot \boldsymbol{S}\boldsymbol{\nu}\,\mathrm{d}s = \iint_M ((\mathrm{Div}_s\,\boldsymbol{S}) \cdot \boldsymbol{u} + \boldsymbol{S} \cdot \mathrm{Grad}_s\,\boldsymbol{u})\,\mathrm{d}a. \tag{2.3}$$

The proof of theorems (2.2) and (2.3) under classical regularity assumptions may be found in many papers and books (see [21] and references cited therein). In general, it is required that the surface M be regular in the sense of [20] while the fields $\boldsymbol{\Psi}$ and $\boldsymbol{u}$ must be of C^1 in the interior of M and have the extensions of the same class to the closure of M.

In the derivation of basic relations for thin irregular shell structures we need to generalize theorems (2.2) and (2.3). We shall admit piecewise smooth surfaces and a network composed of such surfaces, as well as suitably regular fields defined on them.

3. Postulates and General Relations

A consistent formulation of field equations and side conditions (boundary and jump conditions) for thin smooth shells as well as for irregular thin shell-like structures can be based on the following two postulates:

(I) *The deformation of the entire irregular shell structure is determined, within a sufficient accuracy, by the deformation of a distinguished material surface-like continuum, called the shell reference network (the reference surface in the case of smooth shells).*

(II) *The equilibrium equations of the entire thin irregular shell structure are determined, within a sufficient accuracy, by a suitable form of the principle of virtual work involving only kinematic and dynamic fields associated with the reference network.*

The first of these postulates is kinematic in nature and can be regarded as the definition of the general thin irregular shell-like structure. The second one should be regarded as the basic dynamic postulate of the theory. It is important to note that both postulates are independent of constitutive equations needed to specify particular classes of materials. It remains to define precisely the representative surface-like continuum referred to in the kinematic hypothesis, and to lay down a suitable form of the principle of virtual work referred to in the dynamic hypothesis.

Intuitively, the representative, two-dimensional, surface-like continuum should be capable of undergoing motion and deformation through the physical space and of having resistance to stretching and bending. Such a continuum, variously called in the literature on regular shell structures (the shell reference surface, the shell fundamental surface or the shell carrying surface), can be defined as a union of geometric surfaces in the space satisfying suitable regularity assumptions. Such a surface-like continuum in an undeformed configuration will be denoted by M, and the union of all singular curves in M will be denoted by Γ. The deformation of $M \backslash \Gamma$ and Γ itself will be described by two maps: $\boldsymbol{\chi}\colon M \backslash \Gamma \to \mathcal{E}$ and $\boldsymbol{\chi}_\Gamma\colon \Gamma \to \mathcal{E}$. The principle of virtual work can then formally be expressed as a statement which asserts that for all virtual displacements $\mathbb{V}$ the following holds

$$G(M; \mathbb{V}) \equiv G_i(M; \mathbb{V}) - G_e(M; \mathbb{V}) - G_\Gamma = 0. \tag{3.1}$$

Here G_i and G_e are real-valued set functions designated to represent the internal and external virtual works of the shell-like structures while G_Γ stands for an additional virtual work of generalized forces acting along the Γ. The explicit form of these three terms and of the virtual displacement $\mathbb{V}$ must be consistent with the two hypotheses.

The regularity assumptions introduced in Section 2, and also explicitly or implicitly in most papers on thin shell theory, are far too restrictive for many problems of engineering importance if various geometric, material and kinematic irregularities are to be admitted.

A fully satisfactory treatment of those problems, from the mathematical point of view, would call for use of concepts from the geometric measure theory. For example, the configurations of the surface-like continuum representing a thin irregular shell-like structure (in the broadest sense) may be defined as rectifiable currents – the 2-dimensional surfaces of geometric measure theory. Then the deformation of the reference network can be described by Lipschitz continuous mappings. Let us note that the development of the geometric measure theory [12] was motivated in part by problems of minimal surfaces and soap bubbles, which surely we would like to treat as thin shells (actually membranes). However, the formulation of the theory of irregular shells within such a setting seems to be beyond the scope of any single paper at the moment, and we shall not develop such a theory here as well.

Various irregularities encountered in the analysis of irregular shell problems can be grouped into three broad classes, [11]:

(1) The undeformed reference surface of the shell is not smooth (thus it may contain folds), or it is not a surface in the sense of classical differential geometry of surfaces (for example, two smooth intersecting surfaces do not form a surface as a whole). Such irregularities may be called geometric.

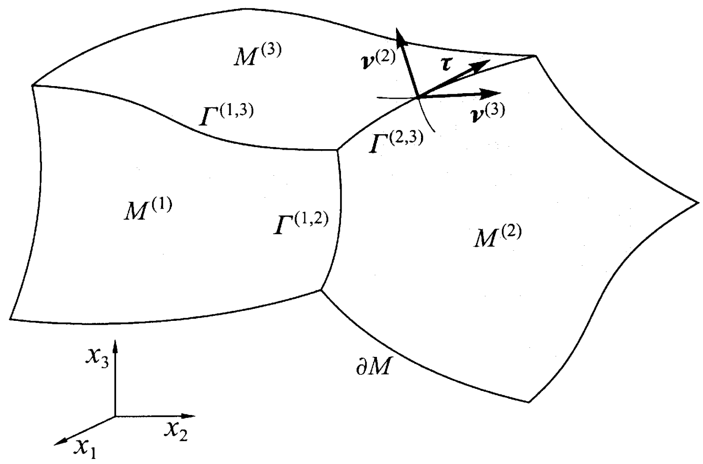

Figure 2.

(2) Deformation of the shell reference surface (smooth or non-smooth in the undeformed configuration) fails to be smooth. We may refer to this kind of irregularities as kinematic, since they are associated with the deformation.
(3) The shell structure cannot be considered as a single shell (smooth or non-smooth and undergoing smooth or non-smooth deformation) but rather as a union of some number of single shells interconnected along junctions. This type of irregularities may be called mechanical, because the junctions themselves may have their own mechanical properties, possibly quite different from properties of the adjacent shells.

The basic assumption made in the three cases is that all the irregularities are restricted to distinct curves and points (i.e. to sets of zero area measure) on the shell reference surface. Under this assumption, all three classes of irregularities can be considered at once as follows.

In the most general case, the undeformed configuration of the reference surface-like continuum of a thin irregular shell-like structure can be defined to be a network $M \in \mathcal{E}$ consisting of a finite number of surface elements $M^{(k)}, k = 1, 2, \ldots, K$, with the following properties:

(1) Each $M^{(k)}$ is a bounded, oriented, connected and smooth surface of class $C^n, n \geqslant 2$, whose boundary $\partial M^{(k)}$ consists of a finite number of closed Jordan curves oriented consistently with $M^{(k)}$ that do not meet in cusps.
(2) No two distinct surface elements $M^{(k)}$ have common interior points.
(3) Two or more distinct surface elements $M^{(k)}$ may have a smooth spatial curve $\Gamma^{(a)}$ as a common part of the boundaries. Such a curve is defined by

$$\Gamma^{(a)} = \partial M^{(k_1)} \cap \partial M^{(k_2)} \cap \cdots \cap \partial M^{(k_m)} \quad \text{if } k_1 \neq k_2 \neq \cdots \neq k_m. \qquad (3.2)$$

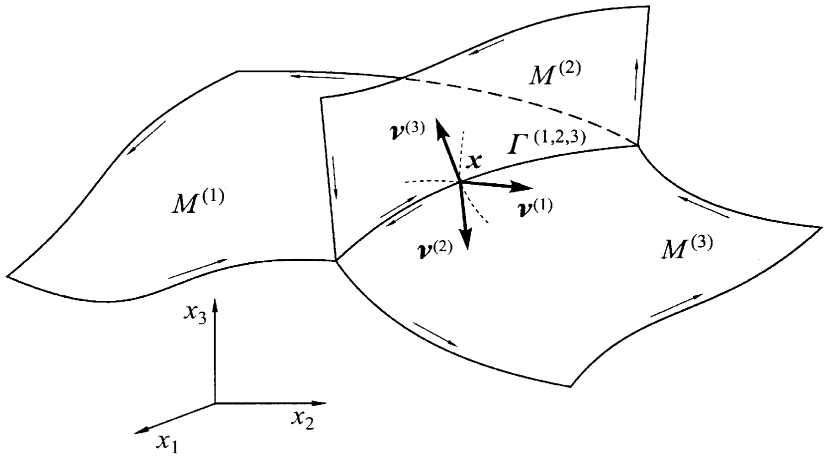

Figure 3.

(4) Two or more distinct curves $\Gamma^{(a)}$ may have in common only single isolated points.

We then define M as the union of all the closed surface elements $\bar{M}^{(k)} = M^{(k)} \cup \partial M^{(k)}$, and by Γ we denote the union of all curves $\Gamma^{(a)}$. It is clear that $\Gamma \subset M$. Moreover, the boundary ∂M of M defined by

$$\partial M = \left(\bigcup_{k=1}^{K} \partial M^{(k)} \right) \backslash \Gamma, \tag{3.3}$$

consists of a finite number of Jordan curves (not necessarily closed). Two rather simple examples of such a network are shown in Figure 2 and Figure 3.

Each $M^{(k)}$ can be regarded as the reference surface of a regular shell-like element. Each $\Gamma^{(a)}$ can be regarded as representing a geometric space curve, but also the corresponding reference axis of a rod-like element, a multiple shell intersection, a technological junction, a plastic hinge developing during the deformation process, etc. Thus $\Gamma^{(a)}$ may also represent a one-dimensional continuum endowed with its own kinematic and/or physical properties. By the requirement (4) each $\Gamma^{(a)}$ coincides with parts of the boundaries of the adjacent surface elements, what is a reasonable assumption in the case of thin shells discussed here. Note also that the requirement (4) excludes rod-like elements, whose reference axes do not coincide with any boundary of the surface elements $M^{(k)}$.

