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On dynamically and kinematically exact theory of shells

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ABSTRACT: In the dynamically and kinematically exact theory of shells, the shell is represented by a material base surface with an attached structure tensor. The gross deformation of the shell cross sections is described by the translation vector of the surface and the rotation tensor work-averaged through the shell thickness. Some relations between 3D fields in the shell-like body and 2D fields modelling the shell behaviour are discussed. We show, in particular, that the exact resultant stress power of the shell does not coincide with the 2D effective stress power following from the exact resultant balance laws. The difference expressed through an intrinsic deformation vector is responsible for approximations in the shell constitutive description. We sketch how to refine the shell model in the case of a simple equilibrium problem of the non-linearly elastic shell.

1 INTRODUCTION

The non-linear theory of shells developed by Libai and Simmonds (1983, 1998) was formulated with regard to a non-material weighted surface of mass taken as the shell base surface.

In a similar approach summarised in Chróścielewski et al. (2004) and Eremeyev & Pietraszkiewicz (2004), a material surface arbitrarily located within the shell-like body was taken as the base surface. Then the two-dimensional (2D) local balance equations of linear and angular momentum as well as the dynamic boundary conditions were derived with regard to the material surface by an exact through-the-thickness integration of 3D balance laws of continuum mechanics. The corresponding shell kinematics was established on the 2D level as an energetically exact dual structure from the virtual work identity. As a result, the gross deformation of the shell cross sections was described by the translation vector field and the through-the-thickness work-averaged rotation tensor field, both defined at the material base surface. Then exact, unique expressions of the shell strain and bending tensors, both defined again only at the base surface, followed as direct consequences of the exact 2D resultant balance laws.

The aim of this report is to show:

a) how the natural shell strain and bending measures are related to deformation of the base surface with an attached structure tensor;
b) how the shell translation and rotation vector fields as well as the strain and bending tensor fields can be interpreted in terms of 3D deformation fields of continuum mechanics;
c) that in such a dynamically and kinematically exact shell theory the resultant stress power density cannot in general be expressed entirely through the exact 2D shell stress and strain measures thus leading to approximations in the constitutive description.

A possible refinement of the effective stress power density is sketched for a simple equilibrium problem of the non-linearly elastic shell.

2 EXACT RESULTANT LAWS FOR SHELLS

The classical continuum mechanics is based on the balance laws of forces and couples

$$\begin{aligned} &\iint_{\partial \mathrm{P}} \mathbf{T}\mathbf{n}\,\mathrm{d}a + \iiint_{\mathrm{P}} \mathbf{f}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{P}} \rho_0 \mathbf{v}\,\mathrm{d}v, \\ &\iint_{\partial \mathrm{P}} \mathbf{y}\times\mathbf{T}\mathbf{n}\,\mathrm{d}a + \iiint_{\mathrm{P}} \mathbf{y}\times\mathbf{f}\,\mathrm{d}v = \frac{\mathrm{d}}{\mathrm{d}t}\iiint_{\mathrm{P}} \mathbf{y}\times(\rho_0\mathbf{v})\,\mathrm{d}v, \end{aligned} \qquad (1)$$

assumed to hold for any sufficiently regular part P of the body identified with its reference placement B. Here $\mathbf{y} = \mathbf{y}(\mathrm{x},t)$ is the position vector at time t of a material particle whose position in the reference placement is $\mathrm{x} \in \mathrm{B}$, $\mathbf{T}(\mathrm{x},t)$ the 1st Piola-Kirchhoff stress tensor, $\mathbf{n}$ the external unit normal to $\partial\mathrm{P}$, $\mathbf{f}(\mathrm{x},t)$ the body force vector, $\mathbf{v}(\mathrm{x},t) = \dot{\mathbf{y}}(\mathrm{x},t)$ the velocity vector, and $\rho_0(\mathrm{x})$ the reference mass density.

