Zastosowania mechaniki w budownictwie lądowym i wodnym

Księga Jubileuszowa poświęcona
70-leciu urodzin Profesora Piotra Wilde

On Refined Intrinsic Shell Equations in the Rotated Basis

Wojciech Pietraszkiewicz

Institute of Fluid-Flow Machinery of the Polish Academy of Sciences,
ul. Fiszera 14, 80–952 Gdańsk

Dedicated to Prof. P. Wilde on his 70th birthday

Abstract

O uściślonych równaniach wewnętrznych powłok w bazie obróconej

Równania równowagi i warunki ciągłości odkształceń powłok cienkich zostały sformułowane względem pośredniej bazy nieholonomicznej, określonej przez sztywny obrót bazy naturalnej powierzchni podstawowej powłoki nieodkształconej. Stosując oszacowania błędu wprowadzanego do tych równań w ramach pierwszego przybliżenia do gęstości energii odkształcenia sprężystego powłoki, wyprowadzono konsekwentnie uproszczony układ sześciu nieliniowych równań typu wewnętrznego. Te równania wyrażone zostały przez trzy wypadkowe siły membranowe i trzy zmiany krzywizny powierzchni podstawowej jako jedyne zmienne niezależne. To pozwoliło na uściślenie czterech z sześciu równań, w porównaniu do podobnego układu sześciu równań wewnętrznych, lecz wyrażonego tylko przez sześć miar odkształceń lub sześć miar naprężeń powierzchni podstawowej powłoki.

1. Introduction

Geometrically non-linear problems of thin isotropic elastic shells can be formulated and analysed using different fields as independent variables of the boundary value problem (BVP), see Pietraszkiewicz (1989, 2001a). The displacement form of shell relations is used most often in the literature, but it is very complex for unrestricted rotations and requires C^1 interelement compatibility if the finite element method is applied in the analysis. Non-linear shell equations expressed through rotations and other fields can effectively be applied to one-dimensional shell problems. However, in two-dimensional problems the configuration space of such a BVP contains the rotation group $SO(3)$ and then a non-standard analysis on $SO(3)$ is required.

Wyd. IBW PAN, Gdańsk 2001, 217–231

Displacements and/or rotations may not be needed in some problems of flexible shells. It may be enough to know only the surface strain and/or stress measures, or even only some global characteristics expressed through these measures such as, for example, the strain energy corresponding to shell buckling or eigen frequencies of vibration of a deformed shell. Such an intrinsic formulation of non-linear shell equations was given first by Chien (1944) through Green type surface strains and changes of curvatures as independent field variables. The intrinsic shell equations were then developed, classified and expressed through various modified surface strain and/or stress measures in several papers reviewed by Pietraszkiewicz (1989). In such intrinsic BVPs the equilibrium equations and compatibility conditions form together the fundamental system of six non-linear partial differential equations. In particular, Danielson (1970) proposed to choose membrane stress resultants and changes of curvatures as primary field variables, which allowed him to derive a refined set of intrinsic shell equations. Modified sets of refined intrinsic shell equations were discussed by Koiter and Simmonds (1973), Pietraszkiewicz (1977, 1979, 1980), Simmonds (1979), and Libai and Simmonds (1983).

In this report an alternative system of refined intrinsic equations for the geometrically non-linear theory of thin isotropic elastic shells is discussed. The shell equations are written in components relative to the rotated base vectors introduced by Alumäe (1949). The equations are expressed through modified membrane stress resultants and changes of curvatures associated with the rotated basis. The system was presented concisely already in Pietraszkiewicz (1989), eqs. (6.24), but the review type style of that paper did not permit to give there many details of the derivation process itself. Here the alternative refined intrinsic shell equations (43) are derived in detail, with all necessary intermediate transformations.

The paper is organised as follows. In Section 2 we remind notation and present basic definitions associated with description of deformation of the surface in the non-holonomic rotated basis. Equilibrium conditions presented in Section 3 are derived from the principle of virtual work (10) postulated for the shell reference surface. Introducing there three constraints (14) with the help of Lagrange multipliers, modified equilibrium conditions associated with the rotated basis are given in the weak (19) and strong (20)–(23) forms, with six scalar equilibrium equations in the rotated basis presented in (24). Compatibility conditions for relative surface strain measures are derived in Section 4, and their component form in the rotated basis is given in (29). In Section 5 we remind constitutive equations of a homogeneous isotropic elastic shell undergoing small strains, and indicate their error estimates within the first approximation to the elastic strain energy density. Finally, in Section 6 the modified refined intrinsic shell equations (43) are derived and their distinguishing properties are discussed.

2. Surface Geometry and Deformation

In this report we shall apply primarily the system of notation used in Pietraszkiewicz (1989) and remind here only basic relations.

Let $\mathcal{M}$ be a connected, oriented and regular surface of class C^n, $n \geq 2$, in the three-dimensional Euclidean point space $\mathcal{E}$ whose translation (three-dimensional vector) space is E. The position vector of a point $M \in \mathcal{M}$ is given by

$$\mathbf{r} = \overrightarrow{OM} = \mathbf{r}(\theta^\alpha), \tag{1}$$

where $O \in \mathcal{E}$ is a reference origin and θ^α, $\alpha = 1, 2$, are surface co-ordinates. At $M \in \mathcal{M}$ we have the natural base vectors $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\theta^\alpha \equiv \mathbf{r},_\alpha$, the dual base vectors $\mathbf{a}^\beta$ such that $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha$, where δ^β_α is the Kronecker symbol, the components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the unit normal vector $\mathbf{n} = (1/\sqrt{a})\mathbf{a}_1 \times \mathbf{a}_2$ orienting $\mathcal{M}$, the components $b_{\alpha\beta} = -\mathbf{n},_\alpha \cdot\mathbf{a}_\beta$ of the surface curvature tensor $\mathbf{b}$, and the components $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ of the surface permutation tensor such that $\varepsilon_{12} = -\varepsilon_{21} = \sqrt{a}$, $\varepsilon_{11} = \varepsilon_{22} = 0$.

The boundary $\partial\mathcal{M}$ of $\mathcal{M}$ consists of a finite number of closed, piecewise smooth curves that do not meet in cusps, each described parametrically by $\mathbf{r}(s) = \mathbf{r}[\theta^\alpha(s)]$, where s is the arc length along any regular part of $\partial\mathcal{M}$. At each regular point $M \in \partial\mathcal{M}$ we have the unit tangent vector $\boldsymbol{\tau} = d\mathbf{r}/ds \equiv \mathbf{r}' = \tau^\alpha\mathbf{a}_\alpha$ and the outward unit normal vector $\boldsymbol{\nu} = \mathbf{r},_\nu = \boldsymbol{\tau} \times \mathbf{n} = \nu^\alpha\mathbf{a}_\alpha$, where $(\cdot),_\nu$ is the external surface derivative normal to $\partial\mathcal{M}$.

The deformed surface $\overline{\mathcal{M}}$ with boundary $\partial\overline{\mathcal{M}}$ is described relative to the same origin $O \in \mathcal{E}$ by the relations

$$\bar{\mathbf{r}}(\theta^\alpha) = \boldsymbol{\chi}[\mathbf{r}(\theta^\alpha)] = \mathbf{r}(\theta^\alpha) + \mathbf{u}(\theta^\alpha), \quad \bar{\mathbf{r}}(s) = \boldsymbol{\chi}[\mathbf{r}(s)] = \mathbf{r}(s) + \mathbf{u}(s), \tag{2}$$

where θ^α and s are convected surface co-ordinates, $\boldsymbol{\chi} : \mathcal{M} \to \overline{\mathcal{M}}$ is the deformation function, and $\mathbf{u} \in E$ is the displacement vector.

