# ON USING ROTATIONS AS PRIMARY VARIABLES IN THE NON-LINEAR THEORY OF THIN IRREGULAR SHELLS

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## 1. Introduction

A non-linear theory of thin shell-like structures with irregularities of geometry, material properties and deformation along singular curves was developed in Makowski *et al.* [1,2]. In these papers an irregular structure was modelled by a reference network being a union of piecewise smooth surfaces and surface curves resisting only the stretching and bending. The resulting boundary value problem was expressed through displacement vector as the only independent field variable.

In the general approach to the non-linear theory of shells presented in the book of Libai and Simmonds [3] the finite rotation field appears naturally as one of primary variables of the boundary value problem. This statically and geometrically exact formulation of shell theory, which grew from early ideas of Reissner [4] and Simmonds [5], allowed one to develop effective computational procedures based on the finite element method for both the regular shells [6] and the irregular shell-like structures [7]. The classical thin shell theory was defined in [3] with the help of the Kirchhoff hypothesis regarded as a constitutive hypothesis and not as a kinematic one. It was confirmed in [3] that in the classical theory the rotations become expressible through displacements and are no longer independent field variables. Thus, in order to regard them again as primary variables some additional constraint conditions with Lagrange multipliers should be imposed.

The rotation angle as one of primary variables of thin shell theory was first introduced by Reissner [8] to describe a one-dimensional axisymmetric deformation state of a thin shell of revolution. Simmonds and Danielson [9,10] formulated two-dimensional thin shell relations in terms of the finite rotation and stress function vectors, and derived an appropriate variational principle. Several alternative forms of relations for thin shells expressed in terms of rotations were developed by Pietraszkiewicz [11-14], Shkutin [15], Valid [16,17], Atluri [18], and Libai and Simmonds [19]. In particular, within the geometrically non-linear theory of thin, regular, isotropic, elastic shells many such relations were summarised in Chapter 5 of [14], where references to earlier papers can be found.

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The aim of this report is to extend the results presented in Chapter 5 of [14] in three directions:

a) In place of the reference surface we introduce the reference network defined in [1,2]. This allows one, also within the rotational formulation, to take into account various irregularities of shell geometry, deformation and mechanical properties along singular curves.

b) In all shell relations large surface strains are admitted. This allows one to discuss within the same formulation also large strain problems of irregular shells made, for example, of a rubber-like material with constitutive equations proposed in [3,20].

c) The boundary terms for each regular surface element are discussed in more detail, which allows one to derive an appropriate form of jump conditions at singular curves and points representing the irregularities.

The deformation of the reference network models entirely the deformation of a thin irregular shell-like structure. The network consists of a finite number of regular surface elements connected together along singular spatial curves. The equilibrium conditions of the entire structure are given in Chapter 3 by the postulated principle of virtual work (PVW) in which the internal surface stress and strain fields are associated only with stretching and bending of the reference network. Then appropriate constraints with Lagrange multipliers are introduced into the PVW in order to regard also the rotations as primary variables. Transforming the so modified PVW we obtain the known, [14], local forms of equilibrium equations and boundary conditions. We also derive the local forms of jump conditions at the singular curves (45) and at the singular points (43) and (46). The jump conditions seem to be new in the literature.

## 2. Geometry and deformation of a regular surface element

In this report we shall apply primarily the system of notation used in [14] and remind here only basic relations.

Let $\mathcal{M}^{(k)}$ be a connected, oriented and regular surface element of class $C^n, n \geq 2$, in the three-dimensional Euclidean point space $\mathcal{E}$ whose translation (three-dimensional vector) space is $E$. The position vector of a point $M \in \mathcal{M}^{(k)}$ is given by

$$\mathbf{r} = \overrightarrow{OM} = \mathbf{r}(\theta^\alpha), \tag{1}$$

where $O \in \mathcal{E}$ is a reference origin and $\theta^\alpha, \alpha = 1,2$, are surface co-ordinates. At $M \in \mathcal{M}^{(k)}$ we have the natural base vectors $\mathbf{a}_\alpha = \partial \mathbf{r} / \partial \theta^\alpha \equiv \mathbf{r},_\alpha$, the dual base vectors $\mathbf{a}^\beta$ such that $\mathbf{a}^\beta \bullet \mathbf{a}_\alpha = \delta^\beta_\alpha$, where $\delta^\beta_\alpha$ is the Kronecker symbol, the components $a_{\alpha\beta} = \mathbf{a}_\alpha \bullet \mathbf{a}_\beta$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \bullet \mathbf{a}^\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the unit normal vector $\mathbf{n} = (1/\sqrt{a})\mathbf{a}_1 \times \mathbf{a}_2$ orienting $\mathcal{M}^{(k)}$, the components $b_{\alpha\beta} = -\mathbf{n},_\alpha \bullet \mathbf{a}_\beta$

of the surface curvature tensor **b**, and the components $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta)\bullet\mathbf{n}$ of the surface permutation tensor such that $\varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

The boundary $\partial\mathcal{M}^{(k)}$ of $\mathcal{M}^{(k)}$ consists of a finite number of closed piecewise smooth curves that do not meet in cusps, each described parametrically by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where $s$ is the arc length along any regular part of $\partial\mathcal{M}^{(k)}$. At each regular point $M \in \partial\mathcal{M}^{(k)}$ we have the unit tangent vector $\boldsymbol{\tau} = d\mathbf{r}/ds \equiv \mathbf{r}' = \tau^\alpha \mathbf{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r},_\nu = \boldsymbol{\tau}\times\mathbf{n} = \nu^\alpha \mathbf{a}_\alpha$, where $(\cdot),_\nu$ is the external surface derivative normal to $\partial\mathcal{M}^{(k)}$.

The deformed regular surface element $\overline{\mathcal{M}}^{(k)}$ with the boundary $\partial\overline{\mathcal{M}}^{(k)}$ is described relative to the same origin $O \in \mathcal{E}$ by the relations

$$\bar{\mathbf{r}}(\theta^\alpha) = \chi[\mathbf{r}(\theta^\alpha)] = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha), \quad \bar{\mathbf{r}}(s) = \chi[\mathbf{r}(s)] = \mathbf{r}(s) + \mathbf{u}(s), \tag{2}$$

where $\theta^\alpha$ and $s$ are convected surface co-ordinates, $\chi: \mathcal{M}^{(k)} \to \overline{\mathcal{M}}^{(k)}$ is the deformation function, and $\mathbf{u} \in E$ is the displacement vector.