A deformation of any regular part of M is described by a map $\boldsymbol{\chi}: M\backslash\Gamma \to \mathcal{E}$ which carries each regular surface point $\boldsymbol{x} \in M\backslash\Gamma$ into its spatial place $\boldsymbol{y} = \boldsymbol{\chi}(\boldsymbol{x}) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x})$ on the deformed network $N = \boldsymbol{\chi}(M)$, where $\boldsymbol{u}$: $M\backslash\Gamma \to E$ is the associated displacement field. It is assumed that for each regular $\boldsymbol{x} \in M\backslash\Gamma$ the map $\boldsymbol{\chi} \circ \pi_{\boldsymbol{x}}$: $T_{\boldsymbol{x}}M \to \mathcal{E}$ is of class $C^m, m \geqslant n \geqslant 2$, has a C^m inverse on the co-domain $N = \chi(M)$ and admits finite extensions of the same classes to the boundaries $\partial M^{(k)}$ and $\partial N^{(k)} = \chi(\partial M^{(k)})$.

Let $\boldsymbol{F} = \text{Grad}_s\, \chi(\boldsymbol{x})\colon T_{\boldsymbol{x}}M \to T_{\boldsymbol{y}}\mathcal{E} \equiv E$ be the surface gradient of the map χ. The surface deformation gradient $\boldsymbol{G}$ is defined by $\boldsymbol{G}(\boldsymbol{x}) = \boldsymbol{P}(\boldsymbol{y})\boldsymbol{F}(\boldsymbol{x})$, where $\boldsymbol{P}(\boldsymbol{y})$ is the perpendicular projection on the tangent plane $T_{\boldsymbol{y}}N$. Under the regularity assumptions given above the surface deformation gradient $\boldsymbol{G}(\boldsymbol{x})$ exists at every point $\boldsymbol{x} \in M\backslash\Gamma$. Since so defined $\boldsymbol{G}(\boldsymbol{x})\colon T_{\boldsymbol{x}}M \to T_{\boldsymbol{y}}N$, two tangential surface tensor fields

$$\boldsymbol{E} = \tfrac{1}{2}(\boldsymbol{G}^T\boldsymbol{A}(\boldsymbol{y})\boldsymbol{G} - \boldsymbol{A}), \qquad \boldsymbol{K} = -(\boldsymbol{G}^T\boldsymbol{B}(\boldsymbol{y})\boldsymbol{G} - \boldsymbol{B}) \tag{3.4}$$

provide the Lagrangian type measures of the local strains and bendings of the regular parts of M during its deformation χ, where $\boldsymbol{A}(\boldsymbol{y})$ and $\boldsymbol{B}(\boldsymbol{y})$ denote the surface metric and curvature tensors of the deformed network $N = \chi(M)$, respectively.

Any deformation of M is described by two maps $\chi\colon M\backslash\Gamma \to \mathcal{E}$ and $\chi_\Gamma\colon \Gamma \to \mathcal{E}$, for the singular curve Γ may be admitted to follow its own deformation, in general. We shall not assume a priori that the deformation $\boldsymbol{y} = \chi(\boldsymbol{x})$ be continuous across the singular curve Γ or some parts thereof. Accordingly, we regard $\boldsymbol{y} = \chi(\boldsymbol{x})$ as being defined for all points of $M^{(k)}$ except possibly for points belonging to Γ. We shall then assume that the deformation map χ restricted to $M^{(k)}$ has a finite limit at every point $\boldsymbol{x} \in \Gamma$,

$$\boldsymbol{y}^{(k)} = \chi^{(k)}(\boldsymbol{x}) = \lim_{\boldsymbol{z}\to\boldsymbol{x}} \chi(\boldsymbol{x}) = \boldsymbol{x} + \lim_{\boldsymbol{z}\to\boldsymbol{x}} \boldsymbol{u}(\boldsymbol{x}), \quad \forall \boldsymbol{z} \in \text{int}\, M^{(k)}, \tag{3.5}$$

whenever Γ is a part of the boundary $\partial M^{(k)}$. In some special cases, the deformation χ_Γ may be assumed to be the restriction of χ to Γ.

In order to describe properly a virtual deformation of the regular part of the deformed network N, let us consider a one-parameter family of deformations $\boldsymbol{y} = \chi(\boldsymbol{x}, t) = \boldsymbol{x} + \boldsymbol{u}(\boldsymbol{x}, t)$, where t is a scalar (time-like) parameter. Then by $\boldsymbol{v}(\boldsymbol{x}, t) \equiv (\partial\chi/\partial t)(\boldsymbol{x}, t) = \dot{\boldsymbol{u}}(\boldsymbol{x}, t)$ we denote the virtual displacement (the velocity field in a real motion) and by $\boldsymbol{w}(\boldsymbol{x}, t) \equiv (\partial\boldsymbol{n}/\partial t)(\boldsymbol{x}, t) = \dot{\boldsymbol{n}}(\boldsymbol{x}, t)$ the virtual change of the unit normal $\boldsymbol{n}$ of $N = \chi(M, t)$. Regarding the generalized virtual displacements, collectively denoted by $\mathbb{V} \equiv (\boldsymbol{v}, \boldsymbol{w})$, and the corresponding rates of change of the strain measures (3.4) as the basic kinematic variables, we can obtain the suitable form of the virtual work expression appearing in the dynamic postulate (3.1).

The internal virtual work is generally defined as the work done by internal stress measures over any instantaneous rate of change of work-conjugate deformation measures. In the formal approach it is further assumed that G_i should be an additive set function defined on a collection of measurable and mutually disjoint parts of the shell. Under the additional assumption that G_i is an absolutely continuous real function with respect to the area measure ($\mathcal{H}^2$-Hausdorff measure) of M, the

Radon–Nikodym theorem implies the existence of the internal virtual work density (stress power density) $W_i(\boldsymbol{x})$ such that

$$G_i(M;\mathbb{V}) = \sum_{k=1}^{K} \iint_{M^{(k)}} W_i(\boldsymbol{x})\,\mathrm{d}a, \tag{3.6}$$

where $M^{(k)}$ denote the regular surface elements as defined above. Here and henceforth, the surface integral is understood as the integral with respect to $\mathcal{H}^2$-Hausdorff measure. For smooth compact surfaces, this integral coincides with the classical integral with respect to surface area in the sense of classical analysis. Since G_i should be linear with respect to the generalized virtual displacements $\mathbb{V} \equiv (\boldsymbol{v}, \boldsymbol{w})$, the relation (3.6) allows one to obtain the internal virtual work density $W_i(\boldsymbol{x})$ as follows.

Let $\boldsymbol{L} = \operatorname{grad}_s \boldsymbol{v}$ be the surface gradient, taken on the deformed network N, of the virtual displacement field $\boldsymbol{v}$. According to [20, 29],

$$\boldsymbol{L} = \dot{\boldsymbol{F}}\boldsymbol{G}^{-1}, \qquad \boldsymbol{w} = -\boldsymbol{L}^T\boldsymbol{n}. \tag{3.7}$$

The surface tangential symmetric tensor field

$$\boldsymbol{D} = \tfrac{1}{2}(\boldsymbol{L}^T\boldsymbol{I} + \boldsymbol{P}\boldsymbol{L}), \tag{3.8}$$

where $\boldsymbol{D}(\boldsymbol{y}) \in T_{\boldsymbol{y}}N \otimes T_{\boldsymbol{y}}N$, is an appropriate measure for the surface virtual strain. Performing transformations on $\boldsymbol{D}$ with the help of (3.7) and $\boldsymbol{G}\boldsymbol{G}^{-1} = \boldsymbol{A}(\boldsymbol{y})$ we obtain

$$\begin{aligned} \boldsymbol{D} &= \tfrac{1}{2}((\dot{\boldsymbol{F}}\boldsymbol{G}^{-1})^T\boldsymbol{I}\boldsymbol{G}\boldsymbol{G}^{-1} + \boldsymbol{G}^{-T}\boldsymbol{G}^T\boldsymbol{P}(\dot{\boldsymbol{F}}\boldsymbol{G}^{-1})) \\ &= \tfrac{1}{2}\boldsymbol{G}^{-T}(\dot{\boldsymbol{F}}^T\boldsymbol{F} + \boldsymbol{F}^T\dot{\boldsymbol{F}})\boldsymbol{G}^{-1} = \boldsymbol{G}^{-T}\dot{\boldsymbol{E}}\boldsymbol{G}^{-1} \\ &= \boldsymbol{G}^{-T}(\boldsymbol{G}^T\boldsymbol{E}_e\boldsymbol{G})^{\cdot}\,\boldsymbol{G}^{-1} = \overset{\triangledown}{\boldsymbol{E}}_e, \end{aligned} \tag{3.9}$$

where $\overset{\triangledown}{\boldsymbol{E}}_e$ is the convective virtual change (the convective time derivative in a real motion) of the Eulerian surface strain measure $\boldsymbol{E}_e = \boldsymbol{G}^{-T}\boldsymbol{E}\boldsymbol{G}^{-1}$. Similarly, the surface virtual bending is given by

$$\begin{aligned} \overset{\triangledown}{\boldsymbol{K}}_e &= \boldsymbol{G}^{-T}(\boldsymbol{G}^T\boldsymbol{K}_e\boldsymbol{G})^{\cdot}\,\boldsymbol{G}^{-1} = -\boldsymbol{G}^{-T}(\boldsymbol{F}^T\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y})\boldsymbol{P}\boldsymbol{F})^{\cdot}\,\boldsymbol{G}^{-1} \\ &= -\boldsymbol{L}^T\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y}) - \boldsymbol{P}(\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y})\boldsymbol{P})^{\cdot}\,\boldsymbol{I} - \boldsymbol{B}(\boldsymbol{y})\boldsymbol{P}\boldsymbol{L}, \end{aligned} \tag{3.10}$$

where $\boldsymbol{K}_e = \boldsymbol{G}^{-T}\boldsymbol{K}\boldsymbol{G}^{-1}$ is the Eulerian surface bending measure, and $\overset{\triangledown}{\boldsymbol{K}}_e$ its convective virtual change.

