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**MULTI-PHASE
AND MULTI-COMPONENT
MATERIALS
UNDER DYNAMIC
LOADING**

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ON QUASI-STATIC PROPAGATION OF THE PHASE INTERFACE IN THIN-WALLED INELASTIC BODIES

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Abstract: Applying the general non-linear theory of shells undergoing phase transitions, we derive the balance equations along the singular curve modelling the phase interface in the shell. From the integral forms of balance laws of linear momentum, angular momentum, and energy as well as the entropy inequality we obtain the local static balance equation along the curvilinear phase interface. We also derive the thermodynamic condition allowing one to determine the interface position within the deformed shell midsurface. The special case of the pure mechanical theory is also considered.

Keywords: non-linear shell, phase transition, continuity conditions, quasi-static loading

1. Introduction

Phase transitions (PTs) play an important role in different problems of continuum mechanics. Behaviour of some modern materials is very sensitive to the stress-induced PTs. See, for example, the experimental data on shape memory alloys given by Pieczyska et al. (2006), and Feng and Sun (2006), where thin-walled specimens such as a thin plate or a thin tube were used.

Equilibrium conditions of elastic thin-walled structures (plates and shells) undergoing PT of martensitic type were formulated by Eremeyev and Pietraszkiewicz (2004) and Pietraszkiewicz et al. (2007) within the dynamically and kinematically exact theory of shells presented in books by Libai and Simmonds (1998) and Chróścielewski et al. (2004). From experimental data we know that PT depending on strain rates and inelastic effects may considerably influence the stress state of the solid.

The aim of this paper is to extend the results of Eremeyev and Pietraszkiewicz (2004) and Pietraszkiewicz et al. (2007) by taking into account some inelastic effects of the shell material phases.

2. Shell equilibrium conditions

Within the general theory of shells, the shell displacement is described by the translation vector $\mathbf{u} \in E$ of the shell base surface M and the proper orthogonal tensor $\mathbf{Q} \in SO(3)$ describing an energetically mean rotation of the shell cross-sections, see Chróscielewski et al. (2004).

Consider a two-phase shell such that in the deformed state both material phases appear on closed complementary subregions N_A and N_B of the deformed base surface $N = \chi(M)$ divided by the curvilinear phase interface D . The position vector of D is $\mathbf{y}_C(s)$, where s is the undeformed arc length parameter. Let \mathbf{x} be the position vector of M . Then within the Lagrangian description for any continuous deformation χ we can introduce in $M = \chi^{-1}(N)$ the singular surface curve $C \subset M$ with the position vector $\mathbf{x}_C(s) = \chi^{-1}(\mathbf{y}_C(s))$. The curve C separates the two subregions $M_A = \chi^{-1}(N_A)$ and $M_B = \chi^{-1}(N_B)$. Note that the position vectors $\mathbf{y}_C(s)$ or $\mathbf{x}_C(s)$ of the phase interface are a priori not known and should be determined together with the vector \mathbf{u} and the proper orthogonal tensor \mathbf{Q} fields.

In quasi-static problems discussed here the global equilibrium conditions require the total force and the total torque of all loads acting on any part $\Pi \subset M$ to vanish

$$\iint_{\Pi} \mathbf{f} da + \int_{\partial\Pi} \mathbf{n}_\nu ds = \mathbf{0}, \quad \iint_{\Pi} (\mathbf{c} + \mathbf{y} \times \mathbf{f}) da + \int_{\partial\Pi} (\mathbf{m}_\nu + \mathbf{y} \times \mathbf{n}_\nu) ds = \mathbf{0}, \quad (2.1)$$

where \mathbf{f} and \mathbf{c} are the external surface resultant force and couple vectors applied at any point of N , but measured per unit area of M . Similarly, \mathbf{n}_ν and \mathbf{m}_ν are the internal surface contact stress resultant and stress couple vectors defined at an arbitrary boundary ∂R , $R = \chi(\Pi)$, and measured per unit length of the boundary $\partial\Pi$.