In the convected surface co-ordinates all geometric relations at any regular $\bar{M} \in \partial\overline{\mathcal{M}}^{(k)}$ are now analogous to those given at $M \in \partial\mathcal{M}^{(k)}$, and are expressed by quantities marked by a dash: $\bar{\mathbf{a}}_\alpha, \bar{\mathbf{a}}^\beta, \bar{a}_{\alpha\beta}, \bar{a}^{\alpha\beta}, \bar{\mathbf{a}}, \bar{a}, \bar{b}_{\alpha\beta}, \bar{\varepsilon}_{\alpha\beta}, \bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\tau}}$, etc. The dashed quantities can be expressed through analogous quantities defined on $\mathcal{M}^{(k)}$ and the displacement field **u** with the help of formulae given in [14,21].

Components of the Green type surface deformation measures are defined by

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}\left(\bar{a}_{\alpha\beta} - a_{\alpha\beta}\right), \quad \kappa_{\alpha\beta}(\mathbf{u}) = -\left(\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right), \tag{3}$$

where $\gamma_{\alpha\beta}(\mathbf{u})$ are quadratic polynomials of $\mathbf{u}, \mathbf{u},_\alpha$, and $\kappa_{\alpha\beta}(\mathbf{u})$ are non-rational functions of $\mathbf{u}, \mathbf{u},_\alpha, \mathbf{u},_{\alpha\beta}$.

In the neighbourhood of the regular surface elements $\mathcal{M}^{(k)}$ and $\overline{\mathcal{M}}^{(k)}$ the space $\mathcal{E}$ can be parameterised by the normal system of convected co-ordinates $(\theta^\alpha, \varsigma)$, where $\varsigma$ is the distance from $\mathcal{M}^{(k)}$ and $\overline{\mathcal{M}}^{(k)}$ along **n** and $\bar{\mathbf{n}}$, respectively. Extending the domain of $\chi$ to the neighbourhood of $\mathcal{M}^{(k)}$, the spatial deformation gradient $\mathbf{F}: E \to E$ taken at the surface element $\mathcal{M}^{(k)}$ has the form

$$\begin{aligned} &\mathbf{F} = \nabla\chi(\mathbf{r} + \varsigma\mathbf{n})\,|_{\varsigma=0} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \det\mathbf{F} = \sqrt{\frac{\bar{a}}{a}} > 0, \\ &\frac{\bar{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2\left(\gamma^\alpha_\alpha \gamma^\beta_\beta - \gamma^\alpha_\beta \gamma^\beta_\alpha\right), \end{aligned} \tag{4}$$

where $\otimes$ is the tensor product.

The left polar decomposition of $\mathbf{F}$ gives

$$\mathbf{F}=\mathbf{V}\mathbf{R}, \quad \mathbf{r}_\alpha=\mathbf{R}\mathbf{a}_\alpha=\mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha. \tag{5}$$

Here $\mathbf{R}\in SO(3)$ is the rotation tensor, $\mathbf{V}$ is the left spatial stretch tensor at $\bar{\mathcal{M}}^{(k)}$, and $\mathbf{r}_\alpha$ are the rotated surface non-holonomic base vectors. These fields satisfy the relations

$$\begin{aligned}
&\mathbf{R}=\mathbf{r}_\alpha\otimes\mathbf{a}^\alpha+\bar{\mathbf{n}}\otimes\mathbf{n}, \quad \mathbf{R}^T=\mathbf{R}^{-1}, \quad \det\mathbf{R}=+1,\\
&\mathbf{V}=\bar{\mathbf{a}}_\alpha\otimes\mathbf{r}^\alpha+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}, \quad \mathbf{V}^T=\mathbf{V}, \quad \det\mathbf{V}=\sqrt{\frac{\bar{a}}{a}}>1.
\end{aligned} \tag{6}$$

The modified surface deformation measures associated with $\mathbf{r}_\alpha$ are introduced through the following formulae, [14]:

$$\begin{aligned}
&\boldsymbol{\eta}=\mathbf{V}-\mathbf{1}=\boldsymbol{\eta}_\beta\otimes\mathbf{r}^\beta, \quad \boldsymbol{\eta}_\beta=\eta_{\alpha\beta}\mathbf{r}^\alpha,\\
&\boldsymbol{\mu}=\left(\bar{\mathbf{n}},_\beta-\mathbf{R}\mathbf{n},_\beta\right)\otimes\mathbf{r}^\beta=\boldsymbol{\mu}_\beta\otimes\mathbf{r}^\beta, \quad \boldsymbol{\mu}_\beta=\mu_{\alpha\beta}\mathbf{r}^\alpha,\\
&\eta_{\alpha\beta}=\eta_{\beta\alpha}, \quad \mu_{\alpha\beta}\neq\mu_{\beta\alpha}.
\end{aligned} \tag{7}$$

Here $\mathbf{1}=\mathbf{a}_\alpha\otimes\mathbf{a}^\alpha+\mathbf{n}\otimes\mathbf{n}=\mathbf{r}_\alpha\otimes\mathbf{r}^\alpha+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}$ is the spatial identity tensor. The surface measures $\eta_{\alpha\beta}$ and $\mu_{\alpha\beta}$ satisfy useful kinematic relations given in [14].

Along the boundary $\partial\bar{\mathcal{M}}^{(k)}$ we have

$$\begin{aligned}
&\bar{\mathbf{a}}_\tau\equiv\bar{\mathbf{r}}'=a_\tau\bar{\boldsymbol{\tau}}, \quad \bar{\mathbf{a}}_\nu=\bar{\mathbf{a}}_\tau\times\bar{\mathbf{n}}=a_\tau\bar{\boldsymbol{\nu}}, \quad \bar{\mathbf{n}}=\sqrt{\frac{a}{\bar{a}}}\bar{\mathbf{r}},_\nu\times\bar{\mathbf{r}}',\\
&a_\tau=|\bar{\mathbf{r}}'|=\sqrt{1+2\gamma_{\tau\tau}}\,, \quad \gamma_{\tau\tau}=\gamma_{\alpha\beta}\tau^\alpha\tau^\beta.
\end{aligned} \tag{8}$$

The transformation of $(\boldsymbol{\nu},\boldsymbol{\tau},\mathbf{n})$ during deformation into $(\bar{\mathbf{a}}_\nu,\bar{\mathbf{a}}_\tau,\bar{\mathbf{n}})$ is performed in two steps: the rotation of $(\boldsymbol{\nu},\boldsymbol{\tau},\mathbf{n})$ into $(\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\tau}},\bar{\mathbf{n}})$ by the total rotation tensor $\mathbf{R}_\tau$ with the subsequent extension of $\bar{\boldsymbol{\nu}},\bar{\boldsymbol{\tau}}$ into $\bar{\mathbf{a}}_\nu,\bar{\mathbf{a}}_\tau$ by the factor $a_\tau$:

$$\begin{aligned}
&\bar{\mathbf{a}}_\nu=a_\tau\mathbf{R}_\tau\boldsymbol{\nu}, \quad \bar{\mathbf{a}}_\tau=a_\tau\mathbf{R}_\tau\boldsymbol{\tau}, \quad \bar{\mathbf{n}}=\mathbf{R}_\tau\mathbf{n},\\
&\mathbf{R}_\tau=\frac{1}{a_\tau}\left(\bar{\mathbf{a}}_\nu\otimes\boldsymbol{\nu}+\bar{\mathbf{a}}_\tau\otimes\boldsymbol{\tau}+\bar{\mathbf{n}}\otimes\mathbf{n}\right).
\end{aligned} \tag{9}$$

Kinematic relations involving the tensors $\mathbf{R}$ and $\mathbf{R}_\tau$ are given in [14].

