DETERMINATION OF THE DEFORMED POSITION OF A THIN SHELL FROM SURFACE STRAINS AND HEIGHT FUNCTION

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Abstract

This paper focuses on the following problem: given the strain tensor of a deformed reference surface of a thin shell and the distances of the points on this surface from some arbitrarily fixed reference plane (the so called height function) find the position of this reference surface. Two alternative procedures supplying the solution are developed. The first one follows from the ideas developed by Darboux (1894), whereas the second one is based on the polar decomposition theorem and techniques developed in continuum mechanics. These procedures are purely kinematic, valid for arbitrary surface geometry and for unrestricted surface strains. Szwabowicz (1999) proposed a relatively simple non-linear boundary value problem (BVP) for thin elastic shells, which was expressed in three surface strains and the height function as basic independent field variables. The results of this paper suggest that this approach to the non-linear problems of thin shells may be an attractive alternative to other BVP's developed in the literature.

Key words: Thin shell, non-linear theory, Darboux equation, kinematics of surface
PACS: 46.70.De, 83.10.Bb
1991 MSC: 74K25, 53A05, 53A17

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1 Introduction

The non-linear theory of thin shells is based on the kinematic hypothesis that deformation of a shell can be described with sufficient accuracy by deformation of its reference surface alone. Determination of the deformed position of this surface in the equilibrium state is the ultimate goal of the shell static analysis. One can achieve this goal directly in one step, tackling all the complexity of the problem in one go. Alternatively, one may divide the process of solution into two or more relatively disjoint and easier stages. Feasibility of the latter plan relies on setting the boundary-value problem (BVP) for a shell in variables that would permit to use methods from modern mathematics.

The second author gave a fairly comprehensive review of various existing formulations of the non-linear theory of thin shells in [1,2]. Roughly speaking, all of them may be divided into three major groups: (i) traditional formulation in displacements [3–6], (ii) in rotations and other fields [1,7], and (iii) in surface strain and/or stress measures [1,8–11]. Each of the BVP's connected with these formulations has some advantages and some limitations, and which of these prevail in a particular case depends on the motivation and goals to be achieved.

Solving the BVP for displacements (i) leads to straightforward determination of the position of the deformed shell. This is presently *the only formulation* (allowing for some hybrid variants with additional intermediate working variables) implemented in the commercial FEM software. Its widespread use results from a large number of easily accessible, ready to use element libraries rather than from any real analytic merits (see the comments by Simmonds [12]). A general BVP of this type — for finite deflections — is very complex and — for problems, where non-linearity is the dominant feature — requires tremendous computational effort, well-paid personnel and costly equipment.

The BVP (ii) in rotations and other fields can effectively be applied to one-dimensional (1D) shell problems, see [13]. Its application to 2D problems is limited by several additional requirements such as, for example, special form of the boundary conditions (see [1]) and the necessity to perform non-standard analysis on the proper orthogonal group SO(3).

The intrinsic shell equations (iii) are relatively simple even for unrestricted deflections. However, they are applicable only to special shell problems which can be formulated entirely in the surface strain and/or stress measures as primary working variables. To establish the deformed shell position from these variables one should complete the second step: to solve the non-linear compatibility conditions for displacements. This problem is still not too well recognized in the non-linear theory of shells (see the papers by the first author [14] and

Ciarlet and Larsonneur [15] on this topic).

A novel formulation of the non-linear BVP for thin elastic shells undergoing small strains was developed by Szwabowicz [16]. It is expressed through three surface strains and the height function of the deformed shell reference surface as basic independent field variables. The BVP posed in this form benefits from relative simplicity of intrinsic shell relations and circumvents complexities of the displacement approach. The corresponding field equations consist of three equilibrium equations and one extended equation of Darboux which plays the rôle of the compatibility condition for the four unknowns. When the surface strains and the height function are found by solving the BVP, the surface curvature changes can be computed from a simple differential relation, and internal surface stress and couple resultants follow thereafter from the constitutive equations. It was also noted in Szwabowicz [16] that the two remaining Cartesian components of the deformed position of the shell reference surface can be determined by quadratures from the surface strains and the height function.

In this paper we develop two different but equivalent procedures allowing one to establish the deformed position of the shell reference surface by quadratures from the undeformed surface metric components, three surface strains and the height function of the deformed reference surface. The first procedure has its roots in old ideas of Darboux [17]. It corrects some results derived for other purposes by Hartman and Wintner [18]. The second procedure is based on the polar decomposition theorem used in continuum mechanics and is our original alternative solution to the problem.

Strictly speaking, the problem considered in this paper may be stated as follows. Suppose we are given a surface $\mathscr{M}$ (possibly with a boundary $\partial\mathscr{M}$) immersed in the Euclidean three-space $\mathbb{R}^3$. The immersion is determined by a known position vector $\mathbf{r} = \mathbf{r}(\vartheta^\alpha)$, where ϑ^α, $\alpha = 1, 2$ are curvilinear coordinates covering $\mathscr{M}$.

Suppose our surface has undergone some C^3 deformation $\mathscr{M} \to \overline{\mathscr{M}}$, and the only data related to the deformed surface $\overline{\mathscr{M}}$ at our disposal are the following four functions:

- three components $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\vartheta^\alpha)$ of the surface strain tensor;
- the distance $\overline{z} = \overline{z}(\vartheta^\alpha)$ of the points of $\overline{\mathscr{M}}$ from some fixed plane in $\mathbb{R}^3$.

Our goal is to determine the position $\overline{\mathbf{r}} = \overline{\mathbf{r}}(\vartheta^\alpha)$ *(and thus the field of displacements) of the deformed surface* $\overline{\mathscr{M}}$ *from just these and only these data.*

Surprising as it may seem, the above data *are in all cases sufficient* to solve the problem up to a restricted class of rigid body motions of $\overline{\mathscr{M}}$. To exclude these rigid motions we need to prescribe three additional initial conditions at some arbitrarily chosen point of $\overline{\mathscr{M}}$. Moreover, the solution itself boils down to three subsequent quadratures of suitable combinations of the foregoing four functions. The integration constants in these quadratures are exactly the required additional initial conditions that exclude the rigid motions.

Although the final solution itself has a quite compact and clear-cut form, the process that leads to it is quite involved and requires substantial support from differential geometry and the theory of total differential equations. Therefore in Sections 2 and 3 we briefly recall some basic facts from theory of surfaces in the Euclidean space.

The key element in the process of solving this problem is the observation, following from the theorem of Minding (see [19] or [20]), that any abstract flat Riemannian two-dimensional metric, i.e. a metric with Gaussian curvature $K \equiv 0$, is always immersible in the Euclidean plane $\mathbb{R}^2$. Translating this statement into the language understandable to an engineer, this means that knowledge of the distances between any two points belonging to some flat region in $\mathbb{R}^2$ suffices to determine absolute positions of these points up to rigid body motions.

The second element is the actual computation of the position vector connected with a given prescribed flat metric. The first solution to this problem can be traced back to the celebrated treatise by Darboux [17]. Later this same problem was treated by Hartman and Wintner in [18] (see page 11 for discussion of their result).

Since the problem we are considering here concerns general — and not only flat — metrics, the above two elements must be supplemented by a third one that will enable to extend the solution for flat metrics to this general case. Note that the projection of any surface on the Euclidean plane is obviously a flat subset of this plane. Of course, the metric of this projection differs from the metric of the projected surface, but relation between them can be established by means of simple algebraic operations on our starting data.

The logical combination of the above three elements supplies the solution to our problem in the general case. The rest of this paper is devoted to systematic realization of this plan. In Section 4 we develop two alternative procedures for embedding a two-dimensional flat Riemannian metric into the Oxy plane. Both solutions are expressed via quadratures. The first of them (see Subsection 4.1) aims at determination of those positions of all coordinate lines $\vartheta^\alpha = const$ that realize the prescribed metric. Consecutive steps consist in: (i) determination of the angle ψ between the projected ϑ^1-coordinate curve and the Ox axis, (30); and (ii) determination of the Cartesian components of the position vector expressed through the components of the metric tensor $g_{\alpha\beta}$ and ψ, formulas (34) and (35), respectively.