According to the Cauchy postulate, the contact vectors \mathbf{n}_ν and \mathbf{m}_ν at any $\partial\Pi$ can be represented through the respective internal surface stress resultant and stress couple tensors \mathbf{N} and \mathbf{M} by the relations $\mathbf{n}_\nu = \mathbf{N}\mathbf{v}$ and $\mathbf{m}_\nu = \mathbf{M}\mathbf{v}$, with $\mathbf{v} \in T_x M$ the unit external normal vector of $\partial\Pi$. The tensors \mathbf{N} and \mathbf{M} are some analogues in shell theory of the first Piola-Kirchhoff stress tensor in 3D continuum mechanics.

From (2.1) we obtain the local 2D equilibrium equations of the shell to be satisfied in the regular points of M

$$\text{Div}_s \mathbf{N} + \mathbf{f} = \mathbf{0}, \quad \text{Div}_s \mathbf{M} + ax(\mathbf{N}\mathbf{F}^T - \mathbf{F}\mathbf{N}^T) + \mathbf{c} = \mathbf{0}, \quad (2.2)$$

where $\mathbf{F} = \text{Grad}_s \mathbf{y}$ is the surface gradient of the shell deformation $\mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}$, $ax(\cdot)$ denotes the axial vector associated with the skew tensor (\cdot) ,

while $Grad_s$ and Div_s are the surface gradient and divergence operators on M , respectively.

If Π contains a part of the phase interface C , which is the singular curve with respect to the surface stress measures, then from (2.1) we also obtain *the static 1D balance equations* along C

$$[\mathbf{N}\mathbf{v}] = \mathbf{0}, \quad [\mathbf{M}\mathbf{v}] = \mathbf{0}, \quad (2.3)$$

where the expression $[\dots] = (\dots)^+ - (\dots)^-$ means the jump at C .

3. Kinematic compatibility conditions

Most curvilinear phase interfaces in shells can be either coherent or incoherent in rotations, see Eremeyev and Pietraszkiewicz (2004). For the *coherent* interface both fields \mathbf{y} and \mathbf{Q} are supposed to be continuous at C and the kinematic compatibility conditions along C are

$$[\mathbf{v}] + V[\mathbf{F}\mathbf{v}] = \mathbf{0}, \quad [\mathbf{w}] + V[\mathbf{K}\mathbf{v}] = \mathbf{0}, \quad (3.1)$$

where $\mathbf{v} = \dot{\mathbf{u}}$ is the translational velocity vector, $V = \dot{\mathbf{x}}_C \cdot \mathbf{v}$ is the normal velocity tangent to $T_x M$ of the phase curve C , and $\mathbf{w} = ax(\dot{\mathbf{Q}}\mathbf{Q}^T)$ is the angular velocity vector.

The phase interface is called *incoherent in rotations* if only \mathbf{y} is continuous at C but the continuity of \mathbf{Q} may be violated. In this case the condition (3.1)₁ is still satisfied, but (3.1)₂ may be violated.

4. Constitutive equations

We assume that the surface stress measures \mathbf{N} and \mathbf{M} depend only on prehistories of the surface natural strain and bending tensors $\mathbf{E}, \mathbf{K} \in E \otimes T_x M$ defined in Chróścielewski et al. (2004).

Let us split the surface stress measures and their constitutive equations into elastic (equilibrium) and inelastic (dissipative) parts

$$\mathbf{N} = \mathbf{N}_E + \mathbf{N}_D, \quad \mathbf{M} = \mathbf{M}_E + \mathbf{M}_D. \quad (4.1)$$