### 3. Principle of virtual work for thin irregular shells

A consistent formulation of the mechanical boundary value problem for thin irregular shell-like structures was developed in Makowski *et al.* [1,2], where the displacements were taken as the only independent field variables. The mechanical modelling of such structures was based on two postulates:

- *The deformation of the entire shell-like structure is determined by deformation of a distinguished surface-like continuum, called the shell reference network.*
- *The equilibrium conditions of the entire structure are determined by a suitable form of the principle of virtual work involving only the fields associated with the stretching and bending of the reference network.*

The undeformed reference network $\mathcal{M} \subset \mathcal{E}$ introduced in [1] consists of a finite number of regular surface elements $\mathcal{M}^{(k)}, k = 1,2,\ldots,K,$ with the following properties:

a) No two distinct elements $\mathcal{M}^{(k)}$ have common interior points.

b) Two or more distinct elements may have a smooth spatial curve $\Gamma^{(a)}$ as a common part of the boundaries, which is defined by

$$\Gamma^{(a)} = \partial\mathcal{M}^{(k_1)} \bigcap \partial\mathcal{M}^{(k_2)} \bigcap \ldots \bigcap \partial\mathcal{M}^{(k_m)} \quad \text{if } k_1 \neq k_2 \neq \ldots \neq k_m. \tag{10}$$

c) Two or more distinct curves $\Gamma^{(a)}$ may have in common only single isolated points.

Each $\mathcal{M}^{(k)}$ represents a reference surface of a regular shell part. Each $\Gamma^{(a)}$ can be a surface curve across which some fields fail to be smooth. Examples of geometric irregularities along $\Gamma^{(a)}$ are surface folds or intersections of two or more regular surfaces. Shell parts can be made of different materials, or there may be stepwise thickness changes at $\Gamma^{(a)}$. However, $\Gamma^{(a)}$ can also represent a reference axis of a rod-like element, a technological junction, a plastic hinge developing during deformation process, etc.

The network $\mathcal{M}$ is then regarded as the union of all the closed elements $\mathcal{M}^{(k)} \bigcup \partial\mathcal{M}^{(k)}$, and the singular curve $\Gamma \in \mathcal{M}$ is regarded as the union of all the curves $\Gamma^{(a)}$:

$$\mathcal{M} = \bigcup_{k=1}^{K} \left(\mathcal{M}^{(k)} \bigcup \partial\mathcal{M}^{(k)}\right), \quad \Gamma = \bigcup_{a=1}^{A} \Gamma^{(a)}. \tag{11}$$

From (11) it is apparent that $\Gamma \subset \mathcal{M}$. The boundary $\partial\mathcal{M}$ of the entire network $\mathcal{M}$ defined by

$$\partial\mathcal{M} = \left(\bigcup_{k=1}^{K} \partial\mathcal{M}^{(k)}\right) \setminus \Gamma \tag{12}$$

consists of a finite number of spatial curves. Several examples of such networks are given in [2].

Deformation of $\mathcal{M}$ can be described by two deformation functions: $\chi : \mathcal{M}\backslash\Gamma \to \mathcal{E}$ and $\chi_\Gamma : \Gamma \to \mathcal{E}$, since the singular curve may be admitted to follow its own deformation, in general. In many cases the deformation $\chi$ may be defined on the entire $\mathcal{M}$, and then $\chi_\Gamma$ is a restriction of $\chi$ at $\Gamma$: $\chi_\Gamma = \chi|_\Gamma$. However, we do not assume such a restricted shell deformation at the moment.

The principle of virtual work compatible with the two postulates given above can be taken in the form

$$G \equiv G_{\text{int}} - G_{\text{ext}} - G_\Gamma = 0, \tag{13}$$

where $G_{\text{int}} = G_{\text{int}}(\mathbf{u};\delta\mathbf{u})$ represents the internal virtual work, $G_{\text{ext}} = G_{\text{ext}}(\mathbf{u};\delta\mathbf{u})$ is the external virtual work, and $G_\Gamma = G_\Gamma(\mathbf{u}_\Gamma;\delta\mathbf{u}_\Gamma)$ is the additional virtual work of the generalised forces acting along $\Gamma$. Here we have explicitly indicated that all the virtual works are functionals of the displacements as the only independent field variables. The individual parts of (13) are defined by

$$\begin{aligned} G_{\text{int}} &= \sum_{k=1}^{K} \iint_{\mathcal{M}^{(k)}} \left( N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} \right) dA, \\ G_{\text{ext}} &= \sum_{k=1}^{K} \iint_{\mathcal{M}^{(k)}} \left( \mathbf{p}\cdot\delta\mathbf{u} + \mathbf{h}\cdot\delta\bar{\mathbf{n}} \right) dA + \int_{\partial\mathcal{M}_f} \left( \mathbf{N}^*\cdot\delta\mathbf{u} + \mathbf{H}^*\cdot\delta\bar{\mathbf{n}} \right) ds, \\ G_\Gamma &= \int_\Gamma \sigma_\Gamma ds + \sum_{P_i \in \Gamma} \sigma_i . \end{aligned} \tag{14}$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric stress resultant and stress couple tensors of the Piola-Kirchhoff type, $\delta$ is the symbol of variation, $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the surface deformation measures (3), $\mathbf{p}$ and $\mathbf{h}$ are the external surface force and moment resultant vectors, $\mathbf{N}^*$ and $\mathbf{H}^*$ are the external boundary force and moment resultant vectors, whereas $\sigma_\Gamma$ and $\sigma_i$ are the external virtual work densities along regular parts of $\Gamma$ and at any singular point $P_i \in \Gamma$, respectively. The explicit forms of $\sigma_\Gamma$ and $\sigma_i$ depend on the type of irregularity assumed along $\Gamma$, [2]. Note that since $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{n}} = 0$, only the surface components of $\mathbf{h}$ and $\mathbf{H}^*$ can explicitly be taken into account in the non-linear theory of thin irregular shells discussed here. Transforming (13) and (14) with the help of Stokes' theorem, the local equilibrium equations, boundary conditions and jump conditions at singular curves were derived, [1,2].

If the rotation tensor $\mathbf{R}$ is supposed to be an independent field variable of the boundary value problem, some constraint conditions have to be introduced into the relations (14) and the virtual densities should be expressed in terms of modified surface deformation and stress measures.

