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ON DETERMINATION OF DISPLACEMENTS FROM GIVEN STRAINS AND HEIGHT FUNCTION IN THE NON-LINEAR THEORY OF THIN SHELLS

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Dedicated to Professor Czesław Woźniak on the occasion of his 70^{th} birthday

The position vector of the deformed shell reference surface is expressed by its projection onto the coordinate plane and the height function over that plane. The projected position is then determined by quadratures entirely from three surface strains and the height function. The latter fields can be found as solutions to the non-linear boundary value problem of thin elastic shells, developed by Szwabowicz (1999). The corresponding displacement field is determined from the deformed position vector by simple algebraic formulae.

Key words: thin shell, non-linear theory, Darboux equation, position of surface

1. Introduction

The non-linear theory of thin shells is based on the kinematic hypothesis that deformation of a shell is described with sufficient accuracy by the deformation of its reference surface alone. Then, the principle of virtual work postulated on the reference surface and surface kinematics allows one to establish various non-linear boundary value problems (BVPs) expressed in different sets of fields as independent variables (see Pietraszkiewicz (1989, 2001) and references given there). Of the three main existing approaches the one, so-called displacement form of shell relations, or some of its simplified versions, enjoys the greatest popularity in the literature, but this formulation is very complex and hardly manageable for unrestricted deflections. The intrinsic shell relations expressed in the surface strain and/or stress measures are relatively simple even for unrestricted deflections. However, they are applicable only to special shell problems, and an additional complex non-linear analysis of the compatibility conditions is required in order to determine the displacement field.

A novel formulation of the non-linear BVP for thin elastic shells undergoing small strains was developed by Szwabowicz (1999). It is expressed in three surface strains and one height function of the deformed shell reference surface as basic independent field variables. The BVP posed in this form benefits from relative simplicity of intrinsic shell relations and circumvents complexities of the displacement approach. The corresponding field equations consist of three equilibrium equations and one extended equation of Darboux (see Darboux, 1894), which is a compatibility condition for the four unknowns. When the surface strains and the height function are found from the BVP, the surface curvature changes can be computed from simple differential relations, and the internal surface stress and couple resultants follow then from the constitutive equations.

It was mentioned in Szwabowicz (1999) that the two remaining Cartesian components of the displacement field can be determined by quadratures from three surface strains and the height function. However, only the final formulae based on the Darboux idea (Darboux, 1894) were presented in Szwabowicz (1999), and the appropriate procedure was outlined in sketchy form. The aim of this paper is to explicitly derive such quadratures for the two displacement components. Our solution is based on methods applied in continuum mechanics, and differs from that suggested by Darboux (1894), developed with some errors by Hartman and Wintner (1951), and presented concisely in Szwabowicz (1999).

The contents of the paper is as follows. Section 2 is devoted to notation and some basic relations valid for the surface geometry. The undeformed shell reference surface is projected onto the coordinate plane Oxy and the surface geometry is described by the Euclidean metric of the projected flat region and the height function over that region. This allows us to concisely derive the Darboux equation in Section 3. In Section 4 we propose an original solution to the problem of embedding of the two-dimensional metric into the Oxy plane. Our approach is based on mapping of a domain in the Oxy plane parameterized by Cartesian coordinates into the flat region of the projected reference surface. The gradient of the map is then polarly decomposed into the right stretch tensor and the rotation tensor, for which explicit formulae (4.12) and (4.24), $(4.13)_1$ are derived in terms of the projected flat metric. Position vector (4.27) of the projected region is obtained by quadratures (4.28), and it describes the position of the surface in space as well. In Section 5 we find the spatial position of the deformed shell reference surface using the results of Section 4 for the undeformed surface. The metric of the deformed surface is described by the undeformed surface metric and three strains. Therefore, the spatial position of the deformed surface is immediately found by its analogous projection (5.4) onto the Oxy plane, with subsequent analogous quadratures for the position vector of the projection. The corresponding displacement field is then determined from the simple formula (5.9) in terms of three surface strains and the height function of the deformed shell reference surface.

2. Notation and surface geometry

The notation here follows that used by Pietraszkiewicz (1989, 2001) and Szwabowicz (1999).

A simply connected regular surface $\mathcal{M}$ in the 3D Euclidean point space $\mathcal{E}$ can locally be described by choosing a fixed orthonormal frame $(O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$, $O \in \mathcal{E}$, and three functions $x(\vartheta^\alpha)$, $y(\vartheta^\alpha)$ and $z(\vartheta^\alpha)$ of class C^2, where ϑ^α, $\alpha = 1,2$, are surface curvilinear coordinates. The position vector of $\mathcal{M}$ is then given by

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k} = \boldsymbol{r}(\vartheta^\alpha) \tag{2.1}$$

