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Determination of the midsurface of a deformed shell from prescribed surface strains and bendings via the polar decomposition

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Abstract

We show how to determine the midsurface of a deformed thin shell from known geometry of the undeformed midsurface as well as the surface strains and bendings. The latter two fields are assumed to have been found independently and beforehand by solving the so-called intrinsic field equations of the non-linear theory of thin shells. By the polar decomposition theorem the midsurface deformation gradient is represented as composition of the surface stretch and 3D finite rotation fields. Right and left polar decomposition theorems are discussed. For each decomposition the problem is solved in three steps: (a) the stretch field is found by pure algebra, (b) the rotation field is obtained by solving a system of first-order PDEs, and (c) position of the deformed midsurface follows then by quadratures. The integrability conditions for the rotation field are proved to be equivalent to the compatibility conditions of the non-linear theory of thin shells. Along any path on the undeformed shell midsurface the system of PDEs for the rotation field reduces to the system of linear tensor ODEs identical to the one that describes spherical motion of a rigid body about a fixed point. This allows one to use analytical and numerical methods developed in analytical mechanics that in special cases may lead to closed-form solutions.

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Keywords: Thin shells; Non-linear theory; Intrinsic formulation; Polar decomposition; Kinematics of surface

1. Introduction

Pietraszkiewicz and Szwabowicz [1] worked out two ways of determining the midsurface of a deformed shell from prescribed fields of surface strains $\gamma_{\alpha\beta}$ and bendings $\kappa_{\alpha\beta}$. The two latter fields were assumed to be known from solving a problem posed for the so-called intrinsic field equations of the geometrically non-linear theory of thin elastic shells. Such intrinsic shell equations, originally proposed by Chien [2], were refined by Danielson [3] and Koiter and Simmonds [4] and worked out in detail by Opoka and Pietraszkiewicz [5].

In this paper we develop an alternative novel approach to the same problem. Our present approach is based on the polar decomposition of the midsurface deformation gradient $\mathbf{F}=\mathbf{RU}=\mathbf{VR}$, where $\mathbf{U}$ and $\mathbf{V}$ are the surface right and left stretch tensors, respectively, whereas $\mathbf{R}$ is a 3D finite rotation tensor. Detailed transformations are provided for the right polar decomposition in which the problem of finding the deformed midsurface is solved in three steps:

(1) From known surface strains $\gamma_{\alpha\beta}$ the stretch field $\mathbf{U}$ is found by purely algebraic operations leading to the explicit formula (36).
(2) From known $\mathbf{U}$ and $\kappa_{\alpha\beta}$ the rotation field $\mathbf{R}$ is calculated by solving the linear system of two PDEs (24) whose integrability conditions are proved to be equivalent to the compatibility conditions of the non-linear theory of thin shells.

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(3) With known $\mathbf{R}$ and $\mathbf{U}$ the deformed shell midsurface is found by the quadrature (46).

The main steps of the analogous solution using the left polar decomposition are also concisely presented in Section 6. In both cases we note, in particular, that along any path on the undeformed shell midsurface the linear system (24) or (52) reduces to a system of ODEs for unknown $\mathbf{R}$ that turns out to be identical to a system describing spherical motion of a rigid body about a fixed point. Many closed-form solutions of this system of ODEs are already known in analytical mechanics of rigid-body motion (see [6,7]). This allows one to expect closed-form solutions also for the position of the deformed shell midsurface for a variety of shell initial geometries and deformation states.

2. Shell geometry and deformation

A shell is a 3D solid body identified in a reference (undeformed) configuration with a region $\mathscr{B}$ of the physical space $\mathscr{E}$ that has E for its 3D translation vector space. In the region $\mathscr{B}$ we introduce the normal system of curvilinear coordinates $\{\theta^1, \theta^2, \zeta\}$ such that $-h/2 \leqslant \zeta \leqslant h/2$ is the distance from the shell midsurface $\mathscr{M}$ to the points in $\mathscr{B}$, and h is the thickness of the undeformed shell, see Fig. 1. In the theory of thin shells discussed here h is assumed to be constant and small in comparison with the other two dimensions of the shell.

The midsurface $\mathscr{M}$ is usually defined (locally) by the position vector $\mathbf{x} = x^k(\theta^\alpha)\mathbf{i}_k$, $\alpha = 1, 2$, $k = 1, 2, 3$, relative to some fixed origin $o \in \mathscr{E}$ and an orthonormal Cartesian frame $\{\mathbf{i}_k\}$. With each point $x \in \mathscr{M}$ we can associate two linearly independent covariant surface base vectors $\mathbf{a}_\alpha = \partial\mathbf{x}/\partial\theta^\alpha \equiv \mathbf{x},_\alpha$, the dual (contravariant) surface base vectors $\mathbf{a}^\alpha$ satisfying $\mathbf{a}^\beta \cdot \mathbf{a}_\alpha = \delta^\beta_\alpha$, where δ^β_α denotes the Kronecker symbol, the covariant $a_{\alpha\beta} = \mathbf{a}_\alpha \cdot \mathbf{a}_\beta$ and contravariant $a^{\alpha\beta} = \mathbf{a}^\alpha \cdot \mathbf{a}^\beta = (a_{\alpha\beta})^{-1}$ components of the surface metric tensor $\mathbf{a}$ with $\det(a_{\alpha\beta}) = a > 0$, and the unit normal vector $\mathbf{n} = (1/\sqrt{a})\mathbf{a}_1 \times \mathbf{a}_2$ locally orienting $\mathscr{M}$, see Fig. 2. We can also introduce the covariant components

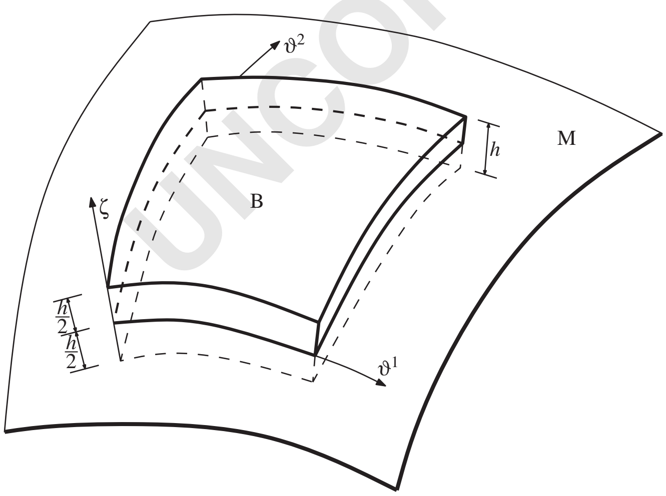

Fig. 1.

$b_{\alpha\beta} = -\mathbf{a}_\alpha \cdot \mathbf{n},_\beta = \mathbf{a}_{\alpha},_\beta \cdot \mathbf{n}$ of the surface curvature tensor $\mathbf{b}$, and the covariant components $\varepsilon_{\alpha\beta} = (\mathbf{a}_\alpha \times \mathbf{a}_\beta) \cdot \mathbf{n}$ of the surface permutation tensor $\boldsymbol{\varepsilon}$ with $\varepsilon_{\alpha\beta} = \sqrt{a}e_{\alpha\beta}$, $e_{12} = -e_{21} = 1$, $e_{11} = e_{22} = 0$.