Let us remind that the components $\gamma_{\alpha\beta}, \kappa_{\alpha\beta}$ and $\eta_{\alpha\beta}, \mu_{\alpha\beta}$ of the surface deformation measures are related by (see [14], formula (5.11))

$$\begin{aligned} \gamma_{\alpha\beta} &= \eta_{\alpha\beta} + \frac{1}{2}\eta^{\lambda}_{\alpha}\eta_{\lambda\beta}, \\ \kappa_{\alpha\beta} &= \frac{1}{2}\left[\left(\delta^{\lambda}_{\alpha} + \eta^{\lambda}_{\alpha}\right)\mu_{\lambda\beta} + \left(\delta^{\lambda}_{\beta} + \eta^{\lambda}_{\beta}\right)\mu_{\lambda\alpha}\right] - \frac{1}{2}\left(b^{\lambda}_{\alpha}\eta_{\lambda\beta} + b^{\lambda}_{\beta}\eta_{\lambda\alpha}\right). \end{aligned} \tag{15}$$

As a result, the internal surface virtual work density appearing in $(14)_1$ can be presented in an alternative form

$$\begin{aligned} & N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} = S^{\alpha\beta}\delta\eta_{\alpha\beta} + H^{\alpha\beta}\delta\mu_{\alpha\beta}, \\ & S^{\alpha\beta} = N^{\alpha\beta} + \frac{1}{2}\left(\eta^{\alpha}_{\lambda}N^{\lambda\beta} + \eta^{\beta}_{\lambda}N^{\alpha\lambda}\right) - \frac{1}{2}\left[\left(b^{\alpha}_{\lambda} - \mu^{\alpha}_{.\lambda}\right)M^{\lambda\beta} + \left(b^{\beta}_{\lambda} - \mu^{\beta}_{.\lambda}\right)M^{\alpha\lambda}\right], \\ & H^{\alpha\beta} = \left(\delta^{\alpha}_{\lambda} + \eta^{\alpha}_{\lambda}\right)M^{\lambda\beta}, \end{aligned} \tag{16}$$

where now $S^{\alpha\beta} = S^{\beta\alpha}$, but $H^{\alpha\beta} \neq H^{\beta\alpha}$, in general.

In the non-linear theory of thin shells the rotation tensor $\mathbf{R}$ is a non-rational function of $\mathbf{u}, \mathbf{u},_{\alpha}$ (explicit formulae are given in [11,13]). This dependence of $\mathbf{R}$ upon $\mathbf{u}$ can also be expressed implicitly through three constraint conditions [22,14]

$$\varepsilon^{\alpha\beta}\mathbf{r}_{\alpha}\cdot\eta_{\lambda\beta}\mathbf{r}^{\lambda} = 0, \quad \bar{\mathbf{n}}\cdot\eta_{\lambda\beta}\mathbf{r}^{\lambda} = 0. \tag{17}$$

These constraints express the known property of the relative surface strain tensor $\boldsymbol{\eta}$, which in thin shell theory is symmetric and does not have out-of-surface components. The property was also confirmed by Libai and Simmonds [3] who used the constitutive Kirchhoff hypothesis to define the classical theory of thin shells as a special case of the general shell theory.

For a virtual deformation the relations (17) put the following constraints on the virtual changes $\delta\eta_{\alpha\beta}$ of $\eta_{\alpha\beta}$:

$$\varepsilon^{\alpha\beta}\mathbf{r}_{\alpha}\cdot\delta\eta_{\lambda\beta}\mathbf{r}^{\lambda} = 0, \quad \bar{\mathbf{n}}\cdot\delta\eta_{\lambda\beta}\mathbf{r}^{\lambda} = 0. \tag{18}$$

Inside of each $\mathcal{M}^{(k)}$ the constraints (18) can be introduced into the surface integral of $(14)_1$ with the help of the respective Lagrange multipliers $S$ and $Q^{\beta}$. It was shown in [14] that in order to express also the boundary terms at each $\partial\mathcal{M}^{(k)}$ explicitly through independent rotations it is necessary to introduce into $(14)_1$ a line integral over $\partial\mathcal{M}^{(k)}$ with the constraints $(18)_2$ multiplied by $B\tau^{\beta}$. Additionally, in $(14)_2$ the external virtual work done by the moments $\mathbf{h}$ and $\mathbf{H}^*$ should be expressed directly in terms of now independent virtual rotations. As a result, $(14)_{1,2}$ can be modified to the form

$$G_{\mathrm{int}}=\sum_{k=1}^{K}\left\{\iint_{\mathcal{M}^{(k)}}\left(\mathbf{N}^{\beta}\bullet\delta\eta_{\lambda\beta}\mathbf{r}^{\lambda}+H^{\alpha\beta}\mathbf{r}_{\alpha}\bullet\delta\mu_{\lambda\beta}\mathbf{r}^{\lambda}\right)dA+\int_{\partial\mathcal{M}^{(k)}}B\tau^{\beta}\overline{\mathbf{n}}\bullet\delta\eta_{\lambda\beta}\mathbf{r}^{\lambda}ds\right\},$$
$$G_{\mathrm{ext}}=\sum_{k=1}^{K}\iint_{\mathcal{M}^{(k)}}\left(\mathbf{p}\bullet\delta\mathbf{u}+\mathbf{m}\bullet\boldsymbol{\omega}\right)dA+\int_{\partial\mathcal{M}_f}\left(\mathbf{N}^{*}\bullet\delta\mathbf{u}+\mathbf{M}^{*}\bullet\boldsymbol{\omega}_{\tau}\right)ds, \tag{19}$$

where now $G_{\mathrm{int}}=G_{\mathrm{int}}(\mathbf{u},\mathbf{R};\delta\mathbf{u},\boldsymbol{\omega})$ and $G_{\mathrm{ext}}=G_{\mathrm{ext}}(\mathbf{u},\mathbf{R};\delta\mathbf{u},\boldsymbol{\omega})$, while other fields are defined by

$$\mathbf{N}^{\beta}=\left(S^{\alpha\beta}+\varepsilon^{\alpha\beta}S\right)\mathbf{r}_{\alpha}+Q^{\beta}\overline{\mathbf{n}},\quad \mathbf{M}^{\beta}=\overline{\mathbf{n}}\times H^{\alpha\beta}\mathbf{r}_{\alpha},$$
$$\mathbf{M}^{*}=\overline{\mathbf{n}}\times\mathbf{H}^{*},\quad \mathbf{m}=\overline{\mathbf{n}}\times\mathbf{h}, \tag{20}$$

$$\boldsymbol{\omega}=\frac{1}{2}(\mathbf{1}\times\mathbf{1})\bullet\left(\delta\mathbf{R}\mathbf{R}^{T}\right)=\frac{1}{2}\left(\mathbf{r}^{\alpha}\times\delta\mathbf{r}_{\alpha}+\overline{\mathbf{n}}\times\delta\overline{\mathbf{n}}\right),$$
$$\boldsymbol{\omega}_{\tau}=\frac{1}{2}(\mathbf{1}\times\mathbf{1})\bullet\left(\delta\mathbf{R}_{\tau}\mathbf{R}_{\tau}^{T}\right)=\frac{1}{2}\left(\overline{\boldsymbol{\nu}}\times\delta\overline{\boldsymbol{\nu}}+\overline{\boldsymbol{\tau}}\times\delta\overline{\boldsymbol{\tau}}+\overline{\mathbf{n}}\times\delta\overline{\mathbf{n}}\right). \tag{21}$$

