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ON THE ALUMÄE TYPE NON-LINEAR THEORY OF THIN IRREGULAR SHELLS

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Non-linear relations for thin shell structures with irregularities of geometry, material properties and deformation are discussed. All the relations are formulated relative to an intermediate non-holonomic surface base. From the modified 2D principle of virtual work the local known equilibrium equations and boundary conditions, as well as the corresponding new jump conditions at singular curves and points are derived. The consistently simplified strain energy density for rubber-like shells undergoing large elastic strains is constructed. Within each regular shell element, the resulting BVP is expressed in terms of displacements, rotations and Lagrange multipliers as the primary field variables.

1. Introduction

An overwhelming majority of non-linear problems of thin regular shells are analysed using either the Lagrangian or the mixed formulation of shell relations, or a variety of their simplified and/or modified versions, [1]. In the Lagrangian boundary value problem (BVP) the only independent field variable is the displacement vector of the shell reference surface, while in the mixed approach the surface deformation and stress measures are the primary independent variables.

Alumäe [2,3] proposed an original set of equilibrium equations and compatibility conditions for thin shells written relative to the intermediate non-holonomic surface base vectors, which are obtained by a rigid-body rotation of the base vectors associated with the undeformed reference surface. The corresponding rotation tensor can be defined from the polar decomposition of the surface deformation gradient, [1]. The same approach was applied by Simmonds and Danielson [4,5] to formulate the BVP for thin elastic shells in terms of the finite rotation and stress function vectors, and to construct an appropriate variational principle in these variables. Several alternative forms of thin shell relations and corresponding BVPs expressed in terms of rotations and other fields as independent variables were discussed by Pietraszkiewicz [6-8], Shkutin [9], Zubov [10], Atluri [11], Badur and Pietraszkiewicz [12], Kayuk [13], Valid [14], and Libai and Simmonds [15,16]. In particular, within the geometrically non-linear theory of thin regular isotropic elastic shells many such relations were summarised in Chapter 5 of [1].

Real shell structures are usually irregular ones and may contain folds, stiffeners, branches, parts made of different materials or thicknesses, technological junctions, plastic hinges developing at some level of bending, etc. This causes some fields to be discontinuous or not differentiable along specified curves on the reference surface. The general non-linear theory of thin shell-like structures with irregularities of geometry, material properties and deformation was developed in the Lagrangian description by Makowski *et al.* [17,18], where the resulting BVP was formulated through the displacement vector as the primary field variable.

The aim of this paper is to extend the Alumäe type geometrically non-linear theory of thin regular shells summarised in Chapter 5 of [1] into the domain of the irregular thin shell-like structures undergoing large strains. As in [17], we represent the irregular 3D shell-like structure by a 2D reference network being a union of piecewise smooth surfaces joined together along parts of their boundaries. The junction curves constitute the singular spatial curves at which some fields may not be continuous or differentiable. The equilibrium conditions are given in Chapter 3 by the postulated principle of virtual work (PVW) of [17], where additional constraints with Lagrange multipliers are introduced in order to regard also the rotation tensor as the primary field variable. Transforming in Chapter 4 the so modified PVW we obtain the known, [1-5], local forms of equilibrium equations and boundary conditions. Additionally, we derive new local relations: the jump conditions at singular curves and points. In Chapter 5 we discuss the 2D strain energy density for a homogeneous isotropic rubber-like shell undergoing large elastic strains. By modifying the corresponding 2D density of the first-approximation theory derived in [19], we construct a consistently reduced simplest approximation to the strain energy density appropriate for the Alumäe type large-strain theory of shells. The resulting BVP is expressed in terms of displacements, rotations and some Lagrange multipliers as the primary field variables.

2. Geometry and deformation of a regular surface element

In this report we shall apply primarily the system of notation used in [1] and remind here only basic relations.

Let $\mathcal{M}^{(k)}$ be a connected, oriented and regular surface element of class C^n, $n \geq 2$, in the three-dimensional Euclidean point space $\mathcal{E}$ whose translation (three-dimensional vector) space is E. The position vector of a point $M \in \mathcal{M}^{(k)}$ is given by

$$\boldsymbol{r} = \overline{OM} = \boldsymbol{r}(\theta^\alpha), \tag{1}$$

where $O \in \mathcal{E}$ is a reference origin and θ^α, $\alpha = 1, 2$, are surface co-ordinates. At $M \in \mathcal{M}^{(k)}$ we have the natural base vectors $\boldsymbol{a}_\alpha = \partial \boldsymbol{r}/\partial\theta^\alpha \equiv \boldsymbol{r}_{,\alpha}$, the dual base vectors $\boldsymbol{a}^\beta$ such that $\boldsymbol{a}^\beta \cdot \boldsymbol{a}_\alpha = \delta^\beta_\alpha$ where δ^β_α is the Kronecker symbol, the components $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ of the surface metric tensor $\boldsymbol{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the unit normal vector $\boldsymbol{n} = (1/\sqrt{a})\boldsymbol{a}_1 \times \boldsymbol{a}_2$ orienting $\mathcal{M}^{(k)}$, the components $b_{\alpha\beta} = -\boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_\beta$ of the surface curvature tensor $\boldsymbol{b}$, and the components $\varepsilon_{\alpha\beta} = (\boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta)\cdot\boldsymbol{n}$ of the surface permutation tensor such that $\varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

The boundary $\partial\mathcal{M}^{(k)}$ of $\mathcal{M}^{(k)}$ consists of a finite number of closed piecewise smooth curves that do not meet in cusps, each described parametrically by $\boldsymbol{r}(s) = \boldsymbol{r}[\theta^\alpha(s)]$, where s is the arc length along any regular part of $\partial\mathcal{M}^{(k)}$. At each regular point $M \in \mathcal{M}^{(k)}$ we have the unit tangent vector $\boldsymbol{\tau} = d\boldsymbol{r}/ds \equiv \boldsymbol{r}' = \tau^\alpha \boldsymbol{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \boldsymbol{r}_{,\nu} = \boldsymbol{\tau} \times \boldsymbol{n} = \nu^\alpha \boldsymbol{a}_\alpha$, where $(\cdot)_{,\nu}$ is the external surface derivative normal to $\partial\mathcal{M}^{(k)}$.

The deformed regular surface element $\overline{\mathcal{M}}^{(k)}$ with the boundary $\partial\overline{\mathcal{M}}^{(k)}$ is described relative to the same origin $O \in \mathcal{E}$ by the relations

$$\bar{\boldsymbol{r}}(\theta^\alpha) = \chi[\boldsymbol{r}(\theta^\alpha)] = \boldsymbol{r}(\theta^\alpha) + \boldsymbol{u}(\theta^\alpha); \tag{2}$$

$$\bar{\boldsymbol{r}}(\theta^\alpha) = \chi[\boldsymbol{r}(s)] = \boldsymbol{r}(s) + \boldsymbol{u}(s),$$

where θ^α and s are convected surface co-ordinates, $\chi : \mathcal{M}^{(k)} \to \overline{\mathcal{M}}^{(k)}$ is the deformation function, and $\boldsymbol{u} \in E$ is the displacement vector.

In the convected surface co-ordinates all geometric relations at any regular $\overline{M} \in \partial\overline{\mathcal{M}}^{(k)}$ are now analogous to those given at $M \in \partial\mathcal{M}^{(k)}$, and are expressed by quantities marked by a dash: $\overline{\boldsymbol{a}}_\alpha, \overline{\boldsymbol{a}}^\beta, \overline{a}_{\alpha\beta}, \overline{a}^{\alpha\beta}, \overline{\boldsymbol{a}}, \overline{a}, \overline{b}_{\alpha\beta}, \overline{\varepsilon}_{\alpha\beta}, \overline{\boldsymbol{\nu}}, \overline{\boldsymbol{\tau}}$, etc. The dashed quantities can be expressed through analogous quantities defined on $\mathcal{M}^{(k)}$ and the displacement field $\boldsymbol{u}$ with the help of formulae given in [1].

Components of the Green type surface deformation measures are defined by

$$\gamma_{\alpha\beta}(\boldsymbol{u}) = \frac{1}{2}(\overline{a}_{\alpha\beta} - a_{\alpha\beta}), \quad \kappa_{\alpha\beta}(\boldsymbol{u}) = -(\overline{b}_{\alpha\beta} - b_{\alpha\beta}), \tag{3}$$

where $\gamma_{\alpha\beta}(\boldsymbol{u})$ are quadratic polynomials of $\boldsymbol{u}, \boldsymbol{u}_{,\alpha}$ and $\kappa_{\alpha\beta}(\boldsymbol{u})$ are non-rational functions of $\boldsymbol{u}, \boldsymbol{u}_{,\alpha}, \boldsymbol{u}_{,\alpha\beta}$.