Taking into account the relations

$$\begin{aligned}\dot{\boldsymbol{F}} &= \operatorname{Grad}_s \boldsymbol{v}, \qquad \boldsymbol{I}\boldsymbol{A}(\boldsymbol{y})\boldsymbol{P} = \mathbf{1}, \quad \dot{\mathbf{1}} = \mathbf{0},\\ (\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y})\boldsymbol{G})^{\cdot} &= -(\operatorname{Grad}_s \boldsymbol{n})^{\cdot} = -\operatorname{Grad}_s \dot{\boldsymbol{n}},\end{aligned} \tag{3.11}$$

we obtain the following Lagrangian surface virtual strain measures

$$\begin{aligned}\dot{\boldsymbol{E}} &= \tfrac{1}{2}(\boldsymbol{F}^T\mathbf{1}\boldsymbol{F})^{\cdot} = \tfrac{1}{2}(\dot{\boldsymbol{F}}^T\boldsymbol{I}\boldsymbol{G} + \boldsymbol{G}^T\boldsymbol{P}\dot{\boldsymbol{F}}),\\ \dot{\boldsymbol{K}} &= -(\boldsymbol{F}^T\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y})\boldsymbol{G})^{\cdot} = -(\dot{\boldsymbol{F}}^T\boldsymbol{I}\boldsymbol{B}(\boldsymbol{y})\boldsymbol{G} + \boldsymbol{F}^T\operatorname{Grad}_s \dot{\boldsymbol{n}}).\end{aligned} \tag{3.12}$$

Within the bending theory of thin shells, the simplest internal virtual work density can be postulated in the form

$$W_i = \boldsymbol{N}\cdot\dot{\boldsymbol{E}} + \boldsymbol{M}\cdot\dot{\boldsymbol{K}} = \boldsymbol{N}_K\cdot\overset{\triangledown}{\boldsymbol{E}}_e + \boldsymbol{M}_K\cdot\overset{\triangledown}{\boldsymbol{K}}_e. \tag{3.13}$$

Here $\boldsymbol{N}_K$ and $\boldsymbol{M}_K$ are the internal surface stress and couple resultant tensors of the Kirchhoff type while $\boldsymbol{N}$ and $\boldsymbol{M}$ are the internal surface stress and couple resultant tensors of the 2nd Piola–Kirchhoff type, respectively, which are related by

$$\boldsymbol{N}_K = \boldsymbol{G}\boldsymbol{N}\boldsymbol{G}^T, \qquad \boldsymbol{M}_K = \boldsymbol{G}\boldsymbol{M}\boldsymbol{G}^T. \tag{3.14}$$

Both pairs of the surface stress measures in (3.13) are tangential symmetric surface tensors: $(\boldsymbol{N}_K, \boldsymbol{M}_K) \in T_{\boldsymbol{y}}N \otimes T_{\boldsymbol{y}}N$ and $(\boldsymbol{N}, \boldsymbol{M}) \in T_{\boldsymbol{x}}M \otimes T_{\boldsymbol{x}}M$.

It should be apparent that we can replace any of the strain measures (3.4) by any other measures of the first and second order of the surface deformation. Because all strain measures are derivable from $\boldsymbol{E}$ and $\boldsymbol{K}$, such a change would not lead to quantitative new results, although in special cases other measures may be more convenient.

The external virtual work expression $G_e(M; \mathbb{V})$ may be obtained along the same line of reasoning. In general, the shell may be subjected to external loading applied on the network $M\backslash\Gamma$ and on a part ∂M_f of the boundary ∂M. Thus, $G_e(M; \mathbb{V})$ must consist of the surface part and the boundary part. Moreover, if any part of M is considered, then the boundary part of $G_e(M; \mathbb{V})$ must be split into an external part and an internal part. Keeping these facts in mind and adopting the same assumptions as for the internal virtual work with corresponding modifications of the boundary parts of $G_e(M; \mathbb{V})$, the consistent external virtual work expression (mechanical power) takes the form

$$G_e(M; \mathbb{V}) = \sum_{k=1}^{K} \iint_{M^{(k)}} W_e \,\mathrm{d}a + \int_{\partial M_f} w_e \,\mathrm{d}s, \tag{3.15}$$

where the virtual work densities of the external surface and boundary loads can be given by

$$W_e = \boldsymbol{p}\cdot\boldsymbol{v} + \boldsymbol{h}\cdot\boldsymbol{w}, \qquad w_e = \boldsymbol{t}^*\cdot\boldsymbol{v} + \boldsymbol{h}^*\cdot\boldsymbol{w}. \tag{3.16}$$

Here $\boldsymbol{p}$ and $\boldsymbol{h}$ are the external surface force and moment resultant vectors referred to the undeformed surface element $M^{(k)}$, while $\boldsymbol{t}^*$ and $\boldsymbol{h}^*$ are the external boundary force and moment resultant vectors referred to the part ∂M_f of the undeformed boundary ∂M.

At this stage of the analysis it is not apparent what form should be postulated for the additional virtual work expression G_Γ along the union of all singular curves in the reference network, and this will be discussed in the Chapter 5.

As for regularity assumptions, we need only to require that all fields appearing in (3.13) and (3.16) be regular enough for the integrals to be meaningfully defined. Thus the fields need not be even continuous. However, stronger assumptions will be needed in order to ensure the existence of field equations and side conditions. In addition we must require that every part $M^{(k)}$ of M be regular enough in the sense that the generalized surface divergence theorem should be applicable. In particular, this is assured if the boundary $\partial M^{(k)}$ consists of a finite number of closed Jordan curves consistently oriented with $M^{(k)}$.

4. Equilibrium Equations and Boundary Conditions

Derivation of the Lagrangian local equilibrium equations and boundary conditions for thin smooth shells may be found in several papers [13, 14], but performed with the explicit use of convected coordinate system and components of various vector and tensor fields. Here we present such a derivation in coordinate-free notation and for the whole thin irregular shell-like structure.

Let us introduce (3.12) into $(3.13)_1$ and use the symmetry of $\boldsymbol{N}$ and $\boldsymbol{M}$, which yields

$$W_i = \boldsymbol{I}(\boldsymbol{G}\boldsymbol{N} - \boldsymbol{B}(\boldsymbol{y})\boldsymbol{G}\boldsymbol{M}) \cdot \dot{\boldsymbol{F}} + (\boldsymbol{F}\boldsymbol{M}) \cdot \operatorname{Grad}_s \dot{\boldsymbol{n}}. \tag{4.1}$$

From $\boldsymbol{F}^T\boldsymbol{n} = \boldsymbol{0}$ it follows that $(\boldsymbol{F}^T\boldsymbol{n})^{\cdot} = \dot{\boldsymbol{F}}^T\boldsymbol{n} + \boldsymbol{F}^T\dot{\boldsymbol{n}} = \boldsymbol{0}$, which multiplied from the left by $\boldsymbol{G}^{-T}$ gives $\boldsymbol{P}\dot{\boldsymbol{n}} = -\boldsymbol{G}^{-T}\dot{\boldsymbol{F}}^T\boldsymbol{n}$. But the relation $\boldsymbol{n} \cdot \boldsymbol{n} = 1$ leads to $\dot{\boldsymbol{n}} \cdot \boldsymbol{n} = 0$ which means that $\dot{\boldsymbol{n}}(\boldsymbol{y}) \in T_{\boldsymbol{y}}N$ and, therefore, $\boldsymbol{P}\dot{\boldsymbol{n}} = \dot{\boldsymbol{n}}$. Taking this into account, we transform the last term in (4.1) into

$$\begin{aligned}(\boldsymbol{F}\boldsymbol{M}) \cdot \operatorname{Grad}_s \dot{\boldsymbol{n}} &= (\boldsymbol{G}\boldsymbol{M}) \cdot \operatorname{Grad}_s \dot{\boldsymbol{n}} \\ &= \operatorname{Div}_s((\boldsymbol{G}\boldsymbol{M})^T\dot{\boldsymbol{n}}) - \operatorname{Div}_s(\boldsymbol{G}\boldsymbol{M})) \cdot \dot{\boldsymbol{n}} \\ &= \operatorname{Div}_s((\boldsymbol{G}\boldsymbol{M})^T\dot{\boldsymbol{n}}) + [(\boldsymbol{n} \otimes \operatorname{Div}_s(\boldsymbol{G}\boldsymbol{M}))\boldsymbol{G}^{-T}] \cdot \dot{\boldsymbol{F}}.\end{aligned} \tag{4.2}$$

As a result, the internal virtual work density $(3.13)_1$ can be represented by

$$W_i = \boldsymbol{T} \cdot \dot{\boldsymbol{F}} + \operatorname{Div}_s(\boldsymbol{H}^T\dot{\boldsymbol{n}}), \tag{4.3}$$

where

$$\boldsymbol{T} = \boldsymbol{I}(\boldsymbol{G}\boldsymbol{N} - \boldsymbol{B}(\boldsymbol{y})\boldsymbol{H}) + \boldsymbol{n} \otimes (\operatorname{Div}_s \boldsymbol{H})\boldsymbol{G}^{-T}, \quad \boldsymbol{H} = \boldsymbol{G}\boldsymbol{M}. \tag{4.4}$$