With each regular point $M \in \mathcal{M}$ we can associate the natural base vectors $\boldsymbol{a}_\alpha = \partial\boldsymbol{r}/\partial\vartheta^\alpha \equiv \boldsymbol{r}_{,\alpha}$, the dual base vectors $\boldsymbol{a}^\beta$ such that $\boldsymbol{a}^\beta \cdot \boldsymbol{a}_\alpha = \delta^\beta_\alpha$ with $\delta^1_1 = \delta^2_2 = 1$, $\delta^1_2 = \delta^2_1 = 0$, the components $a_{\alpha\beta} = \boldsymbol{a}_\alpha \cdot \boldsymbol{a}_\beta$ and $a^{\alpha\beta} = \boldsymbol{a}^\alpha \cdot \boldsymbol{a}^\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the unit normal vector $\boldsymbol{n} = (\boldsymbol{a}_1 \times \boldsymbol{a}_2)/\sqrt{a}$ orienting $\mathcal{M}$, the components $b_{\alpha\beta} = -\boldsymbol{n}_{,\alpha} \cdot \boldsymbol{a}_\beta$ of the second fundamental tensor $\mathbf{b}$, and the components $\epsilon_{\alpha\beta} = (\boldsymbol{a}_\alpha \times \boldsymbol{a}_\beta) \cdot \boldsymbol{n}$ of the surface permutation tensor $\boldsymbol{\epsilon}$ with $\epsilon_{\alpha\beta} = \sqrt{a}e_{\alpha\beta}$, $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$.

The components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfy the Gauss-Mainardi-Codazzi equations

$$b_{\beta\lambda}|_\mu = b_{\beta\mu}|_\lambda \qquad\qquad b_{\alpha\lambda}b_{\beta\mu} - b_{\alpha\mu}b_{\beta\lambda} = R_{\alpha\beta\lambda\mu} \tag{2.2}$$

where $(\cdot)|_\alpha$ denotes the surface covariant derivative, and $R_{\alpha\beta\lambda\mu}$ are components of the Riemann-Christoffel tensor related to the Gauss curvature K and $a_{\alpha\beta}$ by

$$
\begin{aligned}
R_{\alpha\beta\lambda\mu} &= a_{\alpha\kappa}(\Gamma^{\kappa}_{\beta\mu},_{\lambda} - \Gamma^{\kappa}_{\beta\lambda},_{\mu} + \Gamma^{\kappa}_{\rho\lambda}\Gamma^{\rho}_{\beta\mu} - \Gamma^{\kappa}_{\rho\mu}\Gamma^{\rho}_{\beta\lambda}) \\
\Gamma^{\kappa}_{\alpha\beta} &= \frac{1}{2}a^{\kappa\lambda}(a_{\lambda\alpha},_{\beta} + a_{\lambda\beta},_{\alpha} - a_{\alpha\beta},_{\lambda}) = -\boldsymbol{a}_{\alpha} \cdot \boldsymbol{a}^{\kappa},_{\beta} \\
K &= \frac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}R_{\alpha\beta\lambda\mu} \equiv \mathrm{Riem}(a_{\alpha\beta})
\end{aligned}
\tag{2.3}
$$

It is evident from $(2.3)_3$ that K is determined by the $\mathrm{Riem}(\cdot)$ operator entirely from the metric components $a_{\alpha\beta}$.

3. Position of a surface

The position of $\mathcal{M}$ in $\mathcal{E}$ can also be established by prescribing three functions $a_{\alpha\beta}(\vartheta^{\lambda})$ and one coordinate function, say $z(\vartheta^{\alpha})$, called the height function of $\mathcal{M}$, all of class C^2, which satisfy the Darboux equation (Darboux, 1894)

$$
\mathbf{M}(z) - K(1 - z,_{\alpha} z,_{\beta} a^{\alpha\beta}) = 0 \tag{3.1}
$$

where the Monge-Ampère operator $\mathbf{M}(z)$ is defined by

$$
\mathbf{M}(z) \stackrel{def}{=} \frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}z|_{\alpha\lambda}z|_{\beta\mu} \tag{3.2}
$$

Indeed, for a given $z(\vartheta^{\alpha})$, the vector $\boldsymbol{r}$ can be decomposed into

$$
\boldsymbol{r} = \boldsymbol{p} + z\boldsymbol{k} \tag{3.3}
$$

where $\boldsymbol{p}$ is the position vector of $P \in \mathcal{P}$ – the projection of $M \in \mathcal{M}$ onto the coordinate plane Oxy, see Fig. 1. From (3.3) it follows that $z = \boldsymbol{r} \cdot \boldsymbol{k}$ and $z,_{\alpha} = \boldsymbol{a}_{\alpha} \cdot \boldsymbol{k}$, which, if introduced into the identity $\boldsymbol{k} = (\boldsymbol{a}_{\alpha} \cdot \boldsymbol{k})\boldsymbol{a}^{\alpha} + (\boldsymbol{n} \cdot \boldsymbol{k})\boldsymbol{n}$, leads to

$$
\boldsymbol{k} = z,_{\alpha} \boldsymbol{a}^{\alpha} + (\boldsymbol{n} \cdot \boldsymbol{k})\boldsymbol{n} \tag{3.4}
$$