The laws (1) specified for the shell-like body with the base surface $M \subset \mathrm{B}$ arbitrarily located within B can exactly be transformed into the resultant forms

$$
\begin{aligned}
&\int_{\partial\Pi} \boldsymbol{N}\boldsymbol{\nu}\,\mathrm{d}s + \iint_{\Pi} \boldsymbol{f}\,\mathrm{d}a = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Pi} \boldsymbol{p}\,\mathrm{d}a, \\
&\int_{\partial\Pi} (\boldsymbol{M}\boldsymbol{\nu} + \boldsymbol{y}\times\boldsymbol{N}\boldsymbol{\nu})\,\mathrm{d}s + \iint_{\Pi} (\boldsymbol{c} + \boldsymbol{y}\times\boldsymbol{f})\,\mathrm{d}a \\
&\qquad = \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Pi} (\boldsymbol{p} + \boldsymbol{y}\times\boldsymbol{s})\,\mathrm{d}a.
\end{aligned} \tag{2}
$$

Here $\Pi = \mathrm{P} \cap M$ is the part of M corresponding to P, and $\boldsymbol{y}(x,t)$ is the position vector at t of a material particle on the base surface $M(t)$ which position in the reference placement is $x \in M$. Moreover, $\boldsymbol{N}(x,t)$ and $\boldsymbol{M}(x,t)$ are the stress and couple resultant tensors, $\boldsymbol{\nu}$ the unit vector externally normal to $\partial\Pi$ and tangent to $\Pi \subset M$, $\boldsymbol{f}(x,t)$ and $\boldsymbol{c}(x,t)$ the resultant surface force and couple vectors, and $\boldsymbol{p}(x,t)$ and $\boldsymbol{s}(x,t)$ the resultant surface linear and angular momentum vectors, respectively. Details of the exact through-the-thickness integration procedure leading to (2) are given in Libai & Simmonds (1998) and Chróścielewski et al. (2004).

The resultant balance laws (2) hold iff

$$
\begin{aligned}
\iint_{\Pi} \sigma\,\mathrm{d}a = &\iint_{\Pi} \{(\boldsymbol{f} - \boldsymbol{p})\bullet\boldsymbol{v} + [\boldsymbol{c} - (\boldsymbol{p} + \boldsymbol{y}\times\boldsymbol{s})]\bullet\boldsymbol{\omega}\}\,\mathrm{d}a \\
&+ \int_{\partial\Pi} (\boldsymbol{N}\boldsymbol{\nu}\bullet\boldsymbol{v} + \boldsymbol{M}\boldsymbol{\nu}\bullet\boldsymbol{\omega})\,\mathrm{d}s,
\end{aligned} \tag{3}
$$

where

$$
\sigma = \boldsymbol{N}\bullet(\nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}) + \boldsymbol{M}\bullet\nabla\boldsymbol{\omega} \tag{4}
$$

is the 2D effective stress power density. In (3) and (4), $\bullet$ means the scalar product in the tensor space, $\boldsymbol{v}$ the linear velocity vector of the base surface, $\boldsymbol{\omega}$ the angular velocity vector of the shell cross section work-averaged through the shell thickness, $\boldsymbol{\omega} = ax\,\boldsymbol{\Omega}$ the axial vector associated with the skew tensor $\boldsymbol{\Omega}$, and $\boldsymbol{F} = \nabla\boldsymbol{y}$ the shell deformation gradient, with ∇ the surface gradient operator on M.

Let in the reference (undeformed) placement the shell is represented by the position vector $\boldsymbol{x}(x) \in E$ of M and the non-singular structure tensor $\boldsymbol{T}_0(x) \in E\otimes E$, $\det\boldsymbol{T}_0 \neq 0$, attached to any point $x \in M$, where E is the 3D vector space. The tensor $\boldsymbol{T}_0$ can be introduced, in particular, through three directors $\boldsymbol{t}_{0i}(x)$, $i = 1,2,3$, such that $\boldsymbol{t}_{0i} = \boldsymbol{T}_0(x)\mathbf{i}_i$, with $\mathbf{i}_i$ an orthonormal base of a 3D inertial frame. Then the velocity fields $\boldsymbol{v}$ and $\boldsymbol{\Omega}$ in (3) have to satisfy the differential equations

$$
\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{y}(x,t) = \boldsymbol{v}(x,t), \quad \frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{T}(x,t) = \boldsymbol{\Omega}(x,t)\boldsymbol{T}(x,t), \tag{5}
$$

which solutions are

$$
\begin{aligned}
&\boldsymbol{y}(x,t) = \boldsymbol{y}(x,0) + \int_0^t \boldsymbol{v}(x,\tau)\mathrm{d}\tau = \boldsymbol{x}(x) + \boldsymbol{u}(x,t), \\
&\boldsymbol{T}(x,t) = \boldsymbol{Q}(x,t)\boldsymbol{T}_0(x).
\end{aligned} \tag{6}
$$

In (6), $\boldsymbol{u}$ is the translation vector and $\boldsymbol{Q}$ the through-the-thickness work-averaged rotation tensor, $\det\boldsymbol{Q} = +1$, $\boldsymbol{Q}^{-1} = \boldsymbol{Q}^T$, describing the gross motion of the shell cross sections, while the non-singular tensor $\boldsymbol{T}$ is the shell structure tensor in the actual placement.

