On natural strain measures of the non-linear micropolar continuum

W. Pietraszkiewicz [a,*], V. A. Eremeyev [b]

[a] *Institute of Fluid-Flow Machinery of the Polish Academy of Sciences, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland*

[b] *South Scientific Center of RASci & South Federal University, Milchakova str. 8a, 344090 Rostov on Don, Russia*

Abstract

We discuss three different ways of defining the strain measures in the non-linear micropolar continuum: a) by a direct geometric approach, b) considering the strain measures as the fields required by the structure of local equilibrium conditions, and c) requiring the strain energy density of the polar-elastic body to satisfy the principle of invariance under superposed rigid-body deformations. The geometric approach a) generates several two-point deformation measures as well as some Lagrangian and Eulerian strain measures. The ways b) and c) allow one to choose those Lagrangian strain measures which satisfy the additional mechanical requirements. These uniquely selected relative strain measures are called the natural ones. All the strain measures discussed here are formulated in the general coordinate-free form. They are valid for unrestricted translations, stretches and changes of orientations of the micropolar body, and are required to identically vanish in the absence of deformation. The relation of the Lagrangian stretch and wryness tensors derived here to the ones proposed in the literature is thoroughly discussed.

Key words: micropolar continuum, strain measures, nonlinear elasticity, wryness tensor

1 Introduction

The micropolar (or the Cosserat type) continuum differs from the classical (or the Cauchy type) continuum in that in the former one each material particle

* Corresponding author. Tel: +48 58 6995263; fax: +48 58 3416144.
Email addresses: pietrasz@imp.gda.pl (W. Pietraszkiewicz), eremeyev@math.rsu.ru (V. A. Eremeyev).

can translate and independently rotate, that is it has six degrees of freedom of a rigid body. Main ideas leading to the micropolar continuum were discussed already at the end of XIXth century by Kelvin, Helmholtz, Duhem, Voigt and Cosserat and were worked out in detail by Cosserat and Cosserat (1909). Later results obtained within the non-linear micropolar continuum were summarised for example by Toupin (1964), Truesdell and Noll (1965), Kafadar and Eringen (1971) and Pabst (2005) where many references to earlier original papers were given. Nowadays the micropolar continuum is used with success to model various phenomena in many areas of solid and fluid mechanics such as, for example, granular media, composites, polycristalline solids, biomaterials, liquid crystals, foams, magnetic fluids, nano-materials, as well as thin bodies: rods, plates, and shells.

Yet, the representative references collected at the end of this paper and summarised in Table 1 of Chapter 6 indicate that various approaches were used in the literature to introduce the Lagrangian strain measures into the non-linear micropolar continuum. In most papers the strain measures were given simply by definition or referring to Kafadar and Eringen (1971) and Eringen and Kafadar (1976), who referred to Cosserat and Cosserat (1909) and called the measures the Cosserat deformation and wryness tensors. However, the strain measures originally proposed by Cosserat and Cosserat (1909) had been written in an awkward notation through components of some fields in the common Cartesian frame. Today such an approach is hardly readable and it is not apparent that the strain measures used in many contemporary papers are exactly those proposed by Cosserat and Cosserat (1909) indeed. Additionally, the stretch and wryness tensors are defined by different authors in various forms using, for example, a) components in two different curvilinear coordinate systems associated with the undeformed (reference) of deformed (actual) placements of the body, b) components in the convective coordinate system, c) Lagrangian or Eulerian descriptions, d) different representations of the rotation group SO(3) in terms of various finite rotation vectors, Euler angles, quaternions etc., e) formally different tensor operations and sign conventions, as well as f) requiring or not the strain measures to vanish in the undeformed placement of the body. Even the gradient and divergence operators as well as the Cauchy theorem influencing definitions of work-conjugate pairs of the stress and strain measures are not defined in the same way in the literature. As a result, we feel that there is a need to bring some order into definitions of the strain measures to be used in this field.