The tensor fields $\boldsymbol{T}(\boldsymbol{x}) \in E \otimes T_{\boldsymbol{x}}M$ and $\boldsymbol{H}(\boldsymbol{x}) \in T_{\boldsymbol{y}}N \otimes T_{\boldsymbol{x}}M$ are assumed to be of class C^1 in the interior of each smooth surface element $M^{(k)}$ and to have extensions of the same class to the boundary with finite limits $\boldsymbol{T}^{(k)}(\boldsymbol{x})$ and $\boldsymbol{H}^{(k)}(\boldsymbol{x})$ at any $\boldsymbol{x} \in \partial M^{(k)}$, respectively. Then applying the surface divergence theorems (2.2) and (2.3) on each smooth surface element $M^{(k)}$, we obtain

$$\iint_{M^{(k)}} W_i \,\mathrm{d}a = -\iint_{M^{(k)}} (\mathrm{Div}_s\, \boldsymbol{T}) \cdot \boldsymbol{v} \,\mathrm{d}a + \int_{\partial M^{(k)}} (\boldsymbol{t}_\nu^{(k)} \cdot \boldsymbol{v}^{(k)} + \boldsymbol{h}_\nu^{(k)} \cdot \boldsymbol{w}^{(k)}) \,\mathrm{d}s, \tag{4.5}$$

where at each $\boldsymbol{x} \in \partial M^{(k)}$

$$\boldsymbol{t}_\nu^{(k)} = \boldsymbol{T}^{(k)}\boldsymbol{\nu}^{(k)}, \qquad \boldsymbol{h}_\nu^{(k)} = \boldsymbol{H}^{(k)}\boldsymbol{\nu}^{(k)}. \tag{4.6}$$

The external surface virtual work density W_e can be rewritten with the help of (2.3) in the form

$$W_e = \boldsymbol{l} \cdot \boldsymbol{v} + \mathrm{Div}_s((\boldsymbol{n} \cdot \boldsymbol{v})\boldsymbol{G}^{-1}\boldsymbol{h}), \qquad \boldsymbol{l} = \boldsymbol{p} + \mathrm{Div}_s(\boldsymbol{n} \otimes \boldsymbol{G}^{-1}\boldsymbol{h}). \tag{4.7}$$

Assuming again that $\boldsymbol{h}$ is of class C^1 in int $M^{(k)}$ and has a finite extension of the same class to the boundary with a finite limit $\boldsymbol{h}^{(k)}(\boldsymbol{x})$ at any $\boldsymbol{x} \in \partial M^{(k)}$, we apply (2.2) to transform (4.7) into

$$\begin{aligned} &\iint_{M^{(k)}} W_e \,\mathrm{d}a = \iint_{M^{(k)}} \boldsymbol{l} \cdot \boldsymbol{v} \,\mathrm{d}a - \int_{\partial M^{(k)}} \boldsymbol{k}^{(k)} \cdot \boldsymbol{v} \,\mathrm{d}s, \\ &\boldsymbol{k}^{(k)} = \{(\boldsymbol{G}_{(k)}^{-1}\boldsymbol{h}^{(k)}) \cdot \boldsymbol{\nu}^{(k)}\}\boldsymbol{n}^{(k)}. \end{aligned} \tag{4.8}$$

At each regular boundary point $\boldsymbol{x} \in \partial M^{(k)}$ of any smooth surface element $M^{(k)}$ the surface gradient $\mathrm{Grad}_s\, \boldsymbol{u}$ of the displacement field can be decomposed into tangential and normal derivatives (under the assumption that $\mathrm{Grad}_s\, \boldsymbol{u}$ admits a continuous extension to the boundary $\partial M^{(k)}$):

$$\begin{aligned} &\mathrm{Grad}_s\, \boldsymbol{u} = \boldsymbol{u},_\nu \otimes \boldsymbol{\nu} + \boldsymbol{u}' \otimes \boldsymbol{\tau}, \\ &\boldsymbol{u},_\nu \equiv (\mathrm{Grad}_s\, \boldsymbol{u})\boldsymbol{\nu}, \qquad \boldsymbol{u}' \equiv (\mathrm{Grad}_s\, \boldsymbol{u})\boldsymbol{\tau}, \end{aligned} \tag{4.9}$$

where $\boldsymbol{\tau}$ is the unit tangent vector of $\partial M^{(k)}$. From (4.9) it follows that the field of unit normal vectors $\boldsymbol{n}$ along $\partial N^{(k)} = \boldsymbol{\chi}(\partial M^{(k)})$ can be regarded as a function of $\boldsymbol{u},_\nu$ and $\boldsymbol{u}'$, i.e. $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{u},_\nu, \boldsymbol{u}')$, and is subject to two independent constraints

$$\boldsymbol{y}' \cdot \boldsymbol{n} = (\boldsymbol{\tau} + \boldsymbol{u}') \cdot \boldsymbol{n} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{n} = 1. \tag{4.10}$$

As a result, along the boundary $\boldsymbol{n} = \boldsymbol{n}(\boldsymbol{u},_\nu, \boldsymbol{u}')$ is expressible through $\boldsymbol{u}'$ and a scalar function $\phi = \phi(\boldsymbol{u},_\nu, \boldsymbol{u}')$ describing the rotational deformation of the shell lateral boundary surface. The structure of the function $\phi(\boldsymbol{u},_\nu, \boldsymbol{u}')$ was discussed in [27], where the general expression for $\boldsymbol{w} \equiv \dot{\boldsymbol{n}}$ in terms of $\varphi \equiv \dot{\phi}$ and $\boldsymbol{v}' \equiv \dot{\boldsymbol{u}}'$ was derived in the form

$$\boldsymbol{w} = \boldsymbol{q}\varphi + \boldsymbol{L}\boldsymbol{v}', \qquad \boldsymbol{q}(\phi, \boldsymbol{u}') \equiv \partial_\phi \boldsymbol{n}, \qquad \boldsymbol{L}(\phi, \boldsymbol{u}') \equiv \partial_{\boldsymbol{u}'}\boldsymbol{n}. \tag{4.11}$$

Explicit expressions for the vector-valued function $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{u},_\nu, \boldsymbol{u}')$ and the tensor-valued function $\boldsymbol{L} = \boldsymbol{L}(\boldsymbol{u},_\nu, \boldsymbol{u}')$ depend on the particular definition of the scalar-valued function $\phi = \phi(\boldsymbol{u},_\nu, \boldsymbol{u}')$ employed. Three particular cases of ϕ, denoted by n_ν, ϑ_ν and ω_t, were discussed in the literature [13, 24–28]. For the present general considerations no particular choice of ϕ need to be made, and we shall derive the relevant local equilibrium equations as well as boundary and jump conditions for any such a choice.

With the help of (4.11) the second term in the line integral of (4.5) can be transformed further into

$$\int_{\partial M^{(k)}} \boldsymbol{h}_\nu^{(k)} \cdot \boldsymbol{w}^{(k)}\, \mathrm{d}s = \int_{\partial M^{(k)}} (-\boldsymbol{f}^{(k)} \cdot (\boldsymbol{v}^{(k)})' + h^{(k)}\varphi^{(k)})\, \mathrm{d}s, \tag{4.12}$$

where the supplementary internal force $\boldsymbol{f}^{(k)}(\boldsymbol{x})$ and the moment $h^{(k)}(\boldsymbol{x})$ resulting from the internal moment vector $\boldsymbol{h}_\nu^{(k)}$ acting along the boundary $\partial M^{(k)}$ are defined by

$$\boldsymbol{f}^{(k)} = -\boldsymbol{L}^T \boldsymbol{h}_\nu^{(k)}, \qquad h^{(k)} = \boldsymbol{q} \cdot \boldsymbol{h}_\nu^{(k)}. \tag{4.13}$$

Along each $\partial M^{(k)}$ there may be singular points P_a, $a = 1, \ldots, A$, described by $s = s_a$, at which the field $\boldsymbol{f}^{(k)} \cdot \boldsymbol{v}^{(k)}$ is not differentiable. Such singular points are, for example, corners of the closed Jordan curves composing $\partial M^{(k)}$ or points with singularities of the fields $\boldsymbol{L}, \boldsymbol{q}, \boldsymbol{h}_\nu^{(k)}$ and $\boldsymbol{v}^{(k)}$. At such singular points we assume the existence of finite limits of $\boldsymbol{f}^{(k)}$ and $\boldsymbol{v}^{(k)}$ defined by

$$\boldsymbol{f}_a^{(k)\pm} = \lim_{h\to 0} \boldsymbol{f}^{(k)}(s_a \pm h), \qquad \boldsymbol{v}_a^{(k)\pm} = \lim_{h\to 0} \boldsymbol{v}^{(k)}(s_a \pm h). \tag{4.14}$$

Then the line integral (4.5) can be transformed by applying integration by parts which leads to

$$\begin{aligned}\iint_{M^{(k)}} W_i\, \mathrm{d}a \;=\; & -\iint_{M^{(k)}} (\mathrm{Div}_s\, \boldsymbol{T}) \cdot \boldsymbol{v}\, \mathrm{d}a \\ & + \int_{\partial M^{(k)}} (\boldsymbol{p}_\nu^{(k)} \cdot \boldsymbol{v}^{(k)} + h^{(k)}\varphi^{(k)})\, \mathrm{d}s \\ & + \sum_{P_a \in \partial M^{(k)}} (\boldsymbol{f}_a^{(k)+} \cdot \boldsymbol{v}_a^{(k)+} - \boldsymbol{f}_a^{(k)-} \cdot \boldsymbol{v}_a^{(k)-}),\end{aligned} \tag{4.15}$$

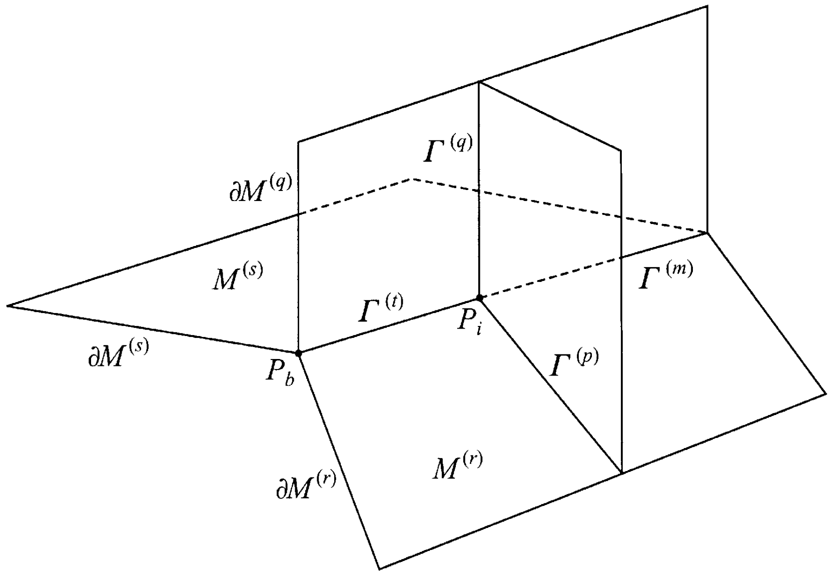

Figure 4.

where

$$p_\nu^{(k)} = t_\nu^{(k)} + (f^{(k)})' \tag{4.16}$$

is the effective internal stress resultant along $\partial M^{(k)}$.