The surface base vector fields $\mathbf{a}_\alpha(\theta^\lambda)$ and $\mathbf{n}(\theta^\lambda)$ satisfy the Gauss–Weingarten equations

$$\mathbf{a}_{\alpha},_\beta = \Gamma^\lambda_{\alpha\beta}\mathbf{a}_\lambda + b_{\alpha\beta}\mathbf{n}, \quad \mathbf{n},_\beta = -b^\lambda_\beta \mathbf{a}_\lambda, \tag{1}$$

where the Christofell symbols $\Gamma^\lambda_{\alpha\beta}$ of the second kind appearing as coefficients in (1) are related to the surface metric components by the formulas

$$\Gamma^\lambda_{\alpha\beta} = \tfrac{1}{2}a^{\lambda\mu}(a_{\mu\alpha},_\beta + a_{\mu\beta},_\alpha - a_{\alpha\beta},_\mu) = -\mathbf{a}_\alpha \cdot \mathbf{a}^\lambda,_\beta. \tag{2}$$

The second covariant derivatives of $\mathbf{a}_\beta$ satisfy the relations

$$\begin{aligned}\mathbf{a}_{\beta|\lambda\mu} - \mathbf{a}_{\beta|\mu\lambda} &= (b^\kappa_\lambda b_{\beta\mu} - b^\kappa_\mu b_{\beta\lambda})\mathbf{a}_\kappa + (b_{\beta\lambda|\mu} - b_{\beta\mu|\lambda})\mathbf{n} \\ &= R^{\kappa}_{.\beta\lambda\mu}\mathbf{a}_\kappa,\end{aligned} \tag{3}$$

where

$$R^\kappa_{.\beta\lambda\mu} = \Gamma^\kappa_{\beta\lambda\mu},_\lambda - \Gamma^\kappa_{\beta\lambda},_\mu + \Gamma^\rho_{\beta\mu}\Gamma^\kappa_{\rho\lambda} - \Gamma^\rho_{\beta\lambda}\Gamma^\kappa_{\rho\mu} \tag{4}$$

are components of the surface Riemann–Christoffel tensor and $(.)_{|\alpha}$ denotes the surface covariant differentiation in the metric of $\mathscr{M}$ defined, for example, in [8–11]. From (3) we obtain the Gauss–Mainardi–Codazzi (GMC) equations

$$b^\kappa_\lambda b_{\beta\mu} - b^\kappa_\mu b_{\beta\lambda} = R^\kappa_{.\beta\lambda\mu}, \quad b_{\beta\lambda|\mu} - b_{\beta\mu|\lambda} = 0. \tag{5}$$

For comprehensive exposition of other definitions and concepts we refer the reader to classical books on differential geometry and tensor calculus, but the references such as [9,10,12,13] explain these questions directly in the context of the theory of thin shells.

Consider a deformation χ of the shell, i.e. a map $\chi: \mathscr{B} \to \overline{\mathscr{B}}$. The theory of thin shells is based on an assumption that the 3D deformation of a shell can be approximated with a sufficient accuracy by deformation of its reference (usually middle) surface. As a result, during deformation the shell is represented by a material surface capable of resisting stretching and bending.

In the deformed configuration the shell is represented by a midsurface $\overline{\mathscr{M}}$. We assume that θ^α are the material (convected) coordinates and that the image of the midsurface $\mathscr{M}$ under χ coincides with $\overline{\mathscr{M}}$, i.e. $\overline{\mathscr{M}} = \chi(\mathscr{M})$. Then the position vector $\mathbf{y} = y^k(\theta^\alpha)\mathbf{i}_k$ of $\overline{\mathscr{M}}$ relative to the same fixed frame $\{o, \mathbf{i}_k\}$ is

$$\mathbf{y}(\theta^\alpha) = \chi[\mathbf{x}(\theta^\alpha)], \tag{6}$$

and the field of displacements can be obtained from

$$\mathbf{u}(\theta^\alpha) = \mathbf{y}(\theta^\alpha) - \mathbf{x}(\theta^\alpha). \tag{7}$$

In convected coordinates all quantities defined and the relations written earlier for $\mathscr{M}$ hold true on $\overline{\mathscr{M}}$ as well. To indicate which of the two configurations is meant, we shall provide all symbols pertaining to the deformed one with a bar above the symbol, e.g. $\bar{\mathbf{a}}_\alpha$, $\bar{a}_{\alpha\beta}$, $\bar{a}$, $\bar{b}_{\alpha\beta}$, $\bar{\varepsilon}_{\alpha\beta}$, $\bar{\mathbf{n}}$, $\bar{\Gamma}^\lambda_{\alpha\beta}$, $\bar{R}^\kappa_{.\beta\lambda\mu}$, etc., and leave those pertaining to the undeformed configuration unmarked, see Fig. 2.

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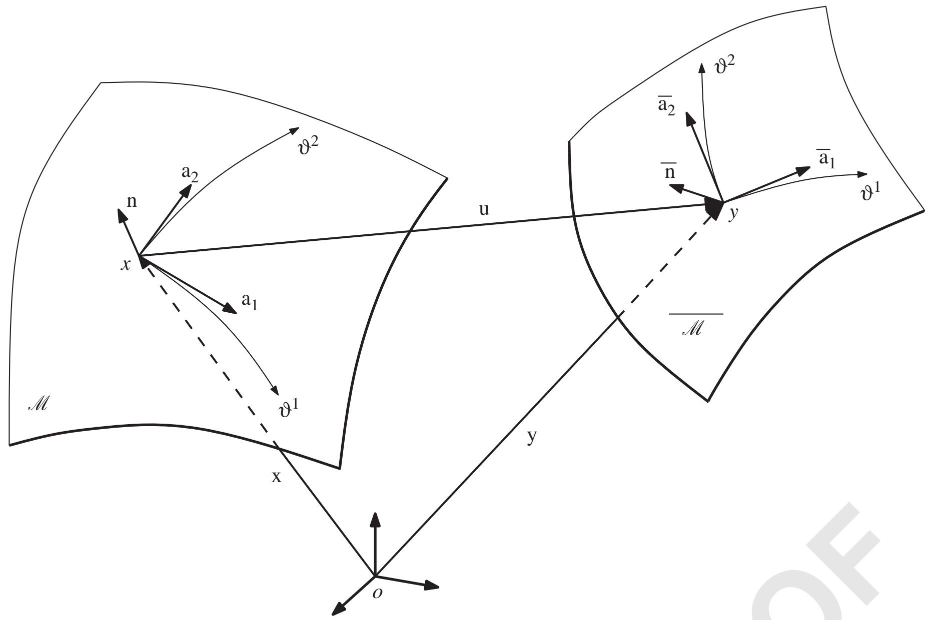

Fig. 2.

The deformation state of the shell midsurface is usually described by two Green type surface strain and bending tensors with covariant components

$$\gamma_{\alpha\beta} = \tfrac{1}{2}(\bar{a}_{\alpha\beta} - a_{\alpha\beta}), \quad \kappa_{\alpha\beta} = -(\bar{b}_{\alpha\beta} - b_{\alpha\beta}). \tag{8}$$

In this paper we want to find the position vector $\mathbf{y} = \mathbf{y}(\theta^\alpha)$ of $\overline{\mathscr{M}}$ and/or the displacement field $\mathbf{u} = \mathbf{u}(\theta^\alpha)$ defined in (7) from the position vector $\mathbf{x} = \mathbf{x}(\theta^\alpha)$ and two fields $\gamma_{\alpha\beta} = \gamma_{\alpha\beta}(\theta^\lambda)$ and $\kappa_{\alpha\beta} = \kappa_{\alpha\beta}(\theta^\lambda)$. The latter fields are assumed to have been found beforehand by solving the so-called intrinsic field equations of the non-linear theory of thin shells worked out by Opoka and Pietraszkiewicz [5]. Two different ways leading to this goal have recently been proposed by Pietraszkiewicz and Szwabowicz [1]. Below we develop an alternative novel approach leading to the solution of this problem.

