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Determination of the midsurface of a deformed shell from prescribed fields of surface strains and bendings

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Abstract

We show how to determine the midsurface of a deformed thin shell from the following set of data: known geometry of the undeformed midsurface, the surface strains and the surface bendings. It is assumed that the two latter fields had been obtained beforehand by solving a problem posed for the so-called intrinsic field equations of the non-linear theory of thin shells. Two different methods of determining the deformed midsurface in space are worked out: (a) directly from its first and second fundamental form using some results from mathematical analysis; (b) integrating the system of first-order PDEs for the surface deformation gradient. In both cases the corresponding integrability conditions are discussed; it is shown that they are equivalent to the compatibility conditions of the non-linear theory of thin shells.

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1. Introduction

The intrinsic formulation of the geometrically non-linear theory of thin, isotropic elastic shells, originally suggested by Chien (1944) in terms of the surface strains $\gamma_{\alpha\beta}$ and bendings $\varkappa_{\alpha\beta}$ of the shell midsurface, was refined by Danielson (1970) and Koiter and Simmonds (1973) and worked out in detail by Opoka and Pietraszkiewicz (2004), where many references related to this topic are given. In the latter paper the governing field equations were expressed via the membrane stress resultants $N^{\alpha\beta}$ and the midsurface bendings $\varkappa_{\alpha\beta}$ as primary unknowns. Compared with the complexity of the field equations formulated in displacements as unknowns, discussed e.g. by Pietraszkiewicz and Szwabowicz (1981) and Pietraszkiewicz (1984), the intrinsic formulation is relatively simple. It consists of six quadratic intrinsic shell equations (ISEs) with four intrinsic

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dynamic and/or kinematic boundary conditions, two of which are quadratic and the other two are linear in the unknowns. Having found $N^{\alpha\beta}$, the midsurface strains $\gamma_{\alpha\beta}$ can then be calculated from inverted linear constitutive relations.

Yet, the intrinsic formulation has never gained popularity within engineering community it deserves. The main reason is that having calculated $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$ from the ISEs we can easily establish the first and second fundamental forms of the deformed shell midsurface, but not its unique deformed position in space itself. One would expect that during over 60 years from the pioneering Chien paper, and after various modified and/or refined versions of the non-linear ISEs proposed in the literature, several methods should have been developed for establishing the deformed shell midsurface from known $\gamma_{\alpha\beta}$ and $\varkappa_{\alpha\beta}$. But after a thorough search of the literature we are surprised to admit that we are not aware of any such a method published elsewhere. The only result which is related to this problem was given by Zubov (1989) in the context of the non-linear theory of dislocations in thin elastic shells. We shall comment on his proposal in Section 5 and compare it with our direct and simpler approach developed there.

In this paper we consider the following complementary problem:

Given the surface strain and bending tensor fields, $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\theta^\lambda)$ and $\varkappa_{\alpha\beta} = \varkappa_{\alpha\beta}(\theta^\lambda)$, respectively, prescribed on some middle surface $\mathscr{M}$ of an undeformed thin shell find the position vector $\mathbf{y} = \mathbf{y}(\theta^\alpha)$ of the midsurface $\overline{\mathscr{M}}$ in the deformed configuration.

The problem discussed here is related to the classical problem of differential geometry: immersion a 2D manifold into the 3D Euclidean space. Indeed, from known $\mathscr{M}$ as well as $\gamma_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ one can easily establish the components of two fundamental forms of $\overline{\mathscr{M}}$ by $\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta}$ and $\bar{b}_{\alpha\beta} = b_{\alpha\beta} - \varkappa_{\alpha\beta}$, where $a_{\alpha\beta}$ and $b_{\alpha\beta}$ are the metric and curvature components of $\mathscr{M}$, respectively. Then according to the theorem of Bonnet (1867), the two fundamental forms of the surface $\overline{\mathscr{M}}$ determine locally its position in the 3D Euclidean space up to a rigid-body motion. From mathematical point of view this solves the problem of existence of such a surface. But in the non-linear theory of shells such a statement for $\overline{\mathscr{M}}$ is not satisfactory, because in engineering one usually needs to know *uniquely* the position of $\overline{\mathscr{M}}$ in space described by the position vector $\mathbf{y} = \mathbf{y}(\theta^\alpha)$. This can be achieved only by formulating an appropriate system of PDEs and solving it with a unique set of boundary and/or initial conditions.

Some elements of determination of a surface in space from prescribed two fundamental forms can be found in books on differential geometry by Spivak (1979), do Carmo (1976), Ciarlet (2005), as well as in the recent papers by Ciarlet and Larsonneur (2002), and Ciarlet and Mardare (2005). In Section 3 we use some of those results and develop the two-step method of unique determination of $\mathscr{M}$ from prescribed $a_{\alpha\beta}$ and $b_{\alpha\beta}$. In the first step we formulate a system of two linear, first-order PDEs for the column vector field $\mathbf{X}$ and show that this system can be converted to an equivalent set of ODEs along curves covering densely the entire domain of the surface coordinates. Then, the set of ODEs is solved by the method of successive approximations, leading to the general formula (14). In the second step the position vector $\mathbf{x}$ of $\mathscr{M}$ is found by quadrature (15). The solution depends on two sets of initial conditions $\mathbf{X}_0$ and $\mathbf{x}_0$ imposed at an arbitrarily chosen point $\mathbf{x}_0 \in \mathscr{M}$ which fix uniquely the position of $\mathscr{M}$ in the ambient 3D Euclidean space. Since $\bar{a}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ are known if $a_{\alpha\beta}$, $b_{\alpha\beta}$ and $\gamma_{\alpha\beta}$, $\varkappa_{\alpha\beta}$ are given, this two-step method is directly applied in Section 4 to establish the unique position of $\overline{\mathscr{M}}$ in the space as well. Although the method developed here is based on known results which are somewhat hidden in the works on classical differential geometry and analysis, its formulation within the non-linear theory of thin shells is new.