For the tensorial angular velocity we obtain

$$
\boldsymbol{\Omega} = \dot{\boldsymbol{Q}}\boldsymbol{Q}^T = -\boldsymbol{Q}\dot{\boldsymbol{Q}}^T = \dot{\boldsymbol{T}}\boldsymbol{T}^{-1}. \tag{7}
$$

3 NATURAL STRAINS AND BENDINGS

Let C be a smooth curve on M given by $x = x(\lambda)$, where λ is a scalar parameter. Then $\boldsymbol{x} = \boldsymbol{x}[x(\lambda)]$ and $\boldsymbol{T}_0 = \boldsymbol{T}_0[x(\lambda)]$ along C and their differentials are

$$
\begin{aligned}
&\mathrm{d}\boldsymbol{x} = \boldsymbol{x}'\,\mathrm{d}\lambda = (\nabla\boldsymbol{x})\mathrm{d}x, \quad \mathrm{d}\boldsymbol{x} \in E, \\
&\mathrm{d}\boldsymbol{T}_0 = \boldsymbol{T}_0'\,\mathrm{d}\lambda = (\nabla\boldsymbol{T}_0)\mathrm{d}x, \quad \mathrm{d}x \in T_xM, \\
&\nabla\boldsymbol{x} = \boldsymbol{I}_0 \in E\otimes T_xM, \quad \nabla\boldsymbol{T}_0 \in E\otimes E\otimes T_xM,
\end{aligned} \tag{8}
$$

where T_xM is the tangent space to M, and $\boldsymbol{I}_0$ the inclusion operator at $x \in M$, see Gurtin & Murdoch (1975) and Murdoch (1990).

The skew part of the tensor $\boldsymbol{D}_0 = (\mathrm{d}\boldsymbol{T}_0)\boldsymbol{T}_0^{-1}$ can be represented by its axial vector $\boldsymbol{b}_0$ depending linearly on $\mathrm{d}x$. Therefore,

$$
\boldsymbol{b}_0 = ax\left[\frac{1}{2}\left(\boldsymbol{D}_0 - \boldsymbol{D}_0^T\right)\right] = \boldsymbol{B}_0\mathrm{d}x, \quad \boldsymbol{B}_0 \in E\otimes T_xM. \tag{9}
$$

The two tensors $\boldsymbol{I}_0$ and $\boldsymbol{B}_0$ are the basic measures of local geometry of the shell base surface M with the attached structure tensor $\boldsymbol{T}_0$.

In the actual (deformed) placement the shell base surface $M(t) = \chi(M,t)$ is described by the position vector $\boldsymbol{y}(x,t)$ and the structure tensor $\boldsymbol{T}(x,t)$. Differentials of $\boldsymbol{y}(x,t)$ and $\boldsymbol{T}(x,t)$ along $C(t) = \chi(C,t)$ are given by

$$
\begin{aligned}
&\mathrm{d}\boldsymbol{y} = \boldsymbol{y}'\,\mathrm{d}\lambda = (\bar{\nabla}\boldsymbol{y})\mathrm{d}y = (\nabla\boldsymbol{y})\mathrm{d}x, \\
&\mathrm{d}\boldsymbol{T} = \boldsymbol{T}'\,\mathrm{d}\lambda = (\bar{\nabla}\boldsymbol{T})\mathrm{d}y = (\nabla\boldsymbol{T})\,\mathrm{d}x, \\
&\bar{\nabla}\boldsymbol{y} = \boldsymbol{I} \in E\otimes T_yM(t), \quad \bar{\nabla}\boldsymbol{T} \in E\otimes E\otimes T_yM(t),
\end{aligned} \tag{10}
$$

where $\bar{\nabla}$ is the surface gradient operator and $\boldsymbol{I}$ the inclusion operator at $M(t)$.