Such a decomposition is widely used, for example in thermoviscoelasticity or plasticity. A simple example of decomposition (4.1) in viscoelastic shells was proposed by Makowski and Pietraszkiewicz (2002) in the form

$$\mathbf{N}_E = \rho \frac{\partial \psi(\mathbf{E}, \mathbf{K})}{\partial \mathbf{E}}, \quad \mathbf{M}_E = \rho \frac{\partial \psi(\mathbf{E}, \mathbf{K})}{\partial \mathbf{K}},$$

$$\mathbf{N}_D = \mathbf{N}_D(\mathbf{E}, \mathbf{K}, \dot{\mathbf{E}}, \dot{\mathbf{K}}), \quad \mathbf{M}_D = \mathbf{M}_D(\mathbf{E}, \mathbf{K}, \dot{\mathbf{E}}, \dot{\mathbf{K}}), \quad (4.2)$$

where ψ is the surface free energy density, and $\mathbf{N}_D = \mathbf{N}_D(\mathbf{E}, \mathbf{K}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$, $\mathbf{M}_D = \mathbf{M}_D(\mathbf{E}, \mathbf{K}, \mathbf{0}, \mathbf{0}) = \mathbf{0}$.

5. Energy balance

To take into account the influence of temperature, let us discuss the balance of energy in the shell. Thermodynamics of shells from various points of view was presented by Green and Naghdi (1970,1979), Murdoch (1976), Zhilin (1976), Simmonds (1984, 2005), and Makowski and Pietraszkiewicz (2002). In the papers various sets of surface fields responsible for temperature were used, and various formulations of the first and second laws of thermodynamics for shells were formulated. In order to present here the reasonably simple results we confine ourselves to the simplest version of these laws suggested by Murdoch (1976). If additionally the temperature changes across the shell thickness are disregarded, at the shell midsurface we have the temperature field which describes some through-the-thickness average temperature.

The energy balance of an arbitrary part $\Pi \subset M$ of the shell midsurface can be described by, see Truesdell (1977),

$$\frac{d}{dt} E = A + Q, \quad (5.1)$$

where $E = \iint_{\Pi} \rho \varepsilon da$, ρ is the undeformed surface mass density, ε the internal energy density, $A = \iint_{\Pi} (\mathbf{f} \cdot \mathbf{v} + \mathbf{c} \cdot \mathbf{w}) da + \int_{\partial \Pi} (\mathbf{n}_v \cdot \mathbf{v} + \mathbf{m}_v \cdot \mathbf{w}) ds$ the power of external loading, $Q = \iint_{\Pi} \rho (q^+ + q^- + q_{\Pi}) da - \int_{\partial \Pi} q_v ds$ the heat supply velocity, q^{\pm} the heat influxes through the upper (+) and lower (-) shell faces, q_{Π} the internal surface heat supply, and q_v is the heat supply through the internal boundary contour. The field q_v is defined through the surface influx vector \mathbf{q} according to $q_v = \mathbf{q} \cdot \mathbf{v}$.

From (5.1) we obtain *the local energy balance*

$$\rho \frac{d}{dt} \varepsilon = \rho (q^+ + q^- + q_{\Pi}) - \text{Div}_s \mathbf{q} + \mathbf{N} \cdot \mathbf{E}^\circ + \mathbf{M} \cdot \mathbf{K}^\circ,$$

where $(\cdot)^\circ$ is the co-rotational time derivative, see Chróscielewski et al. (2004).

When Π contains a part of the singular surface curve C , from (5.1) we also obtain the *local balance equation* along C

$$V[\rho \varepsilon] + [\mathbf{n}_v \cdot \mathbf{v}] + [\mathbf{m}_v \cdot \mathbf{w}] - [\mathbf{q} \cdot \mathbf{v}] = 0. \quad (5.2)$$

6. Clausius-Duhem inequality

We take the second law of thermodynamics in the simple form proposed by Murdoch (1976)

$$\frac{d}{dt} \iint_{\Pi} \rho \eta da \geq \iint_{\Pi} \rho \left(\frac{q^+}{T_{ext}^+} + \frac{q^-}{T_{ext}^-} + \frac{q_{\Pi}}{T} \right) da - \int_{\partial \Pi} \frac{q_v}{T} ds, \quad (6.1)$$

where η is the surface entropy density. In (6.1), by T_{ext}^+ и T_{ext}^- we denote the temperature of the external media surrounding the shell from above and below.