In Section 5 we introduce the surface deformation gradient $\mathbf{F}$ of the shell midsurface and derive the linear system (23) of two PDEs for $\mathbf{F}$ which integrability conditions are equivalent to the compatibility conditions of the non-linear theory of thin shells. The solution of this system is given in the form (31) by the method of successive approximations, and the deformed position of the midsurface $\mathbf{y} = \mathbf{y}(\theta^\alpha)$ follows then by the quadrature (22). The method developed here is direct, compact and is new, as well.

We briefly remind in Section 5 that in a similar method Zubov (1989) used the spatial deformation gradient $\mathbf{G}$ evaluated at the shell midsurface $\mathscr{M}$. It is shown that $\mathbf{G}$ contains redundant part as compared with $\mathbf{F}$, which is not necessary in the non-linear shell problem discussed here. Thus, additional care should be taken to separate the redundant part of $\mathbf{G}$ from the important one.

In both methods of solution discussed here the governing systems of equations, (6) in the first method and (23) in the second one, turn out to be the total differential systems. Existence of local solutions follows in both cases from theorems of Frobenius and Frobenius–Dieudonné (see Maurin, 1980) provided that suitable integrability conditions are satisfied. This latter question is carefully examined. Only for some special geometries of the undeformed midsurface and for particularly simple types of deformation states one might expect to find analytical solutions of the problem. In realistic, highly non-linear problems of engineering importance one will have to rely on numerical methods combined with specialized computer programs which have to be developed.

2. Preliminaries

A shell is a 3D solid body identified in a reference (undeformed) configuration with a region $\mathscr{B}$ of the physical space $\mathscr{E}$ that has E for its 3D translation vector space. In the region $\mathscr{B}$ we introduce the normal system of curvilinear coordinates $\{\theta^1, \theta^2, \zeta\}$, such that $-\frac{h}{2} \leqslant \zeta \leqslant \frac{h}{2}$ is the distance from the shell midsurface $\mathscr{M}$ to the points in $\mathcal{B}$ and h is the thickness of the undeformed shell. In the theory of thin shells discussed here h is assumed to be constant and small in comparison with other shell dimensions.

In the theory of shells the midsurface $\mathscr{M}$ is usually defined (locally) by the position vector $\mathbf{x} = x^k(\theta^\alpha)\mathbf{i}_k$, $\alpha = 1, 2$, $k = 1, 2, 3$, $\mathbf{x} \in E$, relative to some fixed origin $o \in \mathscr{E}$ and an orthonormal Cartesian frame $\{\mathbf{i}_k\}$. At each point $x \in \mathscr{M}$ gradients of the coordinates θ^α constitute the so-called contravariant surface base, $\mathbf{a}^\alpha = \mathrm{grad}(\theta^\alpha)$, and partial derivatives of $\mathbf{x}$, $\frac{\partial \mathbf{x}}{\partial \theta^\alpha} \equiv \mathbf{x}_{,\alpha} = \mathbf{a}_\alpha$, the covariant surface base. We have the relations

$$\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha, \qquad a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta,$$
$$a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = (a_{\alpha\beta})^{-1}, \qquad \det(a_{\alpha\beta}) = a > 0, \tag{99}$$

where δ^β_α denotes the Kronecker symbol, while $a_{\alpha\beta}$ are the covariant and $a^{\alpha\beta}$ the contravariant components of the surface metric tensor $\mathbf{a}$, respectively. The unit normal vector $\mathbf{n} = \frac{1}{\sqrt{a}}\mathbf{a}_1 \times \mathbf{a}_2$ determines locally the orientation of $\mathscr{M}$.

Let $e_{\alpha\beta} = e^{\alpha\beta}$ denote the permutation (Levi-Civita) symbol, i.e. $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$. The covariant components of the surface permutation tensor $\boldsymbol{\varepsilon}$ are given by $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ and the following relations hold true:

$$\varepsilon_{\alpha\beta} = \sqrt{a} e_{\alpha\beta}, \qquad \varepsilon^{\alpha\beta} = \frac{1}{\sqrt{a}} e^{\alpha\beta},$$
$$\varepsilon_{\alpha\beta}\varepsilon^{\lambda\mu} = \delta^\lambda_\alpha\delta^\mu_\beta - \delta^\mu_\alpha\delta^\lambda_\beta, \qquad \varepsilon^{\alpha\lambda}\varepsilon^{\beta\mu}a_{\alpha\beta} = a^{\lambda\mu}, \qquad \varepsilon_{\alpha\beta}a^{\alpha\lambda}a^{\beta\mu} = \varepsilon^{\lambda\mu}. \tag{107}$$

The shape of the surface is described by the second fundamental tensor $\mathbf{b}$, called also the shape operator, through its covariant components

$$b_{\alpha\beta} = -\mathbf{n}_{,\alpha} \cdot \mathbf{a}_\beta = \mathbf{n} \cdot \mathbf{a}_{\alpha,\beta}. \tag{111}$$

For comprehensive exposition of other definitions and concepts we refer the reader to classical books on differential geometry and tensor calculus, but the references such as Chernykh (1964), Green and Zerna (1968), Pietraszkiewicz (1977), Ciarlet (2005) explain these questions directly in the context of the theory of thin shells.