The second procedure is based on mapping a domain in the Oxy plane parameterized by Cartesian coordinates into the flat region of the projected reference surface. The 2D gradient of this map is then decomposed into the right stretch tensor and the rotation tensor, for which explicit formulae (45) and (61), (46) are derived in terms of $g_{\alpha\beta}$. Then the position vector (65) of the projected region is obtained by quadratures (66) and (67) involving only $g_{\alpha\beta}$.

2 Notation and surface geometry

Notation used here follows that of Pietraszkiewicz [1,2] and Szwabowicz [16].

A simply connected regular surface $\mathscr{M}$ in the Euclidean space $\mathbb{R}^3$ may locally be described by choosing a fixed orthonormal frame $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$, $O \in \mathbb{R}^3$, and three functions $x(\vartheta^\alpha)$, $y(\vartheta^\alpha)$ and $z(\vartheta^\alpha)$ of class C^2 [1], where $\vartheta^\alpha, \alpha = 1, 2$, are surface curvilinear coordinates. The position vector of $\mathscr{M}$ is then given by

$$\mathbf{r} = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k} = \mathbf{r}(\vartheta^\alpha)\,. \tag{1}$$

With each regular point $M \in \mathscr{M}$ we can associate the natural basis $\mathbf{a}_\alpha = \partial\mathbf{r}/\partial\vartheta^\alpha \equiv \mathbf{r},_\alpha$, the cobasis $\mathbf{a}^\beta$ such that $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha$ with $\delta^1_1 = \delta^2_2 = 1$, $\delta^1_2 = \delta^2_1 = 0$, the components $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta$ of the surface metric tensor $\mathbf{a}$ with $a = \det(a_{\alpha\beta}) > 0$, the unit normal $\mathbf{n} = \frac{1}{\sqrt{a}}\,\mathbf{a}_1 \times \mathbf{a}_2$ orienting $\mathscr{M}$, the components $b_{\alpha\beta} = -\mathbf{n},_\alpha \cdot \mathbf{a}_\beta$ of the second fundamental tensor $\mathbf{b}$, and the components $\epsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ of the surface permutation tensor $\boldsymbol{\epsilon}$ with $\epsilon_{\alpha\beta} = \sqrt{a}e_{\alpha\beta}$, $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$.

The components $a_{\alpha\beta}$ and $b_{\alpha\beta}$ satisfy the Gauss-Mainardi-Codazzi equations

$$b_{\beta\lambda}|_\mu = b_{\beta\mu}|_\lambda\,, \quad b_{\alpha\lambda}b_{\beta\mu} - b_{\alpha\mu}b_{\beta\lambda} = R_{\alpha\beta\lambda\mu}\,, \tag{2}$$

where $(.)|_\alpha$ denotes the surface covariant derivative, and $R_{\alpha\beta\lambda\mu}$ are components of the Riemann-Christoffel tensor related to the Gauss curvature K and $a_{\alpha\beta}$ by

$$R_{\alpha\beta\lambda\mu} = a_{\alpha\kappa}(\Gamma^\kappa_{\beta\mu},_\lambda - \Gamma^\kappa_{\beta\lambda},_\mu + \Gamma^\kappa_{\rho\lambda}\Gamma^\rho_{\beta\mu} - \Gamma^\kappa_{\rho\mu}\Gamma^\rho_{\beta\lambda})\,, \tag{3}$$

$$\Gamma^\kappa_{\alpha\beta} = \tfrac{1}{2}a^{\kappa\lambda}(a_{\lambda\alpha},_\beta + a_{\lambda\beta},_\alpha - a_{\alpha\beta},_\lambda) = -\mathbf{a}_\alpha \cdot \mathbf{a}^\kappa,_\beta\,, \tag{4}$$

$$K = \tfrac{1}{4}\epsilon^{\alpha\beta}\epsilon^{\lambda\mu}R_{\alpha\beta\lambda\mu} \equiv \mathrm{Riem}(a_{\alpha\beta})\,. \tag{5}$$

[1] Although the standard result from differential geometry actually requires continuity of class C^3 in this context, Hartman and Wintner in [18] lightened this to merely C^2 and even lower in special cases.

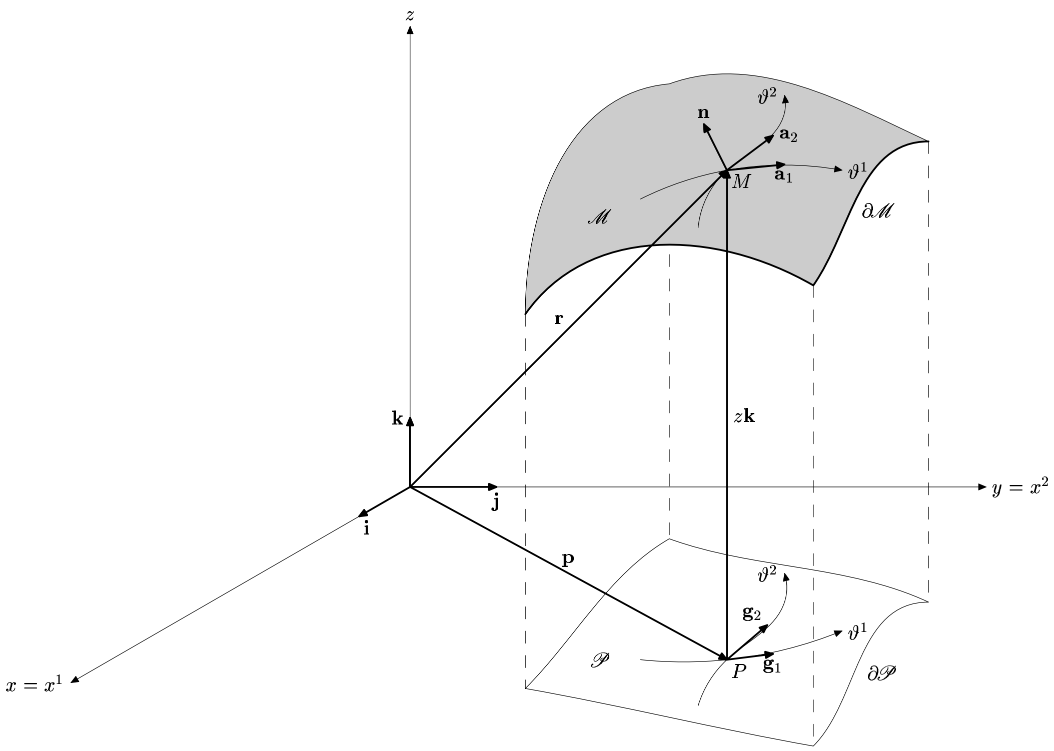

Fig. 1.

It is evident from (5) that K is determined by the Riem(.) operator entirely from the metric components $a_{\alpha\beta}$.