In the convected surface co-ordinates all geometric relations at any regular $\overline{M} \in \partial\overline{\mathcal{M}}$ are now analogous to those given at $M \in \partial\mathcal{M}$, and are expressed by quantities marked by a dash: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{a}}^\beta$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$, $\bar{\mathbf{a}}$, $\bar{a}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\boldsymbol{\nu}}$, $\bar{\boldsymbol{\tau}}$ etc. The dashed quantities can be expressed through analogous quantities defined on $\mathcal{M}$ and the displacement field $\mathbf{u}$ with the help of formulae given in Pietraszkiewicz (1989).

Components of the Green type surface deformation measures are defined by

$$\gamma_{\alpha\beta}(\mathbf{u}) = \frac{1}{2}\left(\bar{a}_{\alpha\beta} - a_{\alpha\beta}\right), \quad \kappa_{\alpha\beta}(\mathbf{u}) = -\left(\bar{b}_{\alpha\beta} - b_{\alpha\beta}\right), \tag{3}$$

where $\gamma_{\alpha\beta}(\mathbf{u})$ are quadratic polynomials of $\mathbf{u}$, $\mathbf{u},_\alpha$, and $\kappa_{\alpha\beta}(\mathbf{u})$ are non-rational functions of $\mathbf{u}$, $\mathbf{u},_\alpha$, $\mathbf{u},_{\alpha\beta}$.

In the neighbourhood of regular surfaces $\mathcal{M}$ and $\overline{\mathcal{M}}$ the space $\mathcal{E}$ can be parameterised by the normal system of convected co-ordinates $(\theta^\alpha, \varsigma)$, where ς is the distance from $\mathcal{M}$ and $\overline{\mathcal{M}}$ along $\mathbf{n}$ and $\bar{\mathbf{n}}$, respectively. Extending the domain of $\boldsymbol{\chi}$ to the neighbourhood of $\mathcal{M}$, the spatial deformation gradient $\mathbf{F} : E \to E$ taken at the surface $\mathcal{M}$ has the form

$$\begin{aligned} &\mathbf{F} = \nabla\boldsymbol{\chi}(\mathbf{r} + \varsigma\mathbf{n})|_{\varsigma=0} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \det\mathbf{F} = \sqrt{\frac{\bar{a}}{a}} > 0, \\ &\frac{\bar{a}}{a} = 1 + 2\gamma^\alpha_\alpha + 2\left(\gamma^\alpha_\alpha\gamma^\beta_\beta - \gamma^\alpha_\beta\gamma^\beta_\alpha\right), \end{aligned} \tag{4}$$

where $\otimes$ is the tensor product.

The left polar decomposition of $\mathbf{F}$ gives

$$\mathbf{F} = \mathbf{V}\mathbf{R}, \quad \mathbf{r}_\alpha = \mathbf{R}\mathbf{a}_\alpha = \mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha. \tag{5}$$

Here $\mathbf{R} \in SO(3)$ is the rotation tensor, $\mathbf{V}$ is the left spatial stretch tensor at $\mathcal{M}$, and $\mathbf{r}_\alpha$ are the rotated surface non-holonomic base vectors. These fields satisfy the relations

$$\begin{aligned} &\mathbf{R} = \mathbf{r}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{R}^T = \mathbf{R}^{-1}, \quad \det\mathbf{R} = +1, \\ &\mathbf{V} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{r}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{V}^T = \mathbf{V}, \qquad \det\mathbf{V} = \sqrt{\frac{\bar{a}}{a}} > 1. \end{aligned} \tag{6}$$

The relative surface strain measures associated with the basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$ are introduced through the following formulae:

$$\begin{aligned} &\boldsymbol{\eta} = \mathbf{V} - \mathbf{1} = \left(\mathbf{a}_\beta + \mathbf{u},_\beta - \mathbf{r}_\beta\right) \otimes \mathbf{r}^\beta = \boldsymbol{\eta}_\beta \otimes \mathbf{r}^\beta, \quad \boldsymbol{\eta}_\beta = \eta_{\alpha\beta}\mathbf{r}^\alpha, \\ &\boldsymbol{\mu} = \left(\bar{\mathbf{n}},_\beta - \mathbf{R}\mathbf{n},_\beta\right) \otimes \mathbf{r}^\beta = \mathbf{R},_\beta \mathbf{n} \otimes \mathbf{r}^\beta = \boldsymbol{\mu}_\beta \otimes \mathbf{r}^\beta, \quad \boldsymbol{\mu}_\beta = \mu_{\alpha\beta}\mathbf{r}^\alpha, \\ &\eta_{\alpha\beta} = \eta_{\beta\alpha}, \quad \mu_{\alpha\beta} \neq \mu_{\beta\alpha}. \end{aligned} \tag{7}$$

Here $\mathbf{1} = \mathbf{a}_\alpha \otimes \mathbf{a}^\alpha + \mathbf{n} \otimes \mathbf{n} = \mathbf{r}_\alpha \otimes \mathbf{r}^\alpha + \bar{\mathbf{n}} \otimes \bar{\mathbf{n}}$ is the spatial identity tensor.

Along the boundary $\partial\overline{\mathcal{M}}$ we have

$$\begin{aligned} &\bar{\mathbf{a}}_\tau \equiv \bar{\mathbf{r}}' = a_\tau\bar{\boldsymbol{\tau}}, \quad \bar{\mathbf{a}}_\nu = \bar{\mathbf{a}}_\tau \times \bar{\mathbf{n}} = a_\tau\bar{\boldsymbol{\nu}}, \quad \bar{\mathbf{n}} = \sqrt{\frac{a}{\bar{a}}}\bar{\mathbf{r}},_\nu \times \bar{\mathbf{r}}', \\ &a_\tau = |\bar{\mathbf{r}}'| = \sqrt{1 + 2\gamma_{\tau\tau}} \;, \quad \gamma_{\tau\tau} = \gamma_{\alpha\beta}\tau^\alpha\tau^\beta. \end{aligned} \tag{8}$$

During shell deformation the transformation of $(\boldsymbol{\nu}, \boldsymbol{\tau}, \mathbf{n})$ into $(\bar{\mathbf{a}}_\nu, \bar{\mathbf{a}}_\tau, \bar{\mathbf{n}})$ is performed in two steps: the rotation of $(\boldsymbol{\nu}, \boldsymbol{\tau}, \mathbf{n})$ into $(\bar{\boldsymbol{\nu}}, \bar{\boldsymbol{\tau}}, \bar{\mathbf{n}})$ by the total rotation tensor $\mathbf{R}_\tau$ with the subsequent extension of $\bar{\boldsymbol{\nu}}$, $\bar{\boldsymbol{\tau}}$ into $\bar{\mathbf{a}}_\nu$, $\bar{\mathbf{a}}_\tau$ by the factor a_τ:

$$\begin{aligned} &\bar{\mathbf{a}}_\nu = a_\tau\mathbf{R}_\tau\boldsymbol{\nu}, \quad \bar{\mathbf{a}}_\tau = a_\tau\mathbf{R}_\tau\boldsymbol{\tau}, \quad \bar{\mathbf{n}} = \mathbf{R}_\tau\mathbf{n}, \\ &\mathbf{R}_\tau = \frac{1}{a_\tau}\left(\bar{\mathbf{a}}_\nu \otimes \boldsymbol{\nu} + \bar{\mathbf{a}}_\tau \otimes \boldsymbol{\tau}\right) + \bar{\mathbf{n}} \otimes \mathbf{n}. \end{aligned} \tag{9}$$

Kinematic relations involving the tensors $\boldsymbol{\eta}$, $\boldsymbol{\mu}$, $\mathbf{R}$ and $\mathbf{R}_\tau$ are given in Pietraszkiewicz (1989).