In the neighbourhood of the regular surface elements $\mathcal{M}^{(k)}$ and $\overline{\mathcal{M}}^{(k)}$ the space $\mathcal{E}$ can be parameterised by the normal system of convected co-ordinates (θ^α, ζ), where ζ is the distance from $\mathcal{M}^{(k)}$ and $\overline{\mathcal{M}}^{(k)}$ along $\boldsymbol{n}$ and $\overline{\boldsymbol{n}}$, respectively. Extending the domain of χ to the neighbourhood of $\mathcal{M}^{(k)}$, the spatial deformation gradient $\boldsymbol{F} : E \to E$ taken at the surface element $\mathcal{M}^{(k)}$ has the form

$$\boldsymbol{F} = \nabla\chi(\boldsymbol{r} + \zeta\boldsymbol{n})\Big|_{\zeta=0} = \overline{\boldsymbol{a}}_\alpha \otimes \boldsymbol{a}^\alpha + \overline{\boldsymbol{n}} \otimes \boldsymbol{n};$$

$$\det \boldsymbol{F} = \sqrt{\frac{\overline{a}}{a}} > 0; \tag{4}$$

$$\frac{\overline{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2(\gamma^\alpha_\alpha\gamma^\beta_\beta - \gamma^\beta_\alpha\gamma^\alpha_\beta),$$

where $\otimes$ is the tensor product.

The left polar decomposition of $\boldsymbol{F}$ gives

$$\boldsymbol{F} = \boldsymbol{V}\boldsymbol{R}, \quad \boldsymbol{r}_\alpha = \boldsymbol{R}\boldsymbol{a}_\alpha = \boldsymbol{V}^{-1}\overline{\boldsymbol{a}}_\alpha. \tag{5}$$

Here $\boldsymbol{R} \in SO(3)$ is the rotation tensor, $\boldsymbol{V}$ is the left spatial stretch tensor at $\mathcal{M}^{(k)}$, and $\boldsymbol{r}_\alpha$ are the rotated surface non-holonomic base vectors. These fields satisfy the relations

$$\boldsymbol{R} = \boldsymbol{r}_\alpha \otimes \boldsymbol{a}^\alpha + \overline{\boldsymbol{n}} \otimes \boldsymbol{n}, \quad \boldsymbol{R}^{\mathrm{T}} = \boldsymbol{R}^{-1};$$

$$\det \boldsymbol{R} = +1, \quad \boldsymbol{V} = \overline{\boldsymbol{a}}_\alpha \otimes \boldsymbol{r}^\alpha + \overline{\boldsymbol{n}} \otimes \overline{\boldsymbol{n}}; \tag{6}$$

$$\boldsymbol{V}^{\mathrm{T}} = \boldsymbol{V}, \quad \det \boldsymbol{V} = \sqrt{\frac{\overline{a}}{a}} > 1.$$

The modified surface deformation measures associated with $\boldsymbol{r}_\alpha$ are introduced through the following formulae, [1]:

$$\boldsymbol{\eta} = \boldsymbol{V} - \boldsymbol{1} = (\boldsymbol{a}_\beta + \boldsymbol{u}_{,\beta} - \boldsymbol{r}_\beta) \otimes \boldsymbol{r}^\beta =$$

$$= \boldsymbol{\eta}_\beta \otimes \boldsymbol{r}^\beta, \quad \boldsymbol{\eta}_\beta = \eta_{\alpha\beta}\boldsymbol{r}^\alpha;$$

$$\boldsymbol{\mu} = (\overline{\boldsymbol{n}}_{,\beta} - \boldsymbol{R}\boldsymbol{n}_{,\beta}) \otimes \boldsymbol{r}^\beta = \boldsymbol{R}_{,\beta}\boldsymbol{n} \otimes \boldsymbol{r}^\beta = \tag{7}$$

$$= \boldsymbol{\mu}_\beta \otimes \boldsymbol{r}^\beta, \quad \boldsymbol{\mu}_\beta = \mu_{\alpha\beta}\boldsymbol{r}^\alpha;$$

$$\eta_{\alpha\beta} = \eta_{\beta\alpha}, \quad \mu_{\alpha\beta} \neq \mu_{\beta\alpha}.$$

Here $\boldsymbol{1} = \boldsymbol{a}_\alpha \otimes \boldsymbol{a}^\alpha + \boldsymbol{n} \otimes \boldsymbol{n} = \boldsymbol{r}_\alpha \otimes \boldsymbol{r}^\alpha + \overline{\boldsymbol{n}} \otimes \overline{\boldsymbol{n}}$ is the spatial identity tensor. The surface measures satisfy useful kinematic relations given in [1].

Along the boundary $\partial\mathcal{M}^{(k)}$ we have

$$\overline{\boldsymbol{a}}_\tau \equiv \overline{\boldsymbol{r}}' = a_\tau \overline{\boldsymbol{\tau}}, \quad \overline{\boldsymbol{a}}_\nu = \overline{\boldsymbol{a}}_\tau \times \overline{\boldsymbol{n}} = a_\tau \overline{\boldsymbol{\nu}};$$

$$\overline{\boldsymbol{n}} = \sqrt{\frac{a}{\overline{a}}}\,\overline{\boldsymbol{r}}_{,\nu} \times \overline{\boldsymbol{r}}'; \tag{8}$$

$$a_\tau = |\overline{\boldsymbol{r}}'| = \sqrt{1 + 2\gamma_{\tau\tau}}, \quad \gamma_{\tau\tau} = \gamma_{\alpha\beta}\tau^\alpha\tau^\beta.$$

The transformation of (ν, τ, n) during deformation into $(\overline{a}_\nu, \overline{a}_\tau, \overline{n})$ is performed in two steps: the rotation of (ν, τ, n) into $(\overline{\nu}, \overline{\tau}, \overline{n})$ by the total rotation tensor R_τ with the subsequent extension of $\overline{\nu}, \overline{\tau}$ into $\overline{a}_\nu, \overline{a}_\tau$ by the factor a_τ :

$$\overline{a}_\nu = a_\tau R_\tau \nu, \quad \overline{a}_\tau = a_\tau R_\tau \tau, \quad \overline{n} = R_\tau n;$$
$$R_\tau = \frac{1}{a_\tau}(\overline{a}_\nu \otimes \nu + \overline{a}_\tau \otimes \tau + \overline{n} \otimes n). \tag{9}$$

Kinematic relations involving the tensors R and R_τ are given in [1].

3. Principle of virtual work for thin irregular shells

A consistent formulation of the mechanical boundary value problem for thin irregular shell-like structures in the Lagrangian description was developed by Makowski *et al.* [17,18], where the displacements were taken as the only independent field variables. The mechanical modelling of such structures was based on two postulates:

- *The deformation of the entire shell-like structure is determined by deformation of a distinguished surface-like continuum, called the shell reference network.*
- *The equilibrium conditions of the entire structure are determined by a suitable form of the principle of virtual work involving only the fields associated with the stretching and bending of the reference network.*

The undeformed reference network $\mathcal{M} \subset \mathcal{E}$ was defined in [17,18] as the union of all the closed elements $\mathcal{M}^{(k)} \cup \partial\mathcal{M}^{(k)}$, and the singular curve $\Gamma \in \mathcal{M}$ as the union of all the junctions of different elements $\mathcal{M}^{(k)}$, $k = 1, 2, ..., K$:

$$\mathcal{M} = \bigcup_{k=1}^{K} (\mathcal{M}^{(k)} \cup \partial\mathcal{M}^{(k)}), \quad \Gamma = \bigcup_{a=1}^{A} \Gamma^{(a)};$$
$$\Gamma^{(a)} = \partial\mathcal{M}^{(k_1)} \cap \partial\mathcal{M}^{(k_1)} \cap ... \cap \partial\mathcal{M}^{(k_m)}; \tag{10}$$
$$\text{if } k_1 \neq k_2 \neq ... \neq k_m.$$

As a result, the boundary $\partial\mathcal{M}$ of the entire network $\mathcal{M}$ defined by

$$\partial\mathcal{M} = \left(\bigcup_{k=1}^{K} \partial\mathcal{M}^{(k)} \right) \backslash \Gamma \tag{11}$$

consists of a finite number of spatial curves. Several examples of such networks are given in [18].