3. Local existence of a surface

According to the classical theorem of Bonnet (1867), called also the theorem of local existence of surfaces, the two fundamental forms: the first $I = a_{\alpha\beta}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta$ and the second $II = b_{\alpha\beta}\mathrm{d}\theta^\alpha\mathrm{d}\theta^\beta$, determine a surface in the 3D Euclidean space up to a rigid-body motion. A more modern version of this theorem answers also the converse question (Spivak, 1979, vol. 3, p. 86): *what conditions must satisfy some pair of fields of quadratic forms, $I = I(\theta^\alpha)$ and $II = II(\theta^\alpha)$, defined on an open, simply-connected two-dimensional domain $\mathscr{U}$ to become a first and second fundamental forms of some surface?*

The answer follows from considerations on local solvability of a system of PDEs, whose coefficients are determined by $a_{\alpha\beta}$ and $b_{\alpha\beta}$. Since these components are by definition the scalar products between all pairs from the set of four vector fields, two covariant base vectors $\mathbf{a}_\alpha = \mathbf{a}_\alpha(\theta^\lambda)$ and two partial derivatives of $\mathbf{n} = \mathbf{n}(\theta^\lambda)$, the above question will be answered if we determine the latter fields in terms of the fields $a_{\alpha\beta} = a_{\alpha\beta}(\theta^\lambda)$ and $b_{\alpha\beta} = b_{\alpha\beta}(\theta^\lambda)$. This problem is governed by two sets of relations: the equations of Gauss

$$\mathbf{a}_{\alpha,\beta} = \Gamma^{\lambda}_{\alpha\beta}\mathbf{a}_{\lambda} + b_{\alpha\beta}\mathbf{n} \tag{1}$$

and the equations of Weingarten

$$\mathbf{n}_{,\beta} = -b^{\lambda}_{\beta}\mathbf{a}_{\lambda}, \qquad \text{where} \qquad b^{\lambda}_{\beta} = b_{\beta\mu}a^{\mu\lambda} \tag{2}$$

and the coefficients appearing in (1) are the Christofell symbols $\Gamma^{\lambda}_{\alpha\beta}$ of the second kind which can be computed from the metric coefficients $a_{\alpha\beta}$ by the formula

$$\Gamma^{\lambda}_{\alpha\beta} = \tfrac{1}{2}a^{\lambda\mu}(a_{\mu\alpha},_{\beta} + a_{\mu\beta},_{\alpha} - a_{\alpha\beta},_{\mu}). \tag{3}$$

It will be easier to analyze this problem if we employ the following matrix notation. At every point $p \in \mathscr{U}$ let us define a column vector

$$\mathbf{X} = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{n} \end{bmatrix}, \tag{4}$$

where for now $\mathbf{n}$ need be neither a unit vector nor orthogonal to the remaining two entries in $\mathbf{X}$, and two square 3 by 3 scalar matrices

$$\mathbb{A}_{\alpha} = \begin{bmatrix} \Gamma^{1}_{1\alpha} & \Gamma^{2}_{1\alpha} & b_{1\alpha} \\ \Gamma^{1}_{2\alpha} & \Gamma^{2}_{2\alpha} & b_{2\alpha} \\ -b^{1}_{\alpha} & -b^{2}_{\alpha} & 0 \end{bmatrix}, \tag{5}$$

Since every entry X_i, $i = 1, 2, 3$, in the column vector $\mathbf{X}$ is an element of a three-dimensional linear vector space $\mathbb{R}^3$, $\mathbf{X}$ is itself an element of the direct sum of three consecutive copies of $\mathbb{R}^3$, i.e. $\mathbf{X} \in \mathbb{R}^3 \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$, and thus belongs to a nine-dimensional linear vector space. Now the differential system that governs the relation between coefficients of the two fundamental forms and the vector fields $\mathbf{a}_{\alpha} = \mathbf{a}_{\alpha}(\theta^{\lambda})$, $\mathbf{n} = \mathbf{n}(\theta^{\lambda})$ may be written down in the form of two vector equations

$$\mathbf{X}_{,\alpha} = \mathbb{A}_{\alpha}\mathbf{X}, \tag{6}$$

where the entries $\Gamma^{\lambda}_{\alpha\beta}$ in the matrices $\mathbb{A}_{\alpha}$ are given by (3). Thus, we are looking for an unknown column vector $\mathbf{X}$ satisfying the linear system (6).

The system (6) is a total differential system. By the theorem of Frobenius, see for example Maurin (1980), local solutions exist if and only if the integrability condition $\varepsilon^{\alpha\beta}\mathbf{X}_{,\alpha\beta} = \mathbf{0}$ is satisfied everywhere in the domain in which the matrices $\mathbb{A}_{\alpha}$ are prescribed. Hence, the system is completely integrable if the matrix equation

$$\varepsilon^{\alpha\beta}(\mathbb{A}_{\alpha,\beta} + \mathbb{A}_{\alpha}\mathbb{A}_{\beta}) = \mathbf{0} \tag{7}$$

holds in $\mathscr{U}$. Therefore, the necessary next step consists in verifying what conditions in terms of $a_{\alpha\beta}$ and $b_{\alpha\beta}$ must be satisfied for the solution to exist.

Straightforward transformations show that after substitution of (5)–(7) one obtains the so-called Gauss-Mainardi-Codazzi (GMC) equations of the surface $\mathscr{M}$

$$R^{\kappa}_{.\beta\lambda\mu} = b^{\kappa}_{\lambda}b_{\beta\mu} - b^{\kappa}_{\mu}b_{\beta\lambda}, \quad b_{\beta\lambda|\mu} - b_{\beta\mu|\lambda} = 0, \tag{8}$$

where $(.)_{|\alpha}$ is the covariant derivative in the metric of $\mathscr{M}$ and the Riemann–Christoffel tensor is defined by

$$R^{\kappa}_{.\beta\lambda\mu} = \Gamma^{\kappa}_{\beta\mu},_{\lambda} - \Gamma^{\kappa}_{\beta\lambda},_{\mu} + \Gamma^{\rho}_{\beta\mu}\Gamma^{\kappa}_{\rho\lambda} - \Gamma^{\rho}_{\beta\lambda}\Gamma^{\kappa}_{\rho\mu}. \tag{9}$$

The GMC equations (8) are presented in various equivalent forms in the literature, depending on the author and intended application, see for example Spivak (1979), do Carmo (1976), Koiter (1966). One of them frequently used in the non-linear theory of thin shells (see Pietraszkiewicz, 1977) is the following:

$$\begin{aligned} &\varepsilon^{\alpha\beta}\varepsilon^{\lambda\mu}(\Gamma_{\alpha.\mu\beta},_{\lambda} + \Gamma^{\kappa}_{\alpha\mu}\Gamma_{\kappa.\beta\lambda} + b_{\alpha\mu}b_{\beta\lambda}) = 0, \\ &\varepsilon^{\lambda\mu}b_{\beta\lambda}|_{\mu} = 0, \end{aligned} \tag{10}$$

where $\Gamma_{\kappa.\alpha\beta} = a_{\kappa\lambda}\Gamma^{\lambda}_{\alpha\beta}$ are the Christoffel symbols of the first kind.