Let us square the relation (3.4), differentiate it with respect to coordinates and multiply the result by $\boldsymbol{a}_{\alpha}$. This yields

$$
\begin{aligned}
(\boldsymbol{n} \cdot \boldsymbol{k})^2 &= 1 - z,_{\alpha} z,_{\beta} a^{\alpha\beta} \\
0 = \boldsymbol{k},_{\beta} \cdot \boldsymbol{a}_{\alpha} &= z,_{\alpha\beta} + z,_{\kappa} \boldsymbol{a}^{\kappa},_{\beta} \cdot \boldsymbol{a}_{\alpha} + (\boldsymbol{n} \cdot \boldsymbol{k})\boldsymbol{n},_{\beta} \cdot \boldsymbol{a}_{\alpha}
\end{aligned}
\tag{3.5}
$$

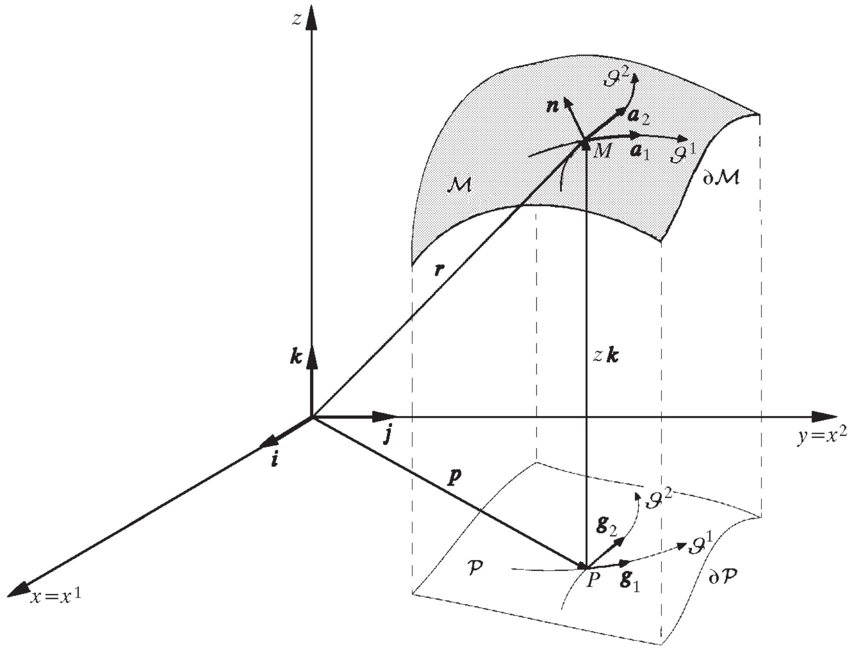

Fig. 1.

From $(3.5)_2$ it follows that

$$b_{\alpha\beta} = \frac{1}{\boldsymbol{n}\cdot\boldsymbol{k}} z|_{\alpha\beta} \tag{3.6}$$

provided that

$$z_{,\alpha}\, z_{,\beta}\, a^{\alpha\beta} < 1 \tag{3.7}$$

In view of $(2.3)_3$, the Gauss equation $(2.2)_2$ can be modified to the form

$$\frac{1}{2}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}b_{\alpha\lambda}b_{\beta\mu} = K \tag{3.8}$$

Introducing now representation (3.6) with $(3.5)_1$ into (3.8) we can easily transform it to the Darboux equation (3.1). Likewise, with representation (3.6) each of the two Mainardi-Codazzi equations $(2.2)_1$ can also be reduced to (3.1) by somewhat more involved transformations. Hence, equation (3.1) is a compatibility condition for the surface metric components $a_{\alpha\beta}$ and the surface position coordinate z.

Four functions $a_{\alpha\beta}(\vartheta^\lambda)$ and $z(\vartheta^\alpha)$ satisfying (3.7) and the Darboux equation (3.1) describe the surface position in space up to the rigid translation and rotation parallel to the Oxy plane.

4. Embedding of two-dimensional flat metrics

Decomposition (3.3) allows one to express the geometry of $\mathcal{M}$ by geometry of its projection $\mathcal{P}$ onto the coordinate plane Oxy. In particular, we obtain

$$\boldsymbol{a}_\alpha = \boldsymbol{g}_\alpha + z_{,\alpha}\boldsymbol{k} \qquad \boldsymbol{g}_\alpha = \boldsymbol{p}_{,\alpha} \tag{4.1}$$

$$a_{\alpha\beta} = g_{\alpha\beta} + z_{,\alpha} z_{,\beta} \qquad g_{\alpha\beta} = \boldsymbol{g}_\alpha \cdot \boldsymbol{g}_\beta \qquad g = \det(g_{\alpha\beta}) \tag{4.2}$$

By decomposition (3.3) and relations (4.2) our problem has been reduced to finding a local embedding of the prescribed flat metric $ds^2 = g_{\alpha\beta}d\vartheta^\alpha d\vartheta^\beta$ into the Euclidean plane Oxy. This differential problem turns out to be completely solvable in terms of quadratures. The principal idea was already outlined by Darboux (1894, p. 216) and developed by Hartmann and Wintner (1951), who presented explicit quadratures for $x[g_{\alpha\beta}(\vartheta^\lambda)]$ and $y[g_{\alpha\beta}(\vartheta^\lambda)]$ expressed by a quadrature for the angle between the x and ϑ^1 coordinate lines. Unfortunately, the formulae in the latter paper contain some errors corrected by Szwabowicz (1999), and the solution itself does not exhibit invariance properties one might expect from that kind of a problem. In the remainder of this Section we develop an alternative, coordinate-invariant method of solution to this problem.