Each $\mathcal{M}^{(k)}$ represents a reference surface of a regular shell part. Each $\Gamma^{(k)}$ can be a surface curve across which some fields fail to be smooth. Examples of geometric irregularities along $\Gamma^{(a)}$ are surface folds, branches and intersections of two or more regular surfaces. Shell parts can be made of different materials, or there may be stepwise thickness changes at $\Gamma^{(a)}$. However, $\Gamma^{(a)}$ can also represent a reference axis of a rod-like element, a technological junction, a plastic hinge developing during deformation process, etc.

Deformation of $\mathcal{M}$ can be described by two deformation functions: $\chi : \mathcal{M}\backslash\Gamma \to \mathcal{E}$ and $\chi_\Gamma : \Gamma \to \mathcal{E}$, for the singular curve may be admitted to follow its own deformation, in general. In many cases the deformation χ may be defined on the entire $\mathcal{M}$, and then χ_Γ is a restriction of χ at Γ : $\chi_\Gamma = \chi\Big|_\Gamma$. However, we do not assume such a restricted shell deformation at the moment.

The principle of virtual work compatible with the two postulates given above can be taken in the form, [17]

$$G \equiv G_{\text{int}} - G_{\text{ext}} - G_\Gamma = 0, \tag{12}$$

where $G_{\text{int}} = G_{\text{int}}(u; \delta u)$ represents the internal virtual work, $G_{\text{ext}} = G_{\text{ext}}(u; \delta u)$ is the external virtual work, and $G_\Gamma = G_\Gamma(u_\Gamma; \delta u_\Gamma)$ is the additional virtual work of the generalised forces acting along Γ. Here we have explicitly indicated that all the virtual works are functionals of the displacements as the only independent field variables. The individual parts of (12) are defined by

$$G_{\text{int}} = \sum_{k=1}^{K} \iint_{\mathcal{M}^{(k)}} (N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta})\, dA;$$
$$G_{\text{ext}} = \sum_{k=1}^{K} \iint_{\mathcal{M}^{(k)}} (p \cdot \delta u + h \cdot \delta\overline{n})\, dA +$$
$$+ \int_{\partial\mathcal{M}_f} (N^* \cdot \delta u + H^* \cdot \delta\overline{n})\, ds; \tag{13}$$
$$G_\Gamma = \int_\Gamma \sigma_\Gamma\, ds + \sum_{P_i \in \Gamma} \sigma_i.$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric stress resultant and stress couple tensors of the Piola-Kirchhoff type, δ is the symbol of variation, $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the surface deformation measures (3), p and h are the external surface force and moment resultant vectors, N^* and H^* are the external boundary force and moment resultant vectors, whereas σ_Γ and σ_i are the external virtual work densities along regular parts of Γ and at any singular point $P_i \in \Gamma$, respectively. The explicit forms of σ_Γ and σ_i depend on the type of irregularity assumed along Γ, [18]. Note that since $\overline{\boldsymbol{n}} \cdot \delta\overline{\boldsymbol{n}} = 0$, only the surface components of $\boldsymbol{h}$ and $\boldsymbol{H}^*$ can explicitly be taken into account in the non-linear theory of thin irregular shells discussed here. Transforming (12) and (13) with the help of Stokes' theorem, the corresponding local equilibrium equations, boundary conditions and jump conditions at singular curves can be derived, [17].

If the rotation tensor $\boldsymbol{R}$ is supposed to be an independent field variable of the boundary value problem, some constraint conditions have to be introduced into the relations (13) and the virtual densities should be expressed in terms of the modified surface deformation and stress measures.

Let us remind that the components $\gamma_{\alpha\beta}, \kappa_{\alpha\beta}$ and $\eta_{\alpha\beta}, \mu_{\alpha\beta}$ of the surface deformation measures are related by (see [1], formula (5.11))

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + \tfrac{1}{2}\eta^\lambda_\alpha \eta_{\lambda\beta};$$

$$\kappa_{\alpha\beta} = \frac{1}{2}\left[(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\mu_{\lambda\beta} + (\delta^\lambda_\beta + \eta^\lambda_\beta)\mu_{\lambda\alpha}\right] - \tag{14}$$

$$-\tfrac{1}{2}(b^\lambda_\alpha \eta_{\lambda_\alpha} + b^\lambda_\beta \eta_{\lambda_\beta}).$$

As a result, the internal surface virtual work density appearing in $(13)_1$ can be presented in an alternative form

$$N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} =$$

$$= S^{\alpha\beta}\delta\eta_{\alpha\beta} + H^{\alpha\beta}\delta\mu_{\alpha\beta};$$

$$S^{\alpha\beta} = N^{\alpha\beta} + \frac{1}{2}(\eta^\alpha_\lambda N^{\lambda\beta} + \eta^\beta_\lambda N^{\alpha\lambda}) - \tag{15}$$

$$-\tfrac{1}{2}\left[(b^\alpha_\lambda - \mu^\alpha_{\cdot\lambda})M^{\lambda\beta} + (b^\beta_\lambda - \mu^\beta_{\cdot\lambda})M^{\alpha\lambda}\right];$$

$$H^{\alpha\beta} = (\delta^\alpha_\lambda + \eta^\alpha_\lambda)M^{\lambda\beta},$$

where now $S^{\alpha\beta} = S^{\beta\alpha}$ but $H^{\alpha\beta} \neq H^{\beta\alpha}$, in general.

In the non-linear theory of thin shells the rotation tensor $\boldsymbol{R}$ is a non-rational function of $\boldsymbol{u}, \boldsymbol{u}_{,\alpha}$ (explicit formulae are given in [6,7]). This dependence of $\boldsymbol{R}$ upon $\boldsymbol{u}$ can also be expressed implicitly through three constraint conditions [1, 12]

$$\varepsilon^{\alpha\beta}\boldsymbol{r}_\alpha \cdot \eta_{\lambda\beta}\boldsymbol{r}^\lambda = 0, \quad \overline{\boldsymbol{n}} \cdot \eta_{\lambda\beta}\boldsymbol{r}^\lambda = 0. \tag{16}$$

These constraints express the known property of the relative surface strain tensor $\boldsymbol{\eta}$, which in thin shell theory is symmetric and does not have out-of-surface components. The property was also confirmed by Libai and Simmonds [15] who used the constitutive Kirchhoff hypothesis to define the classical theory of thin shells as a special case of the general shell theory. For a virtual deformation the relations (16) put the following constraints on the virtual changes $\delta\eta_{\alpha\beta}$ of $\eta_{\alpha\beta}$:

$$\varepsilon^{\alpha\beta}\boldsymbol{r}_\alpha \cdot \delta\eta_{\lambda\beta}\boldsymbol{r}^\lambda = 0, \quad \overline{\boldsymbol{n}} \cdot \delta\eta_{\lambda\beta}\boldsymbol{r}^\lambda = 0. \tag{17}$$

Inside of each $\mathcal{M}^{(k)}$ the constraints (17) can be introduced into the surface integral of $(13)_1$ with the help of the respective Lagrange multipliers S and Q^β. It was shown in [1] that in order to express also the boundary terms at each $\partial\mathcal{M}^{(k)}$ explicitly through independent rotations it is necessary to introduce into $(13)_1$ a line integral over $\partial\mathcal{M}^{(k)}$ with the constraints $(17)_2$ multiplied by $B\tau^\beta$. Additionally, in $(13)_2$ the external virtual work done by the moments $\boldsymbol{h}$ and $\boldsymbol{H}^*$ should be expressed directly in terms of now independent virtual rotations. As a result, $(13)_{1,2}$ can be modified to the form

$$G_{\text{int}} = \sum_{k=1}^{K}\left\{\iint\limits_{\mathcal{M}^{(k)}} (\boldsymbol{N}^\beta \cdot \delta\eta_{\lambda\beta}\boldsymbol{r}^\lambda + H^{\alpha\beta}\boldsymbol{r}_\alpha \cdot \delta\mu_{\lambda\beta}\boldsymbol{r}^\lambda)\, dA + \int\limits_{\partial\mathcal{M}^{(k)}} B\tau^\beta\overline{\boldsymbol{n}} \cdot \delta\eta_{\lambda\beta}\boldsymbol{r}^\lambda\, ds\right\};$$