Here $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_{\tau}$ are the virtual rotation vectors in the interior of each $\mathcal{M}^{(k)}$ and along each $\partial\mathcal{M}^{(k)}$, respectively. Please note that all the couple vectors $\mathbf{M}^{\beta}$, $\mathbf{M}^{*}$ and $\mathbf{m}$ in (20) do not have normal components, that is $\mathbf{M}^{\beta}\bullet\overline{\mathbf{n}}=\mathbf{M}^{*}\bullet\overline{\mathbf{n}}=\mathbf{m}\bullet\overline{\mathbf{n}}\equiv 0$. This is the fundamental property of the theory of thin shells resulting from the two basic postulates given above.

## 4. Local field equations

Let us transform the virtual work principle (13) with (19) keeping in mind that both **R** and **u** are now the independent field variables subject to variation.

According to [14], the virtual deformation measures $\delta\eta_{\alpha\beta}$ and $\delta\mu_{\alpha\beta}$ are expressible through $\delta\mathbf{u}$ and $\boldsymbol{\omega}$ by the relations

$$\delta\eta_{\lambda\beta}\mathbf{r}^{\lambda}=\delta\mathbf{u}_{,\beta}+\overline{\mathbf{a}}_{\beta}\times\boldsymbol{\omega},\quad \delta\mu_{\lambda\beta}\mathbf{r}^{\lambda}=\boldsymbol{\omega}_{,\beta}\times\overline{\mathbf{n}}. \tag{22}$$

Introducing (22) into $(19)_1$, the virtual work principle (13) takes the form

$$\sum_{k=1}^{K}\left\{\iint_{\mathcal{M}^{(k)}}\left[\mathbf{N}^{\beta}\bullet\left(\delta\mathbf{u}_{,\beta}+\overline{\mathbf{a}}_{\beta}\times\boldsymbol{\omega}\right)+\mathbf{M}^{\beta}\bullet\boldsymbol{\omega}_{,\beta}\right]dA+\int_{\partial\mathcal{M}^{(k)}}B\overline{\mathbf{n}}\bullet\left(\delta\mathbf{u}'+\overline{\mathbf{a}}_{\tau}\times\boldsymbol{\omega}\right)ds\right.$$
$$\left.-\iint_{\mathcal{M}^{(k)}}\left(\mathbf{p}\bullet\delta\mathbf{u}+\mathbf{m}\bullet\boldsymbol{\omega}\right)dA\right\}-\int_{\partial\mathcal{M}_f}\left(\mathbf{N}^{*}\bullet\delta\mathbf{u}+\mathbf{M}^{*}\bullet\boldsymbol{\omega}_{\tau}\right)ds-\int_{\Gamma}\sigma_{\Gamma}ds-\sum_{P_i\in\Gamma}\sigma_i=0. \tag{23}$$

The fields $\mathbf{N}^{\beta}$ and $\mathbf{M}^{\beta}$ are assumed to be of class $C^1$ in the interior of each regular surface element $\mathcal{M}^{(k)}$ and to have extensions of the same class to the boundary with finite limits at any $M \in \partial \mathcal{M}^{(k)}$. Then the Stokes theorem allows one to transform the first two surface integrals of (23) for each $\mathcal{M}^{(k)}$ into

$$\begin{aligned} & -\iint_{\mathcal{M}^{(k)}} \left\{ \mathbf{N}^{\beta}\,|_{\beta} \bullet \delta\mathbf{u} + \left( \mathbf{M}^{\beta}\,|_{\beta} + \overline{\mathbf{a}}_{\beta} \times \mathbf{N}^{\beta} \right) \bullet \boldsymbol{\omega} \right\} dA \\ & \qquad + \int_{\partial \mathcal{M}^{(k)}} \left\{ \mathbf{T}_{\nu} \bullet \delta\mathbf{u} + \mathbf{K}_{\nu} \bullet \boldsymbol{\omega} + (B\overline{\mathbf{n}} \bullet \delta\mathbf{u})' \right\} ds, \end{aligned} \tag{24}$$

where

$$\mathbf{T}_{\nu} = \mathbf{N}^{\beta}\nu_{\beta} - (B\overline{\mathbf{n}})', \quad \mathbf{K}_{\nu} = \mathbf{M}^{\beta}\nu_{\beta} - B\overline{\mathbf{a}}_{\nu}. \tag{25}$$

Along each $\partial \mathcal{M}^{(k)}$ there may be singular points $P_c, c = 1,2,\ldots,C,$ described by $s = s_c$, at which the field $B\overline{\mathbf{n}} \bullet \delta\mathbf{u}$ is not differentiable. Such singular points are, for example, corners of the closed curves composing $\partial \mathcal{M}^{(k)}$ or points of singularities of $B$, $\overline{\mathbf{n}}$ and $\delta\mathbf{u}$. At such singular points we assume the existence of finite limits of $B\overline{\mathbf{n}} \bullet \delta\mathbf{u}$ defined by

$$B_c^{\pm}\overline{\mathbf{n}}_c^{\pm} \bullet \delta\mathbf{u}_c^{\pm} = \lim_{h \to 0} \left\{ B(s_c \pm h)\overline{\mathbf{n}}(s_c \pm h) \bullet \delta\mathbf{u}(s_c \pm h) \right\}. \tag{26}$$

Then, the last term in the boundary line integral of (24) can be transformed further to give

$$\int_{\partial \mathcal{M}^{(k)}} (B\overline{\mathbf{n}} \bullet \delta\mathbf{u})' ds = -\sum_{P_c \in \partial \mathcal{M}^{(k)}} \left( B_c^{+}\overline{\mathbf{n}}_c^{+} \bullet \delta\mathbf{u}_c^{+} - B_c^{-}\overline{\mathbf{n}}_c^{-} \bullet \delta\mathbf{u}_c^{-} \right). \tag{27}$$