The skew part of the tensor $\boldsymbol{D} = (\mathrm{d}\boldsymbol{T})\boldsymbol{T}^{-1}$ can again be represented by its axial vector $\boldsymbol{b}$ depending linearly on $\mathrm{d}y$, and this gives

$$
\boldsymbol{b} = ax\left[\frac{1}{2}\left(\boldsymbol{D} - \boldsymbol{D}^T\right)\right] = \boldsymbol{B}\mathrm{d}y, \quad \boldsymbol{B} \in E\otimes T_yM(t). \tag{11}
$$

Since $\mathrm{d}y = \mathsf{F}\mathrm{d}x$, where $\mathsf{F} \in T_yM(t)\otimes T_xM$ is the tangential surface deformation gradient, by subtracting from $(10)_1$ and (11) the respective relations $(8)_1$ and (9) rotated to the actual placement we obtain

$$\begin{aligned} \mathrm{d}\boldsymbol{y} - \boldsymbol{Q}\mathrm{d}\boldsymbol{x} &= (\boldsymbol{I}\mathsf{F} - \boldsymbol{Q}\boldsymbol{I}_0)\mathrm{d}x, \\ \boldsymbol{b} - \boldsymbol{Q}\boldsymbol{b}_0 &= \boldsymbol{B}\mathrm{d}y - \boldsymbol{Q}\boldsymbol{B}_0\mathrm{d}x = (\boldsymbol{B}\mathsf{F} - \boldsymbol{Q}\boldsymbol{B}_0)\mathrm{d}x. \end{aligned} \tag{12}$$

From (12) it is seen that the tensors

$$\boldsymbol{E} = \boldsymbol{I}\mathsf{F} - \boldsymbol{Q}\boldsymbol{I}_0, \quad \boldsymbol{K} = \boldsymbol{B}\mathsf{F} - \boldsymbol{Q}\boldsymbol{B}_0 \tag{13}$$

are the appropriate natural strain and bending measures describing the local deformation of the shell base surface with the attached structure tensor.

4 STRAIN AND BENDING VELOCITIES

We want to show explicitly that the expressions $\nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}$ and $\nabla\boldsymbol{\omega}$ appearing in (4) represent just the co-rotational time derivatives of the natural strain and bending measures (13) defined by

$$\begin{aligned} \overset{\circ}{\boldsymbol{E}} &= \boldsymbol{Q}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^T\boldsymbol{E})\right) = \dot{\boldsymbol{E}} - \boldsymbol{\Omega}\boldsymbol{E}, \\ \overset{\circ}{\boldsymbol{K}} &= \boldsymbol{Q}\left(\frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}^T\boldsymbol{K})\right) = \dot{\boldsymbol{K}} - \boldsymbol{\Omega}\boldsymbol{K}. \end{aligned} \tag{14}$$

Indeed, direct time differentiation of $(13)_1$ with the use of $(13)_1$ yields

$$\begin{aligned} \overset{\circ}{\boldsymbol{E}} &= \dot{\boldsymbol{F}} - \dot{\boldsymbol{Q}}\boldsymbol{I}_0 - \boldsymbol{\Omega}\boldsymbol{E} \\ &= \nabla\dot{\boldsymbol{y}} - \dot{\boldsymbol{Q}}\boldsymbol{Q}^T(\boldsymbol{F} - \boldsymbol{E}) - \boldsymbol{\Omega}\boldsymbol{E} \\ &= \nabla\boldsymbol{v} - \boldsymbol{\Omega}\boldsymbol{F}. \end{aligned} \tag{15}$$

In order to prove that also $\overset{\circ}{\boldsymbol{K}} = \nabla\boldsymbol{\omega}$, let us first explicitly calculate

$$\boldsymbol{D} = (\mathrm{d}\boldsymbol{T})\boldsymbol{T}^{-1} = (\nabla\boldsymbol{Q}\mathrm{d}x)\boldsymbol{Q}^T + \boldsymbol{Q}\boldsymbol{D}_0\boldsymbol{Q}^T \tag{16}$$

and note that the first term of (16) is already a skew tensor $\in E\otimes E$ depending linearly on $\mathrm{d}x$. Therefore, from (11) and (16) we have

$$\begin{aligned} \boldsymbol{b} &= ax\left[(\nabla\boldsymbol{Q}\mathrm{d}x)\boldsymbol{Q}^T\right] + \boldsymbol{Q}\boldsymbol{b}_0 \\ &= (\boldsymbol{K} + \boldsymbol{Q}\boldsymbol{B}_0)\mathrm{d}x. \end{aligned} \tag{17}$$