3. Equilibrium Conditions

Let $\overline{\mathcal{M}}$ be the reference surface of a thin shell in an equilibrium state under the action of the resultant surface force $\mathbf{p}(\theta^\alpha)$ and static moment $\mathbf{h}(\theta^\alpha)$ vectors, both measured per unit area of the undeformed surface $\mathcal{M}$, and the resultant boundary force $\mathbf{N}^*(s)$ and static moment $\mathbf{H}^*(s)$ vectors, both measured per unit length of the undeformed boundary $\partial\mathcal{M}$. We postulate that for all kinematically admissible virtual displacements $\delta\mathbf{u} : \mathcal{M} \to E$ the following principle of virtual work (PVW) is satisfied:

$$\begin{aligned}\iint\limits_{\mathcal{M}} \left(N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta}\right) dA = \iint\limits_{\mathcal{M}} (\mathbf{p}\cdot\delta\mathbf{u} + \mathbf{h}\cdot\delta\bar{\mathbf{n}})\, dA \\ + \int\limits_{\partial\mathcal{M}_f} (\mathbf{N}^*\cdot\delta\mathbf{u} + \mathbf{H}^*\cdot\delta\bar{\mathbf{n}}) ds.\end{aligned} \tag{10}$$

Here $N^{\alpha\beta}$ and $M^{\alpha\beta}$ are components of the symmetric stress resultant and stress couple tensors of the Piola-Kirchhoff type, δ is the symbol of variation, while $\delta\gamma_{\alpha\beta}$ and $\delta\kappa_{\alpha\beta}$ are virtual changes of the surface strain measures (3). Since $\bar{\mathbf{n}}\cdot\delta\bar{\mathbf{n}} = 0$, only the surface components of $\mathbf{h}$ and $\mathbf{H}^*$ can explicitly be taken into account in the non-linear theory of thin shells discussed here. With $\mathbf{u}$ as the only independent field variable we can transform (10) with the help of Stokes' theorem and derive the corresponding three local equilibrium equations in $\mathcal{M}$, four natural boundary conditions on $\partial\mathcal{M}_f$ and three jump conditions at each corner point $M_i \in \partial\mathcal{M}_f$, see Pietraszkiewicz (1989), Chapter 1.

In this report we are interested in deriving non-linear shell relations expressed through modified surface strain and/or stress measures, associated with the rotated basis $\mathbf{r}_\alpha, \bar{\mathbf{n}}$, as independent field variables. This can be done by allowing first the rotation tensor $\mathbf{R}$ to be an additional independent field variable of the BVP. According to Pietraszkiewicz (1989), eqn. (5.11), the surface strain measures (3) and (7) are related by

$$\begin{aligned}\gamma_{\alpha\beta} &= \eta_{\alpha\beta} + \frac{1}{2}\eta^\lambda_\alpha\eta_{\lambda\beta},\\ \kappa_{\alpha\beta} &= \frac{1}{2}\left[\left(\delta^\lambda_\alpha + \eta^\lambda_\alpha\right)\mu_{\lambda\beta} + \left(\delta^\lambda_\beta + \eta^\lambda_\beta\right)\mu_{\lambda\alpha}\right] - \frac{1}{2}\left(b^\lambda_\alpha\eta_{\lambda\beta} + b^\lambda_\beta\eta_{\lambda\alpha}\right).\end{aligned} \tag{11}$$

As a result, the internal virtual work density appearing in the left hand side of (10) can be presented in an alternative form

$$\begin{aligned}N^{\alpha\beta}\delta\gamma_{\alpha\beta} + M^{\alpha\beta}\delta\kappa_{\alpha\beta} &= S^{\alpha\beta}\delta\eta_{\alpha\beta} + H^{\alpha\beta}\delta\mu_{\alpha\beta},\\ S^{\alpha\beta} &= N^{\alpha\beta} + \frac{1}{2}\left(\eta^\alpha_\lambda N^{\lambda\beta} + \eta^\beta_\lambda N^{\alpha\lambda}\right)\\ &\quad -\frac{1}{2}\left[\left(b^\alpha_\lambda - \mu^\alpha_{.\lambda}\right)M^{\lambda\beta} + \left(b^\beta_\lambda - \mu^\beta_{.\lambda}\right)M^{\alpha\lambda}\right],\\ H^{\alpha\beta} &= \left(\delta^\alpha_\lambda + \eta^\alpha_\lambda\right)M^{\lambda\beta},\end{aligned} \tag{12}$$

where now $S^{\alpha\beta} = S^{\beta\alpha}$, but $H^{\alpha\beta} \neq H^{\beta\alpha}$, in general.

The dependence of $\mathbf{R}$ upon $\mathbf{u}$, $\mathbf{u},_\alpha$ can also be expressed implicitly through three constraint conditions, see Badur and Pietraszkiewicz (1986),

$$\varepsilon^{\alpha\beta}\mathbf{r}_\alpha \cdot \eta_{\lambda\beta}\mathbf{r}^\lambda = 0, \quad \bar{\mathbf{n}} \cdot \eta_{\lambda\beta}\mathbf{r}^\lambda = 0. \tag{13}$$

These constraints express the known property of the relative surface strain tensor $\boldsymbol{\eta}$, which in thin shell theory is symmetric and does not have out-of-surface components. The property was also confirmed by Libai and Simmonds (1998) who used the constitutive Kirchhoff hypothesis to define the classical theory of thin shells as a special case of the general shell theory. The relations (13) put the following constraints on the virtual changes $\delta\eta_{\alpha\beta}$ of $\eta_{\alpha\beta}$:

$$\varepsilon^{\alpha\beta}\mathbf{r}_\alpha \cdot \delta\eta_{\lambda\beta}\mathbf{r}^\lambda = 0, \quad \bar{\mathbf{n}} \cdot \delta\eta_{\lambda\beta}\mathbf{r}^\lambda = 0. \tag{14}$$