The second term of the boundary line integral in (24) contains the virtual rotation $\boldsymbol{\omega}$, which should still be expressed through the virtual rotation $\boldsymbol{\omega}_{\tau}$ of the boundary. Let us remind that along each $\partial \mathcal{M}^{(k)}$ the total rotation tensor $\mathbf{R}_{\tau}$ is defined as the superposition of two finite rotations, [14]:

$$\mathbf{R}_{\tau} = \mathbf{Q}_{V}\mathbf{R}, \quad \mathbf{Q}_{V} = \overline{\boldsymbol{\nu}} \otimes \mathbf{r}_{\nu} + \overline{\boldsymbol{\tau}} \otimes \mathbf{r}_{\tau} + \overline{\mathbf{n}} \otimes \overline{\mathbf{n}}, \tag{28}$$

where

$$\begin{aligned} & \mathbf{r}_{\nu} = \mathbf{r}_{\alpha}\nu^{\alpha} = \mathbf{R}\boldsymbol{\nu} = \frac{1}{a_{\tau}}\left\{ (1+\eta_{\tau\tau})\overline{\boldsymbol{\nu}} + \eta_{\nu\tau}\overline{\boldsymbol{\tau}} \right\}, \\ & \mathbf{r}_{\tau} = \mathbf{r}_{\alpha}\tau^{\alpha} = \mathbf{R}\boldsymbol{\tau} = \frac{1}{a_{\tau}}\left\{ -\eta_{\nu\tau}\overline{\boldsymbol{\nu}} + (1+\eta_{\tau\tau})\overline{\boldsymbol{\tau}} \right\}, \\ & \overline{\boldsymbol{\nu}} = \mathbf{Q}_{V}\mathbf{r}_{\nu}, \quad \overline{\boldsymbol{\tau}} = \mathbf{Q}_{V}\mathbf{r}_{\tau}, \quad \eta_{\nu\tau} = \eta_{\alpha\beta}\nu^{\alpha}\tau^{\beta}, \quad \eta_{\tau\tau} = \eta_{\alpha\beta}\tau^{\alpha}\tau^{\beta}. \end{aligned} \tag{29}$$

Therefore, taking variations of $\mathbf{R}_{\tau}$ defined either by $(9)_2$ or by $(28)_1$ we obtain

$$\begin{aligned}\delta\mathbf{R}_\tau\mathbf{R}_\tau^T &= \boldsymbol{\omega}_\tau\times\mathbf{1}\\ &= \left(\delta\mathbf{Q}_V\mathbf{Q}_V^T\right)\mathbf{Q}_V\mathbf{R}\mathbf{R}^T\mathbf{Q}_V^T+\mathbf{Q}_V\left(\delta\mathbf{R}\mathbf{R}^T\right)\mathbf{Q}_V^T\\ &= \left(\mathbf{q}+\mathbf{Q}_V\boldsymbol{\omega}\right)\times\mathbf{1},\end{aligned} \tag{30}$$

where

$$\mathbf{q}=\frac{1}{2}\left(\mathbf{1}\times\mathbf{1}\right)\bullet\left(\delta\mathbf{Q}_V\mathbf{Q}_V^T\right). \tag{31}$$

From $(30)_{1,2}$ it follows that

$$\boldsymbol{\omega}_\tau=\mathbf{q}+\mathbf{Q}_V\boldsymbol{\omega},\quad \boldsymbol{\omega}=\mathbf{Q}_V^T\left(\boldsymbol{\omega}_\tau-\mathbf{q}\right). \tag{32}$$

Let us evaluate more explicitly the formula $(32)_2$ for $\boldsymbol{\omega}$ at the boundary $\partial\overline{\mathcal{M}}^{(k)}$. Keeping in mind that $\mathbf{1}=\overline{\boldsymbol{\nu}}\otimes\overline{\boldsymbol{\nu}}+\overline{\boldsymbol{\tau}}\otimes\overline{\boldsymbol{\tau}}+\overline{\mathbf{n}}\otimes\overline{\mathbf{n}}$ along $\partial\overline{\mathcal{M}}^{(k)}$, and taking variation of $(28)_2$ we establish the relations

$$\begin{aligned}\mathbf{1}\times\mathbf{1} &= -\overline{\boldsymbol{\nu}}\otimes\left(\overline{\boldsymbol{\tau}}\otimes\overline{\mathbf{n}}-\overline{\mathbf{n}}\otimes\overline{\boldsymbol{\tau}}\right)-\overline{\boldsymbol{\tau}}\otimes\left(\overline{\mathbf{n}}\otimes\overline{\boldsymbol{\nu}}-\overline{\boldsymbol{\nu}}\otimes\overline{\mathbf{n}}\right)-\overline{\mathbf{n}}\otimes\left(\overline{\boldsymbol{\nu}}\otimes\overline{\boldsymbol{\tau}}-\overline{\boldsymbol{\tau}}\otimes\overline{\boldsymbol{\nu}}\right),\\ \delta\mathbf{Q}_V\mathbf{Q}_V^T &= \delta\overline{\boldsymbol{\nu}}\otimes\overline{\boldsymbol{\nu}}+\delta\overline{\boldsymbol{\tau}}\otimes\overline{\boldsymbol{\tau}}+\delta\overline{\mathbf{n}}\otimes\overline{\mathbf{n}}+\overline{\boldsymbol{\nu}}\otimes\overline{\boldsymbol{\tau}}\left(\delta\mathbf{r}_\nu\bullet\mathbf{r}_\tau\right)+\overline{\boldsymbol{\nu}}\otimes\overline{\mathbf{n}}\left(\delta\mathbf{r}_\nu\bullet\overline{\mathbf{n}}\right)\\ &\quad+\overline{\boldsymbol{\tau}}\otimes\overline{\boldsymbol{\nu}}\left(\delta\mathbf{r}_\tau\bullet\mathbf{r}_\nu\right)+\overline{\boldsymbol{\tau}}\otimes\overline{\mathbf{n}}\left(\delta\mathbf{r}_\tau\bullet\overline{\mathbf{n}}\right)+\overline{\mathbf{n}}\otimes\overline{\boldsymbol{\nu}}\left(\delta\overline{\mathbf{n}}\bullet\mathbf{r}_\nu\right)+\overline{\mathbf{n}}\otimes\overline{\boldsymbol{\tau}}\left(\delta\overline{\mathbf{n}}\bullet\mathbf{r}_\tau\right).\end{aligned} \tag{33}$$

Introducing (33) into (31) and taking into account that

$$\boldsymbol{\omega}_\tau=\overline{\boldsymbol{\nu}}\left(\boldsymbol{\omega}_\tau\bullet\overline{\boldsymbol{\nu}}\right)+\overline{\boldsymbol{\tau}}\left(\boldsymbol{\omega}_\tau\bullet\overline{\boldsymbol{\tau}}\right)+\overline{\mathbf{n}}\left(\boldsymbol{\omega}_\tau\bullet\overline{\mathbf{n}}\right), \tag{34}$$

after some transformations from $(32)_2$ we obtain

$$\boldsymbol{\omega}=\mathbf{r}_\nu\left(\boldsymbol{\omega}_\tau\bullet\mathbf{r}_\nu\right)+\mathbf{r}_\tau\left(\boldsymbol{\omega}_\tau\bullet\mathbf{r}_\tau\right)-\overline{\mathbf{n}}\left(\delta\mathbf{r}_\tau\bullet\mathbf{r}_\nu\right). \tag{35}$$