Since the vector $\mathrm{d}x$ does not depend on time, from (17) and $(12)_2$ we obtain

$$\begin{aligned} \dot{\boldsymbol{K}}\mathrm{d}x &= ax\left[(\nabla\dot{\boldsymbol{Q}}\mathrm{d}x)\boldsymbol{Q}^T + (\nabla\boldsymbol{Q}\mathrm{d}x)\dot{\boldsymbol{Q}}^T\right] \\ &= ax\left[\nabla\boldsymbol{\Omega}\mathrm{d}x + \boldsymbol{\Omega}(\nabla\boldsymbol{Q}\mathrm{d}x)\boldsymbol{Q}^T - (\nabla\boldsymbol{Q}\mathrm{d}x)\boldsymbol{Q}^T\boldsymbol{\Omega}\right] \\ &= (\nabla\boldsymbol{\omega} + \boldsymbol{\Omega}\boldsymbol{K})\mathrm{d}x, \end{aligned} \tag{18}$$

and from (14) it follows that $\overset{\circ}{\boldsymbol{K}} = \nabla\boldsymbol{\omega}$ indeed.

5 LOCAL 3D DEFORMATION

Any point $\mathrm{y}(t) = \chi(\mathrm{x}, t) \in \mathrm{B}(t)$ outside the deformed base surface $M(t) = \chi(M, t)$ can be described by the spatial position vector

$$\begin{aligned} &\mathbf{y}(\mathrm{x},t) = \boldsymbol{y}(x,t) + \boldsymbol{\zeta}(x,\xi,t), \\ &\boldsymbol{\zeta} = \boldsymbol{Q}\mathbf{z} = \boldsymbol{Q}(\xi\boldsymbol{t}_0 + \mathbf{e}), \quad \mathbf{e}(x,0,t) = \mathbf{0}, \end{aligned} \tag{19}$$

where ξ is the coordinate in the thickness direction, $\boldsymbol{t}_0$ the base vector along ξ at $x \in M$, and $\mathbf{e}(x,\xi,t)$ an intrinsic deformation vector. Therefore,

$$\mathbf{u} = \boldsymbol{u} + \xi(\boldsymbol{Q} - \mathbf{1})\boldsymbol{t}_0 + \boldsymbol{Q}\mathbf{e}. \tag{20}$$

The first two terms of (20) represent the work-averaged linear part of the displacement distribution across the shell thickness. The linear part is entirely defined by two fields $\boldsymbol{u}(x,t)$ and $\boldsymbol{Q}(x,t)$ being solutions of the 2D initial-boundary value problem of the dynamically and kinematically exact shell theory. The last term in (20) represents a deviation of the unknown 3D field $\mathbf{u}(x,\xi,t)$ from this linear distribution.

Let $\mathbf{F}(\mathrm{x},t) = \nabla\mathbf{y}(\mathrm{x},t)$, $\det\mathbf{F} > 0$, be the spatial deformation gradient at $\mathrm{x} \in \mathrm{B}$. By the polar decomposition theorem we have $\mathbf{F} = \mathbf{R}\mathbf{U}$, where $\mathbf{R}$ is the rotation tensor (proper orthogonal) and $\mathbf{U}$ the right stretch tensor (symmetric and positive definite).

In the spatial convected coordinate system ξ^i, $i = 1,2,3$, such that $\xi^3 \equiv \xi = 0$ on the base surface, the spatial deformation gradient at $\mathrm{x} \in \mathrm{B}$ is given by

$$\mathbf{F} = \mathbf{y},_i \otimes \mathbf{g}^i, \quad (.),_i \equiv \frac{\partial}{\partial\xi^i}(.), \tag{21}$$

where $\mathbf{g}^i$ are the contravariant base vectors of the coordinates ξ^i in the reference placement.