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The questions discussed up to now remain within the scope of differential geometry of surfaces. But techniques used for solving systems like (6) belong to the theory of differential equations and are disconnected from the geometric background of the problem. Here, the first step consists in showing that the problem can be converted to an equivalent infinite set of systems of ODEs along curves covering densely the entire domain $\mathcal{U}$.

Suppose that some two fields of quadratic forms, whose coefficients are continuously differentiable in $\mathcal{U}$, satisfy the integrability condition (7). Then, by the theorem of Frobenius–Dieudonné, see Maurin (1980), for every initial condition $\mathbf{X}(\theta_0^\alpha) = \mathbf{X}_0$ prescribed at a point $p_0 \in \mathcal{U}$ with coordinates θ_0^α there exists, possibly in some smaller domain $\hat{\mathcal{U}} \subset \mathcal{U}$, a unique solution $\mathbf{X}(\theta^\alpha)$ satisfying this initial condition, and all such solutions depend continuously on the initial value $\mathbf{X}_0$. More recent results along this line are due to Ciarlet and Larsonneur (2002) and Ciarlet and Mardare (2005). In particular, the latter paper shows how to extend the solution to the closure $\overline{\mathcal{U}}$ of the domain, which permits to establish existence of a surface with a boundary.

Consider a particular solution $\mathbf{X}$ of the system (6) and a curve $\mathcal{C} : [a, b] \ni s \to \theta^\alpha(s)$ leaving from some point $p_0 \in \mathcal{U}$ to another point $p \in \mathcal{U}$. Suppose the value of $\mathbf{X}$ at p_0 is $\mathbf{X}_0$. Note that the restriction $\mathbf{X}|_{\mathcal{C}}$ of this solution to the curve $\mathcal{C}$ satisfies the following system of ODEs:

$$\frac{\mathrm{d}\mathbf{X}|_{\mathcal{C}}}{\mathrm{d}s} = \mathbb{A}^C \mathbf{X}|_{\mathcal{C}}, \tag{11}$$

where the matrix $\mathbb{A}^C$ is given by

$$\mathbb{A}^C = \mathbb{A}_\alpha \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}. \tag{12}$$

Let us reverse the argumentation. Now consider the initial value problem for the system of ODEs

$$\frac{\mathrm{d}\mathbf{X}^*}{\mathrm{d}s} = \mathbb{A}^C \mathbf{X}^*$$

for some abstract vector field $\mathbf{X}^*$ along the same curve $\mathcal{C}$ with the same initial condition $\mathbf{X}^*(0) = \mathbf{X}_0$. By the standard results from the theory of ordinary differential equations this problem has a unique solution $\mathbf{X}^*(s)$. Therefore, $\mathbf{X}^*(s)$ must be identical with the restriction of $\mathbf{X}$ to $\mathcal{C}$ on the interval where it exists, i.e. we must have $\mathbf{X}|_{\mathcal{C}} = \mathbf{X}^*(s)$.

Thus, instead of solving the system (6) directly, we may compute a particular solution $\mathbf{X}(\theta^\alpha)$ corresponding to some initial condition $\mathbf{X}(\theta_0^\alpha) = \mathbf{X}_0$ by covering the domain $\mathcal{U}$ with a set of paths leaving radially from the initial point $p_0 \in \mathcal{U}$ and then solving an initial value problem for the system of ODEs

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} = \mathbb{A}^C \mathbf{X}, \qquad \mathbb{A}^C = \mathbb{A}_\alpha \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} \tag{13}$$

along each of the paths.

There is still the question of the initial conditions themselves and the constraints

$$\begin{aligned}
\langle X_\alpha, X_\beta \rangle &= \mathbf{a}_\alpha \cdot \mathbf{a}_\beta = a_{\alpha\beta}, \\
\langle X_\alpha, X_3 \rangle &= \mathbf{a}_\alpha \cdot \mathbf{n} = 0, \\
\langle X_3, X_3 \rangle &= \mathbf{n} \cdot \mathbf{n} = 1.
\end{aligned}$$

We want any solution $\mathbf{X}$ to satisfy at every point where it exists, the initial points inclusive. It is proved in do Carmo (1976, p. 312), that setting the initial value at some arbitrarily chosen point p_0 with coordinates θ_0^α to, say

$$\mathbf{X}_0 = \begin{bmatrix} \mathbf{v}_1 \\ \mathbf{v}_2 \\ \mathbf{v}_3 \end{bmatrix},$$

such that

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$$\mathbf{v}_\alpha \cdot \mathbf{v}_\beta = a_{\alpha\beta}(\theta_0^\alpha), \qquad \mathbf{v}_3 = \frac{\mathbf{v}_1 \times \mathbf{v}_2}{|\mathbf{v}_1 \times \mathbf{v}_2|},$$

automatically yields solutions satisfying these constraints everywhere. The proof relies on computing the position vector $\mathbf{x}(\theta^\alpha)$ with the use of (15) from a given solution $\mathbf{X}$ and reverse confirmation of the thesis.