Let us consider an open domain $\mathcal{U}$ in the $O\vartheta^1\vartheta^2$ plane. The position vector $\boldsymbol{q}$ of points in $\mathcal{U}$ with the Cartesian coordinates $(\vartheta^1, \vartheta^2)$, the line element, the unit base vectors and the standard metric are as follows

$$\begin{aligned} &\boldsymbol{q} = \vartheta^\alpha \boldsymbol{i}_\alpha \qquad d\boldsymbol{q} = d\vartheta^\alpha \boldsymbol{i}_\alpha \qquad \boldsymbol{i}_\alpha = \boldsymbol{i}^\alpha = (\boldsymbol{i}, \boldsymbol{j}) \\ &ds_q^2 = (d\vartheta^1)^2 + (d\vartheta^2)^2 \end{aligned} \tag{4.3}$$

Consider now a map $f : \mathcal{U} \to \mathcal{P}$, $\boldsymbol{p} = f(\boldsymbol{q})$ of $\mathcal{U}$, whose image is the region $\mathcal{P}$ also lying in the $O\vartheta^1\vartheta^2$ plane. Write down the position vector of points in $\mathcal{P}$ as

$$\boldsymbol{p} = x^\alpha \boldsymbol{i}_\alpha \tag{4.4}$$

Then, two functions $x^\alpha = x^\alpha(\vartheta^\beta)$ establish the location of any point of $\mathcal{U}$ in the image of f. Assuming f to be continuously differentiable, we may write

$$\begin{aligned} &d\boldsymbol{p} = \boldsymbol{\Gamma} d\boldsymbol{q} \qquad \boldsymbol{g}_\alpha = \boldsymbol{\Gamma}\boldsymbol{i}_\alpha \\ &\boldsymbol{\Gamma} = \nabla \boldsymbol{p} = \boldsymbol{p}_{,\alpha} \otimes \boldsymbol{i}^\alpha = \boldsymbol{g}_\alpha \otimes \boldsymbol{i}^\alpha \end{aligned} \tag{4.5}$$

where $\boldsymbol{\Gamma}$ is the 2D gradient of the map f taken in the metric ds_q^2. Assuming f to be orientation-preserving, we also have $\det \boldsymbol{\Gamma} > 0$.

The polar decomposition of $\boldsymbol{\Gamma}$ yields

$$\boldsymbol{\Gamma} = \mathbf{QH} \tag{4.6}$$

where, using the terminology of continuum mechanics, $\mathbf{H}$ is the right stretch tensor (symmetric, $\mathbf{H}^\top = \mathbf{H}$, and positive definite, $\boldsymbol{v} \cdot (\mathbf{H}\boldsymbol{v}) > 0$ for all vectors $\boldsymbol{v} \neq \mathbf{0}$), and $\mathbf{Q}$ is the rotation tensor (proper orthogonal, $\mathbf{Q}^\top = \mathbf{Q}^{-1}$, $\det \mathbf{Q} = +1$). Our goal is to determine $\mathbf{H}$ and $\mathbf{Q}$, and then $\boldsymbol{\Gamma}$, from three components $g_{\alpha\beta}$ alone.

From $(4.5)_1$ and (4.6) it follows that

$$ds^2 = d\boldsymbol{p} \cdot d\boldsymbol{p} = d\boldsymbol{q} \cdot (\mathbf{H}^2 d\boldsymbol{q}) = \boldsymbol{i}_\alpha \cdot (\mathbf{H}^2 \boldsymbol{i}_\beta) d\vartheta^\alpha d\vartheta^\beta \tag{4.7}$$

If we introduce the tensor $\mathbf{G}$ such that

$$\begin{aligned} &\mathbf{G} = g_{\alpha\beta} \boldsymbol{i}^\alpha \otimes \boldsymbol{i}^\beta \qquad\qquad \det \mathbf{G} = g \\ &\operatorname{tr} \mathbf{G} = G = g_{11} + g_{22} \end{aligned} \tag{4.8}$$

then from (4.7) and (4.8) we obtain

$$\mathbf{H}^2 = \mathbf{G} \qquad\qquad \det \mathbf{H} = \sqrt{g} \tag{4.9}$$

The 2D second-order tensor $\mathbf{H}$ satisfies the Cayley-Hamilton equation

$$\mathbf{H}^2 - (\operatorname{tr} \mathbf{H})\mathbf{H} + (\det \mathbf{H})\mathbf{I} = \mathbf{0} \tag{4.10}$$

in which, according to Hoger and Carlson (1984), Eq. (5.2)

$$\operatorname{tr} \mathbf{H} = \sqrt{\operatorname{tr} \mathbf{G} + 2\sqrt{\det \mathbf{G}}} = \sqrt{C} \qquad\qquad C = G + 2\sqrt{g} \tag{4.11}$$

Therefore, solving (4.10) for $\mathbf{H}$ with the use of (4.9) and (4.11) we express $\mathbf{H}$ solely by $g_{\alpha\beta}$

$$\mathbf{H} = \frac{1}{\sqrt{C}}(\mathbf{G} + \sqrt{g}\,\mathbf{I}) \tag{4.12}$$