3 Equation of Darboux

The position of $\mathscr{M}$ in $\mathbb{R}^3$ can also be established by prescribing three functions $a_{\alpha\beta}(\vartheta^\lambda)$ of class C^2 and one coordinate function, say $z(\vartheta^\alpha)$, called the height function of $\mathscr{M}$, satisfying the equation of Darboux [17]

$$\mathcal{M}(z) - K\,(1 - z_{,\alpha}\, z_{,\beta}\, a^{\alpha\beta}) = 0\,, \tag{6}$$

where $\mathcal{M}(z)$ is the Monge-Ampère operator defined by

$$\mathcal{M}(z) \stackrel{def}{=} \tfrac{1}{2}\,\epsilon^{\alpha\beta}\epsilon^{\lambda\mu} z|_{\alpha\lambda} z|_{\beta\mu}\,. \tag{7}$$

To see how the equation (6) is related with the problem of determination of a surface $\mathscr{M}$ in $\mathbb{R}^3$, let us decompose the position vector $\mathbf{r}$ as follows

$$\mathbf{r} = \mathbf{p} + z\,\mathbf{k}\,, \tag{8}$$

where $\mathbf{p}$ is the position vector of $P \in \mathscr{P}$ — the projection of $M \in \mathscr{M}$ onto the coordinate plane Oxy, Fig. 1. From (8) it follows that $z = \mathbf{r} \cdot \mathbf{k}$ and $z,_\alpha = \mathbf{a}_\alpha \cdot \mathbf{k}$, which introduced into the identity $\mathbf{k} = (\mathbf{a}_\alpha \cdot \mathbf{k})\mathbf{a}^\alpha + (\mathbf{n} \cdot \mathbf{k})\mathbf{n}$ leads to

$$\mathbf{k} = z,_\alpha \mathbf{a}^\alpha + n\,\mathbf{n}\,, \quad n = \mathbf{n} \cdot \mathbf{k}\,. \tag{9}$$

Let us square the relation (9), differentiate it with respect to the curvilinear coordinates and multiply the result by $\mathbf{a}_\alpha$. This yields

$$n^2 = 1 - z,_\alpha\, z,_\beta\, a^{\alpha\beta}\,, \tag{10}$$

$$0 = \mathbf{k},_\beta \cdot \mathbf{a}_\alpha = z,_{\alpha\beta} + z,_\kappa\, \mathbf{a}^\kappa,_\beta \cdot \mathbf{a}_\alpha + n\,\mathbf{n},_\beta \cdot \mathbf{a}_\alpha\,. \tag{11}$$

From (11) it follows that

$$b_{\alpha\beta} = \frac{1}{n}\, z|_{\alpha\beta}\,, \tag{12}$$

provided that

$$z,_\alpha\, z,_\beta\, a^{\alpha\beta} < 1\,, \quad n^2 > 0\,. \tag{13}$$

In view of (5), the Gauss equation $(2)_2$ can be modified to the form

$$\tfrac{1}{2}\,\epsilon^{\alpha\beta}\epsilon^{\lambda\mu} b_{\alpha\lambda} b_{\beta\mu} = K\,. \tag{14}$$

Introducing now the representation (12) and (10) into (14) we can easily obtain the Darboux equation (6).

Likewise, using (12) the Mainardi-Codazzi equations $(2)_1$ can be transformed as follows:

$$0 = \epsilon^{\lambda\mu} b_{\beta\lambda}|_\mu = \epsilon^{\lambda\mu}\left(\frac{1}{n} z|_{\beta\lambda}\right)|_\mu = \epsilon^{\lambda\mu}\left[\left(\frac{1}{n}\right),_\mu z|_{\beta\lambda} + \frac{1}{n} z|_{\beta\lambda\mu}\right]\,. \tag{15}$$

But

$$\left(\frac{1}{n}\right)|_\mu = \frac{1}{n^3} z|_{\alpha\mu} a^{\alpha\kappa} z|_\kappa\,,$$

$$\epsilon^{\lambda\mu} z|_{\beta\lambda\mu} = \frac{1}{2} R_{\alpha\beta\lambda\mu}\epsilon^{\lambda\mu} z|^\alpha\,,$$

and therefore

$$0 = \epsilon^{\lambda\mu} b_{\beta\lambda}|_{\mu} = \frac{1}{n}\left[\frac{1}{n^2}\epsilon^{\lambda\mu} z|_{\alpha\mu} z|_{\beta\lambda} + \epsilon_{\alpha\beta} K\right] z|^{\alpha} \,. \tag{16}$$

The equation (16) is identically satisfied either if $z|^{\alpha} = a^{\alpha\kappa} z,_{\kappa} = 0$, that is if $z(\vartheta^{\alpha}) = const$, $b_{\alpha\beta} \equiv 0$, and the surface $\mathscr{M}$ is the plane parallel to the Oxy plane, or if

$$\epsilon^{\lambda\mu} z|_{\alpha\mu} z|_{\beta\lambda} + \epsilon_{\alpha\beta} K n^2 = 0 \,. \tag{17}$$

Note, however, that the latter equation is also satisfied if $z(\vartheta^{\alpha}) = const$ and, thus, covers both above cases. Multiplying (17) by $\epsilon^{\alpha\beta}$ and using (10) we obtain the equation of Darboux (6).

The equation (6) is a compatibility condition for the surface metric components $a_{\alpha\beta}$ and the surface position coordinate z.

Four functions $a_{\alpha\beta}(\vartheta^{\lambda})$ and $z(\vartheta^{\alpha})$ satisfying (13) and the Darboux equation (6) describe the surface position in space up to rigid translation and rotation parallel to the Oxy plane.

4 Embeddings of two-dimensional flat metrics

Let us project the surface $\mathscr{M}$ onto the coordinate plane Oxy, Fig. 1. The projected flat region $\mathscr{P}$ is parameterized by the same curvilinear coordinates ϑ^{α}, and by (8) we have

$$\mathbf{a}_{\alpha} = \mathbf{g}_{\alpha} + z,_{\alpha} \mathbf{k} \,, \qquad \mathbf{g}_{\alpha} = \mathbf{p},_{\alpha} \,, \qquad a_{\alpha\beta} = g_{\alpha\beta} + z,_{\alpha} z,_{\beta} \,, \tag{18}$$

$$g_{\alpha\beta} = \mathbf{g}_{\alpha} \cdot \mathbf{g}_{\beta} \,, \qquad \mathbf{g}^{\beta} \cdot \mathbf{g}_{\alpha} = \delta^{\beta}_{\alpha} \,, \qquad g^{\alpha\beta} = \mathbf{g}^{\alpha} \cdot \mathbf{g}^{\beta} \,, \tag{19}$$

$$g = \det(g_{\alpha\beta}) = g_{11} g_{22} - (g_{12})^2 \,, \qquad g^{\alpha\beta} = \frac{1}{g}\begin{bmatrix} g_{22} & -g_{12} \\ -g_{12} & g_{11} \end{bmatrix} \,, \tag{20}$$

$$\cos\alpha = \frac{\mathbf{g}_1}{\sqrt{g_{11}}} \cdot \frac{\mathbf{g}_2}{\sqrt{g_{22}}} = \frac{g_{12}}{\sqrt{g_{11} g_{22}}} \,, \qquad \sin\alpha = \left| \frac{\mathbf{g}_1}{\sqrt{g_{11}}} \times \frac{\mathbf{g}_2}{\sqrt{g_{22}}} \right| = \frac{\sqrt{g}}{\sqrt{g_{11} g_{22}}} \,,$$

where α is the angle between ϑ^1- and ϑ^2-coordinate curves at $P \in \mathscr{P}$.

The permutation tensor $\boldsymbol{\varepsilon}$ in the flat region $\mathscr{P}$ is defined by

$$\boldsymbol{\varepsilon} = \varepsilon_{\alpha\beta}\, \mathbf{g}^{\alpha} \cdot \mathbf{g}^{\beta} = e_{\alpha\beta}\, \mathbf{i}^{\alpha} \otimes \mathbf{i}^{\beta} = \mathbf{i} \otimes \mathbf{j} - \mathbf{j} \otimes \mathbf{i} \,, \tag{21}$$