The solutions to the initial value problem (13) may be obtained with the use of any of the well-known techniques, numerical techniques inclusive. In particular, using the method of successive approximations one ends up with the solution in the form of the infinite series

$$\mathbf{X} = \sum_{i=0}^{\infty} \hat{\mathbf{X}}_i, \tag{14}$$

where the terms $\hat{\mathbf{X}}_i$ are given by the recursive formulae

$$\begin{aligned} \hat{\mathbf{X}}_0 &= \mathbf{X}_0, \\ &\vdots \\ \hat{\mathbf{X}}_i &= \int_{p_0}^{p} \mathbb{A}^C(s)\hat{\mathbf{X}}_{i-1}(s)\mathrm{d}s, \\ &\vdots \end{aligned}$$

with the first term $\hat{\mathbf{X}}_0$ equal to the given initial value.

Note that for $i > 0$ we have

$$\frac{\mathrm{d}\hat{\mathbf{X}}_i}{\mathrm{d}s} = \mathbb{A}^C\hat{\mathbf{X}}_{i-1}.$$

Therefore,

$$\frac{\mathrm{d}\mathbf{X}}{\mathrm{d}s} = \sum_{i=1}^{\infty} \mathbb{A}^C\hat{\mathbf{X}}_{i-1} = \mathbb{A}^C\sum_{i=1}^{\infty}\hat{\mathbf{X}}_{i-1} = \mathbb{A}^C\mathbf{X},$$

and thereby the series (14) formally satisfies the system (13). By passing to the limit with $p \to p_0$ we obtain $\hat{\mathbf{X}}_i \to \mathbf{0}$ for all $i > 0$ and hence $\mathbf{X} \to \mathbf{X}_0$, so the initial condition is satisfied. For the proof of convergence see Maurin (1980).

Having solved (6) one obtains the position vector of the surface from the quadrature

$$\mathbf{x}(\theta^\alpha) = \mathbf{x}_0 + \int_{p_0}^{p} \mathbf{a}_\alpha \mathrm{d}\theta^\alpha, \tag{15}$$

where $\mathbf{x}_0$ is the initial value of $\mathbf{x}$ at some arbitrarily chosen point $\mathbf{x}_0 \in \mathscr{M}$ labeled by θ_0^α.

Thus, the entire solution depends on two sets of arbitrarily chosen initial conditions: the column vector $\mathbf{X}_0$ and the vector $\mathbf{x}_0$. These two vectors fix uniquely the position of the surface in the ambient Euclidean space. Since they may be chosen arbitrarily, $a_{\alpha\beta}$ and $b_{\alpha\beta}$ really determine a surface only to within a rigid-body motion. In particular applications to shell problems, wherein there exist separate side conditions, imposed for instance along the boundary of $\mathscr{M}$ and used previously for obtaining the fields of strains and bendings, one should carefully choose the values of $\mathbf{X}_0$ and $\mathbf{x}_0$ to ensure that these side conditions are not violated. This can always be achieved if the intrinsic shell problem had been solved correctly.

4. Determination of the deformed midsurface of a thin shell

Consider a deformation $\boldsymbol{\chi}$ of the shell, i.e. a map $\boldsymbol{\chi} : \mathcal{B} \to \overline{\mathcal{B}}$. The theory of thin shells is based on an assumption that the 3D deformation of the shell can be approximated with a sufficient accuracy by deformation of its reference (usually middle) surface. During deformation the thin shell is represented by a material surface capable of resisting to stretching and bending. We assume that θ^α are the material (convected) coordinates and that

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the image of the midsurface $\mathscr{M}$ under $\boldsymbol{\chi}$ coincides with the deformed midsurface $\overline{\mathscr{M}}$, i.e. $\overline{\mathscr{M}} = \boldsymbol{\chi}(\mathscr{M})$. Then, the position vector $\mathbf{y} = y^k(\theta^\alpha)\mathbf{i}_k$ of $\overline{\mathscr{M}}$ relative to the same fixed frame $\{o, \mathbf{i}_k\}$ is

$$\mathbf{y}(\theta^\alpha) = \boldsymbol{\chi}[\mathbf{x}(\theta^\alpha)], \tag{16}$$

and the field of displacements can be obtained as

$$\mathbf{u}(\theta^\alpha) = \mathbf{y}(\theta^\alpha) - \mathbf{x}(\theta^\alpha). \tag{17}$$

In the convected coordinates all quantities defined and the relations written earlier for $\mathscr{M}$ hold true also on $\overline{\mathscr{M}}$. To indicate which of the two configurations is meant, we shall provide all symbols pertaining to the deformed one with a bar above the symbol, e.g. $\bar{a}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{b}_{\alpha\beta}$, $\bar{b}$, $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\overline{\Gamma}^\lambda_{\alpha\beta}$, $\overline{R}^\kappa_{.\beta\lambda\mu}$, etc., and leave those pertaining to the undeformed configuration unmarked.

Deformation of the shell midsurface is described by two Green type surface strain and bending tensors with covariant components

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \quad \varkappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}). \tag{18}$$

Our goal is to find the position $\mathbf{y} = \mathbf{y}(\theta^\alpha)$ of $\overline{\mathscr{M}}$ and/or the displacement field $\mathbf{u} = \mathbf{u}(\theta^\alpha)$ defined in (17) from the position vector $\mathbf{x} = x(\theta^\alpha)$ and two given fields $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\theta^\alpha)$ and $\varkappa_{\alpha\beta} = \varkappa_{\alpha\beta}(\theta^\alpha)$ which have already been found as solutions of the intrinsic shell equations by Opoka and Pietraszkiewicz (2004).