The rotation tensor $\mathbf{Q}$ can be represented in 2D space as

$$\begin{aligned} &\mathbf{Q} = \cos\phi \mathbf{I} - \sin\phi \mathbf{e} \\ &\mathbf{I} = \boldsymbol{g}_\alpha \otimes \boldsymbol{g}^\alpha = \boldsymbol{i}_\alpha \otimes \boldsymbol{i}^\alpha \\ &\mathbf{e} = e_{\alpha\beta} \boldsymbol{i}^\alpha \otimes \boldsymbol{i}^\beta = \boldsymbol{i} \otimes \boldsymbol{j} - \boldsymbol{j} \otimes \boldsymbol{i} \end{aligned} \tag{4.13}$$

where ϕ is the angle of rotation in the Oxy plane. In order to express $\mathbf{Q}$ in terms of $g_{\alpha\beta}$, let us use the integrability condition for $\boldsymbol{p}$

$$e^{\alpha\beta}\boldsymbol{p},_{\alpha\beta} = \mathbf{0} \tag{4.14}$$

which with the help of $(4.5)_1$ and (4.6) takes the form

$$[\mathbf{Q},_\alpha \mathbf{He} + \mathbf{Q}(\mathbf{He}),_\alpha]\boldsymbol{i}^\alpha = \mathbf{0} \tag{4.15}$$

Differentiating $(4.13)_1$ and using $(4.13)_{2,3}$ leads to

$$\mathbf{Q},_\alpha = -\mathbf{Qe}\phi,_\alpha \tag{4.16}$$

When (4.16) is introduced into (4.15) and the result left-multiplied by $\mathbf{H}$, using $\nabla\phi \equiv \phi,_\alpha \boldsymbol{i}^\alpha$ we obtain

$$\mathbf{HeHe}\nabla\phi = \mathbf{H}(\mathbf{He}),_\alpha \boldsymbol{i}^\alpha \tag{4.17}$$

Let us now apply the Cayley-Hamilton theorem (4.10) to the tensor $\mathbf{He}$. Since $\operatorname{tr}(\mathbf{He}) = 0$ and $\det(\mathbf{He}) = \det\mathbf{H} = \sqrt{g}$, the result is

$$\mathbf{HeHe} = -\sqrt{g}\,\mathbf{I} \tag{4.18}$$

which if introduced into (4.17) yields

$$\nabla\phi = -\frac{1}{\sqrt{g}}\mathbf{H}(\mathbf{He}),_\alpha \boldsymbol{i}^\alpha \tag{4.19}$$

By differentiating (4.12) we obtain

$$\begin{aligned}
(\mathbf{He}),_\alpha \boldsymbol{i}^\alpha &= \Big\{-\frac{1}{2C}\mathbf{He}C,_\alpha + \frac{1}{\sqrt{C}}[(\mathbf{Ge}),_\alpha + \mathbf{e}(\sqrt{g}),_\alpha]\Big\}\boldsymbol{i}^\alpha \\
\mathbf{H}(\mathbf{He}),_\alpha \boldsymbol{i}^\alpha &= \frac{1}{C}\Big\{-\frac{1}{2}\mathbf{Ge}C,_\alpha + (\mathbf{G} + \sqrt{g}\,\mathbf{I})[(\mathbf{Ge}),_\alpha + \mathbf{e}(\sqrt{g}),_\alpha]\Big\}\boldsymbol{i}^\alpha = \\
&= \frac{1}{C}\Big[\mathbf{G}(\mathbf{Ge}),_\alpha + \sqrt{g}(\mathbf{Ge}),_\alpha - \frac{1}{2}\mathbf{Ge}G,_\alpha + \mathbf{e}\sqrt{g}(\sqrt{g}),_\alpha\Big]\boldsymbol{i}^\alpha
\end{aligned} \tag{4.20}$$

With $(4.20)_2$ and (4.11) the final relation for the gradient of ϕ is

$$\nabla\phi = \frac{1}{G + 2\sqrt{g}}\Big[\frac{1}{2\sqrt{g}}\mathbf{Ge}G,_\alpha - \frac{1}{\sqrt{g}}\mathbf{G}(\mathbf{Ge}),_\alpha - (\mathbf{Ge}),_\alpha - \mathbf{e}(\sqrt{g}),_\alpha\Big]\boldsymbol{i}^\alpha \tag{4.21}$$

or in a component form

$$\phi,_\alpha = \frac{1}{G+2\sqrt{g}} \Big[\frac{1}{\sqrt{g}} g_{\alpha\beta}\Big(\frac{1}{2}e^{\beta\lambda}G,_\lambda - \delta^{\beta\kappa} g_{\kappa\rho},_\lambda e^{\rho\lambda}\Big) - \\ - g_{\alpha\beta},_\lambda e^{\beta\lambda} - e_{\alpha}^{\cdot\lambda}(\sqrt{g}),_\lambda\Big] \tag{4.22}$$