3. Polar decomposition of the midsurface deformation gradient

Let ∇_s be the surface gradient operator at $x \in \mathscr{M}$. Differentiating the deformation $\mathbf{y} = \chi(\mathbf{x})$ (in the Fréchet sense) we obtain the midsurface deformation gradient field defined by

$$\mathbf{F} = \nabla_s \chi(\mathbf{x}) = \mathbf{y},_\alpha \otimes \mathbf{a}^\alpha. \tag{9}$$

Due to the identity $\mathbf{y},_\alpha = \bar{\mathbf{a}}_\alpha$ the deformation gradient can also be regarded as the two-point tensor field $\mathbf{F} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha \in T_y\overline{\mathscr{M}} \otimes T_x\mathscr{M}$ which maps material elements $\mathrm{d}\mathbf{x} \in T_x\mathscr{M}$ into $\mathrm{d}\mathbf{y} \in T_y\overline{\mathscr{M}}$, so that $\mathrm{d}\mathbf{y} = \mathbf{F}\,\mathrm{d}\mathbf{x}$. For the coordinate-free notation Gurtin and Murdoch [14] as well as Man and Cohen [15] proposed to distinguish the gradients $\mathbf{y},_\alpha \otimes \mathbf{a}^\alpha$ and $\bar{\mathbf{a}}_\alpha \otimes \mathbf{a}^\alpha$ by relating them through the canonical inclusion $\mathbf{I}_y \in E \otimes T_y\overline{\mathscr{M}}$ and perpendicular projection $\mathbf{P}_y \in T_y\overline{\mathscr{M}} \otimes E$ operators. In the present paper there is no need to use such a formal approach, for here we use convected coordinates and tensor analysis in mixed notation. Thus, formal differences between codomains of $\mathbf{y},_\alpha$ and $\bar{\mathbf{a}}_\alpha$ (as well as $\mathbf{x},_\alpha$ and $\mathbf{a}_\alpha$) are apparent from the context.

Since both tangent planes, $T_x\mathscr{M}$ and $T_y\overline{\mathscr{M}}$, lie in the same 3D Euclidean space, there is a rotation $\mathbf{R}$ that takes one to the other. This in conjunction with the theorem of Tissot (see [16]) justifies the following two representations for $\mathbf{F}$:

$$\mathbf{F} = \mathbf{RU} = \mathbf{VR}, \tag{10}$$

where $\mathbf{U} \in T_x\mathscr{M} \otimes T_x\mathscr{M}$ and $\mathbf{V} \in T_y\overline{\mathscr{M}} \otimes T_y\overline{\mathscr{M}}$ are the right and left stretch tensors, respectively, both symmetric and positive definite, and $\mathbf{R} \in E \otimes E$ is a proper orthogonal tensor, so that the relations $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{R}\mathbf{R}^{\mathrm{T}} = \mathbf{I}$ hold and $\mathbf{I}$ is the unit tensor in E. In analogy to continuum mechanics, but with some abuse of this calling, we shall refer to (10) as the right and left polar decompositions of the tensor $\mathbf{F}$, respectively. A comprehensive justification of (10) is given below.

According to the theorem of Tissot an arbitrary map acting between two surfaces immersed in $\mathscr{E}$ preserves orthogonality of either exactly one orthogonal pair of families of curves drawn on these surfaces or preserves orthogonality of all such orthogonal pairs (when the map is a conformal map). Denote the directions tangent to the pair of orthogonal families of curves by $\mathbf{e}_\alpha$ ($\alpha = 1, 2$) on $\mathscr{M}$ and $\bar{\mathbf{e}}_\alpha$ on $\overline{\mathscr{M}}$. Consider the linear map defined by (9) between the planes tangent to $\mathscr{M}$ and $\overline{\mathscr{M}}$ at the point $\mathbf{x}$ and its image $\mathbf{y} = \chi(\mathbf{x})$, respectively. Therefore the following equations hold true:

$$\lambda_1\bar{\mathbf{e}}_1 = \mathbf{F}\mathbf{e}_1, \quad \lambda_2\bar{\mathbf{e}}_2 = \mathbf{F}\mathbf{e}_2, \quad \mathbf{e}_1 \cdot \mathbf{e}_2 = 0, \quad \bar{\mathbf{e}}_1 \cdot \bar{\mathbf{e}}_2 = 0, \tag{11}$$

where λ_α, $\alpha = 1, 2$, are some real numbers. Together with the fields of unit normals $\mathbf{n} = \mathbf{e}_1 \times \mathbf{e}_2$ and $\bar{\mathbf{n}} = \bar{\mathbf{e}}_1 \times \bar{\mathbf{e}}_2$, the fields of directions on both surfaces provide us with two fields of orthonormal 3D frames related by the map χ. Therefore there

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must exist a proper orthogonal tensor $\mathbf{R}$ that transforms (strictly speaking: rotates) the unbarred frame into the barred one

$$\bar{\mathbf{e}}_1 = \mathbf{R}\mathbf{e}_1, \quad \bar{\mathbf{e}}_2 = \mathbf{R}\mathbf{e}_2, \quad \bar{\mathbf{n}} = \mathbf{R}\mathbf{n}, \tag{12}$$

and this tensor has the representation

$$\mathbf{R} = \bar{\mathbf{e}}_1 \otimes \mathbf{e}_1 + \bar{\mathbf{e}}_2 \otimes \mathbf{e}_2 + \bar{\mathbf{n}} \otimes \mathbf{n}. \tag{13}$$

Substituting the right-hand sides of the first two equations (12) for $\bar{\mathbf{e}}_\alpha$ into the first two equations (11) we obtain

$$\lambda_1 \mathbf{R}\mathbf{e}_1 = \mathbf{F}\mathbf{e}_1, \quad \lambda_2 \mathbf{R}\mathbf{e}_2 = \mathbf{F}\mathbf{e}_2,$$

which may be further transformed to

$$\lambda_1 \mathbf{e}_1 = \mathbf{R}^{\mathrm{T}}\mathbf{F}\mathbf{e}_1, \quad \lambda_2 \mathbf{e}_2 = \mathbf{R}^{\mathrm{T}}\mathbf{F}\mathbf{e}_2. \tag{14}$$

By the above and the equations $\mathbf{F}\mathbf{n}=\mathbf{F}^{\mathrm{T}}\bar{\mathbf{n}}=\mathbf{0}$ the tensor $\mathbf{U}=\mathbf{R}^{\mathrm{T}}\mathbf{F}$ is a *surface* tensor whose principal directions are $\mathbf{e}_\alpha$ and the numbers λ_α are the corresponding eigenvalues. We still need to prove that $\mathbf{U}$ is symmetric.

Note that the directions $\mathbf{e}_\alpha$ constitute a Cartesian basis in the plane tangent to $\mathscr{M}$. Therefore there must exist four numbers $U_{\alpha\beta}$ such that

$$\mathbf{U} = U_{11}\mathbf{e}_1 \otimes \mathbf{e}_1 + U_{12}\mathbf{e}_1 \otimes \mathbf{e}_2 + U_{21}\mathbf{e}_2 \otimes \mathbf{e}_1 + U_{22}\mathbf{e}_2 \otimes \mathbf{e}_2.$$

Yet, by the orthogonality of the directions $\mathbf{e}_\alpha$ and by (14), we must have $U_{12}=U_{21}=0$ and it follows that $U_{11}=\lambda_1$ and $U_{22}=\lambda_2$. Hence $\mathbf{U}$ is symmetric and has the spectral representation

$$\mathbf{U} = \lambda_1 \mathbf{e}_1 \otimes \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 \otimes \mathbf{e}_2. \tag{15}$$

Thus, the decomposition $\mathbf{F} = \mathbf{R}\mathbf{U}$ exists.