The constraints (14) can be introduced into the surface integral of (10) with the help of respective Lagrange multipliers S and Q^β. In order to express also boundary terms at each $\partial\mathcal{M}$ explicitly through independent rotations, it is necessary to introduce into (10) an additional line integral over $\partial\mathcal{M}$ with the constraints $(14)_2$ multiplied by $B\tau^\beta$, (Pietraszkiewicz 1989). The external virtual work done by the moments $\mathbf{h}$ and $\mathbf{H}^*$ should be expressed in (10) directly in terms of now independent virtual rotations. As a result, (10) can be modified into the form

$$\begin{aligned}
&\iint\limits_{\mathcal{M}} \left(\mathbf{N}^\beta \cdot \delta\eta_{\lambda\beta}\mathbf{r}^\lambda + H^{\alpha\beta}\mathbf{r}_\alpha \cdot \delta\mu_{\lambda\beta}\mathbf{r}^\lambda\right) dA + \int\limits_{\partial\mathcal{M}} B\tau^\beta \bar{\mathbf{n}} \cdot \delta\eta_{\lambda\beta}\mathbf{r}^\lambda ds \\
&\quad = \iint\limits_{\mathcal{M}} (\mathbf{p} \cdot \delta\mathbf{u} + \mathbf{m} \cdot \boldsymbol{\omega})\, dA + \int\limits_{\partial\mathcal{M}_f} (\mathbf{N}^* \cdot \delta\mathbf{u} + \mathbf{M}^* \cdot \boldsymbol{\omega}_\tau) ds,
\end{aligned} \tag{15}$$

where now

$$\begin{aligned}
&\mathbf{N}^\beta = R^{\alpha\beta}\mathbf{r}_\alpha + Q^\beta\bar{\mathbf{n}}, \quad \mathbf{M}^\beta = \bar{\mathbf{n}} \times H^{\alpha\beta}\mathbf{r}_\alpha, \\
&\mathbf{M}^* = \bar{\mathbf{n}} \times \mathbf{H}^*, \quad \mathbf{m} = \bar{\mathbf{n}} \times \mathbf{h}, \quad R^{\alpha\beta} = S^{\alpha\beta} + \varepsilon^{\alpha\beta} S,
\end{aligned} \tag{16}$$

$$\begin{aligned}
\boldsymbol{\omega} &= \frac{1}{2}(\mathbf{1} \times \mathbf{1}) \cdot \left(\delta\mathbf{R}\mathbf{R}^T\right) = \frac{1}{2}\left(\mathbf{r}^\alpha \times \delta\mathbf{r}_\alpha + \bar{\mathbf{n}} \times \delta\bar{\mathbf{n}}\right), \\
\boldsymbol{\omega}_\tau &= \frac{1}{2}(\mathbf{1} \times \mathbf{1}) \cdot \left(\delta\mathbf{R}_\tau\mathbf{R}_\tau^T\right) = \frac{1}{2}(\bar{\boldsymbol{\nu}} \times \delta\bar{\boldsymbol{\nu}} + \bar{\boldsymbol{\tau}} \times \delta\bar{\boldsymbol{\tau}} + \bar{\mathbf{n}} \times \delta\bar{\mathbf{n}}).
\end{aligned} \tag{17}$$

Here $\boldsymbol{\omega}$ and $\boldsymbol{\omega}_\tau$ are virtual rotation vectors in the interior of $\mathcal{M}$ and along $\partial\mathcal{M}$, respectively.

Please note that all couple vectors $\mathbf{M}^\beta$, $\mathbf{M}^*$ and $\mathbf{m}$ in (15) do not have normal components, that is $\mathbf{M}^\beta \cdot \bar{\mathbf{n}} = \mathbf{M}^* \cdot \bar{\mathbf{n}} = \mathbf{m} \cdot \bar{\mathbf{n}} \equiv 0$. This is the fundamental property of the theory of thin shells resulting from the postulated PVW, (10) or (15).

Let us transform the virtual work principle (15) taking into account that the virtual surface strain measures $\delta\eta_{\alpha\beta}$ and $\delta\mu_{\alpha\beta}$ are expressible through $\delta\mathbf{u}$ and $\boldsymbol{\omega}$ by the relations

$$\delta\eta_{\lambda\beta}\mathbf{r}^\lambda = \delta\mathbf{u},_\beta + \bar{\mathbf{a}}_\beta \times \boldsymbol{\omega}, \quad \delta\mu_{\lambda\beta}\mathbf{r}^\lambda = \boldsymbol{\omega},_\beta \times \bar{\mathbf{n}}. \tag{18}$$

Introducing (18) into (15) the PVW can be transformed into the form (see Pietraszkiewicz 2001b for an additional transformation at the boundary $\partial\mathcal{M}$)

$$
\begin{aligned}
&-\iint\limits_{\mathcal{M}} \left\{ \left(\mathbf{N}^{\beta}|_{\beta} + \mathbf{p}\right) \cdot \delta\mathbf{u} + \left(\mathbf{M}^{\beta}|_{\beta} + \bar{\mathbf{a}}_{\beta} \times \mathbf{N}^{\beta} + \mathbf{m}\right) \cdot \boldsymbol{\omega} \right\} dA \\
&+ \int\limits_{\partial\mathcal{M}_f} \left\{ \left(\mathbf{N}^{\beta}\nu_{\beta} - (B\bar{\mathbf{n}})' - \mathbf{N}^{*}\right) \cdot \delta\mathbf{u} + \left(\mathbf{M}^{\beta}\nu_{\beta} + B\bar{\mathbf{a}}_{\nu} - \mathbf{M}^{*}\right) \cdot \boldsymbol{\omega}_{\tau} \right\} ds \\
&- \sum_{M_i \in \partial\mathcal{M}_f} [B\bar{\mathbf{n}}]_i \cdot \delta\mathbf{u}_i \\
&+ \int\limits_{\partial\mathcal{M}_d} \left\{ \left(\mathbf{N}^{\beta}\nu_{\beta} - (B\bar{\mathbf{n}})'\right) \cdot \delta\mathbf{u} + \left(\mathbf{M}^{\beta}\nu_{\beta} + B\bar{\mathbf{a}}_{\nu}\right) \cdot \boldsymbol{\omega}_{\tau} \right\} ds = 0.
\end{aligned}
\tag{19}
$$

For an arbitrary, but kinematically admissible, virtual deformation the fields $\delta\mathbf{u}$ and $\boldsymbol{\omega}_{\tau}$ vanish identically along $\partial\mathcal{M}_d$, and the fourth line of (19) vanishes as well. Then the principle of virtual work (19) requires the following local relations to be satisfied:

the local equilibrium equations

$$
\mathbf{N}^{\beta}|_{\beta} + \mathbf{p} = \mathbf{0}, \quad \mathbf{M}^{\beta}|_{\beta} + \bar{\mathbf{a}}_{\beta} \times \mathbf{N}^{\beta} + \mathbf{m} = \mathbf{0} \ \text{ at each regular } \ M \in \mathcal{M}; \tag{20}
$$

the static boundary conditions

$$
\begin{gathered}
\mathbf{N}^{\beta}\nu_{\beta} - (B\bar{\mathbf{n}})' - \mathbf{N}^{*} = \mathbf{0}, \quad \mathbf{M}^{\beta}\nu_{\beta} + B\bar{\mathbf{a}}_{\nu} - \mathbf{M}^{*} = \mathbf{0} \\
\text{along regular parts of } \ \partial\mathcal{M}_f;
\end{gathered}
\tag{21}
$$

the jump conditions

$$
[B\bar{\mathbf{n}}]_i \cdot \delta\mathbf{u}_i = 0 \ \text{ at each singular point } \ P_i \in \partial\mathcal{M}_f; \tag{22}
$$

The corresponding work-conjugate geometric boundary conditions are

$$
\mathbf{u} - \mathbf{u}^{*} = \mathbf{0}, \quad \mathbf{R}_{\tau}\mathbf{n} - \mathbf{R}_{\tau}^{*}\mathbf{n} = \mathbf{0} \ \text{ along } \ \partial\mathcal{M}_d. \tag{23}
$$

The relations (20)–(23) extend a similar set of shell relations (5.57)–(5.60) of Pietraszkiewicz (1989) derived for the geometrically non-linear theory of thin elastic shells. Only symmetric parts of $\mu_{\alpha\beta}$ and $H^{\alpha\beta}$ were present in the equilibrium conditions of that paper, for the contribution of the skew-symmetric part of $\mu_{\alpha\beta}$ in the elastic strain energy density was found to be negligible within the error of the first-approximation theory. The relations (20)–(23) are derived here without using any constitutive equations yet. Therefore, they have a wider range of applicability.