Now the angle of rotation arising in the polar decomposition of $\mathbf{\Gamma}$ follows from the quadrature

$$\phi = \phi_0 + \int \phi,_\alpha \, d\vartheta^\alpha \tag{4.23}$$

According to (4.23) with (4.22), the angle ϕ is expressed entirely by the components $g_{\alpha\beta}$ in the explicit relation

$$\begin{aligned}\phi = \phi_0 + \int \frac{1}{G\sqrt{g}+2g}\Big\{ & \Big[\sqrt{g}(g_{12},_1 - g_{11},_2) + g_{11}g_{12},_1 + \\ & + \frac{1}{2}g_{12}(g_{22}-g_{11}),_1 - \frac{1}{2}(g_{11}+g_{22})g_{11},_2\Big]\, d\vartheta^1 + \\ & + \Big[\sqrt{g}(g_{22},_1 - g_{12},_2) - g_{22}g_{12},_2 + \\ & + \frac{1}{2}g_{12}(g_{22}-g_{11}),_2 + \frac{1}{2}(g_{11}+g_{22})g_{22},_1\Big]\, d\vartheta^2\Big\}\end{aligned} \tag{4.24}$$

Let us introduce the rotation tensor $(4.13)_1$ described by the angle (4.24) and the right stretch tensor (4.12) into the polar decomposition formula (4.6), which allows one to calculate the gradient $\mathbf{\Gamma}$

$$\begin{aligned}\mathbf{\Gamma} &= \frac{1}{\sqrt{C}}(\cos\phi\mathbf{I} - \sin\phi\mathbf{e})(\mathbf{G}+\sqrt{g}\,\mathbf{I}) = x^\alpha,_\beta\, \boldsymbol{i}_\alpha \otimes \boldsymbol{i}^\beta \\ x^\alpha,_\beta &= \frac{1}{\sqrt{G+2\sqrt{g}}}\Big[\cos\phi(\delta^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}\,\delta^\alpha_\beta) - \sin\phi(e^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}\,e^{\alpha}_{\cdot\beta})\Big]\end{aligned} \tag{4.25}$$

But

$$\mathbf{\Gamma}\boldsymbol{i}_\beta = \boldsymbol{p},_\beta = x^\alpha,_\beta\, \boldsymbol{i}_\alpha \equiv x,_\beta\, \boldsymbol{i} + y,_\beta\, \boldsymbol{j} \tag{4.26}$$

and the position vector of the point $P \in \mathcal{P}$ follows from the quadrature

$$\boldsymbol{p} = \boldsymbol{p}_0 + \int \boldsymbol{p},_\beta \; d\vartheta^\beta \tag{4.27}$$

or, explicitly, for the Cartesian components of $\boldsymbol{p} = x\boldsymbol{i} + y\boldsymbol{j}$

$$\begin{aligned}
x = x_0 + \int \frac{1}{\sqrt{G + 2\sqrt{g}}} \Big\{ &[\cos\phi(g_{11} + \sqrt{g}) - \sin\phi g_{12}] d\vartheta^1 + \\
&+ [\cos\phi g_{12} - \sin\phi(g_{22} + \sqrt{g})]\ d\vartheta^2 \Big\} \\
y = y_0 + \int \frac{1}{\sqrt{G + 2\sqrt{g}}} \Big\{ &[\cos\phi g_{12} + \sin\phi(g_{11} + \sqrt{g})] d\vartheta^1 + \\
&+ [\cos\phi(g_{22} + \sqrt{g}) + \sin\phi g_{12}]\ d\vartheta^2 \Big\}
\end{aligned} \tag{4.28}$$

By (4.28) the position vector $\boldsymbol{p}(\vartheta^\alpha)$ is expressed in terms of the components $g_{\alpha\beta}$ as well. With known $\boldsymbol{p}(\vartheta^\alpha)$, the position vector $\boldsymbol{r}(\vartheta^\alpha)$ of the surface $\mathcal{M}$ in the space $\mathcal{E}$ follows from relation (3.3).

Thus, in order to find the embedding of a flat metric ds^2 into the 2D Euclidean space one needs to perform three quadratures: first (4.24), which yields the angle of rotation $\phi = \phi(\vartheta^\alpha)$, and two remaining (4.28) yielding two Cartesian coordinates $x = x(\vartheta^\alpha)$ and $y = y(\vartheta^\alpha)$, respectively.

5. Position of the deformed surface

Let $\overline{\mathcal{M}} = \boldsymbol{\chi}(\mathcal{M})$ be the reference surface of the deformed shell obtained from the undeformed surface $\mathcal{M}$ by the deformation map $\boldsymbol{\chi}$. The map is assumed to be single-valued, orientation-preserving and differentiable a sufficient number of times. The position vector of $\overline{\mathcal{M}}$ relative to the same orthonormal frame $(O, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ can be described in analogy to (2.1) by

$$\overline{\boldsymbol{r}}(\vartheta^\alpha) = \boldsymbol{\chi}[\boldsymbol{r}(\vartheta^\alpha)] = \overline{x}(\vartheta^\alpha)\boldsymbol{i} + \overline{y}(\vartheta^\alpha)\boldsymbol{j} + \overline{z}(\vartheta^\alpha)\boldsymbol{k} = \boldsymbol{r}(\vartheta^\alpha) + \boldsymbol{u}(\vartheta^\alpha) \tag{5.1}$$

where $\boldsymbol{u}$ is the displacement vector and ϑ^α are convected surface coordinates.