Furthermore, the following transformation confirms validity of the decomposition $(10)_2$:

$$\mathbf{F} = \mathbf{R}\mathbf{U} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{V}\mathbf{R}$$

and, by (12) and (15), the surface tensor $\mathbf{V} = \mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}$ has the spectral representation

$$\mathbf{V} = \lambda_1 \bar{\mathbf{e}}_1 \otimes \bar{\mathbf{e}}_1 + \lambda_2 \bar{\mathbf{e}}_2 \otimes \bar{\mathbf{e}}_2. \tag{16}$$

For future use it is convenient to introduce the non-holonomic base vectors $\mathbf{s}_\alpha$ and $\mathbf{s}^\beta$ in $T_x\mathscr{M}$, called the stretched base vectors and defined by

$$\mathbf{s}_\alpha = \mathbf{U}\mathbf{a}_\alpha, \quad \mathbf{s}^\alpha = \bar{a}^{\alpha\beta}\mathbf{s}_\alpha, \quad \mathbf{s}_\alpha \cdot \mathbf{s}^\beta = \delta_\alpha^\beta,$$
$$\mathbf{s}_\alpha \cdot \mathbf{s}_\beta = \bar{a}_{\alpha\beta}, \quad \mathbf{s}^\alpha \cdot \mathbf{s}^\beta = \bar{a}^{\alpha\beta}. \tag{17}$$

Using (17) we can write

$$\mathbf{U} = \mathbf{s}_\alpha \otimes \mathbf{a}^\alpha = U^\alpha_\beta \mathbf{a}_\alpha \otimes \mathbf{a}^\beta, \quad \mathbf{U}^{-1} = \mathbf{a}_\alpha \otimes \mathbf{s}^\alpha = (U^{-1})^\alpha_\beta \mathbf{a}_\alpha \otimes \mathbf{a}^\beta,$$

$$\mathbf{R} = \bar{\mathbf{a}}_\alpha \otimes \mathbf{s}^\alpha + \bar{\mathbf{n}} \otimes \mathbf{n}, \quad \mathbf{R}^{-1} = \mathbf{s}_\alpha \otimes \bar{\mathbf{a}}^\alpha + \mathbf{n} \otimes \bar{\mathbf{n}}. \tag{18}$$

Note that $\mathbf{U}$ is non-singular by definition and, as such, invertible. Its inverse can be computed with the use of the formula

$$\mathbf{U}^{-1} = -\frac{1}{\det(\mathbf{U})}\boldsymbol{\varepsilon}\mathbf{U}\boldsymbol{\varepsilon}, \quad (U^{-1})^\alpha_\beta = \sqrt{\frac{a}{\bar{a}}}\varepsilon^{\alpha\lambda}\varepsilon_{\beta\mu}U^\mu_\lambda, \tag{19}$$

which follows from application of the Cayley–Hamilton theorem to the tensor $\mathbf{U}\boldsymbol{\varepsilon}$.

Let us introduce two further surface tensor fields on $\mathscr{M}$: the so-called relative surface strain and bending measures $\boldsymbol{\eta}$ and $\boldsymbol{\mu}$, respectively, defined as

$$\boldsymbol{\eta} = \mathbf{U} - \mathbf{a}, \quad \boldsymbol{\mu} = \mathbf{R}^{\mathrm{T}}(\bar{\mathbf{n}}_{,\beta} \otimes \mathbf{a}^\beta) + \mathbf{b}, \tag{20}$$

$$\boldsymbol{\eta} = \boldsymbol{\eta}_\beta \otimes \mathbf{a}^\beta, \quad \boldsymbol{\eta}_\beta = \mathbf{s}_\beta - \mathbf{a}_\beta = \eta_{\alpha\beta}\mathbf{a}^\alpha, \quad \eta_{\alpha\beta} = \eta_{\beta\alpha}, \tag{21}$$

$$\boldsymbol{\mu} = \boldsymbol{\mu}_\beta \otimes \mathbf{a}^\beta, \quad \boldsymbol{\mu}_\beta = \mathbf{R}^{\mathrm{T}}\bar{\mathbf{n}}_{,\beta} - \mathbf{n}_{,\beta} = \mu_{\alpha\beta}\mathbf{a}^\alpha, \quad \mu_{\alpha\beta} \neq \mu_{\beta\alpha}. \tag{22}$$

These relative measures, introduced already by Alumäe [17] in a descriptive manner, are related to the measures $\boldsymbol{\gamma}$ and $\boldsymbol{\kappa}$ via the following formulas (see [18]):

$$\gamma_{\alpha\beta} = \eta_{\alpha\beta} + \tfrac{1}{2}\eta^\lambda_\alpha \eta_{\lambda\beta},$$
$$\kappa_{\alpha\beta} = \tfrac{1}{2}[(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\mu_{\lambda\beta} + (\delta^\lambda_\beta + \eta^\lambda_\beta)\mu_{\lambda\alpha}] - \tfrac{1}{2}(b^\lambda_\alpha \eta_{\lambda\beta} + b^\lambda_\beta \eta_{\lambda\alpha}). \tag{23}$$

4. Field of rotations

The relation between the field of rotations $\mathbf{R} = \mathbf{R}(\theta^\lambda)$ on $\mathscr{M}$ and partial derivatives of $\mathbf{R}$ is governed by two linear PDEs

$$\mathbf{R}_{,\alpha} = \mathbf{R} \times \mathbf{k}_\alpha, \tag{24}$$

where the two vectors $\mathbf{k}_\alpha$ were introduced by Shamina [19] in the context of deformation of 3D continuum and called the vectors of change of curvature of the coordinate lines.

Let us derive Eq. (24) for completeness. In view of the orthogonality of $\mathbf{R}$ we have $\mathbf{R}^{\mathrm{T}}\mathbf{R} = \mathbf{I}$, which differentiated along the surface coordinates leads to

$$\mathbf{R}_{,\alpha}^{\mathrm{T}}\mathbf{R} + \mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha} = \mathbf{0},$$

or in an equivalent form

$$\mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha} = -(\mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha})^{\mathrm{T}}.$$

Hence, the two tensors $\mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha}$ are skew-symmetric and, therefore, each of them has an axial vector $\mathbf{k}_\alpha$ such that

$$\mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha} = \mathbf{k}_\alpha \times \mathbf{I} = \mathbf{I} \times \mathbf{k}_\alpha. \tag{25}$$

Multiplying (25) by $\mathbf{R}$ from the left-hand side we obtain exactly (24). Solving (25) for $\mathbf{k}_\alpha$ we can express $\mathbf{k}_\alpha$ in terms of rotations

$$\mathbf{k}_\alpha = \tfrac{1}{2}(\mathbf{I} \times \mathbf{I}) \cdot (\mathbf{R}^{\mathrm{T}}\mathbf{R}_{,\alpha}). \tag{26}$$

We shall now consider solvability of the following problem: *given two vector fields $\mathbf{k}_\alpha = \mathbf{k}_\alpha(\theta^\lambda)$ find the corresponding field of rotations $\mathbf{R} = \mathbf{R}(\theta^\lambda)$.*

Given the fields $\mathbf{k}_\alpha = \mathbf{k}_\alpha(\theta^\lambda)$ we obtain the system of two linear PDEs (24) for the unknown field of rotations $\mathbf{R} = \mathbf{R}(\theta^\lambda)$. This is a total differential system whose local solutions exist if and only if the integrability conditions $\boldsymbol{\varepsilon}^{\alpha\beta}\mathbf{R}_{,\alpha\beta} = \mathbf{0}$ are satisfied. To express these conditions in terms of the axial vectors $\mathbf{k}_\alpha$ we need to derive the formula for second derivatives of the rotation

$$\begin{aligned}\mathbf{R}_{,\alpha\beta} &= \mathbf{R}_{,\beta} \times \mathbf{k}_\alpha + \mathbf{R} \times \mathbf{k}_{\alpha,\beta} \\ &= (\mathbf{R} \times \mathbf{k}_\beta) \times \mathbf{k}_\alpha + \mathbf{R} \times \mathbf{k}_{\alpha,\beta} \\ &= \mathbf{R}[(\mathbf{I} \times \mathbf{k}_\beta)(\mathbf{I} \times \mathbf{k}_\alpha) + \mathbf{I} \times \mathbf{k}_{\alpha,\beta}].\end{aligned}$$

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Hence $\varepsilon^{\alpha\beta}\mathbf{R},_{\alpha\beta}=\mathbf{0}$ are satisfied when

$$\varepsilon^{\alpha\beta}[(\mathbf{I}\times\mathbf{k}_\beta)(\mathbf{I}\times\mathbf{k}_\alpha)+\mathbf{I}\times\mathbf{k}_{\alpha},_{\beta}]=\mathbf{0}. \tag{27}$$