Derivatives of $\mathbf{y}$ given by $(19)_1$ with regard to the coordinates allow us to present $\mathbf{F}$ in the form

$$\mathbf{F} = \boldsymbol{Q}\boldsymbol{\Lambda} = \boldsymbol{Q}(\mathbf{1} + \boldsymbol{\Theta}), \quad \boldsymbol{\Lambda} = \boldsymbol{\lambda}_i \otimes \mathbf{g}^i, \quad \boldsymbol{\Theta} = \boldsymbol{\theta}_i \otimes \mathbf{g}^i, \tag{22}$$

where

$$\begin{aligned} &\boldsymbol{\theta}_\alpha = \boldsymbol{\varepsilon}_\alpha + \boldsymbol{\kappa}_\alpha \times (\xi\boldsymbol{t}_0 + \mathbf{e}) + \mathbf{e},_\alpha, \quad \boldsymbol{\theta}_3 = \mathbf{e},_3, \quad \alpha = 1,2, \\ &\boldsymbol{\varepsilon}_\alpha = \boldsymbol{Q}^T\boldsymbol{y},_\alpha - \boldsymbol{x},_\alpha, \quad \boldsymbol{\kappa}_\alpha = ax(\boldsymbol{Q}^T\boldsymbol{Q},_\alpha), \\ &\mathbf{E} = \boldsymbol{\varepsilon}_\alpha \otimes \boldsymbol{a}^\alpha, \quad \mathbf{K} = \boldsymbol{\kappa}_\alpha \otimes \boldsymbol{a}^\alpha. \end{aligned} \tag{23}$$

Here $\boldsymbol{\varepsilon}_\alpha$, $\boldsymbol{\kappa}_\alpha$ and $\mathbf{E}$, $\mathbf{K}$ are the shell strain and bending vectors and tensors, respectively, in the material representation which are related to the tensors defined in (13) according to

$$\boldsymbol{\varepsilon}_\alpha = \boldsymbol{Q}^T \boldsymbol{E} \boldsymbol{a}_\alpha\,, \quad \boldsymbol{\kappa}_\alpha = \boldsymbol{Q}^T \boldsymbol{K} \boldsymbol{a}_\alpha\,, \quad \boldsymbol{x}_{,\alpha} = \boldsymbol{I}_0 \boldsymbol{a}_\alpha\,. \tag{24}$$

The relation $(22)_1$ is just another form of the polar decomposition of $\mathbf{F}$ at $\mathrm{x} \in \mathrm{B}$, where the stretch tensor $\boldsymbol{\Lambda}$ has still the positive determinant, $\det\boldsymbol{\Lambda} > 0$, but is non-symmetric, in general: $\boldsymbol{\Lambda}^T \neq \boldsymbol{\Lambda}$.

In particular, at the shell base surface where $\xi = 0$ we obtain

$$\mathbf{F}_0 = \mathbf{R}_0 \mathbf{U}_0 = \boldsymbol{Q} \boldsymbol{\Lambda}_0 = \boldsymbol{Q}(\mathbf{1} + \boldsymbol{\Theta}_0)\,. \tag{25}$$

From (25) it is apparent that $\boldsymbol{Q}(x, t)$ should not be identified with $\mathbf{R}_0(x, t)$ following from the 3D polar decomposition of $\mathbf{F}_0$ taken at $M \subset \mathrm{B}$, which was used by Pietraszkiewicz (1979). One should keep in mind that $\boldsymbol{Q}(x, t)$ has been introduced through (3) and $(6)_2$ with the help of energetic considerations, while $\mathbf{R}_0(x, t)$ followed from purely geometric and algebraic transformations.

6 RESULTANT STRESS POWER DENSITY

In continuum mechanics the stress power density $\Sigma(\mathrm{x},t)$ per unit volume of B is given by $\Sigma = \mathbf{T} \bullet \dot{\mathbf{F}}$. The resultant stress power density $\mathit{\Sigma}$ per unit area of M can be defined by direct through-the-thickness integration

$$\mathit{\Sigma} = \int_-^+ \Sigma \mu \mathrm{d}\xi\,, \quad \int_-^+ (.) \equiv \int_{-h_0^-}^{+h_0^+} (.)\,, \tag{26}$$

where $-h_0^- \le \xi \le h_0^+$ is the distance from M along ξ, μ the geometric through-the-thickness expansion factor, and $h = h_0^- + h_0^+ > 0$ the shell thickness.