In the convected coordinates all geometric quantities and relations at any regular point $\overline{M} \in \overline{\mathcal{M}}$ are now analogous to those at $M \in \mathcal{M}$ given in Section 2. In this report the quantities corresponding to the deformed surface $\overline{\mathcal{M}}$ will be marked with a dash: $\overline{\boldsymbol{a}}_\alpha$, $\overline{\boldsymbol{a}}^\beta$, $\overline{a}_{\alpha\beta}$, $\overline{a}^{\alpha\beta}$, $\overline{\mathbf{a}}$, $\overline{a}$, $\overline{\boldsymbol{n}}$, $\overline{b}_{\alpha\beta}$, $\overline{\epsilon}_{\alpha\beta}$, $\overline{K}$ etc., while the surface covariant derivative in the deformed metric will be denoted by a double vertical stroke $(\cdot)\|_\alpha$. The dashed quantities can be expressed by analogous undashed quantities defined on $\mathcal{M}$ and the displacement field $\boldsymbol{u}$ with the help of formulae presented in Pietraszkiewicz (1989) and Szwabowicz (1999).

When the field $\boldsymbol{u}$ is known from solving a shell BVP, the spatial position of $\overline{\mathcal{M}}$ is uniquely described by (5.1). It follows from discussion in Sections 3 and 4 that the position of $\overline{\mathcal{M}}$ in space can also be established by the four functions of class C^2: three metric components $\overline{a}_{\alpha\beta}(\vartheta^\lambda)$ and the height function $\overline{z}(\vartheta^\alpha)$ satisfying the Darboux equation for $\overline{\mathcal{M}}$

$$\frac{1}{2}\,\overline{\epsilon}^{\alpha\beta}\,\overline{\epsilon}^{\lambda\mu}\,\overline{z}\|_{\alpha\lambda}\,\overline{z}\|_{\beta\mu} - \overline{K}(1 - \overline{z},_\alpha\,\overline{z},_\beta\,\overline{a}^{\alpha\beta}) = 0 \tag{5.2}$$

The metric components $\overline{a}_{\alpha\beta}$ can be found from the relation

$$\overline{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta} \tag{5.3}$$

where $\gamma_{\alpha\beta}$ are the components of the Green surface strain tensor $\boldsymbol{\gamma}$. Thus, having given the metric of $\mathcal{M}$ and the three functions $\gamma_{\alpha\beta}(\vartheta^\lambda)$ of class C^2 satisfying (5.2) we are able to uniquely establish the metric of $\overline{\mathcal{M}}$.

The BVP formulated in $\gamma_{\alpha\beta}$ and $\overline{z}$ as independent field variables was recently proposed by Szwabowicz (1999) for static analysis of thin shells. It was also assumed that the shell is composed of an isotropic elastic material and that the strains are small everywhere in the shell space. The resulting BVP consists of four non-linear shell equations – three equilibrium equations and one compatibility condition following from the Darboux equation (5.2) – which are linear in $\gamma_{\alpha\beta}$ and non-linear only in $\overline{z}$. Four fields $\gamma_{\alpha\beta}$ and $\overline{z}$ satisfying such a BVP allow us to establish the spatial position of $\overline{\mathcal{M}}$ by carrying out three quadratures analogous to those discussed in Section 4.

With $\overline{z}(\vartheta^\alpha)$ given, the vector $\overline{\boldsymbol{r}}$ can be decomposed, as in (3.3), into

$$\overline{\boldsymbol{r}} = \overline{\boldsymbol{p}} + \overline{z}\boldsymbol{k} \tag{5.4}$$

The geometry of projection $\overline{\mathcal{P}}$ of $\overline{\mathcal{M}}$ onto the Oxy plane is described by

$$\begin{aligned} \overline{\boldsymbol{g}}_\alpha &= \overline{\boldsymbol{p}},_\alpha = \overline{\boldsymbol{a}}_\alpha - \overline{z},_\alpha\,\boldsymbol{k} \\ \overline{g}_{\alpha\beta} &= \overline{\boldsymbol{g}}_\alpha \cdot \overline{\boldsymbol{g}}_\beta = \overline{a}_{\alpha\beta} - \overline{z},_\alpha\,\overline{z},_\beta \\ \overline{g} &= \det(\overline{g}_{\alpha\beta}) \end{aligned} \tag{5.5}$$

Having given the geometry of $\mathcal{M}$, it is seen from (5.3) and $(5.5)_{2,3}$ that three strains $\gamma_{\alpha\beta}$ and the height function $\overline{z}$ uniquely describe the flat metric $d\overline{s}^2 = \overline{g}_{\alpha\beta}d\vartheta^\alpha d\vartheta^\beta$ of the projected region $\overline{\mathcal{P}}$. Therefore, following Section 4, by analogy to (4.27), (4.26) and $(4.25)_1$ we can immediately establish Cartesian components of $\overline{\boldsymbol{p}} = \overline{x}^\alpha \boldsymbol{i}_\alpha = \overline{x}\boldsymbol{i} + \overline{y}\boldsymbol{j}$ expressed by $\overline{g}_{\alpha\beta}$ from the quadrature

$$\overline{\boldsymbol{p}} = \overline{\boldsymbol{p}}_0 + \int \frac{1}{\sqrt{\overline{G} + 2\sqrt{\overline{g}}}} \Big[\cos\overline{\phi}(\delta^{\alpha\lambda}\,\overline{g}_{\lambda\beta} + \sqrt{\overline{g}}\delta^\alpha_\beta) - \\ - \sin\overline{\phi}(e^{\alpha\lambda}\,\overline{g}_{\lambda\beta} + \sqrt{\overline{g}}\,e^{\alpha}_{\cdot\beta})\Big]\boldsymbol{i}_\alpha\, d\vartheta^\beta \tag{5.6}$$