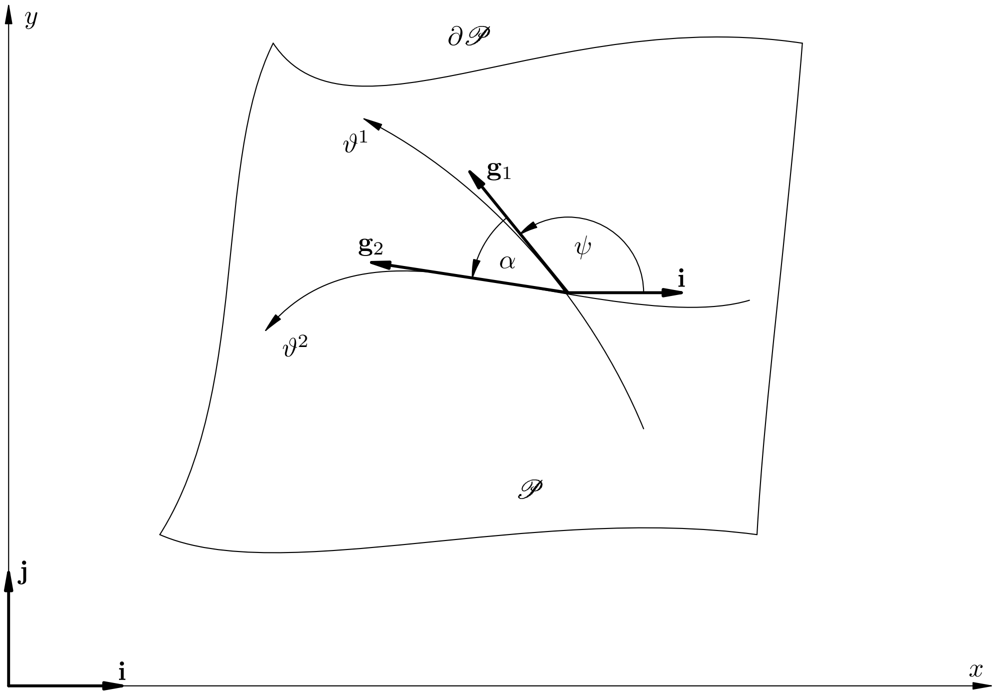

Fig. 2. Projection of $\mathscr{M}$ seen from above.

$$\varepsilon_{\alpha\beta} = \sqrt{g}\, e_{\alpha\beta} \quad , \quad \mathbf{i}_\alpha = \mathbf{i}^\alpha = (\mathbf{i}, \mathbf{j})\,.$$

By the decomposition (8) and geometric relations (18)-(21) the problem of finding the position vector $\mathbf{r}$ of $\mathscr{M}$ has been reduced to finding a local embedding of a prescribed flat metric $ds^2 = g_{\alpha\beta} d\vartheta^\alpha d\vartheta^\beta$ into the Euclidean plane Oxy. This differential problem turns out to be completely solvable via quadratures, which was already noted by Darboux ([17, page 216]).

4.1 Determination of the embedding via coordinate curves

Let ψ denote the oriented angle between the axis Ox and the projection of coordinate curve ϑ^1, and α the oriented angle between the tangents to the coordinate lines ϑ^1 and ϑ^2 (see Fig. 2). Then the following relations between the base vectors $\mathbf{g}_\alpha$ and the angles ψ and α hold true:

$$\mathbf{g}_1 = \sqrt{g_{11}}\,(\cos\psi \mathbf{i} + \sin\psi \mathbf{j}) \ , \tag{22}$$

$$\begin{aligned} \mathbf{g}_2 &= \sqrt{g_{22}}\,[\cos(\psi+\alpha)\mathbf{i} + \sin(\psi+\alpha)\mathbf{j}] \\ &= \sqrt{\frac{g_{22}}{g_{11}}}\,(\cos\alpha\, \mathbf{g}_1 - \sin\alpha\, \boldsymbol{\varepsilon}\mathbf{g}_1) \ , \end{aligned} \tag{23}$$

$$\boldsymbol{\varepsilon}\mathbf{g}_1 = \sqrt{g_{11}}\,(\sin\psi\, \mathbf{i} - \cos\psi\, \mathbf{j}) = -\sqrt{g}\, \mathbf{g}^2\,. \tag{24}$$

Let us differentiate (22) with respect to the coordinates ϑ^λ and use the relations (22), (23) and (24). We get

$$\begin{aligned}\mathbf{g}_1,_\lambda &= \frac{1}{2\sqrt{g_{11}}} g_{11},_\lambda (\cos\psi\,\mathbf{i} + \sin\psi\,\mathbf{j}) + \sqrt{g_{11}}\,(-\sin\psi\,\mathbf{i} + \cos\psi\,\mathbf{j})\,\psi,_\lambda \\ &= \frac{1}{2g_{11}} g_{11},_\lambda \mathbf{g}_1 + \sqrt{g}\mathbf{g}^2\psi,_\lambda \ ,\end{aligned}$$

$$\mathbf{g}_1,_\lambda \cdot \mathbf{g}^2 = \sqrt{g} g^{22} \psi,_\lambda = \frac{g_{11}}{\sqrt{g}} \psi,_\lambda \ . \tag{25}$$

But, as follows from the definition (4) of the Christoffel symbols, we also have

$$\begin{aligned}\mathbf{g}_1,_\lambda \cdot \mathbf{g}^2 &= \frac{1}{2} g^{2\mu} (g_{\mu 1},_\lambda + g_{\mu\lambda},_1 - g_{1\lambda},_\mu) \\ &= -\frac{1}{2g} [g_{12} g_{11},_\lambda - g_{11} (g_{12},_\lambda + g_{2\lambda},_1 - g_{1\lambda},_2)] \ ,\end{aligned} \tag{26}$$

where (20) have been used. Combining now (25) and (26) we obtain a total differential equation for the angle ψ:

$$\psi,_\lambda = -\frac{1}{2\sqrt{g}} \left(\frac{g_{12}}{g_{11}} g_{11},_\lambda - g_{12},_\lambda - g_{2\lambda},_1 + g_{1\lambda},_2 \right) \ , \tag{27}$$

which may also be written as an overdetermined system of two partial differential equations

$$\psi,_1 = -\frac{1}{2\sqrt{g}} \left(\frac{g_{12}}{g_{11}} g_{11},_1 - 2g_{12},_1 + g_{11},_2 \right) \ , \tag{28}$$

$$\psi,_2 = -\frac{1}{2\sqrt{g}} \left(\frac{g_{12}}{g_{11}} g_{11},_2 - g_{22},_1 \right) \ . \tag{29}$$

Verification of the integrability condition $\psi,_{\lambda\mu}\varepsilon^{\lambda\mu} = 0$ shows that the equation (27) is completely integrable provided the metric $ds^2 = g_{\alpha\beta} d\vartheta^\alpha d\vartheta^\beta$ is flat, i.e. the compatibility condition $\mathrm{Riem}(g_{\alpha\beta}) = 0$ is satisfied. This condition is identically satisfied here, because by assumption the components $g_{\alpha\beta}$ describe a flat metric over the region $\mathscr{P}$.

When compared with similar equations given already by Darboux [17, page 216] the equations (28) and (29) have opposite signs on the right-hand sides. This may result from different sign convention applied by Darboux [ibid]. Hartmann and Wintner delivered without derivation the set of equations analogous to (28) and (29) , see [18, eqs. (4) and (5)]. However, their equations differ from ours not only by opposite signs of the right-hand sides, but also by additional terms proportional to g_{12} in parenthesis. Aside from possible differences in sign convention, the relations of [18] would agree with ours if $g_{12} = const$, which is the case for orthogonal coordinates, for example. But there is no indication in [18] that the coordinates used there belong to any special class.

Now the angle ψ can be computed from known $g_{\alpha\beta}$ by straightforward integration

$$\psi = \psi_0 - \int \frac{1}{2\sqrt{g}} \left[\left(\frac{g_{12}}{g_{11}} g_{11},_1 - 2g_{12},_1 + g_{11},_2 \right) d\vartheta^1 + \left(\frac{g_{12}}{g_{11}} g_{11},_2 - g_{22},_1 \right) d\vartheta^2 \right] . \tag{30}$$

The position vector $\mathbf{p}$ of $P \in \mathscr{P}$ and its differential expressed with respect to curvilinear coordinates are given by

$$\mathbf{p} = x^\alpha \, \mathbf{i}_\alpha \, , \qquad \mathbf{dp} = \mathbf{p},_\beta \, d\vartheta^\beta = \mathbf{g}_\beta d\vartheta^\beta = x^\alpha,_\beta \, \mathbf{i}_\alpha d\vartheta^\beta \, . \tag{31}$$

Introducing (22), (23) and (24) into (31), after some transformations we obtain the following system of partial differential equations:

$$x^1,_1 = \sqrt{g_{11}} \cos\, \psi \, , \qquad x^1,_2 = \frac{1}{\sqrt{g_{11}}} \left(g_{12} \cos\, \psi - \sqrt{g} \sin\, \psi \right) \, , \tag{32}$$

$$x^2,_1 = \sqrt{g_{11}} \sin\, \psi \, , \qquad x^2,_2 = \frac{1}{\sqrt{g_{11}}} \left(g_{12} \sin\, \psi + \sqrt{g} \cos\, \psi \right) \, . \tag{33}$$

The equations (32) and (33) differ from those derived by Hartman and Wintner [18] in that we have opposite signs in front of the second terms in parenthesis. We could not find any reasonable explanation for this difference, except that there is a typo or a simple error in the derivation supplied by Hartman and Wintner in [18].