The equilibrium equations (20) seem to be formally identical with those already used by Alumäe (1949, 1956) and Simmonds and Danielson (1970, 1972) in their approach to thin shell theory formulated in the rotated basis. However, the physical meaning of similar symbols in (20) and in those papers is different.

The shell equilibrium equations given in those papers as well as the ones given earlier by Chien (1944) were derived by through-the-thickness integration of 3D equilibrium equations of continuum mechanics. According to Libai and Simmonds (1983, 1998) the work-conjugate shell kinematics associated with such shell equilibrium equations consists of the displacement vector **u** and the rotation tensor **Q** as two independent field variables. In such a general shell theory 2D constitutive equations should be provided for all six components of $\mathbf{N}^{\beta}$ and $\mathbf{M}^{\beta}$. Our equilibrium equations (20) have been generated by the 2D principle of virtual work (10) of thin shell theory, where the displacement vector **u** is the only independent field variable, and $\mathbf{R} = \mathbf{R}(\mathbf{u}, \mathbf{u},_{\alpha})$. In order to treat also **R** as an additional independent field variable, the set of Lagrange multipliers S, Q^{β}, B had to be introduced which are additional independent field variables. In our formulation of thin shell theory 2D constitutive equations should be provided only for three $S^{\alpha\beta}$ and four $H^{\alpha\beta}$.

When expressed by components in the rotated basis $\mathbf{r}_{\alpha}$, $\bar{\mathbf{n}}$ the vector relations (20) lead to six scalar local equilibrium equations

$$\begin{aligned}
&H^{\alpha\beta}|_{\beta} - \varepsilon^{\alpha\lambda} H_{\lambda}^{\cdot\beta} k_{\beta} - Q^{\beta}\left(\delta_{\beta}^{\alpha} + \eta_{\beta}^{\alpha}\right) + \hat{m}^{\alpha} = 0,\\
&\varepsilon_{\alpha\lambda} H^{\alpha\beta}\left(b_{\beta}^{\lambda} - \mu_{\cdot\beta}^{\lambda}\right) - \varepsilon_{\alpha\lambda} R^{\alpha\beta}\left(\delta_{\beta}^{\lambda} + \eta_{\beta}^{\lambda}\right) = 0,\\
&R^{\alpha\beta}|_{\beta} - Q^{\beta}\left(b_{\beta}^{\alpha} - \mu_{\cdot\beta}^{\alpha}\right) - \varepsilon^{\alpha\lambda} R_{\lambda}^{\cdot\beta} k_{\beta} + \hat{p}^{\alpha} = 0,\\
&Q^{\beta}|_{\beta} + R^{\alpha\beta}\left(b_{\alpha\beta} - \mu_{\alpha\beta}\right) + p = 0,
\end{aligned} \tag{24}$$

where now $\mathbf{p} = \hat{p}^{\alpha}\mathbf{r}_{\alpha} + p\bar{\mathbf{n}}, \quad \mathbf{m} = \bar{\mathbf{n}} \times \hat{m}^{\alpha}\mathbf{r}_{\alpha}$.

The set of equations (24) contains only the surface strain measures $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$, k_{α}, stress measures $R^{\alpha\beta}$, $H^{\alpha\beta}$, Q^{α} and components of **p, m** in the rotated basis.

4. Compatibility Conditions

It follows from $\left(\mathbf{R}\mathbf{R}^{T}\right),_{\beta} = \mathbf{0}$ that $\mathbf{R},_{\beta}\mathbf{R}^{T}$ is the skew-symmetric tensor expressible through the axial bending vector $\mathbf{l}_{\beta}$:

$$\mathbf{R},_{\beta}\mathbf{R}^{T} = \mathbf{l}_{\beta} \times \mathbf{1}, \quad \mathbf{l}_{\beta} = \varepsilon^{\alpha\lambda}\mu_{\alpha\beta}\mathbf{r}_{\lambda} + k_{\beta}\bar{\mathbf{n}}. \tag{25}$$

Let us introduce the finite rotation vector $\boldsymbol{\theta} = 2\tan\omega/2\mathbf{e}$, where the unit vector **e** describes the direction of the axis of rotation of **R** and ω is the angle of rotation about **e**. Then in terms of $\boldsymbol{\theta}$ we have

$$\begin{aligned}
&\mathbf{r}_{\beta} = \mathbf{R}\mathbf{a}_{\beta} = \mathbf{a}_{\beta} + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{a}_{\beta} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{a}_{\beta}\right), \quad t = 1 + \frac{1}{4}\boldsymbol{\theta}\cdot\boldsymbol{\theta}\\
&\bar{\mathbf{n}} = \mathbf{R}\mathbf{n} = \mathbf{n} + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{n} + \frac{1}{2}\boldsymbol{\theta} \times \mathbf{n}\right), \quad \mathbf{l}_{\beta} = \frac{1}{t}\left(\boldsymbol{\theta},_{\beta} - \frac{1}{2}\boldsymbol{\theta},_{\beta} \times \boldsymbol{\theta}\right).
\end{aligned} \tag{26}$$

Since $\bar{\mathbf{a}}_\alpha = \mathbf{a}_\alpha + \mathbf{u},_\alpha$, we can solve $(7)_1$ and (26) for $\mathbf{u},_\alpha$ and $\boldsymbol{\theta},_\alpha$, which yields

$$\mathbf{u},_\alpha = \boldsymbol{\eta}_\alpha + \frac{1}{t}\boldsymbol{\theta} \times \left(\mathbf{r}_\alpha - \frac{1}{2}\boldsymbol{\theta} \times \mathbf{r}_\alpha\right), \quad \boldsymbol{\theta},_\alpha = \mathbf{l}_\alpha - \frac{1}{2}\boldsymbol{\theta} \times \mathbf{l}_\alpha + \frac{1}{4}\left(\boldsymbol{\theta} \cdot \mathbf{l}_\alpha\right)\boldsymbol{\theta}. \tag{27}$$

The integrability conditions $\varepsilon^{\alpha\beta}\mathbf{u},_{\alpha\beta} = \mathbf{0}$ and $\varepsilon^{\alpha\beta}\boldsymbol{\theta},_{\alpha\beta} = \mathbf{0}$ of (27) take the forms

$$\varepsilon^{\alpha\beta}\left(\boldsymbol{\eta}_{\alpha|\beta} + \mathbf{l}_\beta \times \mathbf{r}_\alpha\right) = \mathbf{0}, \quad \varepsilon^{\alpha\beta}\left(\mathbf{l}_{\alpha|\beta} + \frac{1}{2}\mathbf{l}_\alpha \times \mathbf{l}_\beta\right) = \mathbf{0}. \tag{28}$$