By virtue of (4.5) and (4.15) the internal virtual work (3.6) for the entire reference network can now be written in two forms

$$\begin{aligned} G_i &= -\iint_{M\backslash\Gamma} (\mathrm{Div}_s\, \boldsymbol{T}) \cdot \boldsymbol{v}\, \mathrm{d}a \\ &\quad + \int_{\partial M} (\boldsymbol{t}_\nu \cdot \boldsymbol{v} + \boldsymbol{h}_\nu \cdot \boldsymbol{w})\, \mathrm{d}s + \int_\Gamma ([\![\boldsymbol{t}_\nu \cdot \boldsymbol{v}]\!] + [\![\boldsymbol{h}_\nu \cdot \boldsymbol{w}]\!])\, \mathrm{d}s \\ &= -\iint_{M\backslash\Gamma} (\mathrm{Div}_s\, \boldsymbol{T}) \cdot \boldsymbol{v}\, \mathrm{d}a \\ &\quad + \int_{\partial M} (\boldsymbol{p}_\nu \cdot \boldsymbol{v} + h\varphi)\, \mathrm{d}s + \int_\Gamma ([\![\boldsymbol{p}_\nu \cdot \boldsymbol{v}]\!] + [\![h\varphi]\!])\, \mathrm{d}s \\ &\quad + \sum_{P_i \in \Gamma} [\boldsymbol{f} \cdot \boldsymbol{v}]_i + \sum_{P_b \in \partial M} [\boldsymbol{f} \cdot \boldsymbol{v}]_b. \end{aligned} \tag{4.17}$$

Here the jumps at each regular point $\boldsymbol{x} \in \Gamma^{(a)} \equiv \partial M^{(1)} \cap \partial M^{(2)} \cap \cdots \cap \partial M^{(n)}$ of the common curve for $n \geqslant 2$ adjacent surface elements are defined by

$$\begin{aligned} [\![\boldsymbol{t}_\nu \cdot \boldsymbol{v}]\!] &= \pm \boldsymbol{t}_\nu^{(1)} \cdot \boldsymbol{v}^{(1)} \pm \boldsymbol{t}_\nu^{(2)} \cdot \boldsymbol{v}^{(2)} \pm \cdots \pm \boldsymbol{t}_\nu^{(n)} \cdot \boldsymbol{v}^{(n)}, \\ [\![\boldsymbol{h}_\nu \cdot \boldsymbol{w}]\!] &= \pm \boldsymbol{h}_\nu^{(1)} \cdot \boldsymbol{w}^{(1)} \pm \boldsymbol{h}_\nu^{(2)} \cdot \boldsymbol{w}^{(2)} \pm \cdots \pm \boldsymbol{h}_\nu^{(n)} \cdot \boldsymbol{w}^{(n)}, \end{aligned} \tag{4.18}$$

$$
\begin{aligned}
[\![\boldsymbol{p}_\nu \cdot \boldsymbol{v}]\!] &= \pm \boldsymbol{p}_\nu^{(1)} \cdot \boldsymbol{v}^{(1)} \pm \boldsymbol{p}_\nu^{(2)} \cdot \boldsymbol{v}^{(2)} \pm \cdots \pm \boldsymbol{p}_\nu^{(n)} \cdot \boldsymbol{v}^{(n)}, \\
[\![h\varphi]\!] &= \pm h^{(1)}\varphi^{(1)} \pm h^{(2)}\varphi^{(2)} \pm \cdots \pm h^{(n)}\varphi^{(n)}.
\end{aligned} \tag{4.19}
$$

The signs in the definitions (4.18) and (4.19) of the jumps must be chosen consistently with a fixed orientation of the curve $\Gamma^{(a)}$. If the orientation of $\Gamma^{(a)}$ coincides with the orientation of the boundary curve $\partial M^{(k)}$, then the sign '$-$' must be chosen for the corresponding term and the sign '$+$' otherwise. If we denote by $\boldsymbol{\tau}_\Gamma$ the unit tangent vector specifying the orientation of the curve $\Gamma^{(a)}$, then $\boldsymbol{\nu}^{(k)} = \pm\boldsymbol{\tau}_\Gamma \times \boldsymbol{m}^{(k)}$, and the sign must be chosen in such a way that the boundary $\partial M^{(k)}$ be consistently oriented with $M^{(k)}$.

The jumps at all singular points of M have been divided in (4.17) into the jumps $[\boldsymbol{f} \cdot \boldsymbol{v}]_i$ at the internal points $P_i \in \Gamma, i = 1, \dots, I$, and the jumps $[\boldsymbol{f} \cdot \boldsymbol{v}]_b$ at the boundary points $P_b \in \partial M$ (Figure 4). At each internal point P_i being the common point of $m \geqslant 2$ adjacent branches $\Gamma^{(m)}$ of Γ, and at each boundary point P_b being the common point of $t \geqslant 2$ adjacent parts $\partial M^{(t)}$ of ∂M and q adjacent branches $\Gamma^{(q)}$ of Γ approaching P_b from inside M, the jumps are defined by

$$
\begin{aligned}
[\boldsymbol{f} \cdot \boldsymbol{v}]_i &= \pm \boldsymbol{f}_i^{(1)\pm} \cdot \boldsymbol{v}_i^{(1)\pm} \pm \boldsymbol{f}_i^{(2)\pm} \cdot \boldsymbol{v}_i^{(2)\pm} \pm \cdots \pm \boldsymbol{f}_i^{(m)\pm} \cdot \boldsymbol{v}_i^{(m)\pm}, \\
[\boldsymbol{f} \cdot \boldsymbol{v}]_b &= \pm \boldsymbol{f}_b^{(1)\pm} \cdot \boldsymbol{v}_b^{(1)\pm} \pm \boldsymbol{f}_b^{(2)\pm} \cdot \boldsymbol{v}_b^{(2)\pm} \pm \cdots \pm \boldsymbol{f}_b^{(t)\pm} \cdot \boldsymbol{v}_b^{(t)\pm} \\
&\quad \pm \boldsymbol{f}_i^{(1)\pm} \cdot \boldsymbol{v}_b^{(1)\pm} \pm \boldsymbol{f}_i^{(2)\pm} \cdot \boldsymbol{v}_b^{(2)\pm} \pm \cdots \pm \boldsymbol{f}_i^{(q)\pm} \cdot \boldsymbol{v}_b^{(q)\pm}.
\end{aligned} \tag{4.20}
$$

Similar transformations can be applied to the external virtual work (3.15) with (4.8), $(3.16)_2$ and (4.11), which gives

$$
\begin{aligned}
G_e &= \iint_{M\backslash\Gamma} \boldsymbol{l} \cdot \boldsymbol{v}\, \mathrm{d}a - \int_\Gamma [\![\boldsymbol{k} \cdot \boldsymbol{v}]\!]\, \mathrm{d}s \\
&\quad + \int_{\partial M_f} \{(\boldsymbol{t}^* - \boldsymbol{k}) \cdot \boldsymbol{v} + \boldsymbol{h}^* \cdot \boldsymbol{w}\}\, \mathrm{d}s - \int_{\partial M_d} \boldsymbol{k} \cdot \boldsymbol{v}\, \mathrm{d}s \\
&= \iint_{M\backslash\Gamma} \boldsymbol{l} \cdot \boldsymbol{v}\, \mathrm{d}a - \int_\Gamma [\![\boldsymbol{k} \cdot \boldsymbol{v}]\!]\, \mathrm{d}s + \int_{\partial M_f} \{(\boldsymbol{p}^* - \boldsymbol{k}) \cdot \boldsymbol{v} + h^*\varphi\}\, \mathrm{d}s \\
&\quad - \int_{\partial M_d} \boldsymbol{k} \cdot \boldsymbol{v}\, \mathrm{d}s + \sum_{P_b \in \partial M_f} [\boldsymbol{f}^* \cdot \boldsymbol{v}]_b,
\end{aligned} \tag{4.21}
$$

where $\partial M_d = \partial M \backslash \partial M_f$ is the complementary part of ∂M, and

$$
\begin{aligned}
\boldsymbol{k} &= \{(\boldsymbol{G}^{-1}\boldsymbol{h}) \cdot \boldsymbol{\nu}\}\boldsymbol{n}, \qquad \boldsymbol{f}^* = -\boldsymbol{L}^T\boldsymbol{h}^*, \\
h^* &= \boldsymbol{q} \cdot \boldsymbol{h}^*, \qquad \boldsymbol{p}^* = \boldsymbol{t}^* + (\boldsymbol{f}^*)'.
\end{aligned} \tag{4.22}
$$

The jumps $[\![\boldsymbol{k} \cdot \boldsymbol{v}]\!]$ along the common curve $\Gamma^{(n)}$ for $n \geqslant 2$ adjacent surface elements are defined analogously to $(4.18)_1$ and $(4.19)_1$. However, the jumps $[\boldsymbol{f}^* \cdot \boldsymbol{v}]_b$

in (4.21) take into account only those limiting values which are obtained approaching P_b along branches of the boundary ∂M_f. If the boundary point P_b is a common point of $t \geqslant 2$ adjacent parts $\partial M^{(t)}$ of ∂M and q adjacent branches $\Gamma^{(q)}$ of Γ, then

$$[\boldsymbol{f}^* \cdot \boldsymbol{v}]_b = \pm \boldsymbol{f}_b^{*(1)\pm} \cdot \boldsymbol{v}_b^{(1)\pm} \pm \boldsymbol{f}_b^{*(2)\pm} \cdot \boldsymbol{v}_b^{(2)\pm} \pm \cdots \pm \boldsymbol{f}_b^{*(t)\pm} \cdot \boldsymbol{v}_b^{(t)\pm}. \qquad (4.23)$$