$$G_{\text{ext}} = \sum_{k=1}^{K}\iint\limits_{\mathcal{M}^{(k)}} (\boldsymbol{p} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \boldsymbol{\omega})\, dA + \int\limits_{\partial\mathcal{M}_f} (\boldsymbol{N}^* \cdot \delta\boldsymbol{u} + \boldsymbol{M}^* \cdot \boldsymbol{\omega}_\tau)\, ds, \tag{18}$$

where now $G_{\text{int}} = G_{\text{int}}(\boldsymbol{u}, \boldsymbol{R}; \delta\boldsymbol{u}, \boldsymbol{\omega})$ and $G_{\text{ext}} = G_{\text{ext}}(\boldsymbol{u}, \boldsymbol{R}; \delta\boldsymbol{u}, \boldsymbol{\omega}_\tau)$, while other fields are defined by

$$\boldsymbol{N}^\beta = (S^{\alpha\beta} + \varepsilon^{\alpha\beta}S)\boldsymbol{r}_\alpha + Q^\beta\overline{\boldsymbol{n}};$$

$$\boldsymbol{M}^\beta = \overline{\boldsymbol{n}} \times H^{\alpha\beta}\boldsymbol{r}_\alpha; \tag{19}$$

$$\boldsymbol{M}^* = \overline{\boldsymbol{n}} \times \boldsymbol{H}^*, \quad \boldsymbol{m} = \overline{\boldsymbol{n}} \times \boldsymbol{h};$$

$$\boldsymbol{\omega} = \frac{1}{2}(\boldsymbol{1} \times \boldsymbol{1}) \cdot (\delta\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}) = \frac{1}{2}(\boldsymbol{r}^\alpha \times \delta\boldsymbol{r}_\alpha + \overline{\boldsymbol{n}} \times \delta\overline{\boldsymbol{n}});$$

$$\boldsymbol{\omega}_\tau = \frac{1}{2}(\boldsymbol{1} \times \boldsymbol{1}) \cdot (\delta\boldsymbol{R}_\tau\boldsymbol{R}_\tau^{\mathrm{T}}) = \frac{1}{2}(\overline{\boldsymbol{\nu}} \times \delta\overline{\boldsymbol{\nu}} + \overline{\boldsymbol{\tau}} \times \delta\overline{\boldsymbol{\tau}} + \overline{\boldsymbol{n}} \times \delta\overline{\boldsymbol{n}}). \tag{20}$$

Here $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\tau$ are the virtual rotation vectors in the interior of each $\mathcal{M}^{(k)}$ and along each $\partial\mathcal{M}^{(k)}$, respectively. Please note that all the couple vectors $\boldsymbol{M}^\beta, \boldsymbol{M}^*$ and $\boldsymbol{m}$ in (19) do not have normal components, that is $\boldsymbol{M}^\beta \cdot \overline{\boldsymbol{n}} = \boldsymbol{M}^* \cdot \overline{\boldsymbol{n}} = \boldsymbol{m} \cdot \overline{\boldsymbol{n}} \equiv 0$. This is the fundamental property of the theory of thin shells resulting from the two basic postulates given above.

4. Local field equations

Let us transform the virtual work principle (12) with (18) keeping in mind that both $\boldsymbol{R}$ and $\boldsymbol{u}$ are now the independent field variables subject to variation.

According to [1], the virtual deformation measures $\delta\eta_{\alpha\beta}$ and $\delta\mu_{\alpha\beta}$ are expressible through $\delta\boldsymbol{u}$ and $\boldsymbol{\omega}$ by the relations

$$\delta\eta_{\lambda\beta}\boldsymbol{r}^\lambda = \delta\boldsymbol{u}_{,\beta} + \overline{\boldsymbol{a}}_\beta \times \boldsymbol{\omega}, \quad \delta\mu_{\lambda\beta}\boldsymbol{r}^\lambda = \boldsymbol{\omega}_{,\beta} \times \overline{\boldsymbol{n}}. \quad (21)$$

Introducing (21) into $(18)_1$, the virtual work principle (12) takes the form

$$\sum_{k=1}^{K}\Bigg\{\iint\limits_{\mathcal{M}^{(k)}} \Big[\boldsymbol{N}^\beta \cdot (\delta\boldsymbol{u}_{,\beta} + \overline{\boldsymbol{a}}_\beta \times \boldsymbol{\omega}) + \boldsymbol{M}^\beta \cdot \boldsymbol{\omega}_{,\beta}\Big]\, dA + \int\limits_{\partial\mathcal{M}^{(k)}} B\overline{\boldsymbol{n}} \cdot (\delta\boldsymbol{u}' + \overline{\boldsymbol{a}}_\tau \times \boldsymbol{\omega})\, ds - \iint\limits_{\mathcal{M}^{(k)}} (\boldsymbol{p} \cdot \delta\boldsymbol{u} + \boldsymbol{m} \cdot \boldsymbol{\omega})\, dA\Bigg\} - \int\limits_{\partial\mathcal{M}_f} (\boldsymbol{N}^* \cdot \delta\boldsymbol{u} + \boldsymbol{M}^* \cdot \boldsymbol{\omega}_\tau)\, ds - \int\limits_{\Gamma} \sigma_\Gamma\, ds - \sum_{P_i \in \Gamma} \sigma_i = 0. \quad (22)$$

The fields $\boldsymbol{N}^\beta$ and $\boldsymbol{M}^\beta$ are assumed to be of class C^1 in the interior of each regular surface element $\mathcal{M}^{(k)}$ and to have extensions of the same class to the boundary with finite limits at any $M \in \partial\mathcal{M}^{(k)}$. Then the Stokes theorem allows one to transform the first two surface integrals of (22) for each $\mathcal{M}^{(k)}$ into

$$-\iint\limits_{\mathcal{M}^{(k)}} \{\boldsymbol{N}^\beta\big|_\beta \cdot \delta\boldsymbol{u} + (\boldsymbol{M}^\beta\big|_\beta + \overline{\boldsymbol{a}}_\beta \times \boldsymbol{N}^\beta) \cdot \boldsymbol{\omega}\}\, dA + \int\limits_{\partial\mathcal{M}^{(k)}} \{\boldsymbol{T}_\nu \cdot \delta\boldsymbol{u} + \boldsymbol{K}_\nu \cdot \boldsymbol{\omega} + (B\overline{\boldsymbol{n}} \cdot \delta\boldsymbol{u})'\}\, ds, \quad (23)$$

where

$$\boldsymbol{T}_\nu = \boldsymbol{N}^\beta \nu_\beta - (B\overline{\boldsymbol{n}})', \quad \boldsymbol{K}_\nu = \boldsymbol{M}^\beta \nu_\beta - B\overline{\boldsymbol{a}}_\nu. \quad (24)$$

Along each $\partial\mathcal{M}^{(k)}$ there may be singular points P_c, $c = 1, 2, ..., C$, described by $s = s_c$, at which the field $B\overline{\boldsymbol{n}} \cdot \delta\boldsymbol{u}$ is not differentiable. Such singular points are, for example, corners of the closed curves composing $\partial\mathcal{M}^{(k)}$ or points of singularities of $B, \overline{\boldsymbol{n}}$ and $\delta\boldsymbol{u}$. At such singular points we assume the existence of finite limits of $B\overline{\boldsymbol{n}} \cdot \delta\boldsymbol{u}$ defined by

$$B_c^\pm \overline{\boldsymbol{n}}_c^\pm \cdot \delta\boldsymbol{u}_c^\pm = \lim_{h\to 0}\{B(s_c \pm h)\overline{\boldsymbol{n}}(s_c \pm h) \cdot \delta\boldsymbol{u}(s_c \pm h)\}. \quad (25)$$

Then, the last term in the boundary line integral of (23) can be transformed further to give

$$\int\limits_{\partial\mathcal{M}^{(k)}} (B\overline{\boldsymbol{n}} \cdot \delta\boldsymbol{u})'\, ds = -\sum_{P_c \in \partial\mathcal{M}^{(k)}} (B_c^+ \overline{\boldsymbol{n}}_c^+ \cdot \delta\boldsymbol{u}_c^+ - B_c^- \overline{\boldsymbol{n}}_c^- \cdot \delta\boldsymbol{u}_c^-). \quad (26)$$