These are vector forms of compatibility conditions for the non-linear deformation of the shell reference surface. They were derived first by Alumäe (1949, 1956) and independently by Simmonds and Danielson (1970) in components relative to the rotated basis $\mathbf{r}_\alpha$, $\bar{\mathbf{n}}$ in the form equivalent to

$$\begin{aligned}
\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta} + \varepsilon^{\alpha\beta}\left(\delta^\kappa_\alpha + \eta^\kappa_\alpha\right)\varepsilon_{\kappa\lambda}k_\beta &= 0, \\
\varepsilon^{\alpha\beta}\eta^\lambda_\alpha\left(b_{\lambda\beta} - \mu_{\lambda\beta}\right) - \varepsilon^{\alpha\beta}\mu_{\alpha\beta} &= 0, \\
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\mu_{\lambda\alpha|\beta} - \varepsilon^{\alpha\beta}k_\alpha\left(b^\kappa_\beta - \mu^\kappa_{.\beta}\right) &= 0, \\
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\mu_{\lambda\alpha}\left(b_{\kappa\beta} - \frac{1}{2}\mu_{\kappa\beta}\right) + \varepsilon^{\alpha\beta}k_{\alpha|\beta} &= 0.
\end{aligned} \tag{29}$$

The relations (29) assure that nine functions $\eta_{\alpha\beta}$, $\mu_{\alpha\beta}$, k_α of class C^1 define the regular reference surface $\overline{\mathcal{M}}$ of the deformed shell with accuracy up to its rigid motion in space $\mathcal{E}$.

5. Constitutive Equations

Let us discuss in more detail thin shells made of a homogeneous, isotropic elastic material undergoing small strains. In such a case the strain energy density of the shell, within the consistent first approximation, is the sum of two quadratic functions describing stretching and bending energy densities of the shell reference surface. When expressed by relative surface strain measures, the strain energy density takes the form (Pietraszkiewicz 1989)

$$\begin{aligned}
W &= \frac{h}{2}H^{\alpha\beta\lambda\mu}\left(\eta_{\alpha\beta}\eta_{\lambda\mu} + \frac{h^2}{12}\rho_{\alpha\beta}\rho_{\lambda\mu}\right) + O\left(Eh\eta^2\theta^2\right), \\
H^{\alpha\beta\lambda\mu} &= \frac{E}{2(1+\nu)}\left(a^{\alpha\lambda}a^{\beta\mu} + a^{\alpha\mu}a^{\beta\lambda} + \frac{2\nu}{1-\nu}a^{\alpha\beta}a^{\lambda\mu}\right).
\end{aligned} \tag{30}$$

Here E is the Young modulus, ν is the Poisson ratio, and $\rho_{\alpha\beta}$ are symmetric parts of $\mu_{\alpha\beta}$ defined by

$$\mu_{\alpha\beta} = \rho_{\alpha\beta} + \varepsilon_{\alpha\beta}\rho, \quad \rho_{\alpha\beta} = \frac{1}{2}\left(\mu_{\alpha\beta} + \mu_{\beta\alpha}\right), \quad \rho = \frac{1}{2}\varepsilon^{\alpha\beta}\mu_{\alpha\beta}. \tag{31}$$

The error of W at any point of the shell is indicated in (30) in terms of the small parameter θ defined by Koiter (1980)

$$\theta = \max\left(\frac{h}{b},\ \frac{h}{L},\ \frac{h}{l},\ \sqrt{\frac{h}{R}},\ \sqrt{\eta}\right), \tag{32}$$

where

- η – the largest strain in the shell space,
- h – the constant shell thickness,
- R – the smallest radius of curvature of $\mathcal{M}$,
- l – the smallest wavelength of geometric patterns of $\mathcal{M}$,
- L – the smallest wavelength of deformation patterns of $\mathcal{M}$,
- b – the distance from the lateral shell boundary.

Differentiating $(30)_1$ we obtain corresponding constitutive equations

$$\begin{gathered}
S^{\alpha\beta} = \frac{\partial W}{\partial \eta_{\alpha\beta}} = C\left[(1-\nu)\,\eta^{\alpha\beta} + \nu a^{\alpha\beta}\eta^{\kappa}_{\kappa}\right] + O\left(Eh\eta\theta^2\right), \\
G^{\alpha\beta} = \frac{\partial W}{\partial \rho_{\alpha\beta}} = D\left[(1-\nu)\,\rho^{\alpha\beta} + \nu a^{\alpha\beta}\rho^{\kappa}_{\kappa}\right] + O\left(Eh^2\eta\theta^2\right), \\
C = \frac{Eh}{1-\nu^2}, \qquad D = \frac{Eh^3}{12(1-\nu^2)}\ .
\end{gathered} \tag{33}$$

where $G^{\alpha\beta}$ are symmetric parts of $H^{\alpha\beta}$ defined by

$$H^{\alpha\beta} = G^{\alpha\beta} + \varepsilon^{\alpha\beta}G, \quad G^{\alpha\beta} = \frac{1}{2}\left(H^{\alpha\beta} + H^{\beta\alpha}\right), \quad G = \frac{1}{2}\varepsilon_{\alpha\beta}H^{\alpha\beta}. \tag{34}$$

It follows from $(30)_1$ that $G = O\left(Eh^2\eta\theta^2\right)$.

The inverse constitutive equations are

$$\begin{gathered}
\eta_{\alpha\beta} = A\left[(1+\nu)\,S_{\alpha\beta} - \nu a_{\alpha\beta}S^{\kappa}_{\kappa}\right] + O\left(\eta\theta^2\right), \\
\rho_{\alpha\beta} = B\left[(1+\nu)\,G_{\alpha\beta} - \nu a_{\alpha\beta}G^{\kappa}_{\kappa}\right] + O\left(\frac{\eta\theta^2}{h}\right), \\
A = \frac{1}{Eh}\ , \qquad B = \frac{12}{Eh^3}.
\end{gathered} \tag{35}$$

6. Refined Intrinsic Shell Equations

Let us introduce representations (31) into the compatibility conditions (29). Taking into account errors of the constitutive equations (35), the compatibility conditions can be reduced to

$$\begin{aligned}
&\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta}+k_\lambda=O\left(\frac{\eta\theta^2}{\lambda}\right),\\
&2\rho-\varepsilon^{\alpha\beta}\eta_{\alpha\kappa}\left(b^\kappa_\beta-\rho^\kappa_\beta\right)=O\left(\frac{\eta\theta^4}{h}\right),\\
&\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\rho_{\lambda\alpha|\beta}+\varepsilon^{\kappa\beta}\rho_{,\beta}-\varepsilon^{\alpha\beta}k_\alpha\left(b^\kappa_\beta-\rho^\kappa_\beta\right)=O\left(\frac{\eta\theta^4}{h\lambda}\right),\\
&\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\left(b_{\lambda\alpha}-\frac{1}{2}\rho_{\lambda\alpha}\right)\rho_{\kappa\beta}+\varepsilon^{\alpha\beta}k_{\alpha|\beta}=O\left(\frac{\eta\theta^2}{\lambda^2}\right).
\end{aligned}\tag{36}$$

Orders of covariant surface derivatives of some fields in (36) are estimated dividing their maximal values by the large parameter λ defined by

$$\lambda=\frac{h}{\theta}=\min\left(b,\ L,\ l,\ \sqrt{hR},\ \frac{1}{\sqrt{\eta}}\right).\tag{37}$$

It follows from $(36)_{1,2}$ that $k_\lambda=O(\eta/\lambda)$ and $\rho=O(\eta\theta/\lambda)$.