Let us now introduce (4.17) and (4.21) into the principle of virtual work (3.1) leading to

$$\begin{aligned}
&-\iint_{M\setminus\Gamma} (\mathrm{Div}_s\, \boldsymbol{T} + \boldsymbol{l}) \cdot \boldsymbol{v}\, \mathrm{d}a \\
&+ \int_{\partial M_f} \{(\boldsymbol{p}_\nu - \boldsymbol{p}^* + \boldsymbol{k}) \cdot \boldsymbol{v} + (h - h^*)\varphi\}\, \mathrm{d}s + \sum_{P_b \in \partial M_f} [(\boldsymbol{f} - \boldsymbol{f}^*) \cdot \boldsymbol{v}]_b \\
&+ \int_{\partial M_d} \{(\boldsymbol{p}_\nu + \boldsymbol{k}) \cdot \boldsymbol{v} + h\varphi\}\, \mathrm{d}s + \sum_{P_b \in \partial M_d} [\boldsymbol{f} \cdot \boldsymbol{v}]_b \\
&+ \int_{\Gamma} \{[\![(\boldsymbol{p}_\nu + \boldsymbol{k}) \cdot \boldsymbol{v}]\!] + [\![h\varphi]\!]\}\, \mathrm{d}s + \sum_{P_i \in \Gamma} [\boldsymbol{f} \cdot \boldsymbol{v}]_i - G_\Gamma = 0. \qquad (4.24)
\end{aligned}$$

For arbitrary but kinematically admissible virtual deformations, the fields $\boldsymbol{v}$ and φ vanish identically along ∂M_d, thus causing the third line of (4.24) to vanish as well. Then, from (4.24) we obtain:

The local equilibrium equations

$$\mathrm{Div}_s\, \boldsymbol{T} + \boldsymbol{l} = \boldsymbol{0} \quad \text{at each regular } \boldsymbol{x} \in M. \qquad (4.25)$$

The static boundary and corner conditions

$$\begin{aligned}
&\boldsymbol{p}_\nu - \boldsymbol{p}^* + \boldsymbol{k} = \boldsymbol{0}, \qquad h - h^* = 0 \ \text{ along regular parts of } \partial M_f, \\
&\boldsymbol{f}_b - \boldsymbol{f}_b^* = \boldsymbol{0} \quad \text{at each singular point } P_b \in \partial M_f.
\end{aligned} \qquad (4.26)$$

Correspondingly, the work-conjugate geometric boundary and corner conditions take the form

$$\boldsymbol{u} - \boldsymbol{u}^* = \boldsymbol{0}, \qquad \phi - \phi^* = 0 \ \text{ along regular parts of } \partial M_d. \qquad (4.27)$$

As it has been expected, the local Lagrangian equilibrium equations as well as the boundary and corner conditions for thin irregular shell-like structures are the same as in the classical theory of thin smooth shells, [13, 14].

5. Jump Conditions

The principal new element of the theory of thin irregular shells appears in the concept of jump conditions along a singular curve Γ. Such conditions are needed, besides the constitutive relations, to complete the set of field equations and boundary conditions of this shell theory.

The mathematical structure of thin shell theory imposes definite restrictions on the possible form of the jump conditions. These restrictions have implicitly been introduced by assuming expressions (3.6) and (3.13) for the internal virtual work density. As a result, if the local equilibrium equations (4.25) and boundary conditions (4.26–7) are satisfied, the transformed PVW (4.24) requires that

$$\int_{\Gamma}(\llbracket(\boldsymbol{p}_{\nu}+\boldsymbol{k})\cdot\boldsymbol{v}\rrbracket+\llbracket h\varphi\rrbracket)\,\mathrm{d}s+\sum_{P_i\in\Gamma}[\boldsymbol{f}\cdot\boldsymbol{v}]_i-G_{\Gamma}=0, \tag{5.1}$$

where the jumps at regular and singular points of Γ are defined by (4.19) and $(4.20)_1$, respectively.

Equation (5.1) represents the weak form of the jump conditions compatible with the basic postulates of the non-linear theory of thin irregular shells. It should be satisfied for any type of geometric and kinematic irregularities, and for any mechanical properties prescribed along regular parts of Γ and at each singular point $P_i \in \Gamma$. From (5.1) it follows that the most general form of the virtual work expression G_Γ allowed within the non-linear theory of thin irregular shells is

$$G_{\Gamma}=\int_{\Gamma}\sigma_{\Gamma}(\boldsymbol{x})\,\mathrm{d}s+\sum_{P_i\in\Gamma}\sigma_i, \tag{5.2}$$

where σ_Γ is the virtual work density along regular parts of Γ, and σ_i is the virtual work expression at any singular point $P_i \in \Gamma$. The functions σ_Γ and σ_i must be specified in each particular case of the irregularity.

Since (5.1) has to be satisfied for any part of Γ, we obtain the corresponding local forms of the jump conditions:

$$\begin{aligned}&\llbracket(\boldsymbol{p}_{\nu}+\boldsymbol{k})\cdot\boldsymbol{v}\rrbracket+\llbracket h\varphi\rrbracket-\sigma_{\Gamma}=0\quad\text{at regular points of }\ \Gamma,\\&[\boldsymbol{f}\cdot\boldsymbol{v}]_i-\sigma_i=0\quad\text{at each singular point }\ P_i\in\Gamma.\end{aligned} \tag{5.3}$$

The jump conditions (5.3) constitute the additional set of basic relations which should be satisfied at various geometric, kinematic and mechanical irregularities of thin shell structures discussed in this paper.

Effects of geometric, kinematic and mechanical irregularities contained in the jump conditions (5.3) are mutually coupled. For special types of irregularities the jump conditions can be considerably simplified and presented in a more explicit uncoupled form. As examples, we discuss below some simpler forms of the jump conditions appropriate for particular types of irregularities.

5.1. Geometric Irregularities

Let us assume that there are no kinematic and mechanical irregularities along Γ and at $P_i \in \Gamma$, but only geometric ones. This means that Γ is a geometric singular curve, and the field $\boldsymbol{u}$: $M \to E$ is of class C^n, $n \geqslant 2$ not only within each $M^{(k)}$ of M, but also over the entire reference network M, including the Γ . The simplest example of such an irregularity is the fold shown in Figure 2, and a more complex example is illustrated in Figure 3.

Let $\boldsymbol{u}^{(k)}$ denote a finite limit of $\boldsymbol{u}$ on the boundary $\partial M^{(k)}$ of any $M^{(k)}$, and let $\phi^{(k)}$ be the corresponding rotational parameter along $\partial M^{(k)}$. Within the non-linear theory of thin irregular shells discussed here the junction along $\Gamma = \partial M^{(k)} \cap \partial M^{(l)} \cap \cdots \cap \partial M^{(n)}$ and at $P \in \Gamma$ is called rigid if the values of fields $\boldsymbol{u}^{(k)}$ and $\phi^{(k)}$ belonging to all adjacent surface elements are the same and equal to the values associated with Γ and P_i themselves

$$\begin{aligned} &\boldsymbol{u}^{(k)} = \boldsymbol{u}^{(l)} = \cdots = \boldsymbol{u}^{(n)} = \boldsymbol{u}_\Gamma = \boldsymbol{u}|_\Gamma, \\ &\phi^{(k)} = \phi^{(l)} = \cdots = \phi^{(n)} = \phi_\Gamma, \qquad \boldsymbol{u}_i = \boldsymbol{u}|_{P_i}. \end{aligned} \tag{5.4}$$

Then the corresponding virtual fields satisfy the relations

$$\begin{aligned} &\boldsymbol{v}^{(k)} = \boldsymbol{v}^{(l)} = \cdots = \boldsymbol{v}^{(n)} = \boldsymbol{v}_\Gamma = \boldsymbol{v}|_\Gamma, \\ &\varphi^{(k)} = \varphi^{(l)} = \cdots = \varphi^{(n)} = \varphi_\Gamma, \qquad \boldsymbol{v}_{\boldsymbol{i}} = \boldsymbol{v}|_{P_i}. \end{aligned} \tag{5.5}$$

The functions σ_Γ and σ_i in (5.3) representing the virtual works along Γ and at $P_i \in \Gamma$ have to be linear with respect to the common virtual deformation parameters (5.5)

$$\sigma_\Gamma = \boldsymbol{f}_\Gamma \cdot \boldsymbol{v}_\Gamma + h_\Gamma \varphi_\Gamma, \qquad \sigma_i = \boldsymbol{f}_i \cdot \boldsymbol{v}_i. \tag{5.6}$$

Here $\boldsymbol{f}_\Gamma$ and h_Γ can be interpreted as external loads distributed along regular parts of Γ, and $\boldsymbol{f}_i$ as external concentrated forces applied at each singular point $P_i \in \Gamma$.