The second term of the boundary line integral in (23) contains the virtual rotation $\boldsymbol{\omega}$, which should still be expressed through the virtual rotation $\boldsymbol{\omega}_\tau$ of the boundary. Let us remind that along each $\partial\mathcal{M}^{(k)}$ the total rotation tensor $\boldsymbol{R}_\tau$ is defined as the superposition of two finite rotations, [1]:

$$\boldsymbol{R}_\tau = \boldsymbol{Q}_V \boldsymbol{R}, \quad \boldsymbol{Q}_V = \overline{\boldsymbol{\nu}} \otimes \boldsymbol{r}_\nu + \overline{\boldsymbol{\tau}} \otimes \boldsymbol{r}_\tau + \overline{\boldsymbol{n}} \otimes \overline{\boldsymbol{n}}, \quad (27)$$

where

$$\begin{gathered}\boldsymbol{r}_\nu = \boldsymbol{r}_\alpha \nu^\alpha = \boldsymbol{R}\boldsymbol{\nu} = \frac{1}{a_\tau}\{(1 + \eta_{\tau\tau})\overline{\boldsymbol{\nu}} + \eta_{\nu\tau}\overline{\boldsymbol{\tau}}\}; \\ \boldsymbol{r}_\tau = \boldsymbol{r}_\alpha \tau^\alpha = \boldsymbol{R}\boldsymbol{\tau} = \frac{1}{a_\tau}\{-\eta_{\nu\tau}\overline{\boldsymbol{\nu}} + (1 + \eta_{\tau\tau})\overline{\boldsymbol{\tau}}\}; \\ \overline{\boldsymbol{\nu}} = \boldsymbol{Q}_V \boldsymbol{r}_\nu, \quad \overline{\boldsymbol{\tau}} = \boldsymbol{Q}_V \boldsymbol{r}_\tau; \\ \eta_{\nu\tau} = \eta_{\alpha\beta}\nu^\alpha\tau^\beta, \quad \eta_{\tau\tau} = \eta_{\alpha\beta}\tau^\alpha\tau^\beta.\end{gathered} \quad (28)$$

Therefore, taking variations of $\boldsymbol{R}_\tau$ defined either by $(9)_2$ or by $(27)_1$ we obtain

$$\begin{aligned}\delta\boldsymbol{R}_\tau \boldsymbol{R}_\tau^{\mathrm{T}} &= \boldsymbol{\omega}_\tau \times \boldsymbol{1} = \\ &= (\delta\boldsymbol{Q}_V \boldsymbol{Q}_V^{\mathrm{T}})\boldsymbol{Q}_V \boldsymbol{R}\boldsymbol{R}^{\mathrm{T}}\boldsymbol{Q}_V^{\mathrm{T}} + \boldsymbol{Q}_V(\delta\boldsymbol{R}\boldsymbol{R}^{\mathrm{T}})\boldsymbol{Q}_V^{\mathrm{T}} = \\ &= (\boldsymbol{q} + \boldsymbol{Q}_V\boldsymbol{\omega}) \times \boldsymbol{1},\end{aligned} \quad (29)$$

where

$$q = \frac{1}{2}(1 \times 1) \cdot (\delta Q_V Q_V^{\mathrm{T}}). \tag{30}$$

From $(30)_{1,2}$ it follows that

$$\omega_\tau = q + Q_V \omega, \quad \omega = Q_V^{\mathrm{T}}(\omega_\tau - q). \tag{31}$$

Let us evaluate more explicitly the formula $(31)_2$ for ω at the boundary $\partial\overline{\mathcal{M}}^{(k)}$. Keeping in mind that $1 = \overline{\nu} \otimes \overline{\nu} + \overline{\tau} \otimes \overline{\tau} + \overline{n} \otimes \overline{n}$ along $\partial\overline{\mathcal{M}}^{(k)}$, and taking variation of $(27)_2$ we establish the relations

$$\begin{aligned} 1 \times 1 = &-\overline{\nu} \otimes (\overline{\tau} \otimes \overline{n} - \overline{n} \otimes \overline{\tau}) - \\ &-\overline{\tau} \otimes (\overline{n} \otimes \overline{\nu} - \overline{\nu} \otimes \overline{n}) - \\ &-\overline{n} \otimes (\overline{\nu} \otimes \overline{\tau} - \overline{\tau} \otimes \overline{\nu}); \\ \delta Q_V Q_V^{\mathrm{T}} = &\,\delta\overline{\nu} \otimes \overline{\nu} + \delta\overline{\tau} \otimes \overline{\tau} + \delta\overline{n} \otimes \overline{n} + \\ &+\overline{\nu} \otimes \overline{\tau}(\delta r_\nu \cdot r_\tau) + \overline{\nu} \otimes \overline{n}(\delta r_\nu \cdot \overline{n}) + \\ &+\overline{\tau} \otimes \overline{\nu}(\delta r_\tau \cdot r_\nu) + \overline{\tau} \otimes \overline{n}(\delta r_\tau \cdot \overline{n}) + \\ &+\overline{n} \otimes \overline{\nu}(\delta\overline{n} \cdot r_\nu) + \overline{n} \otimes \overline{\tau}(\delta\overline{n} \cdot r_\tau). \end{aligned} \tag{32}$$

Introducing (32) into (30) and taking into account that

$$\omega_\tau = \overline{\nu}(\omega_\tau \cdot \overline{\nu}) + \overline{\tau}(\omega_\tau \cdot \overline{\tau}) + \overline{n}(\omega_\tau \cdot \overline{n}), \tag{33}$$

after some transformations from $(31)_2$ we obtain

$$\omega = r_\nu(\omega_\tau \cdot r_\nu) + r_\tau(\omega_\tau \cdot r_\tau) - \overline{n}(\delta r_\tau \cdot r_\nu). \tag{34}$$

The relation (34) means that the virtual rotations ω and ω_τ differ only by their normal components. But K_ν in (24) does not have a normal component at all. Thus, using $(8)_1$, $(19)_1$, $(24)_2$ and (34) we are able to show that at the boundary

$$K_\nu \cdot \omega = K_\nu \cdot \omega_\tau. \tag{35}$$

The simple relation (35) just confirms that the theory of thin shells discussed here is insensitive to the virtual works done on the normal drilling components of ω and ω_τ, for the corresponding drilling components of the couples are indefinite in this shell model. The virtual works done by the drilling couples can be taken into account only in the general theory of shells, [10,16].

With the help of (23), (26) and (35) the internal virtual work for the entire reference network $\mathcal{M}$ can be put in the form

$$\begin{aligned} G_{\mathrm{int}} = &-\iint\limits_{\mathcal{M}\setminus\Gamma} \left\{ N^\beta\big|_\beta \cdot \delta u + \right. \\ &+\left. \left(M^\beta\big|_\beta + \overline{a}_\beta \times N^\beta \right) \cdot \omega \right\} dA + \\ &+\int\limits_{\partial\mathcal{M}} (T_\nu \cdot \delta u + K_\nu \cdot \omega_\tau)\, ds + \\ &+\int\limits_{\Gamma} ([[T_\nu \cdot \delta u]] + [[K_\nu \cdot \omega_\tau]])\, ds + \\ &+\sum_{P_i \in \Gamma} [B\overline{n} \cdot \delta u]_i + \sum_{P_b \in \partial\mathcal{M}} [B\overline{n} \cdot \delta u]_b. \end{aligned} \tag{36}$$

In (36) the jumps at each regular point $P \in \Gamma^{(a)}$ of the common curve $\Gamma^{(a)} = \partial\mathcal{M}^{(1)} \bigcap \partial\mathcal{M}^{(2)} \bigcap \ldots \bigcap \partial\mathcal{M}^{(n)}$ for $n \geq 2$ adjacent surface elements are defined by

$$\begin{aligned} [[T_\nu \cdot \delta u]] &= \pm T_\nu^{(1)\pm} \cdot \delta u^{(1)\pm} \pm \\ &\pm T_\nu^{(2)\pm} \cdot \delta u^{(2)\pm} \pm \ldots \pm T_\nu^{(n)\pm} \cdot \delta u^{(n)\pm}; \\ [[K_\nu \cdot \omega_\tau]] &= \pm K_\nu^{(1)\pm} \cdot \omega_\tau^{(1)\pm} \pm \\ &\pm K_\nu^{(2)\pm} \cdot \omega_\tau^{(2)\pm} \pm \ldots \pm K_\nu^{(n)\pm} \cdot \omega_\tau^{(n)\pm}. \end{aligned} \tag{37}$$

The numerical superscripts (n) introduced into the right hand sides of (37) indicate explicitly that those functions are defined only along the particular $\partial\mathcal{M}^{(n)}$.