The local form of (6.1) is

$$\rho \frac{d}{dt} \eta \geq \rho \left(\frac{q^+}{T_{ext}^+} + \frac{q^-}{T_{ext}^-} + \frac{q_{\Pi}}{T} \right) - Div_s \left(\frac{1}{T} \mathbf{q} \right).$$

If a part of $C \in \Pi$ then from (6.1) follows the relation along C

$$V[\rho\eta] - \left[\frac{1}{T} \mathbf{q} \cdot \mathbf{v} \right] \equiv -\delta^2 \leq 0.$$

The quantity δ^2 represents creation of entropy at the interface C . If $\delta = 0$ then the phase transition is called reversible. In such a case the balance equation along C reduces to

$$V[\rho\eta] - \left[\frac{1}{T} \mathbf{q} \cdot \mathbf{v} \right] = 0. \quad (6.2)$$

7. Thermodynamic continuity condition

The relations (2.3), (4.1), (5.2) and (6.2) derived above can be transformed further by excluding the normal velocity V of the interface. If the field T is continuous at C , that is when $[T] = 0$, one can eliminate the field \mathbf{q} from (5.2) and (6.2) which leads to

$$V[\rho\psi] + [\mathbf{n}_\nu \cdot \mathbf{v}] + [\mathbf{m}_\nu \cdot \mathbf{w}] = 0, \quad (7.1)$$

where $\psi = \varepsilon - T\eta$ is the surface free energy density.

Further transformations of (7.1) are similar to those given in Eremeyev and Pietraszkiewicz (2004). Using (3.1) from (7.1) we obtain

$$[\mathbf{v} \cdot \mathbf{C}\mathbf{v}] = 0, \quad (7.2)$$

where \mathbf{C} is the surface Eshelby tensor.

At the coherent curvilinear interface

$$\mathbf{C} \equiv \mathbf{C}_c = \rho\psi \mathbf{A} - \mathbf{N}^T \mathbf{F} - \mathbf{M}^T \mathbf{K},$$

while at the interface incoherent in rotations

$$\mathbf{C} \equiv \mathbf{C}_i = \rho\psi \mathbf{A} - \mathbf{N}^T \mathbf{F}.$$

As a result, in the case of quasi-static loading along the interface C we obtain the same thermodynamic condition as was given earlier by Eremeyev and Pietraszkiewicz (2004) for the case of phase equilibrium. But contrary to the latter paper, here we do not assume that the shell is elastic. In our case the constitutive equations involve an arbitrary dependence of the surface stress measures on prehistories of deformation measures and temperature.

8. Transition to mechanics

To construct the purely mechanical theory let us discuss an isothermal process in which the temperature field is constant, $T = T_{ext}^+ = T_{ext}^- = const$. Then from the relations (5.1) and (6.1) it follows that

$$\frac{d}{dt} \Psi \leq A. \quad (8.1)$$

Here $\Psi = \iint_{\Pi} \rho \psi da$ means the internal stored energy, while A represents the power of external loading. It is possible to show that for elastic shells with phase transitions the inequality (8.1) transforms into the equality, which becomes equivalent to the weak formulation of the problem based on the variational principle of stationary total potential energy, see Eremeyev and Pietraszkiewicz (2004). However, when the constitutive equations of the viscoelastic shells are used the variational inequality (8.1) leads to the additional balance relation (7.2).

8. Conclusions

We have derived the balance conditions at the curvilinear phase boundaries in the non-linear theory of shells. The conditions have followed from the integral balance laws of linear momentum, angular momentum, and energy as well as from the entropy inequality. It has been shown that the additional relation at the curvilinear phase interface derived by Eremeyev and Pietraszkiewicz (2004) for elastic shells should also be satisfied for quasi-static processes in thermoelastic shells. In the case of thermoviscoelastic shells the relation contains not only the equilibrium parts but also the dissipative parts of the stress resultant and stress couple tensors. The relation is valid when there is no entropy creation at the phase interface, that is when the phase transition is assumed to be reversible.

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