It is straightforward to show with the use of vector algebra that the first component in (27) may be transformed as follows:

$$\begin{aligned}(\mathbf{I}\times\mathbf{k}_\beta)(\mathbf{I}\times\mathbf{k}_\alpha)&=[(\mathbf{a}^\lambda\otimes\mathbf{a}_\lambda+\mathbf{n}\otimes\mathbf{n})\times\mathbf{k}_\beta]\times\mathbf{k}_\alpha\\&=\mathbf{a}^\lambda\otimes[\mathbf{k}_\beta(\mathbf{a}_\lambda\cdot\mathbf{k}_\alpha)-\mathbf{a}_\lambda(\mathbf{k}_\beta\cdot\mathbf{k}_\alpha)]\\&\quad+\mathbf{n}\otimes[\mathbf{k}_\beta(\mathbf{n}\cdot\mathbf{k}_\alpha)-\mathbf{n}(\mathbf{k}_\beta\cdot\mathbf{k}_\alpha)]\\&=\mathbf{k}_\alpha\otimes\mathbf{k}_\beta-(\mathbf{k}_\alpha\cdot\mathbf{k}_\beta)\mathbf{I},\end{aligned}$$

so that (27) becomes

$$\varepsilon^{\alpha\beta}\mathbf{I}\times\mathbf{k}_{\alpha},_{\beta}+\varepsilon^{\alpha\beta}\mathbf{k}_\alpha\otimes\mathbf{k}_\beta-\varepsilon^{\alpha\beta}(\mathbf{k}_\alpha\cdot\mathbf{k}_\beta)\mathbf{I}=\mathbf{0}.$$

Here the term $\varepsilon^{\alpha\beta}(\mathbf{k}_\alpha\cdot\mathbf{k}_\beta)\mathbf{I}$ vanishes identically, and the last term is a skew-symmetric tensor whose axial vector is $-\frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{k}_\alpha\times\mathbf{k}_\beta$. Hence, the system (24) may have solutions if and only if

$$\varepsilon^{\alpha\beta}(\mathbf{k}_{\alpha},_{\beta}-\tfrac{1}{2}\mathbf{k}_\alpha\times\mathbf{k}_\beta)=\mathbf{0}. \tag{28}$$

In the context of the theory of thin shells the integrability condition (28) was derived independently by Chernykh and Shamina [8] and Pietraszkiewicz [20].

Let us reveal the geometric meaning of the integrability conditions (28). Differentiating $(10)_1$ twice, and remembering that the left-hand side represents the integrability conditions for $\mathbf{F}$, which was proved in [1], we obtain

$$\mathbf{F},_{\alpha\beta}-\mathbf{F},_{\beta\alpha}=\mathbf{0}=(\mathbf{R},_{\alpha\beta}-\mathbf{R},_{\beta\alpha})\mathbf{U}+\mathbf{R}(\mathbf{U},_{\alpha\beta}-\mathbf{U},_{\beta\alpha}). \tag{29}$$

The left-hand side of (29) was explicitly calculated in [1]. Differentiating twice $\mathbf{F}=\bar{\mathbf{a}}_\lambda\otimes\mathbf{a}^\lambda$ term by term to obtain $\mathbf{F},_{\alpha\beta}$, then exchanging the indices $\alpha\rightleftarrows\beta$ and calculating the difference $\mathbf{F},_{\alpha\beta}-\mathbf{F},_{\beta\alpha}$, we obtained

$$\begin{aligned}\mathbf{F},_{\alpha\beta}-\mathbf{F},_{\beta\alpha}=(&\bar{R}^{\kappa}_{.\lambda\beta\alpha}-\bar{b}^{\kappa}_{\beta}\bar{b}_{\lambda\alpha}+\bar{b}^{\kappa}_{\alpha}\bar{b}_{\lambda\beta}\\&-R^{\kappa}_{.\lambda\beta\alpha}+b^{\kappa}_{\beta}b_{\lambda\alpha}-b^{\kappa}_{\alpha}b_{\lambda\beta})\bar{\mathbf{a}}_\kappa\otimes\mathbf{a}^\lambda\\&+(b^{\kappa}_{\alpha|\beta}-b^{\kappa}_{\beta|\alpha})\bar{\mathbf{a}}_\kappa\otimes\mathbf{n}\\&+(\bar{b}_{\lambda\alpha||\beta}-\bar{b}_{\lambda\beta||\alpha})\bar{\mathbf{n}}\otimes\mathbf{a}^\lambda=\mathbf{0},\end{aligned} \tag{30}$$

where $(.)_{||\alpha}$ denotes the surface covariant derivative in the metric of $\overline{\mathscr{M}}$.

It is apparent that vanishing components in the conditions (30) represent exactly the differences between the GMC equations of the deformed and undeformed shell midsurfaces. If we introduce here the relations (8) and perform transformations given in detail by Koiter [21], the conditions (30) become identical to the compatibility conditions of the non-linear theory of thin shells.

One immediately notices that the second term $\mathbf{U},_{\alpha\beta}-\mathbf{U},_{\beta\alpha}$ in the right-hand side of (29) vanishes due to interchangeability of the second partial derivatives of $\mathbf{U}\in T_x\mathscr{M}\otimes T_x\mathscr{M}$. The only term left, the first one in the right-hand side of (29), can equivalently be written as

$$\mathbf{R}\times[(\mathbf{k}_{\alpha},_{\beta}-\tfrac{1}{2}\mathbf{k}_\alpha\times\mathbf{k}_\beta)-(\mathbf{k}_{\beta},_{\alpha}-\tfrac{1}{2}\mathbf{k}_\beta\times\mathbf{k}_\alpha)]\mathbf{U}=\mathbf{0}. \tag{31}$$

Since both $\mathbf{R}$ and $\mathbf{U}$ are non-singular it immediately follows from (31), (30) and (29) that the integrability conditions (28) are equivalent to the compatibility conditions of the non-linear theory of thin shells.

Given the fields of stretches $\mathbf{U}$ (or $\boldsymbol{\eta}$) and rotations $\mathbf{R}$, from (9), (10) and (17) we obtain the system of two linear, vector first-order PDEs for the deformed position vector $\mathbf{y}$,

$$\mathbf{y},_\alpha=\mathbf{R}\mathbf{s}_\alpha=\mathbf{R}\mathbf{U}\mathbf{a}_\alpha. \tag{32}$$

The local solutions of (32) exist provided that the integrability conditions $\varepsilon^{\alpha\beta}\mathbf{y},_{\alpha\beta}=\mathbf{0}$ hold true. These conditions can be transformed as follows:

$$\begin{aligned}\varepsilon^{\alpha\beta}\mathbf{y},_{\alpha\beta}&=\varepsilon^{\alpha\beta}(\mathbf{R},_\beta\mathbf{s}_\alpha+\mathbf{R}\mathbf{s}_{\alpha},_\beta)\\&=\varepsilon^{\alpha\beta}[(\mathbf{R}\times\mathbf{k}_\beta)\mathbf{s}_\alpha+\mathbf{R}(\mathbf{a}_{\alpha},_\beta+\boldsymbol{\eta}_{\alpha},_\beta)]\\&=\varepsilon^{\alpha\beta}\mathbf{R}(\mathbf{k}_\beta\times\mathbf{s}_\alpha+\boldsymbol{\eta}_{\alpha},_\beta)=\mathbf{0}.\end{aligned}$$

Multiplying the above from the left-hand side by $\mathbf{R}^{\mathrm{T}}$ we obtain the integrability conditions coinciding with those derived in [18]

$$\varepsilon^{\alpha\beta}(\boldsymbol{\eta}_{\alpha},_\beta+k_\beta\times\mathbf{s}_\alpha)=\mathbf{0}. \tag{33}$$

We can also calculate the second partial derivatives of $\mathbf{y}$ in an equivalent way as follows:

$$\varepsilon^{\alpha\beta}\mathbf{y},_{\alpha\beta}=\varepsilon^{\alpha\beta}\bar{\mathbf{a}}_{\alpha},_\beta=\varepsilon^{\alpha\beta}(\bar{\Gamma}_{\alpha\beta}{}^{\lambda}\bar{\mathbf{a}}_\lambda+\bar{b}_{\alpha\beta}\bar{\mathbf{n}})=\mathbf{0}. \tag{34}$$

Therefore, the integrability condition (33) is equivalent to the identities following from the symmetry of $\bar{\Gamma}^{\lambda}_{\alpha\beta}$ and $\bar{b}_{\alpha\beta}$ in lower indices. These identities will be used in Section 5.2 to modify the components of $\mathbf{k}_\alpha$.