Here

$$\overline{G} = \overline{g}_{11} + \overline{g}_{22} \qquad\qquad \overline{g} = \overline{g}_{11}\overline{g}_{22} - (\overline{g}_{12})^2 \tag{5.7}$$

and the angle of rotation $\overline{\phi}$ is established from $\overline{g}_{\alpha\beta}$ by a quadrature analogous to (4.23) and (4.22)

$$\overline{\phi} = \overline{\phi}_0 + \int \frac{1}{\overline{G} + 2\sqrt{\overline{g}}}\Big[\frac{1}{\sqrt{\overline{g}}}\overline{g}_{\alpha\beta}\Big(\frac{1}{2}e^{\beta\lambda}\overline{G}_{,\lambda} - \delta^{\beta\kappa}\overline{g}_{\kappa\rho,\lambda}\,e^{\rho\lambda}\Big) - \\ -\overline{g}_{\alpha\beta,\lambda}\,e^{\beta\lambda} - e^{\cdot\lambda}_{\alpha}(\sqrt{\overline{g}})_{,\lambda}\Big]\,d\vartheta^\alpha \tag{5.8}$$

The displacement field $\boldsymbol{u}$, if necessary, can now be calculated from $\overline{\boldsymbol{p}}$ and $\overline{z}$ by the simple relation

$$\boldsymbol{u} = (\overline{\boldsymbol{p}} - \boldsymbol{p}) + (\overline{z} - z)\boldsymbol{k} \tag{5.9}$$

Since all the metric components $\overline{g}_{\alpha\beta}$ in (5.6) and (5.7), and thus also in (5.8), are uniquely expressed by $\gamma_{\alpha\beta}$, $\overline{z}$ and the position of $\mathcal{M}$, we have explicitly determined here the field $\boldsymbol{u}$ in terms of these quantities. The result is purely kinematic and valid for an arbitrary geometry of $\mathcal{M}$ satisfying (3.7) as well as for unrestricted surface strains. It does not depend on the material of which the shell is composed as well.

6. Conclusions

We have explicitly shown that in order to find the spatial position of the reference surface of a deformed shell it is sufficient to know only the position of the undeformed shell reference surface, three surface strains and one height function of the deformed surface over the coordinate plane. The two remaining position components of the deformed surface can be found by quadratures (5.6) and (5.8). This purely kinematic result may have important implications for the non-linear theory of thin shells. It suggests, in particular, that the BVP formulated in the surface strains and the height function by Szwabowicz (1999) may be an attractive alternative to other BVPs reviewed by Pietraszkiewicz (1989, 2001) and expressed in displacements, or in rotations and other fields, or in surface strain and/or stress measures as independent variables.

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References

1. DARBOUX G., 1894, *Leçons sur la Théorie Général des Surfaces, Troisiéme Partie*, Gauthier-Villars, Paris
2. HARTMAN P., WINTNER A., 1951, Gaussian curvature and local embedding, *American Journal of Mathematics*, **73**, 876-884
3. HOGER A., CARLSON D E., 1984, Determination of the stretch and rotation in polar decomposition of the deformation gradient, *Quarterly of Applied Mathematics*, **42**, 1, 113-117
4. PIETRASZKIEWICZ W., 1989, Geometrically nonlinear theories of thin elastic shells, *Advances in Mechanics*, **12**, 1, 52-130
5. PIETRASZKIEWICZ W., 2001, Teorie nieliniowe powłok, Part 4.2 of *Mechanika Sprężystych Płyt i Powłok*, Ed. by C. Woźniak, 424-497, PWN, Warszawa
6. SZWABOWICZ M.L., 1999, Deformable surfaces and almost inextensional deflections of thin shells, Rozprawa habilitacyjna, *Zeszyty Naukowe IMP PAN*, Nr. 501/1490/99, Gdańsk

O wyznaczaniu przemieszczeń ze znanych odkształceń i funkcji wysokości w nieliniowej teorii powłok

Streszczenie

Wektor wodzący powierzchni podstawowej powłoki odkształconej wyrażono przez jego rzut na płaszczyznę odniesienia i funkcję wysokości ponad tę płaszczyznę. Zrzutowane składowe tego wektora wyznaczono za pomocą kwadratur zależnych od odkształceń powierzchni i funkcji wysokości, które można wyznaczyć, rozwiązując zagadnienie

brzegowe nieliniowej teorii cienkich powłok sprężystych w postaci zaproponowanej przez Szwabowicza (1999). Odnośne pole przemieszczeń wyznaczono przy pomocy prostych wzorów algebraicznych.

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