The Cartesian components of the position vector $\mathbf{p} = x\,\mathbf{i} + y\,\mathbf{j}$ can now be obtained from the quadratures

$$x = x_0 + \int \left[\sqrt{g_{11}} \cos\, \psi \, d\vartheta^1 + \frac{1}{\sqrt{g_{11}}} \left(g_{12} \cos\, \psi - \sqrt{g} \sin\, \psi \right) d\vartheta^2 \right] \, , \tag{34}$$

$$y = y_0 + \int \left[\sqrt{g_{11}} \sin\, \psi \, d\vartheta^1 + \frac{1}{\sqrt{g_{11}}} \left(g_{12} \sin\, \psi + \sqrt{g} \cos\, \psi \right) d\vartheta^2 \right] \, . \tag{35}$$

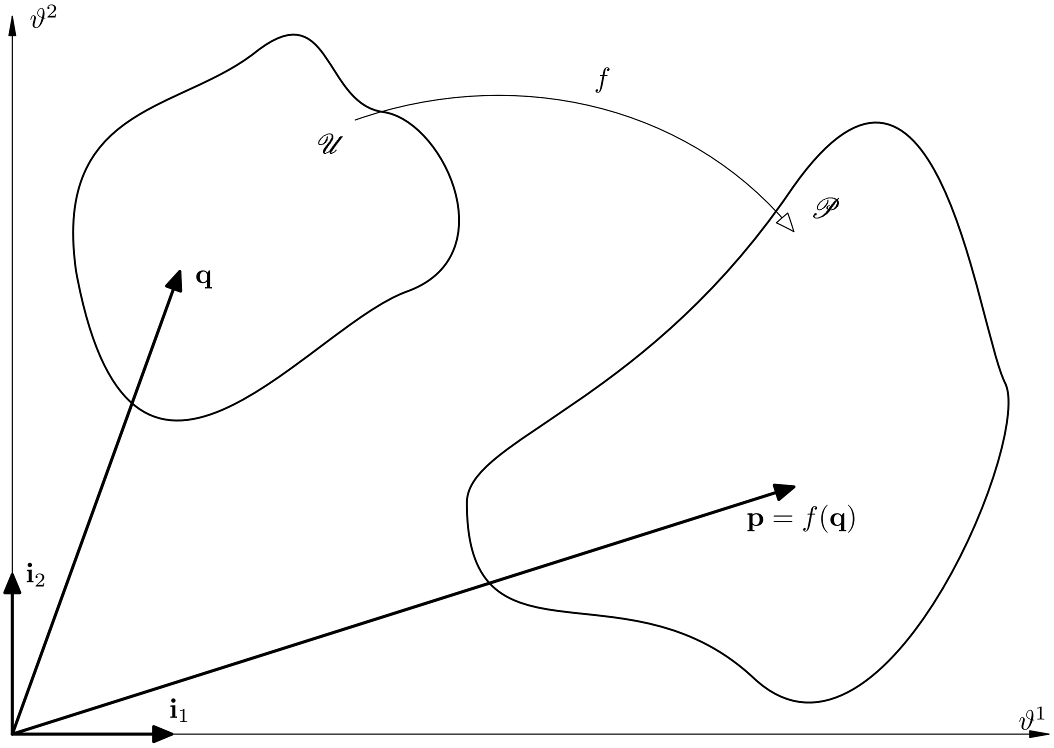

Fig. 3.

By the quadratures (34), (35) and (30) the position vector $\mathbf{p}$ of the flat region $\mathscr{P}$ is expressed entirely in the components $g_{\alpha\beta}$. The solution depends explicitly, through definition of the angle ψ, upon the choice of coordinates used to parameterize the surface $\mathscr{M}$.

4.2 Determining embedding via the polar decomposition theorem

Instead of pursuing the positions of the individual coordinate lines, the question of realization of a given flat metric in the Euclidean plane may be treated as a problem for determination of some map $f : \mathscr{U} \to \mathscr{P}$, where $\mathscr{U} \subset \mathbb{R}^2$ is the domain in the $O\vartheta^1\vartheta^2$ plane over which the given flat metric $ds^2 = g_{\alpha\beta}d\vartheta^\alpha d\vartheta^\beta$ is prescribed (see Fig. 3). We want the map f to place the points of $\mathscr{U}$ in the image $\mathscr{P}$ in such a way that the distances between them coincide with the distances arising from the embedded metric $ds^2 = g_{\alpha\beta}d\vartheta^\alpha d\vartheta^\beta$. Under additional requirement that f be orientation-preserving one faces the classical problem for deformation of a flat body in the Euclidean plane, although the problem is solvable under fairly general conditions and f needs neither be one-to-one nor preserve the orientation.

The position vector $\mathbf{q}$ of points in $\mathscr{U}$ with the Cartesian coordinates $(\vartheta^1, \vartheta^2)$, the line element, and the standard metric are

$$\mathbf{q} = \vartheta^\alpha \mathbf{i}_\alpha \,, \qquad d\mathbf{q} = d\vartheta^\alpha \mathbf{i}_\alpha \,, \qquad ds_q^2 = (d\vartheta^1)^2 + (d\vartheta^2)^2 \,. \tag{36}$$

The position vector of the points in the image $\mathscr{P}$ is given by (31). Two functions $x^\alpha = x^\alpha(\vartheta^\lambda)$ establish the location of any point of $\mathscr{U}$ in the image of f. Assuming f to be continuously differentiable, we may write

$$d\mathbf{p} = \mathbf{\Gamma} d\mathbf{q} \,, \qquad \mathbf{g}_\alpha = \mathbf{\Gamma} \mathbf{i}_\alpha \,, \tag{37}$$

$$\mathbf{\Gamma} = \nabla \mathbf{p} = \mathbf{p}_{,\alpha} \otimes \mathbf{i}^\alpha = \mathbf{g}_\alpha \otimes \mathbf{i}^\alpha \,, \tag{38}$$

where $\mathbf{\Gamma}$ is the 2D gradient of the map f taken in the metric ds_q^2. Assuming f to be orientation-preserving, we also have $\det \mathbf{\Gamma} > 0$.

The polar decomposition of $\mathbf{\Gamma}$ yields

$$\mathbf{\Gamma} = \mathbf{Q}\mathbf{H} \,, \tag{39}$$

where, using the terminology of continuum mechanics, $\mathbf{H}$ is the right stretch tensor (symmetric, $\mathbf{H}^T = \mathbf{H}$, and positive definite, $\mathbf{v} \cdot (\mathbf{H}\mathbf{v}) > 0$ for all vectors $\mathbf{v} \neq \mathbf{0}$), and $\mathbf{Q}$ is the rotation tensor (proper orthogonal, $\mathbf{Q}^T = \mathbf{Q}^{-1}$, $\det \mathbf{Q} = +1$). Our goal is to determine $\mathbf{H}$ and $\mathbf{Q}$, and then $\mathbf{\Gamma}$, from three components $g_{\alpha\beta}$ alone.