Summarizing, the position vector $\mathbf{y}$ of the deformed midsurface $\overline{\mathscr{M}}$ can be found in three consecutive steps:

(1) Find $\mathbf{U}$ from known γ by pure algebra in $T_x\mathscr{M}\otimes T_x\mathscr{M}$.
(2) Calculate $\mathbf{R}$ from known $\mathbf{U}$ and $\kappa_{\alpha\beta}$ by solving the system of two linear PDEs (24) whose integrability conditions are (28).
(3) Find $\mathbf{y}$ from known $\mathbf{R}$ and $\mathbf{U}$ by integrating the system of two linear PDEs (32) whose integrability conditions are (33).

In Section 5 we perform in detail all transformations necessary to complete these three steps.

5. Determination of deformed position of the shell midsurface

5.1. Determination of the surface stretch

From $(10)_1$, (20) and (8) it follows that

$$\mathbf{F}^{\mathrm{T}}\mathbf{F}=\mathbf{U}^2=\mathbf{a}+2\gamma,$$

and the invariants of $\mathbf{U}^2$ in terms of those of γ are

$$\mathrm{tr}(\mathbf{U}^2)=2+2\,\mathrm{tr}(\gamma),$$

$$\det(\mathbf{U}^2)=1+2\,\mathrm{tr}(\gamma)+4\,\det(\gamma).$$

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The Cayley–Hamilton theorem for $\mathbf{U}$ reads

$$\mathbf{U}^2 - \mathrm{tr}(\mathbf{U})\mathbf{U} + \det(\mathbf{U})\mathbf{a} = \mathbf{0},$$

from which we obtain

$$\mathbf{U} = \frac{1}{\mathrm{tr}(\mathbf{U})}[\mathbf{U}^2 + \det(\mathbf{U})\mathbf{a}]. \tag{35}$$

Taking the trace of (35) we can express it through the invariants of $\mathbf{U}^2$ by

$$\mathrm{tr}(\mathbf{U}) = \sqrt{\mathrm{tr}(\mathbf{U}^2) + 2\sqrt{\det(\mathbf{U}^2)}} > 0, \quad \det\mathbf{U} = \sqrt{\det\mathbf{U}^2} > 0.$$

Therefore, introducing all the above results into (35) we obtain $\mathbf{U}$ expressed explicitly in terms of $\boldsymbol{\gamma}$

$$\mathbf{U} = \frac{\{1 + \sqrt{1 + 2\,\mathrm{tr}(\boldsymbol{\gamma}) + 4\,\det(\boldsymbol{\gamma})}\}\mathbf{a} + 2\boldsymbol{\gamma}}{\sqrt{2\{1 + \mathrm{tr}(\boldsymbol{\gamma}) + \sqrt{1 + 2\mathrm{tr}(\boldsymbol{\gamma}) + 4\det(\boldsymbol{\gamma})}\}}}. \tag{36}$$

5.2. Determination of the rotation

The vectors $\mathbf{k}_\alpha$ can be represented through the components in the base $\mathbf{a}_\kappa$, $\mathbf{n}$ according to [18] by

$$\mathbf{k}_\alpha = \varepsilon^{\lambda\kappa}\mu_{\lambda\alpha}\mathbf{a}_\kappa + k_\alpha\mathbf{n}. \tag{37}$$

In (37) there are six components $\mu_{\lambda\alpha}, k_\alpha$ which should be expressed through our data: three U^λ_α (or η^λ_α) and three $\kappa_{\alpha\beta}$.

By the definition (20), by $(18)_2$ and (1) we can express the four tangential components $\mu_{\alpha\beta}$ of $\mathbf{k}_\alpha$ through U^λ_α (or η^λ_α) and $\kappa_{\alpha\beta}$,

$$\begin{aligned}\mu_{\alpha\beta} &= \mathbf{a}_\alpha \cdot (\mathbf{s}^\lambda \otimes \bar{\mathbf{a}}_\lambda + \mathbf{n} \otimes \bar{\mathbf{n}})(-\bar{b}_{\rho\beta}\bar{\mathbf{a}}^\rho) + b_{\alpha\beta} \\ &= b_{\alpha\beta} - (U^{-1})^\lambda_\alpha(b_{\lambda\beta} - \kappa_{\lambda\beta}).\end{aligned} \tag{38}$$

Two normal components k_α of $\mathbf{k}_\alpha$ can be expressed through U^λ_α (or η^λ_α) with the help of integrability conditions (33) which in components in the base $\mathbf{a}_\alpha$, $\mathbf{n}$ read

$$\begin{aligned}&\varepsilon^{\alpha\beta}\eta_{\lambda\alpha|\beta} + \varepsilon^{\alpha\beta}(\delta^\kappa_\alpha + \eta^\kappa_\alpha)\varepsilon_{\kappa\lambda}k_\beta = 0, \\ &\varepsilon^{\alpha\beta}\eta^\lambda_\alpha(b_{\lambda\mu} - \mu_{\lambda\mu}) - \varepsilon^{\alpha\beta}\mu_{\alpha\beta} = 0.\end{aligned} \tag{39}$$

Multiplying the first of (39) by $\varepsilon^{\lambda\sigma}(U^{-1})^\rho_\sigma\varepsilon_{\rho\mu}$, using (19) and performing some transformations we can solve it for k_α and obtain

$$k_\alpha = -\sqrt{\frac{a}{\bar{a}}}\varepsilon^{\kappa\rho}(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\eta_{\lambda\kappa|\rho}. \tag{40}$$

It is easy to show by direct analysis that when $\mu_{\alpha\beta}$ and k_α are expressed by (38) and (40), respectively, the third integrability condition of (39) is identically satisfied.

The system of two linear PDEs (24) can now be integrated provided that the integrability conditions (28) are satisfied. In the intrinsic formulation of non-linear shell equations by Opoka and Pietraszkiewicz [5] three compatibility conditions were used as the principal part of six intrinsic shell equations for $N^{\alpha\beta}$ and $\kappa_{\alpha\beta}$. The fields U^λ_α (or η^λ_α) as linear functions of $N^{\alpha\beta}$, together with $\kappa_{\alpha\beta}$ through which we formulate the problem, satisfy the compatibility conditions within the accuracy of the first approximation to the elastic strain energy density of the shell. Therefore, the integrability conditions (28) are satisfied with the same accuracy in any geometrically non-linear problem of thin elastic shells. As a result, the system (24) is completely integrable.

The first step in solving the system (24) consists in showing that the problem can be converted to an equivalent infinite set of systems of ODEs along curves covering densely the entire domain $\mathscr{M}$. If the integrability condition (28) is satisfied then by the theorem of Frobenius–Dieudonné (see [22]) for every initial value $\mathbf{R}(\theta^\alpha_0) = \mathbf{R}_0$ prescribed at some point $x_0 \in \mathscr{M}$ with coordinates θ^α_0 there exists a unique solution $\mathbf{R}(\theta^\alpha)$ satisfying this initial value, and all such solutions depend continuously on $\mathbf{R}_0$.