Let us introduce the representations $(16)_2$, (31) and (34) into the equilibrium equations (24) and take into account the error of constitutive equations (33) and (34) as well as estimates for $G,\ k_\lambda$ and ρ given above. As a result, the equilibrium equations can be reduced to

$$\begin{aligned}
G^{\alpha\beta}|_\beta-Q^\alpha+\hat{m}^\alpha&=O\left(Eh^2\frac{\eta\theta^2}{\lambda}\right),\\
2S+\varepsilon_{\alpha\lambda}S^{\alpha\beta}\eta^\lambda_\beta-\varepsilon_{\alpha\lambda}G^{\alpha\beta}\left(b^\lambda_\beta-\rho^\lambda_\beta\right)&=O\left(Eh\eta\theta^4\right),\\
S^{\alpha\beta}|_\beta+\varepsilon^{\alpha\beta}S_{,\beta}-Q^\beta\left(b^\alpha_\beta-\rho^\alpha_\beta\right)-\varepsilon^{\alpha\lambda}S^\beta_\lambda k_\beta+\hat{p}^\alpha&=O\left(Eh\frac{\eta\theta^4}{\lambda}\right),\\
S^{\alpha\beta}\left(b_{\alpha\beta}-\rho_{\alpha\beta}\right)+Q^\alpha|_\alpha+p&=O\left(Eh^2\frac{\eta\theta^2}{\lambda^2}\right).
\end{aligned}\tag{38}$$

It is easy to note that the equations $(36)_{1,2}$ can be solved for k_λ, ρ leading to

$$\begin{aligned}
k_\lambda&=-\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta}+O\left(\frac{\eta\theta^2}{\lambda}\right),\\
\rho&=\frac{1}{2}\varepsilon^{\alpha\beta}\eta_{\alpha\kappa}\left(b^\kappa_\beta-\rho^\kappa_\beta\right)+O\left(\frac{\eta\theta^4}{h}\right).
\end{aligned}\tag{39}$$

Likewise, the equations $(38)_{1,2}$ can be solved for Q^β, S which yields

$$\begin{aligned}
Q^\alpha&=G^{\alpha\beta}|_\beta+\hat{m}^\alpha+O\left(Eh^2\frac{\eta\theta^2}{\lambda}\right),\\
S&=\frac{1}{2}\varepsilon_{\alpha\lambda}\left[-S^{\alpha\beta}\eta^\lambda_\beta+G^{\alpha\beta}\left(b^\lambda_\beta-\rho^\lambda_\beta\right)\right],
\end{aligned}\tag{40}$$

Eliminating k_λ, ρ, Q^β, S with the help of (39) and (40) from the remaining equilibrium equations $(38)_{3,4}$ and compatibility conditions $(36)_{3,4}$ we obtain

$$\begin{aligned}
S^{\alpha\beta}|_\beta + \varepsilon^{\alpha\lambda}\varepsilon^{\kappa\rho}S^\beta_\lambda\eta_{\beta\kappa|\rho} + \frac{1}{2}\varepsilon^{\alpha\beta}\varepsilon_{\kappa\rho}\left[\eta^\kappa_\lambda S^{\lambda\rho} + G^{\kappa\lambda}\left(b^\rho_\lambda - \rho^\rho_\lambda\right)\right]|_\beta & \\
-\left(G^{\beta\lambda}|_\lambda + \hat{m}^\beta\right)\left(b^\alpha_\beta - \rho^\alpha_\beta\right) + \hat{p}^\alpha = O\left(Eh\frac{\eta\theta^4}{\lambda}\right), & \qquad (41) \\
S^{\alpha\beta}\left(b_{\alpha\beta} - \rho_{\alpha\beta}\right) + G^{\alpha\beta}|_{\alpha\beta} + \hat{m}^\alpha|_\alpha + p = O\left(Eh^2\frac{\eta\theta^2}{\lambda^2}\right). &
\end{aligned}$$

$$\begin{aligned}
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\rho_{\lambda\alpha|\beta} + \frac{1}{2}\varepsilon^{\kappa\beta}\varepsilon^{\alpha\lambda}\left[\eta_{\alpha\rho}\left(b^\rho_\lambda - \rho^\rho_\lambda\right)\right]|_\beta + \varepsilon^{\alpha\beta}\varepsilon^{\lambda\rho}\eta_{\alpha\lambda|\rho}\left(b^\kappa_\rho - \rho^\kappa_\rho\right) & \\
= O\left(\frac{\eta\theta^2}{h\lambda}\right), & \qquad (42) \\
\varepsilon^{\alpha\beta}\varepsilon^{\lambda\kappa}\left[\left(b_{\lambda\alpha} - \frac{1}{2}\rho_{\lambda\alpha}\right)\rho_{\kappa\beta} - \eta_{\lambda\alpha|\kappa\beta}\right] = O\left(\frac{\eta\theta^2}{\lambda^2}\right). &
\end{aligned}$$

The six scalar equations (41) and (42) are now expressed entirely in terms of twelve components of the surface strain and stress measures related by six constitutive equations (33) or (35). Using (33) or (35) we can eliminate any six of the fields to obtain a definite system of six differential equations for the remaining six surface measures. However, elimination of $S^{\alpha\beta}$ would introduce the error $O\left(Eh\eta\theta^2/\lambda\right)$ into $(41)_1$, while eliminating $\rho_{\alpha\beta}$ we would make the error $O\left(\eta\theta^2/\lambda h\right)$ in $(42)_1$. Both errors are larger than those indicated in $(41)_1$ and $(42)_2$, respectively, which may lead to some loss of accuracy of the solution. This was the main reason for Danielson (1970) to choose membrane stress resultants and changes of curvature as independent field variables of his intrinsic set of six shell equations. We also leave $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ as independent field variables of (41) and (42), and eliminate $G^{\alpha\beta}$ and $\rho_{\alpha\beta}$ using $(33)_2$ and $(35)_1$, respectively, which leads to

$$\begin{aligned}
S^\beta_\alpha|_\beta + A\left[(1+\nu)S^\lambda_\alpha - \nu\delta^\lambda_\alpha S^\kappa_\kappa\right]|_\beta S^\beta_\lambda - \frac{1}{2}A\left[(1+\nu)S^\beta_\lambda S^\lambda_\beta - \nu S^\lambda_\lambda S^\beta_\beta\right]|_\alpha & \\
-\frac{1}{2}D(1-\nu)\left(b^\lambda_\alpha\rho^\beta_\lambda - b^\beta_\lambda\rho^\lambda_\alpha\right)|_\beta - D\left(b^\beta_\alpha - \rho^\beta_\alpha\right)\rho^\lambda_\lambda|_\beta & \\
+\hat{p}_\alpha - \left(b^\beta_\alpha - \rho^\beta_\alpha\right)\hat{m}_\beta = O\left(Eh\frac{\eta\theta^4}{\lambda}\right), & \\
D\rho^\alpha_\alpha|^\beta_\beta + \left(b^\beta_\alpha - \rho^\alpha_\beta\right)S^\alpha_\beta + p + \hat{m}^\alpha|_\alpha = O\left(Eh^2\frac{\eta\theta^2}{\lambda^2}\right), &
\end{aligned}$$