Having solved (15) for $\mathbf{x}(\theta^\alpha)$ we can use definitions of the strain and bending components (18) for determination of covariant components of the metric and curvature tensors of $\overline{\mathscr{M}}$

$$\bar{a}_{\alpha\beta} = a_{\alpha\beta} + 2\gamma_{\alpha\beta}, \quad \bar{b}_{\alpha\beta} = b_{\alpha\beta} - \varkappa_{\alpha\beta} \tag{19}$$

and then mimic the procedure described in Section 3. This leads to the system

$$\overline{\mathbf{X}}_{,\alpha} = \overline{\mathbb{A}}_\alpha \overline{\mathbf{X}}$$

analogous to (6), where $\overline{\mathbf{X}}$ is now defined through $\bar{\mathbf{a}}_\alpha$, $\bar{\mathbf{n}}$ and $\overline{\mathbb{A}}_\alpha$ through $\overline{\Gamma}^\lambda_{\alpha\beta}$, $\bar{b}_{\alpha\beta}$ in analogy to (4) and (5), respectively. One should then repeat all arguments and steps of Section 3 which then lead to determination of the position vector $\mathbf{y}$ in the form analogous to (15)

$$\mathbf{y}(\theta^\alpha) = \mathbf{y}_0 + \int_{p_0}^{p} \bar{\mathbf{a}}_\alpha \mathrm{d}\theta^\alpha,$$

where $\mathbf{y}_0$ is the initial value of $\mathbf{y}$ at any point $y_0 = \chi[x_0(\theta^\alpha_0)] \in \overline{\mathscr{M}}$. Then, the displacements follow naturally from (17).

5. Surface deformation gradient

Closer to the spirit of mechanics, let us employ in this Section the concepts describing local deformation of the shell midsurface. The surface gradient ∇_s of deformation $\mathbf{y} = \boldsymbol{\chi}(\mathbf{x})$ of the shell midsurface, taken relative to the undeformed midsurface $\mathscr{M}$, allows us to introduce the tensor field $\mathbf{F} \in E \otimes T_x\mathscr{M}$ defined by

$$\mathbf{F} = \nabla_s \boldsymbol{\chi}(\mathbf{x}) = \mathbf{y}_{,\alpha} \otimes \mathbf{a}^\alpha, \tag{20}$$

which allows one to write the relations

$$\mathbf{y}_{,\alpha} = \mathbf{F}\mathbf{a}_\alpha. \tag{21}$$

Mathematically, $\mathbf{F}$ so defined is the Frechét derivative of the deformation $\boldsymbol{\chi}$. Thus, given $\mathbf{F}(\theta^\alpha)$ we can determine position of the deformed shell midsurface by the quadrature

$$\mathbf{y} = \mathbf{y}_0 + \int_{p_0}^{p} \mathbf{F}\mathbf{a}_\alpha \mathrm{d}\theta^\alpha, \tag{22}$$

where again $\mathbf{y}_0 = \mathbf{y}(x_0)$, and the corresponding displacement field follows then from (17). Because $\mathbf{y}_{,\alpha} = \bar{\mathbf{a}}_\alpha \in T_y\overline{\mathscr{M}} \subset E$, partial derivatives of $\mathbf{F} = \bar{\mathbf{a}}_\lambda \otimes \mathbf{a}^\lambda$ can be written as

$$\mathbf{F}_{,\alpha} = \mathbf{F}\mathbf{A}_\alpha, \tag{23}$$

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where the two tensors $\mathbf{A}_\alpha$ are given by

$$\mathbf{A}_\alpha = (\overline{\Gamma}^\kappa_{\lambda\alpha} - \Gamma^\kappa_{\lambda\alpha})\mathbf{a}_\kappa \otimes \mathbf{a}^\lambda + b^\kappa_\alpha \mathbf{a}_\kappa \otimes \mathbf{n} + \bar{b}_{\lambda\alpha}\frac{1}{\sqrt{\bar{a}}}(\mathbf{a}_1 \times \mathbf{a}_2) \otimes \mathbf{a}^\lambda. \tag{24}$$

When geometry of $\mathscr{M}$ and components of the surface strains $\gamma_{\alpha\beta}$ and bendings $\kappa_{\alpha\beta}$ are known, the tensors $\mathbf{A}_\alpha$ defined in (24) are known as well. Thus, our problem is governed by the linear system (23) of two PDEs for the unknown $\mathbf{F}$. This is again a total differential system whose integrability conditions $\mathbf{F}_{,\alpha\beta} - \mathbf{F}_{,\beta\alpha} = \mathbf{0}$ yield the tensor equation

$$\varepsilon^{\alpha\beta}(\mathbf{A}_{\alpha,\beta} - \mathbf{A}_\alpha \mathbf{A}_\beta) = \mathbf{0}. \tag{25}$$

Let us reveal the geometric meaning of (25). Taking the second partial derivatives of (23) we obtain