Using the relations $(22)_1$ and $\mathbf{T} = \mathbf{FS}$, where $\mathbf{S} = \mathbf{s}^i \otimes \mathbf{g}_i$ is the 2nd Piola-Kirchhoff stress tensor, the 3D density Σ can be transformed as follows:

$$\begin{aligned} \Sigma &= (\boldsymbol{Q}\boldsymbol{\Lambda}\mathbf{S}) \bullet (\dot{\boldsymbol{Q}}\boldsymbol{\Lambda} + \boldsymbol{Q}\dot{\boldsymbol{\Lambda}}) \\ &= (\boldsymbol{\Lambda}\mathbf{S}\boldsymbol{\Lambda}^T) \bullet (\boldsymbol{Q}^T\dot{\boldsymbol{Q}}) + (\boldsymbol{\Lambda}\mathbf{S}) \bullet (\boldsymbol{Q}^T\boldsymbol{Q}\dot{\boldsymbol{\Lambda}}) \\ &= (\boldsymbol{\Lambda}\mathbf{S}) \bullet \dot{\boldsymbol{\Theta}} = \boldsymbol{\Lambda}\mathbf{s}^\alpha \bullet \dot{\boldsymbol{\theta}}_\alpha + \boldsymbol{\Lambda}\mathbf{s}^3 \bullet \dot{\boldsymbol{\theta}}_3 \\ &= \boldsymbol{\Lambda}\mathbf{s}^\alpha \bullet \dot{\boldsymbol{\varepsilon}}_\alpha + (\mathbf{z} \times \boldsymbol{\Lambda}\mathbf{s}^\alpha) \bullet \dot{\boldsymbol{\kappa}}_\alpha + (\boldsymbol{\Lambda}\mathbf{s}^\alpha \times \boldsymbol{\kappa}_\alpha) \bullet \dot{\mathbf{e}} + (\boldsymbol{\Lambda}\mathbf{S}) \bullet \nabla\dot{\mathbf{e}}. \end{aligned} \tag{27}$$

Now we can integrate $(27)_4$ through the shell thickness as in (26) and obtain

$$\mathit{\Sigma} = \mathbf{n}^\alpha \bullet \dot{\boldsymbol{\varepsilon}}_\alpha + \mathbf{m}^\alpha \bullet \dot{\boldsymbol{\kappa}}_\alpha + \sigma_r\,, \tag{28}$$

where

$$\begin{aligned} \mathbf{n}^\alpha &= \int_-^+ \boldsymbol{\Lambda}\mathbf{s}^\alpha \mu \mathrm{d}\xi = \boldsymbol{Q}^T \boldsymbol{N} \boldsymbol{a}^\alpha\,, \\ \mathbf{m}^\alpha &= \int_-^+ (\mathbf{z} \times \boldsymbol{\Lambda}\mathbf{s}^\alpha) \mu \mathrm{d}\xi = \boldsymbol{Q}^T \boldsymbol{M} \boldsymbol{a}^\alpha, \\ \sigma_r &= \left\{ \int_-^+ (\dot{\mathbf{e}} \times \boldsymbol{\Lambda}\mathbf{s}^\alpha) \mu \mathrm{d}\xi \right\} \bullet \boldsymbol{\kappa}_\alpha + \int_-^+ (\boldsymbol{\Lambda}\mathbf{S}) \bullet \nabla\dot{\mathbf{e}} \mu \mathrm{d}\xi\,. \end{aligned} \tag{29}$$

The first two terms of (28) correspond to the 2D effective stress power density σ defined in (4), but expressed now through the stress and couple resultant vectors as well as the respective strain and bending vectors in the material representation. The additional term σ_r represents that part of the resultant stress power density of the shell-like body which is not expressible entirely through the surface fields included in σ. As a result, the dynamically and kinematically exact theory of shells without σ_r is still approximate in the corresponding constitutive description.

In order to refine the constitutive description of the shell model based on (3) and (4), Makowski & Pietraszkiewicz (2002) introduced into the shell mechanical power an additional interstitial part requiring special constitutive equations. Another possible way of refining the shell model is to express the intrinsic deformation vector field $\mathbf{e}$ through the shell strain and bending fields. Then the two terms present in σ_r can be used for refinement of the constitutive equations. Unfortunately, in many types of shell motion such a relation may not exist or may not be unique. We sketch below a simple problem of the non-linearly elastic shell for which such a relation may be found.