Consider a particular solution $\mathbf{R}$ of the system (24) and a curve $\mathscr{C}: [a, b] \ni s \to \theta^\alpha(s)$ leaving from some point $x_0 \in \mathscr{M}$, labeled by s_0, to another point $x \in \mathscr{M}$, labeled by s. Suppose the value of $\mathbf{R}$ at s_0 be $\mathbf{R}_0$. Note that the restriction $\mathbf{R}|_\mathscr{C}$ of this solution to the curve $\mathscr{C}$ satisfies the following system of ODEs:

$$\frac{\mathrm{d}\mathbf{R}|_\mathscr{C}}{\mathrm{d}s} = \mathbf{R}|_\mathscr{C} \times \mathbf{k}^C, \tag{41}$$

where the vector $\mathbf{k}^C$ is given by

$$\mathbf{k}^C = \mathbf{k}_\alpha\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}. \tag{42}$$

Let us reverse the argumentation. Now consider the initial value problem for the system of ODEs

$$\frac{\mathrm{d}\mathbf{R}^*}{\mathrm{d}s} = \mathbf{R}^* \times \mathbf{k}^C$$

along the same curve $\mathscr{C}$ with the same initial condition $\mathbf{R}^*(s_0) = \mathbf{R}_0$. By the standard results from the theory of ODEs this problem has a unique solution $\mathbf{R}^*(s)$. Therefore, it must be identical with the restriction of $\mathbf{R}$ to $\mathscr{C}$ on the interval where it exists, i.e. we must have $\mathbf{R}|_\mathscr{C} = \mathbf{R}^*(s)$.

This way, instead of solving the system (24) directly, we may compute a particular solution $\mathbf{R}(\theta^\alpha)$ corresponding to some initial condition $\mathbf{R}(\theta^\alpha_0) = \mathbf{R}_0$ by covering the domain $\mathscr{M}$ with a dense set of paths leaving radially from the initial point x_0 and solving the initial value problem for the system of ODEs

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{R}\mathbf{K}, \quad \mathbf{K} = \mathbf{I} \times \mathbf{k}, \quad \mathbf{k} = \mathbf{k}_\alpha\frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s}, \tag{43}$$

$$\begin{aligned}\mathbf{k}_\alpha = \varepsilon^{\kappa\rho}&[b_{\kappa\alpha} - (U^{-1})^\lambda_\kappa(b_{\lambda\alpha} - \kappa_{\lambda\alpha})]\mathbf{a}_\rho \\ &- \sqrt{\frac{a}{\bar{a}}}\varepsilon^{\kappa\rho}U^\lambda_\alpha\eta_{\lambda\kappa|\rho}\mathbf{n}.\end{aligned}$$

Solution to the initial value problem (43) may be obtained with the use of any of the well-known techniques, numerical techniques inclusive. In particular, applying the method of successive approximations (see [22]) the general solution of (43) can

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be presented in the form

$$\mathbf{R}=\mathbf{R}_0\mathbf{R}_s,\quad \mathbf{R}_s=\sum_{i=0}^{\infty}\mathbf{O}_i,\tag{44}$$

$$\mathbf{O}_0(s)=\mathbf{I},\quad \mathbf{O}_i(s)=\int_{s_0}^{s}\mathbf{O}_{i-1}(t)\mathbf{K}(t)\,\mathrm{d}t,\quad i\geqslant 1,$$

where $\mathbf{R}_0=\mathbf{R}(s_0)$ is the rotation tensor at $s=s_0$.

Introducing (44) into (43) we can directly show that the infinite series $\mathbf{R}_s$ solves Eq. (43) with the initial value $\mathbf{R}(s_0)=\mathbf{R}_0$. The series is convergent and it can be proved (see [22]) that in our case it converges to a rotation field $\mathbf{R}_s$ along $\mathscr{C}$, and that the solution is unique for any prescribed initial value.

One may point out a number of special cases when Eq. (43) has the solution in closed form. In particular, when $\mathbf{k}=k(s)\mathbf{i}$, i.e. when $\mathbf{k}$ has a constant direction along $\mathscr{C}$, then $\mathrm{d}\mathbf{i}/\mathrm{d}s=\mathbf{0}$ and the tensors $\mathbf{O}_i$ satisfy the conditions $\mathbf{O}_i\mathbf{O}_j=\mathbf{O}_j\mathbf{O}_i$ for any i,j. Then the solution (44) can be presented in the exponential form

$$\mathbf{R}(s)=\exp\left(\mathbf{I}\times\mathbf{i}\int_{s_0}^{s}k(t)\,\mathrm{d}t\right).\tag{45}$$

A still simpler solution may be obtained if $\mathbf{k}$ itself is constant along $\mathscr{C}$, i.e. when $\mathrm{d}\mathbf{k}/\mathrm{d}s=\mathbf{0}$. Then from (45) it follows that

$$\mathbf{R}(s)=\exp(s\mathbf{I}\times\mathbf{k}).$$

Note that the tensor equation (43) is identical with the one describing the spherical motion of a rigid body about a fixed point, where s is time and $\mathbf{k}$ is the angular velocity vector in the spatial representation (see for example [23–25]). In analytical mechanics many ingenious analytical and numerical methods of integration of Eq. (43) have been devised for various special classes of the function $\mathbf{k}=\mathbf{k}(s)$. A number of such closed-form solutions were summarized, for example, by Gorr et al. [6]. Thus, the results already known in analytical mechanics of rigid-body motion may be of great help when analyzing problems discussed here for thin elastic shells.

5.3. Determination of deformed position of the midsurface

With $\mathbf{R}$ and $\mathbf{U}$ already known, the system of two vector PDEs (32) for the deformed position $\mathbf{y}$ is well defined. Since the integrability conditions (34) are identically satisfied, we can solve the system by quadratures and obtain

$$\mathbf{y}=\mathbf{y}_0+\int_{x_0}^{x}\mathbf{R}\mathbf{s}_\alpha\,\mathrm{d}\theta^\alpha,\tag{46}$$

where $\mathbf{y}_0=\mathbf{y}(x_0)$.

6. Determining the deformed midsurface via the left polar decomposition

Transformations analogous to the ones presented above can also be applied to the left polar decomposition of $\mathbf{F}$,

$$\mathbf{F}=\mathbf{V}\mathbf{R},\tag{47}$$

where now

$$\mathbf{R}=\mathbf{r}_\alpha\otimes\mathbf{a}^\alpha+\bar{\mathbf{n}}\otimes\mathbf{n},\quad \mathbf{R}^{\mathrm{T}}=\mathbf{R}^{-1},\quad \det(\mathbf{R})=+1,$$

$$\mathbf{V}=\bar{\mathbf{a}}_\alpha\otimes\mathbf{r}^\alpha=U^\alpha_\beta\mathbf{r}_\alpha\otimes\mathbf{r}^\beta=\mathbf{V}^{\mathrm{T}},$$

$$\mathbf{V}^{-1}=\mathbf{r}_\alpha\otimes\bar{\mathbf{a}}^\alpha=(U^{-1})^\alpha_\beta\mathbf{r}_\alpha\otimes\mathbf{r}^\beta=\mathbf{V}^{-\mathrm{T}},\tag{48}$$

and the non-holonomic rotated base vectors $\mathbf{r}_\alpha$ and $\mathbf{r}^\beta$ of $T_y\overline{\mathscr{M}}$ are defined by

$$\mathbf{r}_\alpha=\mathbf{R}\mathbf{a}_\alpha=\mathbf{V}^{-1}\bar{\mathbf{a}}_\alpha,\quad \mathbf{r}_\alpha\cdot\mathbf{r}_\beta=a_{\alpha\beta},$$

$$\mathbf{r}^\alpha=a^{\alpha\beta}\mathbf{r}_\beta,\quad \mathbf{r}^\alpha\cdot\mathbf{r}^\beta=a^{\alpha\beta},\quad \mathbf{r}^\beta\cdot\mathbf{r}_\alpha=\delta^\beta_\alpha.\tag{49}$$

Given the fields of rotation $\mathbf{R}=\mathbf{R}(\theta^\lambda)$ and stretch $\mathbf{V}=\mathbf{V}(\theta^\lambda)$, we obtain from (9) and (47) the system of two linear, vector first-order PDEs for the position vector of the deformed midsurface

$$\mathbf{y},_\alpha=\mathbf{V}\mathbf{r}_\alpha=\mathbf{V}\mathbf{R}\mathbf{a}_\alpha.\tag{50}$$

Therefore, the vector $\mathbf{y}$ can be found from (50) in three consecutive steps analogous to those discussed in Section 5.