First let us compute the right stretch $\mathbf{H}$ in terms of $g_{\alpha\beta}$. It follows from (37) and (39) that

$$ds^2 = d\mathbf{p} \cdot d\mathbf{p} = d\mathbf{q} \cdot (\mathbf{H}^2 d\mathbf{q}) = \mathbf{i}_\alpha \cdot (\mathbf{H}^2 \mathbf{i}_\beta) d\vartheta^\alpha d\vartheta^\beta. \tag{40}$$

If we introduce the tensor $\mathbf{G}$ such that

$$\mathbf{G} = g_{\alpha\beta} \mathbf{i}^\alpha \otimes \mathbf{i}^\beta \,, \qquad \det \mathbf{G} = g \,, \qquad \operatorname{tr} \mathbf{G} = G = g_{11} + g_{22} \,, \tag{41}$$

then from (40) and (41) we obtain

$$\mathbf{H}^2 = \mathbf{G} \,, \qquad \det \mathbf{H} = \sqrt{g} \,. \tag{42}$$

The 2D second-order tensor $\mathbf{H}$ satisfies the Cayley-Hamilton equation

$$\mathbf{H}^2 - (\operatorname{tr} \mathbf{H})\mathbf{H} + (\det \mathbf{H})\mathbf{I} = \mathbf{0} \,, \tag{43}$$

and, according to Hoger and Carlson ([21, eq. (5.2)]), we have

$$\operatorname{tr}\mathbf{H} = \sqrt{\operatorname{tr}\mathbf{G} + 2\sqrt{\det\mathbf{G}}} = \sqrt{C}\,, \qquad C = G + 2\sqrt{g}\,. \tag{44}$$

Having solved (43) for $\mathbf{H}$ with the use of (42) and (44), we express $\mathbf{H}$ entirely through $g_{\alpha\beta}$:

$$\mathbf{H} = \frac{1}{\sqrt{C}}(\mathbf{G} + \sqrt{g}\,\mathbf{I})\,. \tag{45}$$

Now our goal is to determine the rotation tensor $\mathbf{Q}$. In 2D space $\mathbf{Q}$ can be represented as

$$\mathbf{Q} = \cos\phi\,\mathbf{I} - \sin\phi\,\boldsymbol{\varepsilon}\,, \tag{46}$$

$$\mathbf{I} = \mathbf{g}_\alpha \otimes \mathbf{g}^\alpha = \mathbf{i}_\alpha \otimes \mathbf{i}^\alpha\,, \tag{47}$$

where ϕ is the oriented angle of rotation in the $O\vartheta^1\vartheta^2$ plane. In order to relate $\mathbf{Q}$ with the coefficients $g_{\alpha\beta}$, we have to resort to the integrability condition for $\mathbf{p}$:

$$\varepsilon^{\alpha\beta}\mathbf{p},_{\alpha\beta} = \mathbf{0}\,. \tag{48}$$

Using (37) and (39) this condition may be transformed to the form

$$[\mathbf{Q},_\alpha \mathbf{H}\boldsymbol{\varepsilon} + \mathbf{Q}(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha]\mathbf{i}^\alpha = \mathbf{0}\,. \tag{49}$$

Note that differentiating (46) and using (47) leads to the following general formula for the gradient of the in-plane rotation:

$$\mathbf{Q},_\alpha = -\mathbf{Q}\boldsymbol{\varepsilon}\,\phi,_\alpha\,. \tag{50}$$

When (50) is introduced into (49) and the result left-multiplied by $\mathbf{H}$, using $\nabla\phi \equiv \phi,_\alpha \mathbf{i}^\alpha$ we obtain

$$\mathbf{H}\boldsymbol{\varepsilon}\mathbf{H}\boldsymbol{\varepsilon}\nabla\phi = \mathbf{H}(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha \mathbf{i}^\alpha\,. \tag{51}$$

Let us now apply the Cayley-Hamilton theorem (43) to the tensor $\mathbf{H}\boldsymbol{\varepsilon}$. Since $\operatorname{tr}(\mathbf{H}\boldsymbol{\varepsilon}) = 0$ and $\det(\mathbf{H}\boldsymbol{\varepsilon}) = \det\mathbf{H} = \sqrt{g}$, the result is

$$\mathbf{H}\boldsymbol{\varepsilon}\mathbf{H}\boldsymbol{\varepsilon} = -\sqrt{g}\,\mathbf{I}\,, \tag{52}$$

which introduced into (51) yields

$$\nabla\phi = -\frac{1}{\sqrt{g}}\mathbf{H}(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha \mathbf{i}^\alpha \,. \tag{53}$$

By differentiating (45) we obtain

$$(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha \mathbf{i}^\alpha = \left\{-\frac{1}{2C}\mathbf{H}\boldsymbol{\varepsilon}C,_\alpha + \frac{1}{\sqrt{C}}\left[(\mathbf{G}\boldsymbol{\varepsilon}),_\alpha + \boldsymbol{\varepsilon}(\sqrt{g}),_\alpha\right]\right\}\mathbf{i}^\alpha \,. \tag{54}$$

Hence, the right-hand side of (53) is

$$\mathbf{H}(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha \mathbf{i}^\alpha = \frac{1}{C}\left\{-\frac{1}{2}\mathbf{G}\boldsymbol{\varepsilon}C,_\alpha + (\mathbf{G}+\sqrt{g}\mathbf{I})\left[(\mathbf{G}\boldsymbol{\varepsilon}),_\alpha + \boldsymbol{\varepsilon}(\sqrt{g}),_\alpha\right]\right\}\mathbf{i}^\alpha \tag{55}$$

$$= \frac{1}{C}\left[\mathbf{G}(\mathbf{G}\boldsymbol{\varepsilon}),_\alpha + \sqrt{g}(\mathbf{G}\boldsymbol{\varepsilon}),_\alpha - \frac{1}{2}\mathbf{G}\boldsymbol{\varepsilon}G,_\alpha + \boldsymbol{\varepsilon}\sqrt{g}(\sqrt{g}),_\alpha\right]\mathbf{i}^\alpha \,. \tag{56}$$

With (56) and (44) we obtain the following total differential equation for the gradient of ϕ:

$$\nabla\phi = \frac{1}{G+2\sqrt{g}}\left[\frac{1}{2\sqrt{g}}\mathbf{G}\boldsymbol{\varepsilon}G,_\alpha - \frac{1}{\sqrt{g}}\mathbf{G}(\mathbf{G}\boldsymbol{\varepsilon}),_\alpha - (\mathbf{G}\boldsymbol{\varepsilon}),_\alpha - \boldsymbol{\varepsilon}(\sqrt{g}),_\alpha\right]\mathbf{i}^\alpha, \tag{57}$$

which in component form reads

$$\phi,_\alpha = \frac{1}{G+2\sqrt{g}}\left[\frac{1}{\sqrt{g}}g_{\alpha\beta}\left(\frac{1}{2}e^{\beta\lambda}G,_\lambda - \delta^{\beta\kappa}g_{\kappa\rho},_\lambda e^{\rho\lambda}\right)\right. \tag{58}$$
$$\left. - g_{\alpha\beta},_\lambda e^{\beta\lambda} - e_{\alpha}^{\cdot\lambda}(\sqrt{g}),_\lambda\right] .$$

To guarantee the existence of solutions to the above two PDE's one should now verify the compatibility condition $\phi,_{\alpha\beta}\, e^{\alpha\beta} = 0$. By (53), this condition reads

$$0 = \left[\frac{1}{\sqrt{g}}\mathbf{H}(\mathbf{H}\boldsymbol{\varepsilon}),_\alpha \mathbf{i}^\alpha\right],_\beta \mathbf{i}^\beta \,. \tag{59}$$

Yet, instead of plunging into involved computations inevitably connected with this test, we may resort here to a general result, valid for arbitrary deformation of a curved surface, obtained by the first author in [14]. According to this result the alteration of the Gaussian curvature caused by a given arbitrary stretch field $\mathbf{U}$ obeys the formula

$$\overline{K}\,\det(\mathbf{U}) - K = \operatorname{div}\left\{\frac{1}{\det(\mathbf{U})}\boldsymbol{\varepsilon}\mathbf{U}[\operatorname{div}(\mathbf{U}\boldsymbol{\varepsilon})]\right\} .$$

Note that upon the substitution $\mathbf{U} = \mathbf{H}$ in the above, the right-hand side becomes identical with the right-hand side of (59) and, since $\overline{K} \equiv K \equiv 0$ in our case, the compatibility condition is satisfied and, thereby, the total differential equation (53) is completely integrable.