$$\begin{aligned}
\mathbf{F}_{,\alpha\beta} &= (\overline{\Gamma}^\kappa_{\lambda\alpha,\beta} - \Gamma^\kappa_{\lambda\alpha,\beta})\bar{\mathbf{a}}_\kappa \otimes \mathbf{a}^\lambda + (\overline{\Gamma}^\rho_{\lambda\alpha} - \Gamma^\rho_{\lambda\alpha})\left(\overline{\Gamma}^\kappa_{\rho\beta}\bar{\mathbf{a}}_\kappa + \bar{b}_{\rho\beta}\bar{\mathbf{n}}\right) \otimes \mathbf{a}^\lambda \\
&\quad + \left(\overline{\Gamma}^\kappa_{\rho\alpha} - \Gamma^\kappa_{\rho\alpha}\right)\bar{\mathbf{a}}_\kappa \otimes \left(-\Gamma^\rho_{\lambda\alpha}\mathbf{a}^\lambda + b^\rho_\beta \mathbf{n}\right) + \bar{b}_{\lambda\alpha,\beta}\bar{\mathbf{n}} \otimes \mathbf{a}^\lambda - \bar{b}_{\lambda\alpha}\bar{b}^\kappa_\beta \bar{\mathbf{a}}_\kappa \otimes \mathbf{a}^\lambda \\
&\quad + \bar{b}_{\rho\alpha}\bar{\mathbf{n}} \otimes \left(-\Gamma^\rho_{\lambda\beta}\mathbf{a}^\lambda + b^\rho_\beta \mathbf{n}\right) + b^\kappa_{\alpha,\beta}\bar{\mathbf{a}}_\kappa \otimes \mathbf{n} + b^\lambda_\alpha\left(\overline{\Gamma}^\kappa_{\lambda\beta}\bar{\mathbf{a}}_\kappa + \bar{b}_{\lambda\beta}\bar{\mathbf{n}}\right) \otimes \mathbf{n} - b^\kappa_\alpha b_{\beta\lambda}\bar{\mathbf{a}}_\kappa \otimes \mathbf{a}^\lambda \\
&= \left(\overline{\Gamma}^\kappa_{\lambda\alpha,\beta} + \overline{\Gamma}^\rho_{\lambda\alpha}\overline{\Gamma}^\kappa_{\rho\beta} - \Gamma^\rho_{\beta\lambda}\overline{\Gamma}^\kappa_{\rho\alpha} - \bar{b}^\kappa_\beta \bar{b}_{\lambda\alpha} - \Gamma^\kappa_{\lambda\alpha,\beta} - \Gamma^\rho_{\lambda\alpha}\overline{\Gamma}^\kappa_{\rho\beta} + \Gamma^\rho_{\lambda\beta}\Gamma^\kappa_{\rho\alpha} - b^\kappa_\alpha b_{\beta\lambda}\right)\bar{\mathbf{a}}_\kappa \otimes \mathbf{a}^\lambda \\
&\quad + \left(\overline{\Gamma}^\kappa_{\rho\alpha}b^\rho_\beta - \Gamma^\kappa_{\rho\alpha}b^\rho_\beta + b^\kappa_{\alpha,\beta} + \overline{\Gamma}^\kappa_{\rho\beta}b^\rho_\alpha\right)\bar{\mathbf{a}}_\kappa \otimes \mathbf{n} \\
&\quad + \left(\Gamma^\rho_{\lambda\alpha}\bar{b}_{\rho\beta} - \Gamma^\rho_{\lambda\alpha}\bar{b}_{\rho\alpha} + \bar{b}_{\lambda\alpha,\beta} - \Gamma^\rho_{\lambda\beta}\bar{b}_{\rho\alpha}\right)\bar{\mathbf{n}} \otimes \mathbf{a}^\lambda + \left(\bar{b}_{\rho\alpha}b^\rho_\beta + b^\rho_\alpha \bar{b}_{\rho\beta}\right)\bar{\mathbf{n}} \otimes \mathbf{n}.
\end{aligned} \tag{26}$$

The second partial derivatives $\mathbf{F}_{,\beta\alpha}$ follow from (26) by interchanging indices $\alpha \rightleftarrows \beta$. As a result, in the expression $\mathbf{F}_{,\alpha\beta} - \mathbf{F}_{,\beta\alpha}$ some terms cancel out while others can be grouped using definitions (9) and notions of the surface covariant derivatives, so that the integrability conditions (25) become equivalent to

$$\begin{aligned}
\mathbf{F}_{,\alpha\beta} - \mathbf{F}_{,\beta\alpha} = &\left(\overline{R}^\kappa_{.\lambda\beta\alpha} - \bar{b}^\kappa_\beta \bar{b}_{\lambda\alpha} + \bar{b}^\kappa_\alpha \bar{b}_{\lambda\beta} - R^\kappa_{.\lambda\beta\alpha} + b^\kappa_\beta b_{\lambda\alpha} - b^\kappa_\alpha b_{\lambda\beta}\right)\bar{\mathbf{a}}_\kappa \otimes \mathbf{a}^\lambda \\
&+ \left(b^\kappa_{\alpha|\beta} - b^\kappa_{\beta|\alpha}\right)\bar{\mathbf{a}}_\kappa \otimes \mathbf{n} + (\bar{b}_{\lambda\alpha\|\beta} - \bar{b}_{\lambda\beta\|\alpha})\bar{\mathbf{n}} \otimes \mathbf{a}^\lambda = \mathbf{0},
\end{aligned} \tag{27}$$

where $(.)_{\|\alpha}$ means covariant derivative in the metric of $\overline{\mathscr{M}}$.

Vanishing of the tensor conditions (27) is equivalent to vanishing of their components

$$\overline{R}^\kappa_{.\lambda\beta\alpha} - \bar{b}^\kappa_\beta \bar{b}_{\lambda\alpha} + \bar{b}^\kappa_\alpha \bar{b}_{\lambda\beta} - \left(R^\kappa_{.\lambda\beta\alpha} - b^\kappa_\beta b_{\lambda\alpha} + b^\kappa_\alpha b_{\lambda\beta}\right) = 0, \tag{28}$$

$$\bar{b}_{\lambda\alpha\|\beta} - \bar{b}_{\lambda\beta\|\alpha} = 0, \quad b^\kappa_{\alpha|\beta} - b^\kappa_{\beta|\alpha} = 0. \tag{29}$$

According to (9) and (8), the conditions (28) represent difference between the Gauss equation of the deformed and undeformed shell midsurfaces $\overline{\mathscr{M}}$ and $\mathscr{M}$, respectively, while (29) may be analogically viewed for the Mainardi-Codazzi equations. If we introduce ()(18)–(29) and perform transformations given in detail by Pietraszkiewicz (1977), the conditions become identical to the compatibility conditions of the non-linear theory of thin shells expressed in terms of the strains $\gamma_{\alpha\beta}$ and bendings $\kappa_{\alpha\beta}$, which were derived first by Chien (1944) and rederived by Galimov (1953) and Koiter (1966).