The relation (35) means that the virtual rotations $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\tau$ differ only by their normal components. But $\mathbf{K}_\nu$ in (25) does not have a normal component at all. Thus, using $(8)_1$, $(20)_1$, $(25)_2$ and (35) we are able to show that at the boundary

$$\mathbf{K}_\nu\bullet\boldsymbol{\omega}=\mathbf{K}_\nu\bullet\boldsymbol{\omega}_\tau. \tag{36}$$

The simple relation (36) just confirms that the theory of thin shells discussed here is insensitive to the virtual works done on the normal, drilling components of $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\tau$, for the corresponding drilling components of the couples are indefinite in this shell model. The virtual works done by the drilling couples can be taken into account only in the general theory of shells, [3,6,7].

With the help of (24), (27) and (36) the internal virtual work for the entire reference network $\mathcal{M}$ can be put in the form

$$G_{\mathrm{int}} = -\iint\limits_{\mathcal{M}\setminus\Gamma} \left\{ \mathbf{N}^{\beta}\,|_{\beta} \bullet \delta\mathbf{u} + \left( \mathbf{M}^{\beta}\,|_{\beta} + \bar{\mathbf{a}}_{\beta} \times \mathbf{N}^{\beta} \right) \bullet \boldsymbol{\omega} \right\} dA + \int\limits_{\partial\mathcal{M}} \left( \mathbf{T}_{\nu} \bullet \delta\mathbf{u} + \mathbf{K}_{\nu} \bullet \boldsymbol{\omega}_{\tau} \right) ds$$
$$+ \int\limits_{\Gamma} \left( [\![\mathbf{T}_{\nu} \bullet \delta\mathbf{u}]\!] + [\![\mathbf{K}_{\nu} \bullet \boldsymbol{\omega}_{\tau}]\!] \right) ds + \sum_{P_i \in \Gamma} [B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_i + \sum_{P_b \in \partial\mathcal{M}} [B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_b . \tag{37}$$

In (37) the jumps at each regular point $P \in \Gamma^{(a)}$ of the common curve $\Gamma^{(a)} = \partial\mathcal{M}^{(1)} \cap \partial\mathcal{M}^{(2)} \cap \ldots \cap \partial\mathcal{M}^{(n)}$ for $n \geq 2$ adjacent surface elements are defined by

$$[\![\mathbf{T}_{\nu} \bullet \delta\mathbf{u}]\!] = \pm \mathbf{T}_{\nu}^{(1)\pm} \bullet \delta\mathbf{u}^{(1)\pm} \pm \mathbf{T}_{\nu}^{(2)\pm} \bullet \delta\mathbf{u}^{(2)\pm} \pm \ldots \pm \mathbf{T}_{\nu}^{(n)\pm} \bullet \delta\mathbf{u}^{(n)\pm},$$
$$[\![\mathbf{K}_{\nu} \bullet \boldsymbol{\omega}_{\tau}]\!] = \pm \mathbf{K}_{\nu}^{(1)\pm} \bullet \boldsymbol{\omega}_{\tau}^{(1)\pm} \pm \mathbf{K}_{\nu}^{(2)\pm} \bullet \boldsymbol{\omega}_{\tau}^{(2)\pm} \pm \ldots \pm \mathbf{K}_{\nu}^{(n)\pm} \bullet \boldsymbol{\omega}_{\tau}^{(n)\pm}. \tag{38}$$

The numerical superscripts $(n)$ introduced into the right hand sides of (38) indicate explicitly that those functions are defined only along the particular $\partial\mathcal{M}^{(n)}$.

The signs in the definitions (38) must be chosen consistently with a fixed orientation of the curve $\Gamma^{(a)}$. If the orientation of $\Gamma^{(a)}$ coincides with the orientation of $\partial\mathcal{M}^{(n)}$, that is when the unit tangent vector $\boldsymbol{\tau}_{\Gamma}$ specifying the orientation of $\Gamma^{(a)}$ is related to $\boldsymbol{\nu}^{(n)}$ of $\partial\mathcal{M}^{(n)}$ by $\boldsymbol{\nu}^{(n)} = +\boldsymbol{\tau}_{\Gamma} \times \mathbf{n}^{(n)}$, the minus sign must be chosen in front of the corresponding term in (38), and the plus sign otherwise.

The jumps at all singular points of $\mathcal{M}$ have been divided in (37) into the jumps $[B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_i$ at the internal points $P_i \in \Gamma$ and the jumps $[B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_b$ at the boundary points $P_b \in \partial\mathcal{M}$. At each internal point $P_i$ being the common point of $m \geq 2$ adjacent branches $\Gamma^{(m)}$, as well as at each boundary point $P_b$ being the common point of $t \geq 2$ adjacent parts $\partial\mathcal{M}^{(t)}$ and $q$ adjacent branches $\Gamma^{(q)}$ approaching $P_b$ from inside of $\mathcal{M}$, the jumps are defined by

$$[B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_i = \pm B_i^{(1)\pm} \bar{\mathbf{n}}_i^{(1)\pm} \bullet \delta\mathbf{u}_i^{(1)\pm} \pm B_i^{(2)\pm} \bar{\mathbf{n}}_i^{(2)\pm} \bullet \delta\mathbf{u}_i^{(2)\pm} \pm \ldots \pm B_i^{(m)\pm} \bar{\mathbf{n}}_i^{(m)\pm} \bullet \delta\mathbf{u}_i^{(m)\pm},$$
$$[B\bar{\mathbf{n}} \bullet \delta\mathbf{u}]_b = \pm B_b^{(1)\pm} \bar{\mathbf{n}}_b^{(1)\pm} \bullet \delta\mathbf{u}_b^{(1)\pm} \pm B_b^{(2)\pm} \bar{\mathbf{n}}_b^{(2)\pm} \bullet \delta\mathbf{u}_b^{(2)\pm} \pm \ldots \pm B_b^{(t)\pm} \bar{\mathbf{n}}_b^{(t)\pm} \bullet \delta\mathbf{u}_b^{(t)\pm}$$
$$\pm B_i^{(1)\pm} \bar{\mathbf{n}}_i^{(1)\pm} \bullet \delta\mathbf{u}_b^{(1)\pm} \pm B_i^{(2)\pm} \bar{\mathbf{n}}_i^{(2)\pm} \bullet \delta\mathbf{u}_b^{(2)\pm} \pm \ldots \pm B_i^{(q)\pm} \bar{\mathbf{n}}_i^{(q)\pm} \bullet \delta\mathbf{u}_b^{(q)\pm}. \tag{39}$$

Here the numerical superscripts indicate that these functions are defined only either on a particular internal branches $\Gamma^{(m)}$ and $\Gamma^{(q)}$, or on a particular $\partial\mathcal{M}^{(t)}$ composing a part of the boundary $\partial\mathcal{M}$.