Now the angle of rotation arising in the polar decomposition of $\mathbf{\Gamma}$ follows from the quadrature

$$\phi = \phi_0 + \int \phi,_\alpha \, d\vartheta^\alpha . \tag{60}$$

According to (60) and (58), the angle ϕ is expressed entirely through the components $g_{\alpha\beta}$ by the explicit relation

$$\begin{aligned}\phi = \phi_0 + \int \frac{1}{G\sqrt{g} + 2g} \Big\{ \Big[& \sqrt{g}(g_{12},_1 - g_{11},_2) + g_{11}g_{12},_1 \\ & + \frac{1}{2}g_{12}(g_{22} - g_{11}),_1 - \frac{1}{2}(g_{11} + g_{22})g_{11},_2 \Big] \, d\vartheta^1 \\ & + \Big[\sqrt{g}(g_{22},_1 - g_{12},_2) - g_{22}g_{12},_2 \\ & + \frac{1}{2}g_{12}(g_{22} - g_{11}),_2 + \frac{1}{2}(g_{11} + g_{22})g_{22},_1 \Big] \, d\vartheta^2 \Big\} .\end{aligned} \tag{61}$$

Finally, in order to determine the position vector, introduce the rotation tensor (46) into the polar decomposition formula (39). This permits to compute the gradient $\mathbf{\Gamma}$:

$$\mathbf{\Gamma} = \frac{1}{\sqrt{C}}\,(\cos\phi\mathbf{I} - \sin\phi\boldsymbol{\varepsilon})\,(\mathbf{G} + \sqrt{g}\mathbf{I}) = x^\alpha,_\beta \, \mathbf{i}_\alpha \otimes \mathbf{i}^\beta , \tag{62}$$

$$x^\alpha,_\beta = \frac{1}{\sqrt{G + 2\sqrt{g}}} \left[\cos\phi(\delta^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}\delta^\alpha_\beta) - \sin\phi(e^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}e^{\alpha}_{.\beta})\right] . \tag{63}$$

But

$$\mathbf{\Gamma}\mathbf{i}_\beta = \mathbf{p},_\beta = x^\alpha,_\beta \, \mathbf{i}_\alpha \equiv x,_\beta \, \mathbf{i} + y,_\beta \, \mathbf{j} , \tag{64}$$

and the position vector of the point $P \in \mathcal{P}$ follows from the quadrature

$$\mathbf{p} = \mathbf{p}_0 + \int \mathbf{p}_{,\beta}\, d\vartheta^\beta \,, \tag{65}$$

or explicitly for Cartesian components of $\mathbf{p} = x\,\mathbf{i} + y\,\mathbf{j}$:

$$\begin{aligned} x = x_0 + \int \frac{1}{\sqrt{G + 2\sqrt{g}}} \Big\{ &[\cos\phi (g_{11} + \sqrt{g}) - \sin\phi\, g_{12}] d\vartheta^1 \\ &+ [\cos\phi\, g_{12} - \sin\phi (g_{22} + \sqrt{g})] d\vartheta^2 \Big\} \,, \end{aligned} \tag{66}$$

$$\begin{aligned} y = y_0 + \int \frac{1}{\sqrt{G + 2\sqrt{g}}} \Big\{ &[\cos\phi\, g_{12} + \sin\phi (g_{11} + \sqrt{g})] d\vartheta^1 \\ &+ [\cos\phi (g_{22} + \sqrt{g}) + \sin\phi\, g_{12}] d\vartheta^2 \Big\} \,. \end{aligned} \tag{67}$$

By (66), (67) and (58), the position vector $\mathbf{p}(\vartheta^\alpha)$ is expressed entirely in terms of components $g_{\alpha\beta}$ as well. With known $\mathbf{p}(\vartheta^\alpha)$, the position vector $\mathbf{r}(\vartheta^\alpha)$ of the surface $\mathscr{M}$ in space $\mathbb{R}^3$ follows from the relation (8).

Summarizing, we have shown that in order to find the embedding of a flat metric ds^2 into the 2D Euclidean space one needs to perform three quadratures. We have given two different but equivalent solutions to this problem. According to the first solution presented in Subsection 4.1 one should calculate the quadrature (30) for the angle ψ and then two quadratures (34) and (35) for the Cartesian coordinates x and y of the position vector $\mathbf{p}$, respectively. In the second solution worked out in Subsection 4.2 one calculates the quadrature (61) that yields the angle of rotation $\phi = \phi(\vartheta^\alpha)$, and then two quadratures (66) and (67) yielding two Cartesian coordinates of $\mathbf{p}$. With known $\mathbf{p}$ and z, the position vector $\mathbf{r}$ of the surface $\mathscr{M}$ can be determined from the simple relation (8).

5 Position of the deformed surface

Let $\overline{\mathscr{M}} = \boldsymbol{\chi}(\mathscr{M})$ be the reference surface of the deformed shell arisen from the undeformed surface $\mathscr{M}$ under some map (deformation) $\boldsymbol{\chi}$. This map is assumed to be single-valued, orientation-preserving and differentiable sufficient number of times. The position vector of $\overline{\mathscr{M}}$ relative to the same orthonormal frame $(O, \mathbf{i}, \mathbf{j}, \mathbf{k})$ can be described, in analogy to (1), by

$$\bar{\mathbf{r}}(\vartheta^\alpha) = \boldsymbol{\chi}[\mathbf{r}(\vartheta^\alpha)] = \bar{x}(\vartheta^\alpha)\mathbf{i} + \bar{y}(\vartheta^\alpha)\mathbf{j} + \bar{z}(\vartheta^\alpha)\mathbf{k} \\ = \mathbf{r}(\vartheta^\alpha) + \mathbf{u}(\vartheta^\alpha)\,, \tag{68}$$

where $\mathbf{u}$ is the displacement vector and ϑ^α are convected surface coordinates.

In the convected coordinates all geometric quantities and relations at any regular point $\overline{M} \in \overline{\mathscr{M}}$ are now analogous to those at $M \in \mathscr{M}$ given in Section 2. In this paper quantities corresponding to the deformed surface $\overline{\mathscr{M}}$ will be marked with an overbar: $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{a}}^\beta$, $\bar{a}_{\alpha\beta}$, $\bar{a}^{\alpha\beta}$, $\bar{\mathbf{a}}$, $\bar{a}$, $\bar{\mathbf{n}}$, $\bar{b}_{\alpha\beta}$, $\bar{\epsilon}_{\alpha\beta}$, $\bar{K}$ etc., while the surface covariant derivative in the deformed metric will be denoted by a double vertical stroke $(.)||_\alpha$. The barred quantities can be expressed through analogous unbarred quantities defined on $\mathscr{M}$ and the displacement field $\mathbf{u}$ with the help of formulae presented in Pietraszkiewicz [1] and Szwabowicz [16].

If the shell problem is solved for displacements $\mathbf{u}$, the position of $\overline{\mathscr{M}}$ in space is uniquely determined by (68). It follows from discussion in Sections 3 and 4 that the position of $\overline{\mathscr{M}}$ in space can also be determined from the four functions of class C^2: three metric components $\bar{a}_{\alpha\beta}(\vartheta^\lambda)$ and the height function $\bar{z}(\vartheta^\alpha)$ satisfying the equation of Darboux connected with $\overline{\mathscr{M}}$:

$$\tfrac{1}{2}\,\bar{\epsilon}^{\alpha\beta}\bar{\epsilon}^{\lambda\mu}\bar{z}\|_{\alpha\lambda}\bar{z}\|_{\beta\mu} - \bar{K}(1 - \bar{z},_\alpha \bar{z},_\beta \bar{a}^{\alpha\beta}) = 0\,. \tag{69}$$

The surface metric components $\bar{a}_{\alpha\beta}$ can be found from the relation

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta}\,, \tag{70}$$

where $\gamma_{\alpha\beta}$ are the components of the surface Green strain tensor $\boldsymbol{\gamma}$. Thus, given the metric of $\mathscr{M}$ and three functions $\gamma_{\alpha\beta}(\vartheta^\lambda)$ of class C^2 satisfying (69), we are able to uniquely establish the metric of $\overline{\mathscr{M}}$.