If (5.5) and (5.6) are introduced into (5.3) then

$$\begin{aligned} &([\![\boldsymbol{p}_\nu + \boldsymbol{k}]\!] - \boldsymbol{f}_\Gamma) \cdot \boldsymbol{v}_\Gamma + ([\![h]\!] - h_\Gamma)\varphi_\Gamma = 0, \\ &([\boldsymbol{f}]_i - \boldsymbol{f}_i) \cdot \boldsymbol{v}_i = 0. \end{aligned} \tag{5.7}$$

But (5.7) should hold for arbitrary virtual parameters. This is assured if the following static jump conditions are satisfied

$$\begin{aligned} &[\![\boldsymbol{p}_\nu + \boldsymbol{k}]\!] - \boldsymbol{f}_\Gamma = \boldsymbol{0}, \qquad [\![h]\!] - h_\Gamma = 0 \quad \text{at regular points of } \Gamma, \\ &[\boldsymbol{f}]_i - \boldsymbol{f} = \boldsymbol{0} \quad \text{at each singular point } P_i \in \Gamma. \end{aligned} \tag{5.8}$$

If additionally the external loads $\boldsymbol{f}_\Gamma$, h_Γ and $\boldsymbol{f}_i$ are not applied, (5.8) can be reduced further into the form

$$\begin{aligned} &[\![\boldsymbol{p}_\nu + \boldsymbol{k}]\!] = \boldsymbol{0}, \qquad [\![h]\!] = 0 \text{ at regular points of } \Gamma, \\ &[\boldsymbol{f}]_i = \boldsymbol{0} \text{ at each singular point } P_i \in \Gamma. \end{aligned} \tag{5.9}$$

The reduced static jump conditions (5.9) describe just the jumps in static variables at Γ caused by either the non-smoothness of the reference network M or the lack of smoothness of the fields entering definitions of $\boldsymbol{p}_\nu, \boldsymbol{k}, h$ and $\boldsymbol{f}$. The latter ones are, for example, the abrupt changes in shell thickness or in material properties at the singular curve Γ.

The kinematic rigidity conditions (5.4) and the general forms of the static jump conditions (5.8) have been formulated here without any restrictions on magnitudes of the displacements, rotations, strains and/or bendings of the shell reference network M. They are valid for multi-fold or multi-intersection junctions along Γ admitting isolated singular points $P_i \in \Gamma$, and are independent of any constitutive relations of the material the shell is composed of.

In the literature the rigidity (continuity) and static jump conditions were formulated only within the simplest possible setting – the classical linear theory of thin isotropic elastic shells – with usually only two smooth shell elements rigidly connected along Γ, and without admitting singular points on Γ. Within such a setting the conditions were given first by Byrne [3], and various equivalent formulations can be found in books by Chernykh [30], Baker et al. [31], Novozhilov et al. [32] and Bernadou [33], where references to original papers are given.

5.2. ELASTIC JUNCTIONS

In many irregular shell structures some or all of the rigidity (continuity) conditions (5.4) may not be satisfied, or we would not like them to be satisfied for various formal (mathematical) or mechanical reasons. In such a case the junction is called non-rigid or deformable. The formulation of jump conditions for such a junction needs an additional explicit description of its deformability.

Let us note that the static jump conditions (5.8) express just the equilibrium conditions along Γ and have to be always satisfied independently of the kinematic conditions assumed along Γ. For a deformable junction we shall always assume that there are no external loads $\boldsymbol{f}_\Gamma$, h_Γ and $\boldsymbol{f}_i$ acting along Γ, so that the static jump conditions take the reduced form (5.9).

The lack of smoothness of the deformation along a part of Γ should necessarily be associated with a high concentration of energy at the corresponding part of Γ. This energy may be taken into account by introducing suitable forms of the virtual work densities σ_Γ and σ_i appearing in (5.2) and (5.3).

As a simplest illustration, let us discuss a deformable junction of only two shell elements modelled by the surface elements $M^{(1)}$ and $M^{(2)}$ having the curve $\Gamma^{(1,2)}$ as their common boundary. Let the orientation of $\Gamma^{(1,2)}$ coincide with the orientation of $\partial M^{(1)}$, Figure 5. According to (4.19)

$$\begin{aligned} &[\![\boldsymbol{p}_\nu + \boldsymbol{k}) \cdot \boldsymbol{v}]\!] = [\![\boldsymbol{p}_\nu + \boldsymbol{k}]\!] \cdot \langle \boldsymbol{v} \rangle + \langle \boldsymbol{p}_\nu + \boldsymbol{k} \rangle \cdot [\![\boldsymbol{v}]\!], \\ &[\![h\varphi]\!] = [\![h]\!]\langle \varphi \rangle + \langle h \rangle [\![\varphi]\!], \end{aligned} \tag{5.10}$$

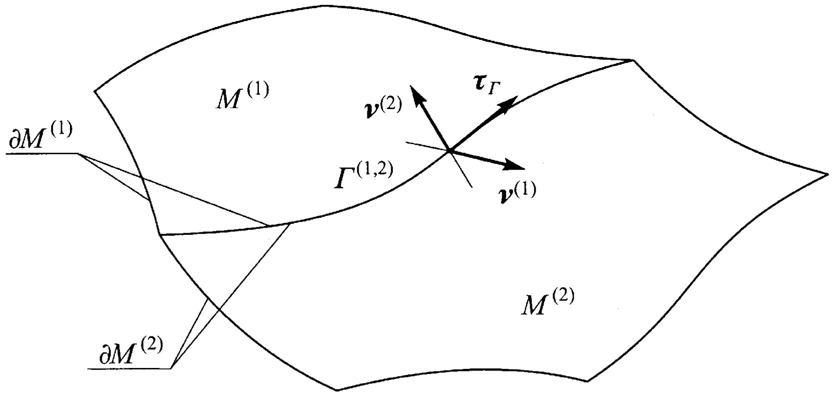

Figure 5.

where for any surface field $\boldsymbol{\varphi}$ the jump and the mean value are defined by

$$[\![\boldsymbol{\varphi}]\!] \equiv \boldsymbol{\varphi}^{(2)} - \boldsymbol{\varphi}^{(1)}, \qquad \langle\boldsymbol{\varphi}\rangle \equiv \tfrac{1}{2}(\boldsymbol{\varphi}^{(2)} + \boldsymbol{\varphi}^{(1)}). \tag{5.11}$$

With the help of (5.10) and (5.11) we can rewrite the general jump conditions $(5.3)_1$ in the form

$$[\![\boldsymbol{p}_\nu + \boldsymbol{k}]\!] \cdot \langle\boldsymbol{v}\rangle + \langle\boldsymbol{p}_\nu + \boldsymbol{k}\rangle \cdot [\![\boldsymbol{v}]\!] + [\![h]\!]\langle\varphi\rangle + \langle h\rangle[\![\varphi]\!] - \sigma_\Gamma = 0. \tag{5.12}$$

Taking into account $(5.9)_1$ the relation (5.12) can be reduced to

$$(\boldsymbol{p}_\nu + \boldsymbol{k}) \cdot [\![\boldsymbol{v}]\!] + h[\![\varphi]\!] - \sigma_\Gamma = 0, \tag{5.13}$$

where now

$$\boldsymbol{p}_\nu^{(1)} + \boldsymbol{k}^{(1)} = \boldsymbol{p}_\nu^{(2)} + \boldsymbol{k}^{(2)} \equiv \boldsymbol{p}_\nu + \boldsymbol{k}, \qquad h^{(1)} = h^{(2)} \equiv h. \tag{5.14}$$

From (5.13) it is apparent that the virtual work density σ_Γ has to be linear with respect to the jumps of virtual displacement $[\![\boldsymbol{v}]\!]$ and virtual rotation parameter $[\![\varphi]\!]$

$$\sigma_\Gamma = \boldsymbol{b} \cdot [\![\boldsymbol{v}]\!] + b[\![\varphi]\!]. \tag{5.15}$$

The junction is called locally elastic if there exists a function $\Sigma_\Gamma = \Sigma_\Gamma([\![\boldsymbol{u}]\!], [\![\phi]\!])$ such that $\sigma_\Gamma = \dot{\Sigma}_\Gamma$. Then the variation of Σ_Γ yields

$$\sigma_\Gamma = (\partial_{[\![\boldsymbol{u}]\!]}\Sigma_\Gamma) \cdot [\![\boldsymbol{v}]\!] + (\partial_{[\![\phi]\!]}\Sigma_\Gamma)[\![\varphi]\!], \tag{5.16}$$

which substituted into (5.13) gives

$$\{(\boldsymbol{p}_\nu + \boldsymbol{k}) - \partial_{[\![\boldsymbol{u}]\!]}\Sigma_\Gamma\} \cdot [\![\boldsymbol{v}]\!] + \{h - \partial_{[\![\phi]\!]}\Sigma_\Gamma\}[\![\varphi]\!] = 0. \tag{5.17}$$

In the case of a deformable junction the values of $[\![\boldsymbol{v}]\!]$ and $[\![\varphi]\!]$ are arbitrary, in general. Therefore, (5.17) is satisfied if the following static jump conditions are satisfied along $\Gamma^{(1,2)}$

$$(\boldsymbol{p}_\nu + \boldsymbol{k}) - \partial_{[\![\boldsymbol{u}]\!]}\Sigma_\Gamma = 0, \qquad h - \partial_{[\![\phi]\!]}\Sigma_\Gamma = 0. \tag{5.18}$$

In particular, Σ_Γ can be assumed to be a symmetric quadratic function with respect to the jumps $[\![\boldsymbol{u}]\!]$ and $[\![\boldsymbol{\phi}]\!]$

$$\Sigma_\Gamma = \tfrac{1}{2}\{\boldsymbol{C}\cdot([\![\boldsymbol{u}]\!]\otimes[\![\boldsymbol{u}]\!]) + c[\![\phi]\!]^2\}, \tag{5.19}$$

where $\boldsymbol{C}$ is a tensor of material constants and c is a material scalar. In this case we can call the junction linearly elastic. With (5.19) the static jump conditions (5.17) take the explicit form

$$(\boldsymbol{p}_\nu + \boldsymbol{k}) - \boldsymbol{C}[\![\boldsymbol{u}]\!] = \boldsymbol{0}, \qquad h - c[\![\phi]\!] = 0. \tag{5.20}$$

If, in addition, the deformation is continuous over $\Gamma^{(1,2)}$, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$, but the deformation gradient is not, then the jump condition reduce further to

$$\boldsymbol{p}_\nu + \boldsymbol{k} = \boldsymbol{0}, \qquad h - c[\![\phi]\!] = 0. \tag{5.21}$$

Upon linearization relative to displacements the conditions (5.21) reduce to those discussed in [8] and [33] for the junction of two plates and shells, respectively.