The signs in the definitions (37) must be chosen consistently with a fixed orientation of the curve $\Gamma^{(a)}$. If the orientation of $\Gamma^{(a)}$ coincides with the orientation of $\partial\mathcal{M}^{(n)}$, that is when the unit tangent vector τ_Γ specifying the orientation of $\Gamma^{(a)}$ is related to $\nu^{(n)}$ of $\partial\mathcal{M}^{(n)}$ by $\nu^{(n)} = +\tau_\Gamma \times n^{(n)}$, the minus sign must be chosen in front of the corresponding term in (38), and the plus sign otherwise.

The jumps at all singular points of $\mathcal{M}$ have been divided in (36) into the jumps $[B\overline{n} \cdot \delta u]_i$ at the internal points $P_i \in \Gamma$ and the jumps $[B\overline{n} \cdot \delta u]_b$ at the boundary points $P_b \in \partial\mathcal{M}$. At each internal point P_i being the common point of $m \geq 2$ adjacent branches $\Gamma^{(m)}$, as well as at each boundary point P_b being the common point of $t \geq 2$ adjacent parts $\partial\mathcal{M}^{(t)}$ and q adjacent branches $\Gamma^{(q)}$ approaching P_b from inside of $\mathcal{M}$, the jumps are defined by

$$
\begin{aligned}
[B\overline{n}\cdot\delta u]_i &= \pm B_i^{(1)\pm}\overline{n}_i^{(1)\pm}\cdot u_i^{(1)\pm}\pm \\
&\pm B_i^{(2)\pm}\overline{n}_i^{(2)\pm}\cdot u_i^{(2)\pm}\pm\ldots \\
&\ldots\pm B_i^{(m)\pm}\overline{n}_i^{(m)\pm}\cdot u_i^{(m)\pm}; \\
[B\overline{n}\delta u]_b &= \pm B_b^{(1)\pm}\overline{n}_b^{(1)\pm}\cdot u_b^{(1)\pm}\pm \\
&\pm B_b^{(2)\pm}\overline{n}_b^{(2)\pm}\cdot u_b^{(2)\pm}\pm\ldots \\
&\ldots\pm B_b^{(t)\pm}\overline{n}_b^{(t)\pm}\cdot u_b^{(t)\pm} \\
&\pm B_i^{(1)\pm}\overline{n}_i^{(1)\pm}\cdot u_b^{(1)\pm}\pm \\
&\pm B_i^{(2)\pm}\overline{n}_i^{(2)\pm}\cdot u_b^{(2)\pm}\pm\ldots \\
&\ldots\pm B_i^{(q)\pm}\overline{n}_i^{(q)\pm}\cdot u_b^{(q)\pm}.
\end{aligned}
\qquad (38)
$$

Here the numerical superscripts indicate that these functions are defined only either on a particular internal branches $\Gamma^{(m)}$ and $\Gamma^{(q)}$, or on a particular $\partial\mathcal{M}^{(t)}$ composing a part of the boundary $\partial\mathcal{M}$.

Introducing (36) with (37) and (38) into (22) we obtain

$$
\begin{aligned}
&\iint\limits_{\mathcal{M}\backslash\Gamma}\Big\{(N^\beta\big|_\beta+p)\cdot\delta u+ \\
&+(M^\beta\big|_\beta+\overline{a}_\beta\times N^\beta+m)\cdot\omega)\Big\}\,ds+ \\
&+\int\limits_{\partial\mathcal{M}_f}\{(T_\nu-N^*)\cdot\delta u+ \\
&+(K_\nu-M^*)\cdot\omega_\tau\}\,ds+ \\
&+\sum_{P_b\in\partial\mathcal{M}_f}[B\overline{n}\cdot\delta u]_b+ \\
&+\int\limits_{\partial\mathcal{M}_f}(T_\nu\cdot\delta u+K_\nu\cdot\omega_\tau)\,ds+ \\
&+\sum_{P_b\in\partial\mathcal{M}_d}[B\overline{n}\cdot\delta u]_b+ \\
&+\int\limits_\Gamma\{[[T_\nu\cdot\delta u]]+[[K_\nu\cdot\omega_\tau]]\}\,ds+ \\
&+\sum_{P_i\in\Gamma}[B\overline{n}\cdot\delta u]_i-\int\limits_\Gamma\sigma_\Gamma\,ds-\sum_{P_i\in\Gamma}\sigma_i=0.
\end{aligned}
\qquad (39)
$$

For an arbitrary, but kinematically admissible, virtual deformation the fields δu and ω_τ vanish identically along $\partial\mathcal{M}_d$, and the third line of (39) vanishes as well. Then the virtual work principle (39) requires the following local relations to be satisfied:

the local equilibrium equations

$$N^\beta\big|_\beta+p=0,\quad M^\beta\big|_\beta+\overline{a}_\beta\times N^\beta+m=0 \qquad (40)$$

at each regular $M\in\mathcal{M}\backslash\Gamma$;

the static boundary conditions

$$T_\nu-N^*=0,\quad K_\nu-M^*=0 \qquad (41)$$

along regular parts of $\partial\mathcal{M}_f$;

the jump conditions

$$[B\overline{n}\cdot\delta u]_b=0\quad\text{at each singular point } P_b\in\partial\mathcal{M}_f; \qquad (42)$$

the jump conditions:

$$[[T_\nu\cdot\delta u]]+[[K_\nu\cdot\omega_\tau]]-\sigma_\Gamma=0 \qquad (43)$$

at regular points of Γ;

$$[B\overline{n}\cdot\delta u]_i-\sigma_i=0 \qquad (44)$$

at each internal singular point $P_i\in\Gamma$.

The corresponding work-conjugate geometric boundary conditions are:

$$u-u^*=0,\quad R_\tau n-R_\tau^* n=0 \qquad (45)$$

along regular parts of $\partial\mathcal{M}_d$.

As it has been expected, the local equilibrium equations (40) as well as the boundary conditions (41) and (45) for thin irregular shell-like structures coincide with those derived within the same formulation for thin regular shells (see [1], Section 5.2). The equilibrium equations (40) were derived first by Alumäe [2,3] and rederived by Simmonds and Danielson [4,5]. The boundary conditions (41) and (45) were proposed first in [1].

Please note that in our jump conditions (42) the virtual displacements still remain coupled with the generalised forces, for in case of the general irregularity of deformation we may not be able to define a common δu_b associated with a singular boundary point $P_b\in\partial\mathcal{M}_f$. The jump conditions (44) and (45) constitute the additional set of basic relations that should be satisfied at the singular curves representing the irregularities of shell geometry, deformation, material properties and loading. All the conditions (40)-(45) are valid for unrestricted displacements, rotations, strains and/or bendings of the reference network $\mathcal{M}$.

The singular curves $\Gamma^{(a)}$ embedded into the shell reference network $\mathcal{M}$ may be of either geometric or physical type, in general. At the geometric curve some fields in the relations (44) or (45) fail to be continuous or smooth of the required class. With the physical curve we can additionally associate some mechanical properties by prescribing appropriate functions $\sigma_\Gamma=$

$= \sigma_\Gamma(u_\Gamma, R_\Gamma; \delta u_\Gamma, \omega_\Gamma)$ and $\sigma_i = \sigma_i(u_i; \delta u_i)$ along $\Gamma^{(a)}$. For special types of irregularities the jump conditions (44) and (45) can be simplified or presented in a more explicit uncoupled form along the lines suggested in [18] for the Lagrangian non-linear theory of thin irregular shells.

5. Constitutive equations of thin elastic shells

For each regular thin shell element made of an elastic material the surface virtual work density $(15)_1$ requires the existence of a surface strain energy density $\Sigma(\eta_{\alpha\beta}, \mu_{\alpha\beta}; \boldsymbol{r})$ defined on $\mathcal{M}^{(k)}$ such that

$$\delta\Sigma = S^{\alpha\beta}\delta\eta_{\alpha\beta} + H^{\alpha\beta}\delta\mu_{\alpha\beta};$$
$$S^{\alpha\beta} = \frac{\partial\Sigma}{\partial\eta_{\alpha\beta}}, \quad H^{\alpha\beta} = \frac{\partial\Sigma}{\partial\mu_{\alpha\beta}}. \tag{46}$$

If the shell element is made of a homogeneous isotropic elastic material and maximal strains in the shell space are small everywhere, the corresponding linear constitutive equations were given already by Alumäe [2], eqs (3.1), see also [1], eqs (5.78).