$$
\begin{aligned}
&\rho_{\alpha}^{\beta}|_{\beta} - \rho_{\beta}^{\beta},_{\alpha} + \frac{1}{2}A(1+\nu)\left[\left(b_{\alpha}^{\lambda} - \rho_{\alpha}^{\lambda}\right)S_{\lambda}^{\beta} - \left(b_{\lambda}^{\beta} - \rho_{\lambda}^{\beta}\right)S_{\alpha}^{\lambda}\right]|_{\beta} \\
&\quad - A\left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta}\right)S_{\lambda}^{\lambda},_{\beta} - A(1+\nu)\left(b_{\alpha}^{\beta} - \rho_{\alpha}^{\beta}\right)\hat{p}_{\beta} = O\left(\frac{\eta\theta^{4}}{h\lambda}\right), \\
&AS_{\alpha}^{\alpha}|_{\beta}^{\beta} + \left(b_{\alpha}^{\beta} - \frac{1}{2}\rho_{\alpha}^{\beta}\right)\rho_{\beta}^{\alpha} - \left(b_{\alpha}^{\alpha} - \frac{1}{2}\rho_{\alpha}^{\alpha}\right)\rho_{\beta}^{\beta} + A(1+\nu)\,\hat{p}^{\alpha}|_{\alpha} \\
&\quad = O\left(\frac{\eta\theta^{2}}{\lambda^{2}}\right).
\end{aligned}
\tag{43}
$$

The set of refined intrinsic shell equations (43) is written in components relative to the rotated basis $\mathbf{r}_{\alpha}, \bar{\mathbf{n}}$. Within the indicated errors, the equations (43) are equivalent to alternative forms of refined intrinsic shell equations proposed by Danielson (1970), Koiter and Simmonds (1973) and Pietraszkiewicz (1977, 1979, 1980). However,

a) our system (43) is expressed through fields $S^{\alpha\beta}$, $\rho_{\alpha\beta}$ appearing naturally in thin shell theory and needing no modifications;
b) when linearised, the system (43) leads to equations of the 'best' linear shell theory according to Budiansky and Sanders (1963), (see also Koiter 1980);
c) our system (43) follows from the equations (36), (38) and (41), (42) which obey a static-geometric duality in the non-linear range of deformation, (see Pietraszkiewicz 1989, 2001a);
d) corresponding sets of work-conjugate static and deformational boundary conditions have been provided in Pietraszkiewicz (2001a).

From (43) it is possible to derive many reduced systems of intrinsic shell equations valid under additional simplifying assumptions about curvatures and variability of the reference surface, stretching-to-bending ratio, variability of stretching and bending deformations etc. Some special cases were discussed in Pietraszkiewicz (1989, 2001a) and Simmonds (1979). However, in our computer age it seems more appropriate to apply direct numerical methods to the full system (43) which would allow one to analyse all possible cases of the non-linear behaviour of thin elastic shells.

References

Alumäe N. A. (1949), Differential equations of the equilibrium states of thin-walled elastic shells in the post-buckling stage (in Russian), *Prikladnaya Matematika i Mekhanika* **13**, 95–106.

Alumäe N. A. (1956), On the expression of basic relations of the non-linear theory of shells (in Russian), *Prikladnaya Matematika i Mekhanika* **20**, 136–139.

Badur J. and Pietraszkiewicz W. (1986), On geometrically non-linear theory of shells derived from pseudo-Cosserat continuum with constrained micro-rotations, [in:] W. Pietraszkiewicz (ed.), *Finite Rotations in Structural Mechanics*, 19–32; Springer-Verlag, Berlin.

Budiansky B. and Sanders J. L. (1963), On the 'best' first-order linear shell theory, [in:] *Progress in Applied Mechanics* (Prager Anniversary Volume), 129–140; MacMillan, New York.

Chien W. Z. (1944), The intrinsic theory of thin shells and plates, *Quart. Appl. Math.* **1**, 4, 297–327; **2**, 1, 43–59; **2**, 2, 120–135.

Danielson D. A. (1970), Simplified intrinsic equations for arbitrary elastic shells, *Int. J. Engng Sci.* **8**, 251–259.

Koiter W. T. and Simmonds J. G. (1973), Foundations of shell theory, [in:] E. Becker and G. K. Mikhailov (eds.), *Theoretical and Applied Mechanics*, 150–175; Springer-Verlag, Berlin.

Koiter W. T. (1980), The intrinsic equations of shell theory with some applications, [in:] S. Nemat-Nasser (ed.), *Mechanics Today* **5,** 139–154; Pergamon Press, Oxford.

Libai A. and Simmonds J. G. (1983), Nonlinear elastic shell theory, [in:] J. W. Hutchinson and T. Y. Wu (eds.), *Advances in Applied Mechanics* **23**, 271–371; Academic Press, New York.

Libai A. and Simmonds J. G. (1998), *The Non-linear Theory of Elastic Shells,* 2nd ed., Cambridge University Press, Cambridge.

Pietraszkiewicz W. (1977), *Introduction to the Non-linear Theory of Shells*, Ruhr-Universität, Mitt. IfMech. No 10, Bochum.

Pietraszkiewicz W. (1979), *Finite Rotations and Lagrangean Description in the Non-linear Theory of Shells*, Polish Sci. Publ., Warszawa - Poznań.

Pietraszkiewicz W. (1980), Finite rotations in the non-linear theory of thin shells, [in:] W. Olszak (ed.), *Thin Shell Theory, New Trends and Applications*, 151–208; Springer-Verlag, Wien - New York.

Pietraszkiewicz W. (1989), Geometrically nonlinear theories of thin elastic shells, *Advances in Mechanics* **12**, 51–130.

Pietraszkiewicz W. (2001a), Nieliniowe teorie powłok, [in:] Cz. Woźniak (ed.) *Mechanika Sprężystych Płyt i Powłok*, 424–497; PWN, Warszawa (in print).

Pietraszkiewicz W. (2001b), On using rotations as primary variables in the non-linear theory of thin irregular shells, [in:] D. Durban et al. (eds.), *Advances in the Mechanics of Plates and Shells*, 245–258; Kluwer Acad. Publ.

Simmonds J. G. and Danielson D. A. (1970), Nonlinear shell theory with a finite rotation vector, *Proc. Koninkl. Nederl. Akademie van Wetenschappen – Amsterdam*, Series B **73**, 460–478.

Simmonds J. G. and Danielson D. A. (1972), Nonlinear shell theory with finite rotation and stress function vectors, *Trans. ASME, J. Applied Mechanics* **39**, 1085–1090.

Simmonds J. G. (1979), Special cases of the nonlinear shell equations, [in:] J. F. Besseling and A. M. A. van der Heiden (eds.), *Trends in Solid Mechanics*, 211–223; Sijthoff and Noordhoff, Delft.

Abstract

Equilibrium equations and compatibility conditions of thin shells are formulated relative to a non-holonomic intermediate basis defined through rigid rotation of the natural undeformed shell basis. Applying error estimates valid within the first approximation to the elastic strain energy density of a shell, the consistently reduced system of six intrinsic shell equations is derived. The equations are expressed through three membrane stress resultants and three changes of curvatures of the shell reference surface as basic independent field variables. This allows one to refine four of six equations as compared with intrinsic shell equations expressed entirely through either of six surface strain or stress measures alone.