The coefficients $\boldsymbol{b}$ and b in (5.15) may depend not only on $[\![\boldsymbol{u}]\!]$ and $[\![\phi]\!]$ but also on their first $[\![\boldsymbol{u}']\!]$ and $[\![\phi']\!]$, second $[\![\boldsymbol{u}'']\!]$ and $[\![\boldsymbol{\phi}'']\!]$ and higher derivatives. This allows to discuss junctions non-locally elastic of the first-, second- and higher-order, respectively.

5.3. Unelastic junctions

Let us again discuss the same deformable junction of only two shell elements, Figure 5, for which the virtual work density σ_Γ takes the general form (5.15). The coefficients $\boldsymbol{b}$ and b in (5.15) may be allowed to depend not only on $[\![\boldsymbol{u}]\!]$, $[\![\phi]\!]$ and their spatial derivatives, as in Section 5.2. They may additionally be allowed to depend on time-like derivatives of those variables, which allows one to account for unelastic properties of junctions such as viscoelastic effects. In the simplest case we can take

$$\boldsymbol{b} = \boldsymbol{b}([\![\boldsymbol{u}]\!], [\![\phi]\!], [\![\dot{\boldsymbol{u}}]\!], [\![\dot{\phi}]\!]), \qquad b = b([\![\boldsymbol{u}]\!], [\![\phi]\!], [\![\dot{\boldsymbol{u}}]\!], [\![\dot{\phi}]\!]). \tag{5.22}$$

The junction may be called locally linearly viscoelastic if $\boldsymbol{b}$ and b are linear forms with respect to the jumps and their first time-like derivatives:

$$\boldsymbol{b} = \boldsymbol{C}[\![\boldsymbol{u}]\!] + \boldsymbol{D}[\![\dot{\boldsymbol{u}}]\!], \qquad b = c[\![\phi]\!] + d[\![\dot{\phi}]\!], \tag{5.23}$$

where $\boldsymbol{C}$ and $\boldsymbol{D}$ are tensors of material constants, while c and d are material scalars. With (5.23) the static jump conditions (5.20) are generalized into

$$\boldsymbol{p}_\nu + \boldsymbol{k} - (\boldsymbol{C}[\![\boldsymbol{u}]\!] + \boldsymbol{D}[\![\dot{\boldsymbol{u}}]\!]) = \boldsymbol{0}, \qquad h - (c[\![\phi]\!] + d[\![\dot{\phi}]\!]) = 0. \tag{5.24}$$

In analogy to Section 5.2, we can still reduce (5.24) in the case of continuous deformation, $[\![\boldsymbol{u}]\!] = \boldsymbol{0}$. We may also discuss various non-locally viscoelastic junctions, by allowing $\boldsymbol{b}$ and b to depend on higher-order spatial derivatives and their time-like derivatives.

In Sections 5.2 and 5.3 we have discussed a deformable junction of only two shell elements. When more than two shell elements meet at the common junction, different expressions of σ_Γ can be prescribed for any adjacent pair of the surface elements $M^{(k)}$ and $M^{(l)}$ connected with the common singular curve Γ. In such a case, a variety of forms of the jump conditions, suitable to model complex behaviour of such a junction, may be discussed.

Finally, we also observe that the additional deformation mapping $\boldsymbol{\chi}_\Gamma\colon \Gamma \to \mathcal{E}$ introduced in Chapter 3 has not appeared explicitly in the three classes of junctions considered above. It is then apparent that a more general class of junctions may be considered in which $\boldsymbol{\chi}_\Gamma\colon \Gamma \to \mathcal{E}$ plays a role of an additional kinematic variable.

6. Conclusions

In this paper we have given the mathematical structure of the general jump conditions along singular geometric and physical curves, which are compatible with the two basic postulates of the theory of thin irregular shells. The derived local forms of the jump conditions are quite complex due to the appearance of the second-order displacement derivatives in the definition of the effective internal stress resultant along the boundary of smooth surface elements. This is an unavoidable feature of the class of thin shell theories, in which the displacements are the only independent field variables. We have also discussed in more detail some simple forms and their simplified versions of the jump conditions appropriate for rigid, elastic and viscoelastic shell junctions. The general approach developed in this report also suggests how the jump conditions for other special kinds of geometric, kinematic and mechanical irregularities can be derived within the class of thin shell theories.

References

1. A.K. Norr, Bibliography of monographs and surveys on shells, *Appl. Mech. Rev.* **43** (1990) 223–234.
2. W. Pietraszkiewicz, Addendum to: Bibliography of monographs and surveys on shells, *Appl. Mech. Rev.* **45** (1992) 249–250.
3. R. Byrne, *Theory of Small Deformations of a Thin Elastic Shell.* Seminar Reports in Mathematics, University of California Press (1944).

4. A.K. Kulkarni, K.W. Neale and F. Ellyin, Consistent theories for intersecting shells. *Nucl. Engng Design* **35** (1975) 377–385.
5. F. Kang and Shi Zhong-Ci, *Mathematical Theory of Elastic Structures*. Science Press Beijing, Springer (1994).
6. J.E. Lagnese, G. Leugering and E.J.P.G. Schmidt, *Modeling, Analysis and Control of Dynamic Elastic Multi-Link Structures*. Birkhäuser, Boston (1994).
7. E. Stein, A. Berg and W. Wagner, Different levels of non-linear shell theory in finite element stability analysis. In *Buckling of Shells*, Springer-Verlag, New York (1982).
8. M. Bernadou, S. Fayolle and F. Lene, Numerical analysis of junctions between plates. *Comp. Meth. Appl. Mech. Eng.* **74** (1989) 307–326.
9. J. Chróścielewski, J. Makowski and H. Stumpf, Finite element analysis of smooth, folded and multi-shell structures. *Comput. Methods Appl. Mech. Eng.* **141** (1997) 1–46.
10. J. Chróścielewski, *Rodzina Elementów Skończonych Klasy* C^0 *w Nieliniowej Sześcioparametrowej Teorii Powłok*. Zesz. Nauk. Politechniki Gdańskiej Nr. 540, Bud. Ląd. LIII, 1–291 (1996).
11. J. Makowski and H. Stumpf, *Mechanics of Irregular Shell Structures*. Mitt. Institut für Mechanik, Nr. 95, Ruhr-Universität Bochum (1994).
12. F. Morgan, *Geometric Measure Theory. A Beginner's Guide*. Academic Press, New York (1988).
13. W. Pietraszkiewicz, Lagrangian description and incremental formulation in the non-linear theory of thin shells. *Int. J. Non-Linear Mech.* **19** (1984) 115–140.
14. W. Pietraszkiewicz, Geometrically nonlinear theories of thin elastic shells, *Advances in Mechanics* **12** (1989) 51–130.
15. J.G. Simmonds, The strain energy density of rubber-like shells, *Int. J. Solids Struct.* **21** (1985) 67–77.
16. H. Stumpf and J. Makowski, On large strain deformation of shells, *Acta Mech.* **65** (1986) 153–168.
17. B. Schieck, W. Pietraszkiewicz and H. Stumpf, Theory and numerical analysis of shells made of hyperelastic incompressible material, *Int. J. Solids Struct.* **29** (1992) 689–709.
18. A. Sawczuk, On plastic analysis of shells. In: W.T. Koiter and G.K. Mikhailov (eds), *Theory of Shells*, North-Holland P.Co, Amsterdam (1980) pp. 27–63.
19. H. Stumpf and B. Schieck, Theory and analysis of shells undergoing finite elastic–plastic strains and rotations, *Acta Mech.* **106** (1994) 1–21.
20. M.E. Gurtin and A.I. Murdoch, A continuum theory of elastic material surfaces, *Arch. Rat. Mech. Anal.* **57** (1975) 291–323.
21. S.S. Antman, *Nonlinear Problems of Elasticity*, Springer-Verlag, New York (1995).
22. M.H. Protter and C.B. Morrey, *A First Course in Real Analysis* (2nd edn). Springer-Verlag, New York (1991).
23. W. Pietraszkiewicz, Explicit Lagrangian incremental and buckling equations for the non-linear theory of thin shells, *Int. J. Non-Linear Mech.* **28** (1993) 209–220.
24. V.V. Novozhilov and V.A. Shamina, On kinematic boundary conditions in non-linear elasticity problems, *Mechanics of Solids* (translation Izvestiia AN SSSR, MTT) **5** (1975) 63–74.
25. W. Pietraszkiewicz and M. Szwabowicz, Entirely Lagrangian non-linear theory of thin shells, *Arch. Mech.* **33** (1981) 273–288.
26. H. Stumpf, General concept of the analysis of thin elastic shells, *Z. Angew. Math. Mech.* **66** (1986) 337–350.
27. J. Makowski and W. Pietraszkiewicz, Work-conjugate boundary conditions in the nonlinear theory of thin shells, *ASME J. Appl. Mech.* **56** (1989) 395–401.
28. W. Pietraszkiewicz, Work-conjugate boundary conditions associated with the total rotation angle of the shell boundary, *ASME J. Appl. Mech.* **60** (1993) 785–786.

29. A.I. Murdoch, A coordinate-free approach to surface kinematics, *Glasgow Math. J.* **32** (1990) 299–307.
30. K.F. Chernykh, *Linear Theory of Shells*, Vol. 2 (in Russian). Leningrad Univ. Press (1964) (English translation by NASA-TT-F 11562, 1968).
31. E.H. Baker, L. Kovalevsky and F.L. Rish, *Structural Analysis of Shells*. Krieger Publ. Comp., Huntington, New York (1981).
32. V.V. Novozilov, K.F. Chernykh and E.I. Mikhailovskii, *Linear Theory of Thin Shells*. Politekhnika, Leningrad (1991) (in Russian).
33. M. Bernadou, *Finite Element Methods for Thin Shell Problems.* Wiley-Masson, Paris (1996).