If large elastic strains are admitted, the density $\Sigma(\eta_{\alpha\beta}, \mu_{\alpha\beta}; \boldsymbol{r})$ can be derived by a consistent reduction to the reference surface $\mathcal{M}^{(k)}$ of the known 3D strain energy density $W(E_{ij}; \boldsymbol{x})$. Alternatively, the suitable form of $\Sigma(\eta_{\alpha\beta}, \mu_{\alpha\beta}; \boldsymbol{r})$ can be established directly from 2D symmetry and invariance requirements. The necessary material functions and constants should then be found from appropriate experiments. Using different methodologies and various surface deformation measures several such 2D strain energy densities were proposed, for example, by Chernykh [20,21], Zubov [10,22], Simmonds [23], Libai and Simmonds [15,16] and Schieck *et al.* [19].

Let us now briefly discuss a possible form of the strain energy density for a thin shell element made of a homogeneous isotropic rubber-like material undergoing large elastic strains. Such a density $\Phi(\gamma_{\alpha\beta}, \kappa_{\alpha\beta}; \boldsymbol{r})$ expressed through the classical Green type surface deformation measures was constructed in [19] under the assumptions of a 3D material incompressibility, a relaxed normality hypothesis, a moderate undeformed thickness $h/R = O(\theta)$, moderate bending strains $h\kappa \leq O(\theta)$ and large membrane strains not exceeding the unity $\gamma \leq O(1)$. Here R is the smallest principal radius of curvature of $\mathcal{M}^{(k)}$, γ and κ are the greatest eigenvalues of the surface deformation measures $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$, respectively, and θ is a small parameter such that $1 + \theta^2 \approx 1$. Under these mild assumptions, with an additional requirement used here that each shell element is thin, $h/R = O(\theta^2)$, the 3D strain energy density $W(E_{ij}; \boldsymbol{x})$ can be reduced, to within the first approximation, to the following 2D form defined on $\mathcal{M}^{(k)}$ (see [19], f.(40)):

$$\Phi \approx hW_{(0)}(\gamma_{\kappa\rho}) + \frac{h^3}{24}\left\{\underline{W^{\alpha\beta}_{(1)}(\gamma_{\kappa\rho})(\kappa^\lambda_\alpha\kappa_{\lambda\beta} - \kappa^\lambda_\lambda\kappa_{\alpha\beta})} + \right. \tag{47}$$
$$\left. + W^{\alpha\beta\lambda\mu}_{(2)}(\gamma_{\kappa\rho})\chi_{\alpha\beta}\chi_{\lambda\mu}\right\}.$$

In the expression (47) the functions $W_{(0)}$, $W^{\alpha\beta}_{(1)}$ and $W^{\alpha\beta\lambda\mu}_{(2)}$ are the modified 3D strain energy density, its first and second derivatives relative to tangent strains, respectively, all taken at the reference surface $\mathcal{M}^{(k)}$, and $\chi_{\alpha\beta} = -(\sqrt{a/\bar{a}}\,\bar{b}_{\alpha\beta} - b_{\alpha\beta})$.

It was shown [19] that the contribution of the terms underlined in (47) is relatively small and can be neglected within an engineering accuracy. It should also be noted that for a thin shell element the estimated membrane and bending energies can be of a comparable order if the greatest eigenvalue γ of $\gamma_{\alpha\beta}$ is at most moderate, so that the approximation $1 + \gamma^2 \approx 1$ holds. Otherwise, Φ can be approximated by only the first term of (47) describing the membrane theory of rubber-like shells. Within the moderate membrane strains $\chi_{\alpha\beta}$ can be approximated by

$$\chi_{\alpha\beta} \approx \kappa_{\alpha\beta}(1 - \gamma^\kappa_\kappa). \tag{48}$$

In this case the simplest approximation to the elastic strain energy density of the rubber-like shell element takes the reduced form

$$\Phi \approx hW_{(0)}(\gamma_{\kappa\rho}) +$$
$$+ \frac{h^3}{24}W^{\alpha\beta\lambda\mu}_{(2)}(\gamma_{\kappa\rho})\left\{\kappa_{\alpha\beta}\kappa_{\lambda\mu}(1 - 2\gamma^\kappa_\kappa)\right\}, \tag{49}$$

where the symmetry of $W^{\alpha\beta\lambda\mu}_{(2)}$ relative to the change of indices $a \leftrightarrow b$, $\lambda \leftrightarrow \mu$ and $(\alpha\beta) \leftrightarrow (\lambda\mu)$ has been taken into account.

Let us express the density (49) in terms of the modified deformation measures $\eta_{\alpha\beta}, \mu_{\alpha\beta}$ using the transformations (14). Decomposing $\mu_{\alpha\beta}$ into the symmetric and skew parts, $\mu_{\alpha\beta} = \rho_{\alpha\beta} + \varepsilon_{\alpha\beta}\rho$, where $\rho_{\alpha\beta} = 1/2(\mu_{\alpha\beta} + \mu_{\beta\alpha})$, the relation $(14)_2$ can be modified into

$$\kappa_{\alpha\beta} = \rho_{\alpha\beta} -$$
$$-\frac{1}{2}[\eta^\lambda_\alpha(b_{\lambda\beta} - \rho_{\lambda\beta}) + \eta^\lambda_\beta(b_{\alpha\lambda} - \rho_{\alpha\lambda})] + \tag{50}$$
$$+\frac{1}{2}(\eta^\lambda_\alpha\varepsilon_{\lambda\beta} + \eta^\lambda_\beta\varepsilon_{\lambda\alpha})\rho.$$

From the surface compatibility conditions it follows that (see [1], f. $(5.76)_2$)

$$\rho = \frac{1}{2+\eta^\kappa_\kappa}\varepsilon^{\alpha\beta}\eta^\lambda_\alpha(b_{\lambda\beta}-\rho_{\lambda\beta}). \qquad (51)$$

The relation (51) allows one to estimate ρ to be $\max(\eta/R, \eta\mu)$, where η and μ are maximal eigenvalues of $\eta_{\alpha\beta}$ and $\rho_{\alpha\beta}$, respectively. Therefore, with the relative error $O(\theta^2)$ the relation (50) can be approximated by

$$\kappa_{\alpha\beta} \approx \rho_{\alpha\beta} + \frac{1}{2}(\eta^\lambda_\alpha\rho_{\lambda\beta} + \eta^\lambda_\beta\rho_{\alpha\lambda}). \qquad (52)$$

Introducing $(14)_1$ and (52) into (49) with the relative error $O(\theta^2)$ we obtain

$$\Phi = \Phi(\gamma_{\alpha\beta}(\eta_{\kappa\rho}), \kappa_{\alpha\beta}(\eta_{\kappa\rho}, \mu_{\kappa\rho})) = \Sigma(\eta_{\kappa\rho}, \mu_{\kappa\rho}) \approx$$
$$\approx h\hat{W}_{(0)}(\eta_{\kappa\rho}) + \qquad (53)$$
$$+\frac{h^3}{24}\hat{W}^{\alpha\beta\lambda\mu}_{(2)}(\eta_{\kappa\rho})\{\rho_{\alpha\beta}\rho_{\lambda\mu} + 2\rho_{\alpha\beta}(\eta^\kappa_\lambda\rho_{\kappa\mu} - \eta^\kappa_\kappa\rho_{\lambda\mu})\}.$$