Differentiating the identity $\mathbf{R}\mathbf{R}^{\mathrm{T}}=\mathbf{I}=\mathbf{r}_\lambda\otimes\mathbf{r}^\lambda+\bar{\mathbf{n}}\otimes\bar{\mathbf{n}}$ along the surface coordinate lines we find that $\mathbf{R},_\alpha\mathbf{R}^{\mathrm{T}}=-(\mathbf{R},_\alpha\mathbf{R}^{\mathrm{T}})^{\mathrm{T}}$. Therefore, $\mathbf{R},_\alpha\mathbf{R}^{\mathrm{T}}$ are also the skew-symmetric tensors expressible through their axial vectors $\mathbf{l}_\alpha$ according to

$$\mathbf{R},_\alpha\mathbf{R}^{\mathrm{T}}=\mathbf{l}_\alpha\times\mathbf{I}=\mathbf{I}\times\mathbf{l}_\alpha,$$

$$\mathbf{l}_\alpha=\mathbf{R}\mathbf{k}_\alpha=\varepsilon^{\lambda\kappa}\mu_{\lambda\alpha}\mathbf{r}_\kappa+k_\alpha\bar{\mathbf{n}}.\tag{51}$$

Given the fields $\mathbf{l}_\alpha=\mathbf{l}_\alpha(\theta^\lambda)$ from $(51)_1$ we obtain the system of two linear PDEs

$$\mathbf{R},_\alpha=\mathbf{l}_\alpha\times\mathbf{R}\tag{52}$$

for the field $\mathbf{R}=\mathbf{R}(\theta^\lambda)$. This is again the total differential system and its local solutions exist iff the integrability conditions $\varepsilon^{\alpha\beta}\mathbf{R},_{\alpha\beta}=\mathbf{0}$ are satisfied, that is when

$$\begin{aligned}\varepsilon^{\alpha\beta}\mathbf{R},_{\alpha\beta}&=\varepsilon^{\alpha\beta}[\mathbf{l}_\alpha,_\beta\times\mathbf{R}+\mathbf{l}_\alpha\times(\mathbf{l}_\beta\times\mathbf{R})]\\&=\varepsilon^{\alpha\beta}[\mathbf{l}_\alpha,_\beta\times\mathbf{I}+(\mathbf{l}_\alpha\times\mathbf{I})(\mathbf{l}_\beta\times\mathbf{I})]\mathbf{R}\\&=\varepsilon^{\alpha\beta}(\mathbf{l}_\alpha,_\beta\times\mathbf{I}+\mathbf{l}_\beta\otimes\mathbf{l}_\alpha)\mathbf{R}=\mathbf{0}.\end{aligned}\tag{53}$$

But $\varepsilon^{\alpha\beta}\mathbf{l}_\beta\otimes\mathbf{l}_\alpha$ is a skew-symmetric tensor whose axial vector is $-\frac{1}{2}\varepsilon^{\alpha\beta}\mathbf{l}_\beta\times\mathbf{l}_\alpha$. Since $\mathbf{R}$ is non-singular, the integrability conditions of (52) are equivalent to

$$\varepsilon^{\alpha\beta}(\mathbf{l}_\alpha,_\beta+\tfrac{1}{2}\mathbf{l}_\alpha\times\mathbf{l}_\beta)=\mathbf{0}.\tag{54}$$

Note the opposite sign of the second term of (54) as compared with (28).

Performing transformations analogous to (3)–(31) one can show that (54) is also equivalent to the compatibility conditions of the non-linear theory of thin shells.

The solution to (52) can be found analogously to the one presented in Section 5.2. We again cover the domain $\mathscr{M}$ with a dense set of paths leaving radially from any initial point $x_0 \in \mathscr{M}$ and then solve the initial value problem for the system of ODEs

$$\frac{\mathrm{d}\mathbf{R}}{\mathrm{d}s} = \mathbf{L}\mathbf{R}, \quad \mathbf{L} = \mathbf{l} \times \mathbf{I}, \quad \mathbf{l} = \mathbf{l}_\alpha \frac{\mathrm{d}\theta^\alpha}{\mathrm{d}s},$$

$$\mathbf{l}_\alpha = \varepsilon^{\kappa\rho}[b_{\kappa\alpha} - (U^{-1})^\lambda_\kappa (b_{\lambda\alpha} - \kappa_{\lambda\alpha})]\mathbf{r}_\rho - \sqrt{\frac{a}{\bar{a}}}\varepsilon^{\kappa\rho}(\delta^\lambda_\alpha + \eta^\lambda_\alpha)\eta_{\lambda\kappa|\rho}\bar{\mathbf{n}}. \tag{55}$$

The general solution to $(55)_1$ can be given in the form

$$\mathbf{R} = \mathbf{R}_0\mathbf{R}_s, \quad \mathbf{R} = \sum_{i=0}^{\infty} \mathbf{P}_i,$$

$$\mathbf{P}_0(s) = \mathbf{I}, \quad \mathbf{P}_i(s) = \int_{s_0}^{s} \mathbf{L}(t)\mathbf{P}_{i-1}(t)\,\mathrm{d}t, \quad i \geqslant 1. \tag{56}$$

The tensor ODE $(55)_1$ is also equivalent to the one describing spherical motion of a rigid body about a fixed point, but now written in the material representation. From mathematical point of view, both representations $(55)_1$ and $(43)_1$ are equivalent and can be transformed to each other by the rotation tensor $\mathbf{R}$. Therefore, their solutions are also equivalent.

Because $\mathbf{V}=\mathbf{R}\mathbf{U}\mathbf{R}^{\mathrm{T}}$, the left stretch tensor $\mathbf{V}$ can be calculated through $\boldsymbol{\gamma}$ and $\mathbf{R}$ by the relation

$$\mathbf{V} = \frac{\{1 + \sqrt{1 + 2\,\mathrm{tr}(\gamma) + 4\,\det(\gamma)}\}\mathbf{r}_\alpha \otimes \mathbf{r}^\alpha + 2\gamma_{\alpha\beta}\mathbf{r}^\alpha \otimes \mathbf{r}^\beta}{\sqrt{2\{1 + \mathrm{tr}(\gamma) + \sqrt{1 + 2\,\mathrm{tr}(\gamma) + 4\,\det(\gamma)}\}}}. \tag{57}$$

When $\mathbf{R}$ and $\mathbf{V}$ are known the position vector $\mathbf{y}$ can be found by integrating directly the system of two PDEs (50). Since the integrability conditions (34) of (50) are identically satisfied, the position vector of the deformed shell midsurface follows from the quadratures

$$\mathbf{y} = \mathbf{y}_0 + \int_{x_0}^{x} \mathbf{V}\mathbf{R}\mathbf{a}_\alpha\,\mathrm{d}\theta^\alpha. \tag{58}$$

7. Conclusions

We have worked out two novel, alternative, three-step methods of determining the deformed shell midsurface from known geometry of the undeformed midsurface as well as the prescribed surface strains and bendings. The methods have been based on the right and/or left polar decompositions of the deformation gradient of the shell midsurface. In both cases the corresponding surface stretch fields are obtained by pure algebra, the 3D rotation fields are calculated by solving the linear systems of first-order PDEs, and positions of the deformed shell midsurface are then found by quadratures.

Along any path on the undeformed shell midsurface the system of PDEs for the rotation field has been reduced to the dense set of linear ODEs which are identical with the ones describing motion of a rigid body about a fixed point. It is expected that the two methods proposed here will be more efficient in applications than those developed in [1], for it should be possible here to use ingenious theoretical and numerical methods developed in analytical mechanics, which in special cases may lead to the analytical solution in closed form.

We also note that this approach has recently been successfully used in a similar problem of classical differential geometry: determination of the surface from components of its two fundamental forms, see Pietraszkiewicz and Vallée [26].

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