7 ELASTIC EQUILIBRIUM PROBLEM

Static equilibrium problem of a non-linearly elastic 3D body without body forces can be described by

$$\begin{aligned} &\mathrm{Div}\left(\mathbf{F}\tilde{\mathbf{S}}(\mathbf{C})\right) = \mathbf{0} \quad \text{in B}, \\ &\mathbf{F}\tilde{\mathbf{S}}(\mathbf{C})\mathbf{n} = \mathbf{t}^* \quad \text{on } \partial\mathrm{B}_\mathrm{f}\,, \end{aligned} \tag{30}$$

where $\mathbf{S} = \tilde{\mathbf{S}}(\mathbf{C})$ is the constitutive equation, $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ the right Cauchy-Green deformation tensor, and $\mathbf{t}^*$ the force vector prescribed on the part $\partial\mathrm{B}_\mathrm{f} \subset \partial\mathrm{B}$.

If B is the shell-like body then introducing $(22)_1$ into $(30)_1$ and using $\mathbf{P} = \boldsymbol{\Lambda}\tilde{\mathbf{S}}(\boldsymbol{\Lambda})$ we have

$$\begin{aligned} \mathrm{Div}(\boldsymbol{Q}\mathbf{P}) &= \boldsymbol{Q}_{,\alpha}\mathbf{P}\mathbf{g}^\alpha + \boldsymbol{Q}\,\mathrm{Div}\mathbf{P} \\ &= \boldsymbol{Q}(\mathbf{K}_\alpha \mathbf{P}\mathbf{g}^\alpha + \mathrm{Div}\mathbf{P}). \end{aligned} \tag{31}$$

Since according to (22) and (23), $\boldsymbol{\Lambda} = \tilde{\boldsymbol{\Lambda}}(\mathbf{E}, \mathbf{K}, \mathbf{e}, \nabla\mathbf{e})$ for the traction-free shell faces, the problem reduces to

$$\begin{aligned} &\mathrm{Div}\mathbf{P} + \mathbf{K}_\alpha \mathbf{P}\mathbf{g}^\alpha = \mathbf{0} \quad \text{in B}, \\ &(\mathbf{P}\mathbf{n})^\pm = \mathbf{0} \quad \text{on } M^\pm \subset \partial\mathrm{B}, \\ &\mathbf{P}\mathbf{n} = \boldsymbol{Q}^T\mathbf{t}^* \quad \text{on } \partial\mathrm{B}_\mathrm{f}' \subset \partial\mathrm{B}. \end{aligned} \tag{32}$$

Please note that the field $\mathbf{P} = \tilde{\mathbf{P}}(\mathbf{E}, \mathbf{K}, \mathbf{e}, \nabla \mathbf{e})$ will also depend on the base surface M through its curvature and on the transverse coordinate ξ. At any fixed point $x \in M$ the equation $(32)_1$ becomes an ordinary differential equation with regard to ξ for the vector field $\mathbf{e}(x, \xi)$. Solving (32) we can establish the relation between $\mathbf{e}(x, \xi)$ and **E, K** as well as their surface gradients. Such a relation would allow one to take in (28) into account the part σ_r of the stress power and to improve accuracy of the expression for the resultant stress power density Σ. This would allow one for a refinement of the corresponding 2D constitutive equations of the non-linearly elastic shell.

8 CONCLUSIONS

Within the dynamically and kinematically exact shell theory the shell is represented by the material base surface with an attached structure tensor. The shell motion is represented by the surface translation vector and rotation tensor fields. The shell deformation is described by the surface strain and bending tensor fields.

It has been shown that the resultant stress power density of the shell depends on time derivatives of the surface strain measures, but also on time derivative of an additional intrinsic deformation field describing a deviation of the unknown 3D displacement field from its linear distribution following from the 2D shell theory. As a result, the shell model without proper account of the intrinsic deformation field should always lead to approximations in the constitutive shell description. A refinement is possible if the intrinsic deformation field can be expressed explicitly through the shell strain measures. Such a relation has been shown to exist in the case of an equilibrium problem of the non-linearly elastic shell.

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