The constitutive equations following from (53) and $(46)_2$ for a thin rubber-like shell element with a reference surface $\mathcal{M}^{(k)}$ are

$$S^{\alpha\beta} = hR^{\alpha\beta} +$$
$$+\frac{h^3}{24}R^{\alpha\beta\lambda\mu\sigma\tau}\{\rho_{\alpha\beta}\rho_{\lambda\mu} + 2\rho_{\alpha\beta}(\eta^\kappa_\lambda\rho_{\kappa\mu} - \eta^\kappa_\kappa\rho_{\lambda\mu})\} +$$
$$+\frac{h^3}{24}(R^{\alpha\kappa\lambda\mu}\rho^\beta_\kappa + R^{\kappa\beta\lambda\mu}\rho^\alpha_\kappa)\rho_{\lambda\mu}; \qquad (54)$$
$$G^{\alpha\beta} = \frac{h^3}{12}R^{\alpha\beta\lambda\mu}\{(1-2\eta^\kappa_\kappa)\rho_{\lambda\mu} + \eta^\kappa_\lambda\rho_{\kappa\mu}\} +$$
$$+\frac{h^3}{24}(R^{\alpha\kappa\lambda\mu}\eta^\beta_\kappa + R^{\kappa\beta\lambda\mu}\eta^\alpha_\kappa)\rho_{\lambda\kappa},$$

where the components of the tangent elasticity tensors are defined by

$$R^{\alpha\beta} = \frac{\partial\hat{W}_{(0)}(\eta_{\kappa\rho})}{\partial\eta_{\alpha\beta}}, \quad R^{\alpha\beta\lambda\mu} = \frac{\partial\hat{W}_{(0)}(\eta_{\kappa\rho})}{\partial\eta_{\alpha\beta}\partial\eta_{\lambda\mu}}, \qquad (55)$$
$$R^{\alpha\beta\lambda\mu\sigma\tau} = \frac{\partial\hat{W}_{(0)}(\eta_{\kappa\rho})}{\partial\eta_{\alpha\beta}\partial\eta_{\lambda\mu}\partial\eta_{\sigma\tau}}.$$

Please note that only the symmetric part $G^{\alpha\beta}$ of $H^{\alpha\beta}$ is given by the constitutive equations $(54)_2$. It means that within the error of (49) the skew part of $H^{\alpha\beta}$ becomes indefinite within this version of thin shell theory.

The structure of our consistently reduced strain energy density (53) is relatively simple. It is quadratic in $\rho_{\alpha\beta}$, and the type of non-linearity in $\eta_{\alpha\beta}$ is governed entirely by the 3D material law. Yet, for rubber-like thin shells our constitutive equations (54) describe a broad class of problems in which bending effects may become important.

6. Conclusions

In this report we have developed the new weak (22) and strong (40)–(45),(46),(7),(16) forms of the BVP for the Alumäe type non-linear theory of thin shell-like structures with irregularities of geometry, material properties and deformation. The formulations are valid for unrestricted displacements, rotations, strains and/or bendings of the reference network, and for an arbitrary external loading. For a rubber-like structure undergoing large elastic strains we have constructed a simplest approximation (53) to the surface elastic strain energy density. Within each surface element $\mathcal{M}^k$ this allows one to express the BVPs in terms of the displacement vector $\boldsymbol{u}$, the rotation tensor $\boldsymbol{R}$ and the Lagrange multipliers S, Q^α, B as the primary field variables. However, we can formulate BVPs in terms of other fields as primary variables as well proceeding as in Sections 5.6 and 5.7 of [1].

In order to analyse irregular shell problems, our formulation should still be completed by definite expressions for the external virtual work densities $\sigma_\Gamma(\boldsymbol{u}_\Gamma, \boldsymbol{R}_\Gamma; \delta\boldsymbol{u}_\Gamma, \omega_\Gamma)$ and $\sigma_i(\boldsymbol{u}_i; \delta\boldsymbol{u}_i)$. A discussion given in [18] within the Lagrangian formulation indicates that a large variety of expressions can be associated with the densities σ_Γ and σ_i. The expressions depend on the type of geometric irregularity of the network $\mathcal{M}$, on the mechanical properties prescribed along Γ and at $P_i \in \Gamma$, as well as on the type of non-smooth deformations $\boldsymbol{\chi}$ and $\boldsymbol{\chi}_\Gamma$ analysed, which can additionally influence the continuity properties of the virtual fields $\delta\boldsymbol{u}$, ω, $\delta\boldsymbol{u}_\Gamma$, ω_Γ, $\delta\boldsymbol{u}_i$. Possible explicit forms of σ_Γ and σ_i will be discussed separately.

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References

1. *Pietraszkiewicz W.* Geometrically nonlinear theories of thin elastic shells// Advances in Mechanics. 1989. Vol. 12. P. 51–130.

2. *Alumäe N.A.* Differential equations of the equilibrium states of thin-walled elastic shells in the post-buckling stage (in Russian)// Prikladnaya Matematika i Mekhanika. 1949. Vol. 13. P. 95–106.

3. *Alumäe N.A.* On the expression of basic relations of the non-linear theory of shells (in Russian) // Prikladnaya Matematika i Mekhanika. 1956. Vol. 20. P. 136–139.

4. *Simmonds J.G., Danielson D.A.* Nonlinear shell theory with a finite rotation vector// Proc. Koninkl. Nederl. Akademie van Wetenschappen. Amsterdam, 1970. Series B. Vol. 73. P. 460–478.

5. *Simmonds J.G., Danielson D.A.* Nonlinear shell theory with finite rotation and stress function vectors// Trans. ASME. J. Applied Mechanics. 1972. Vol. 39. P. 1085–1090.

6. *Pietraszkiewicz W.* Introduction to the Non-linear Theory of Shells// Ruhr-Universität, Mitt. IfMech. № 10. Bochum, 1977.

7. *Pietraszkiewicz W.* Finite Rotations and Lagrangean Description in the Non-linear Theory of Shells. Polish Sci. Publ., Warszawa - Poznań, 1979.

8. *Pietraszkiewicz W.* Finite rotations in the non-linear theory of thin shells/In Thin Shell Theory. New Trends and Applications. Ed. W. Olszak. Springer-Verlag, Wien, New York, 1980. P. 151–208.

9. *Shkutin L.I.* Exact formulation of equations for non-linear deformability of thin shells// Applied Problems of Strength and Plasticity (in Russian). 1977. Vol. 7. P. 3–9; 1978. Vol. 8. P. 38–43; 1978. Vol. 9. P. 19–25. Gorkii University Press.

10. *Zubov L.M.* Methods of Non-linear Elasticity in Theory of Shells (in Russian). Rostov-on-Don, 1982.

11. *Atluri S.N.* Alternate stress and conjugate strain measures, and mixed variational formulations involving rigid rotations, for computational analyses of finitely deformed solids, with applications to plates and shells. I. Theory // Computers and Structures. 1984. Vol. 18. P. 93–116.

12. *Badur J., Pietraszkiewicz W.* On geometrically non-linear theory of elastic shells derived from pseudo-Cosserat continuum with constrained micro-rotations / In Finite Rotations in Structural Mechanics. Ed. W. Pietraszkiewicz. Springer-Verlag. Berlin, 1986. P. 19–32.

13. *Kayuk Ya.F.* Geometrically Non-linear Problems of the Theory of Plates and Shells (in Russian). Kiev, 1987.

14. *Valid R.* The Nonlinear Theory of Shells through Variational Principles. Chichester et al., John Wiley and Sons, 1995.

15. *Libai A., Simmonds J.G.* Nonlinear elastic shell theory/ In Advances in Applied Mechanics 23. Eds J.W. Hutchinson and T.Y. Wu. Academic Press. 1983. P. 271–371.

16. *Libai A., Simmonds J.G.* The Non-linear Theory of Elastic Shells, 2nd ed. Cambridge University Press, Cambridge, 1998.

17. *Makowski J., Pietraszkiewicz W., Stumpf H.* On the general form of jump conditions for thin irregular shells// Archives of Mechanics. 1998. Vol. 50. P. 483–495.

18. *Makowski J., Pietraszkiewicz W., Stumpf H.* Jump conditions in the non-linear theory of thin irregular shells// J. Elasticity. 1999. Vol. 54. P. 1–26.

19. *Schieck B., Pietraszkiewicz W., Stumpf H.* Theory and numerical analysis of shells undergoing large elastic strains// Int. J. Solids Struct. 1992. Vol. 29. P. 689–709.

20. *Chernykh K.F.* Nonlinear theory of isotropically elastic thin shells// Mechanics of Solids (Izv. AN SSSR, MTT). 1980. Vol. 15. P. 118-127.

21. *Chernykh K.F.* Theory of thin shells made of elastomers (rubber-like materials) (in Russian)// Advances in Mechanics. 1983. Vol. 6. P. 111–147.

22. *Zubov L.M.* Description of finite deformations of thin shells through co-ordinates of the reference and actual configurations (in Russian)// Izv. AN SSSR. Mekhanika Tverdogo Tela. 1983. P. 128–135.

23. *Simmonds J.G.* The strain-energy density of rubber-like shells// Int. J. Solids Struct. 1985. Vol. 21. P. 67–77.

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