The solution to the system of equations (23) can again be given by choosing arbitrarily two points p_0, $p \in \mathscr{U}$, so that paths drawn on $\mathscr{U}$ between such points cover the entire domain $\mathscr{U}$. In a local chart any path $\mathscr{C} \in \mathscr{U}$ may be specified by two equations $\theta^\alpha|_\mathscr{C} = \theta^\alpha(s)$, where s denotes the arc length chosen so that $s(p_0) = s_0$. The system (23), when restricted to $\mathscr{C}$, reduces to an ODE of the form

$$\frac{\mathrm{d}\mathbf{F}}{\mathrm{d}s} = \mathbf{F}\mathbf{A}, \qquad \mathbf{A} = \mathbf{A}_\alpha \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s} \tag{30}$$

for an unknown tensor field $\mathbf{F}$.

General solution of (30) can again be given by the method of in the form

$$\mathbf{F} = \mathbf{F}_0\mathbf{F}_s, \quad \mathbf{F}_0 = \mathbf{F}(s_0), \qquad \mathbf{F}_s = \sum_{i=0}^{\infty} \mathbf{H}_i, \tag{31}$$
$$\mathbf{H}_0(s) = \mathbf{I}, \qquad \mathbf{H}_i(s) = \int_{s_0}^{s} \mathbf{H}_{i-1}(t)\mathbf{A}(t)\mathrm{d}t, \quad i \geqslant 1.$$

The tensor field $\mathbf{F}_s = \mathbf{F}(s)$ was called the matricant by Gantmakher (1959).

A somewhat similar approach to the one presented in this section was proposed by Zubov (1989, 1997) in the context of the non-linear theory of dislocations in thin elastic shells. In those works the spatial deformation gradient $\mathbf{G}$ evaluated at the shell midsurface was applied, not the surface deformation gradient $\mathbf{F}$ used in our method. To reveal the difference, let the 3D neighborhood of the midsurfaces $\mathscr{M}$ and $\overline{\mathscr{M}}$ be parametrized by the normal coordinates so that the corresponding position vectors are

$$\mathbf{p} = \mathbf{x} + \zeta\mathbf{n}, \quad \mathbf{q} = \mathbf{y} + \zeta\bar{\mathbf{n}},$$

where ζ is the distance from the corresponding midsurfaces to points in the shell space. This parametrization implies assumption of the Kirchhoff–Love kinematic constraints, such that material fibers that are normal to $\mathscr{M}$ remain normal to $\overline{\mathscr{M}}$ and do not change their lengths. The spatial gradient ∇ of the 3D deformation $\mathbf{q} = \chi(\mathbf{p})$, evaluated at the midsurface $\mathscr{M}$, leads to the tensor field $\mathbf{G} \in E \otimes E$ introduced by Pietraszkiewicz (1977)

$$\mathbf{G} = \nabla\chi(\mathbf{x} + \zeta\mathbf{n})|_{\zeta=0} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \det(\mathbf{G}) = \sqrt{\frac{\bar{a}}{a}} > 0, \tag{32}$$

which implies the relations

$$\bar{\mathbf{a}}_\alpha = \mathbf{G}\mathbf{a}_\alpha, \quad \bar{\mathbf{n}} = \mathbf{G}\mathbf{n}. \tag{33}$$

The tensor field $\mathbf{G}(\theta^\alpha)$ supplies first-order approximation of the three-dimensional state of shell deformation under the Kirchhoff–Love constraints in the neighborhood of its midsurface. Thus, given $\mathbf{G}(\theta^\alpha)$ we can also determine from $(33)_1$ position of the deformed midsurface by the same quadrature (22), and the corresponding displacement field follows then from (17). Please note that in this approach the relation $(33)_2$ is not necessary at all to determine $\mathbf{y}$ and $\mathbf{u}$.

Partial derivatives of $\mathbf{G}$ can also be written in the form similar to (23) with somewhat more complex definition of the tensor analogous to $\mathbf{A}_\alpha$, and the general solution for $\mathbf{G}$ can also be found by the method of successive approximations. However, the 3D tensor $\mathbf{G}$ contains some excessive information as compared with the tensor $\mathbf{F}$, what is associated with the additional term $\bar{\mathbf{n}} \otimes \mathbf{n}$ present in $(32)_1$. Within the non-linear theory of thin shells additional care should be taken to separate the excessive part of $\mathbf{G}$ from the important one. For example, in the right polar decomposition $\mathbf{G} = \mathbf{R}\mathbf{U}$ used by Pietraszkiewicz (1989) it became necessary to represent the 3D stretch tensor as $\mathbf{U} = \mathbf{a} + \boldsymbol{\eta} + \mathbf{n} \otimes \mathbf{n}$. It was found that in shell theory only the tangential part $\mathbf{a} + \boldsymbol{\eta}$ is important, where $\boldsymbol{\eta} \in T_x\mathscr{M} \otimes T_x\mathscr{M}$ is the relative surface stretch tensor. The normal part $\mathbf{n} \otimes \mathbf{n}$ of $\mathbf{U}$ does not play any role here. Our method developed in terms of $\mathbf{F}$ is direct, more compact and therefore should be more efficient in applications.

6. Conclusions

We have presented explicitly two different methods to determine the deformed position of the shell middle surface from the known undeformed midsurface as well as the surface strains and bendings. The first method consists of extending to the deformed midsurface an approach based on some results given in differential geometry for determination of the surface position from components of its first and second fundamental forms. In the second approach the same goal has been achieved by integrating the linear system of PDEs for a surface deformation gradient tensor and then the deformed position of the shell midsurface has been obtained by quadrature.

Our results are complementary to the intrinsic formulation of the geometrically non-linear theory of this elastic shells given by Opoka and Pietraszkiewicz (2004) in terms of the membrane stress resultants and bendings as primary variables of the BVP. Now we want to work out a numerical algorithm based on the results

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given here and an appropriate computer program to solve some realistic examples of highly non-linear problems of the flexible shells. It is expected that the results will show some advantages of using the general and relatively simple intrinsic formulation of the non-linear theory of thin shells in solving such shells problems.

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