The BVP formulated in $\gamma_{\alpha\beta}$ and $\bar{z}$ as independent field variables was recently proposed by Szwabowicz [16] for static analysis of thin shells. It was also assumed in [16] that the shell is composed of an isotropic elastic material and that strains are small everywhere in the shell space. The resulting BVP consists of four non-linear shell equations — three equilibrium equations and one compatibility condition following from the Darboux equation (69) — which are linear in $\gamma_{\alpha\beta}$ and non-linear only in $\bar{z}$. Four fields $\gamma_{\alpha\beta}$ and $\bar{z}$ satisfying such a BVP allow us to establish the spatial position of $\overline{\mathscr{M}}$ by performing three quadratures analogous to those discussed in Section 4.

With $\bar{z}(\vartheta^\alpha)$ given, the vector $\bar{\mathbf{r}}$ can be decomposed, as in (8), into

$$\bar{\mathbf{r}} = \bar{\mathbf{p}} + \bar{z}\mathbf{k}\,. \tag{71}$$

The geometry of projection $\overline{\mathscr{P}}$ of $\overline{\mathscr{M}}$ onto the Oxy plane is described by

$$\bar{\mathbf{g}}_\alpha = \bar{\mathbf{p}}_{,\alpha} = \bar{\mathbf{a}}_\alpha - \bar{z}_{,\alpha}\,\mathbf{k}\,, \tag{72}$$

$$\bar{g}_{\alpha\beta} = \bar{\mathbf{g}}_\alpha \cdot \bar{\mathbf{g}}_\beta = \bar{a}_{\alpha\beta} - \bar{z}_{,\alpha}\,\bar{z}_{,\beta}\ , \qquad \bar{g} = \det(\bar{g}_{\alpha\beta})\,. \tag{73}$$

Given the geometry of $\mathscr{M}$, it is apparent from (70) and (73) that three strains $\gamma_{\alpha\beta}$ and the height function $\bar{z}$ uniquely determine the flat metric $d\bar{s}^2 = \bar{g}_{\alpha\beta}d\vartheta^\alpha d\vartheta^\beta$ of the projected region $\overline{\mathscr{P}}$. Therefore, rewriting the results of Section 4 for the deformed surface $\overline{\mathscr{M}}$ and its projection $\overline{\mathscr{P}}$ onto the Oxy plane we can establish the position vector $\bar{\mathbf{p}} = \bar{x}^\alpha \mathbf{i}_\alpha = \bar{x}\mathbf{i} + \bar{y}\mathbf{j}$ by quadratures using two different solutions.

Applying the first solution given in Subsection 4.1, by analogy to (30), (34) and (35), for the components of $\bar{\mathbf{p}} = \bar{x}\,\mathbf{i} + \bar{y}\,\mathbf{j}$ we obtain

$$\bar{\psi} = \bar{\psi}_0 - \int \frac{1}{2\sqrt{\bar{g}}}\left[\left(\frac{\bar{g}_{12}}{\bar{g}_{11}}\bar{g}_{11,1} - 2\bar{g}_{12,1} + \bar{g}_{11,2}\right)d\vartheta^1 + \left(\frac{\bar{g}_{12}}{\bar{g}_{11}}\bar{g}_{11,2} - \bar{g}_{22,1}\right)d\vartheta^2\right], \tag{74}$$

$$\bar{x} = \bar{x}_0 + \int\left[\sqrt{\bar{g}_{11}}\cos\bar{\psi}\,d\vartheta^1 + \frac{1}{\sqrt{\bar{g}_{11}}}\left(\bar{g}_{12}\cos\bar{\psi} - \sqrt{\bar{g}}\sin\bar{\psi}\right)d\vartheta^2\right], \tag{75}$$

$$\bar{y} = \bar{y}_0 + \int\left[\sqrt{\bar{g}_{11}}\sin\bar{\psi}\,d\vartheta^1 + \frac{1}{\sqrt{\bar{g}_{11}}}\left(\bar{g}_{12}\sin\bar{\psi} + \sqrt{\bar{g}}\cos\bar{\psi}\right)d\vartheta^2\right], \tag{76}$$

where all metric components $\bar{g}_{\alpha\beta}$ are expressed in terms of $a_{\alpha\beta}$, $\gamma_{\alpha\beta}$ and $\bar{z}$ by the relations (73) and (70).

Applying the second solution worked out in Subsection 4.2, by analogy to (60) with (58) and (65) with (64) and (63) we obtain

$$\bar{\phi} = \bar{\phi}_0 + \int \frac{1}{\bar{G} + 2\sqrt{\bar{g}}}\left[\frac{1}{\sqrt{\bar{g}}}\bar{g}_{\alpha\beta}\left(\frac{1}{2}e^{\beta\lambda}\bar{G}_{,\lambda} - \delta^{\beta\kappa}\bar{g}_{\kappa\rho,\lambda}\,e^{\rho\lambda}\right) - \bar{g}_{\alpha\beta,\lambda}\,e^{\beta\lambda} - e_{\alpha}^{\cdot\lambda}(\sqrt{\bar{g}})_{,\lambda}\right]d\vartheta^\alpha\,, \tag{77}$$

$$\bar{\mathbf{p}} = \bar{\mathbf{p}}_0 + \int \frac{1}{\sqrt{G + 2\sqrt{g}}}\left[\cos\phi\,(\delta^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}\,\delta^\alpha_\beta) - \sin\phi\,(e^{\alpha\lambda}g_{\lambda\beta} + \sqrt{g}\,e^{\alpha}_{\cdot\beta})\mathbf{i}_\alpha\,d\vartheta^\beta\right], \tag{78}$$

where

$$\bar{G} = \bar{g}_{11} + \bar{g}_{22}\,, \qquad \bar{g} = \bar{g}_{11}\bar{g}_{22} - (\bar{g}_{12})^2\,.$$

The relations (74)-(76) and (77), (78) allow us to establish explicitly the position $\bar{\mathbf{p}}$ of the deformed reference surface $\overline{\mathscr{M}}$ through the position of the undeformed surface $\mathscr{M}$, three surface strains $\gamma_{\alpha\beta}$ and the height function $\bar{z}$. The displacement field $\mathbf{u}$, if necessary, can now be calculated from $\bar{\mathbf{p}}$ and $\bar{z}$ by the simple relation

$$\mathbf{u} = (\bar{\mathbf{p}} - \mathbf{p}) + (\bar{z} - z)\mathbf{k}\,. \tag{79}$$

The results given above are purely kinematic and valid for an arbitrary geometry of $\mathscr{M}$ satisfying (13) as well as for unrestricted surface strains. They do not depend on material of which the shell is composed as well.

6 Conclusions

The results of this paper should be considered in the context of the approach to the non-linear theory of shells developed by the first author in [16]. Formulation of the shell problem contained therein exhibits some advantages over the traditional ones, but its principal working variables are surface strains and the height function, and such are also the final results obtained as a solution to the boundary value problem. If displacements are of interest in a particular shell problem, procedures worked out here may be directly applied. This, we hope, explains the value and importance of this paper.

In particular, we have explicitly shown that in order to find the deformed position of the reference surface of a thin shell it is enough to know only the position of the undeformed shell reference surface, three surface strains and one height function of the deformed surface over the coordinate plane. The two remaining position components of the deformed surface can be found by quadratures. We have developed here explicitly two different but equivalent forms of the quadratures. The results are purely kinematic and valid for arbitrary deformation of the shell reference surface.

Acknowledgements

The paper was supported by the State Committee for Scientific Research grant KBN No 7 T07A 003 16.

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