Introducing (37) with (38) and (39) into (23) we obtain

$$
\begin{aligned}
&-\iint_{\mathcal{M}\setminus\Gamma}\left\{\left(\mathbf{N}^{\beta}\,|_{\beta}+\mathbf{p}\right)\boldsymbol{\cdot}\delta\mathbf{u}+\left(\mathbf{M}^{\beta}\,|_{\beta}+\overline{\mathbf{a}}_{\beta}\times\mathbf{N}^{\beta}+\mathbf{m}\right)\boldsymbol{\cdot}\boldsymbol{\omega}\right\}ds\\
&+\int_{\partial\mathcal{M}_f}\left\{\left(\mathbf{T}_{\nu}-\mathbf{N}^{*}\right)\boldsymbol{\cdot}\delta\mathbf{u}+\left(\mathbf{K}_{\nu}-\mathbf{M}^{*}\right)\boldsymbol{\cdot}\boldsymbol{\omega}_{\tau}\right\}ds+\sum_{P_b\in\partial\mathcal{M}_f}[B\overline{\mathbf{n}}\boldsymbol{\cdot}\delta\mathbf{u}]_b\\
&+\int_{\partial\mathcal{M}_d}\left(\mathbf{T}_{\nu}\boldsymbol{\cdot}\delta\mathbf{u}+\mathbf{K}_{\nu}\boldsymbol{\cdot}\boldsymbol{\omega}_{\tau}\right)ds+\sum_{P_b\in\partial\mathcal{M}_d}[B\overline{\mathbf{n}}\boldsymbol{\cdot}\delta\mathbf{u}]_b\\
&+\int_{\Gamma}\left\{[\![\mathbf{T}_{\nu}\boldsymbol{\cdot}\delta\mathbf{u}]\!]+[\![\mathbf{K}_{\nu}\boldsymbol{\cdot}\boldsymbol{\omega}_{\tau}]\!]\right\}ds+\sum_{P_i\in\Gamma}[B\overline{\mathbf{n}}\boldsymbol{\cdot}\delta\mathbf{u}]_i-\int_{\Gamma}\sigma_{\Gamma}ds-\sum_{P_i\in\Gamma}\sigma_i=0.
\end{aligned}
\tag{40}
$$

For an arbitrary, but kinematically admissible, virtual deformation the fields $\delta\mathbf{u}$ and $\boldsymbol{\omega}_{\tau}$ vanish identically along $\partial\mathcal{M}_d$, and the third line of (40) vanishes as well. Then the virtual work principle (40) requires the following local relations to be satisfied:

the local equilibrium equations

$$\mathbf{N}^{\beta}\,|_{\beta}+\mathbf{p}=\mathbf{0},\quad \mathbf{M}^{\beta}\,|_{\beta}+\overline{\mathbf{a}}_{\beta}\times\mathbf{N}^{\beta}+\mathbf{m}=\mathbf{0}\quad \text{at each regular } M\in\mathcal{M}\setminus\Gamma\,; \tag{41}$$

the static boundary conditions

$$\mathbf{T}_{\nu}-\mathbf{N}^{*}=\mathbf{0},\quad \mathbf{K}_{\nu}-\mathbf{M}^{*}=\mathbf{0}\quad \text{along regular parts of } \partial\mathcal{M}_f\,; \tag{42}$$

the jump conditions

$$[B\overline{\mathbf{n}}\boldsymbol{\cdot}\delta\mathbf{u}]_b=0\quad \text{at each singular boundary point } P_b\in\partial\mathcal{M}_f. \tag{43}$$

The corresponding work-conjugate geometric boundary conditions are:

$$\mathbf{u}-\mathbf{u}^{*}=\mathbf{0},\quad \mathbf{R}_{\tau}\mathbf{n}-\mathbf{R}_{\tau}^{*}\mathbf{n}=\mathbf{0}\quad \text{along regular parts of } \partial\mathcal{M}_d. \tag{44}$$

As it has been expected, the local equilibrium equations (41) as well as the boundary conditions (42) and (44) for thin irregular shell-like structures coincide with those derived within the theory of thin regular shells expressed in terms of displacements and rotations as the primary variables (see [14], Section 5.2). However, in the jump conditions (43) the virtual displacements still remain coupled with the generalised forces, for in case of the general irregularity of deformation we may not be able to define a common $\delta\mathbf{u}_b$ associated with a singular boundary point $P_b\in\partial\mathcal{M}_f$.

## 5. Jump conditions along singular curves

If the local relations (41)-(44) are satisfied, the principle of virtual work still requires the last line of (40) to be satisfied identically for any part of $\Gamma$. This leads to the following local forms of the jump conditions:

$$[\![\mathbf{T}_{\nu}\boldsymbol{\cdot}\delta\mathbf{u}]\!]+[\![\mathbf{K}_{\nu}\boldsymbol{\cdot}\boldsymbol{\omega}_{\tau}]\!]-\sigma_{\Gamma}=0\quad \text{at regular points of } \Gamma\,; \tag{45}$$

$$[B\overline{\mathbf{n}}\boldsymbol{\cdot}\delta\mathbf{u}]_i-\sigma_i=0\quad \text{at each internal singular point } P_i\in\Gamma. \tag{46}$$

The jump conditions (45) and (46) constitute the additional set of basic relations that should be satisfied at the singular curves representing the irregularities of shell geometry, deformation, material properties and loading. The conditions are valid for unrestricted displacements, rotations, strains and/or bendings of the reference network $\mathcal{M}$.

The singular curves $\Gamma^{(a)}$ embedded into the shell reference network $\mathcal{M}$ may be of either geometric or physical type, in general. At the geometric curve some fields in the relations (45) or (46) fail to be continuous or smooth of the required class. With the physical curve we can additionally associate some mechanical properties by prescribing appropriate functions $\sigma_\Gamma = \sigma_\Gamma(\mathbf{u}_\Gamma, \mathbf{R}_\Gamma; \delta\mathbf{u}_\Gamma, \boldsymbol{\omega}_\Gamma)$ and $\sigma_i = \sigma_i(\mathbf{u}_i; \delta\mathbf{u}_i)$ along $\Gamma^{(a)}$. For special types of irregularities the jump conditions (45) and (46) can be simplified or presented in a more explicit uncoupled form along the lines suggested in [2] for the displacement formulation of the non-linear theory of thin irregular shells. Such particular forms of the functions $\sigma_\Gamma$ and $\sigma_i$ as well as special cases of the jump conditions will be discussed separately.

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