The aim of this paper is to discuss three different methods of defining the strain measures of the non-linear micropolar continuum: a) by a direct geometric approach, b) defining the strain measures as the fields work-conjugate to the respective internal stress and couple-stress tensor fields, and c) applying the principle of invariance under superposed rigid-body deformations to the strain energy density of the polar-elastic body. Each of the three ways allows one to associate different geometric and/or physical interpretations to the corresponding strain measures. In the discussion we use mainly the coordinate-free vector and tensor notation. Orientations of material particles in the reference and deformed placements, respectively, as well as their changes during deformation are described in the most general way by the proper orthogonal tensors. Our primary strain measures called the natural ones are of the relative type, for they are required to vanish in the reference placement.

The geometric approach presented in Chapter 3 consists of analysing differences between the deformed (actual) and undeformed (reference) placements of the position and orientation differentials of the micropolar continuum, respectively. Elements of geometric approach in Cartesian components were used already by Cosserat and Cosserat (1909) and more recently by Merlini (1997) who took explicitly into account the microstructure curvature tensors describing spatial changes of orientations of the material particles in the reference and actual placements. These tensors were independently introduced also by Zubov and Eremeev (1996) and Yeremeyev and Zubov (1999) within the theory of viscoelastic micropolar fluids, and by Chróścielewski et al. (2004) within the general theory of shells. The microstructure curvature tensors were extensively used in discussion of the local symmetry group of elastic shells by Eremeyev and Pietraszkiewicz (2006).

The basic two-point deformation measures as well as the Lagrangian and Eulerian strain measures are defined in $(15)_{2,3}$ and $(17)_{2-5}$, and their transformations by an orthogonal tensor leading to other deformation or strain measures are indicated. The relative Lagrangian $\mathbf{E}$, $\mathbf{\Gamma}$ and Eulerian $\mathbf{G}$, $\mathbf{\Delta}$ stretch and wryness tensors, having several important features as well as satisfying additional mechanical requirements discussed in Chapters 4 and 5, are called the natural strain measures of the micropolar continuum. The strain measures are valid for unrestricted deformation of the micropolar continuum, are non-symmetric in general, vanish in the reference placement of the body and in the rigid-body deformation of the micropolar continuum. Our derivation process itself is concise, direct and seems to be most complete in the literature.

In an alternative approach developed in Chapter 4 the local equilibrium conditions derived in Appendix are regarded as primary relations of the micropolar continuum. These conditions are formally multiplied by the kinematically admissible virtual translation and virtual rotation fields, and after transformations the principle of virtual work for the micropolar continuum is formulated. In particular, it is found that the resulting internal virtual work density (32) requires some referential stress and couple stress tensors to perform virtual work on variations of the Lagrangian strain measures established in Chapter 3. As a result, we prove that the natural strain measures are the required kinematic fields work-conjugate to the appropriate stress measures of the micropolar continuum indeed. This alternative way of defining the strain measures as those required by the structure of the local equilibrium conditions seems not to have been often used in the literature on micropolar continuum, except in the early papers by Reissner (1973, 1975). However, such an approach was used in the general theory of shells, see for example Simmonds (1984), Makowski and Stumpf (1990), Libai and Simmonds (1998), Chróścielewski et al. (2004), Pietraszkiewicz et al. (2005) and Eremeyev and Pietraszkiewicz (2006).

In the third approach discussed in Chapter 5 we seek a reduced form of the strain energy density of the polar-elastic body following from the principle of invariance under superposed rigid-body deformations. This way of introducing the Lagrangian strain measures is most common in the literature and various such procedures were used, for example, by Kafadar and Eringen (1971), Stojanović (1972), Zubov (1990), Zubov and Eremeev (1996), and Nikitin and Zubov (1998). Using the results by Svendsen and Bertram (1999) we confirm again that invariance of the strain energy density is assured when it is the function of the Lagrangian strain measures defined in Chapter 3.

In Chapter 6 we provide a thorough review of various definitions of the Lagrangian strain measures of the non-linear micropolar continuum proposed in several representative papers in the field. In those works different notation, sign conventions, notions of gradient and divergence operators, coordinate systems, form of the Cauchy theorem, description of rotations, etc. are applied. In most papers the measures are introduced simply by definition. To compare them with our measures we bring the strain measures defined in the papers into the common coordinate-free form. The results summarised in Table 1 show that the stretch and wryness tensors used in many papers do not agree with each other and with our Lagrangian strain tensors defined in (13), (17) and/or (20). Most definitions differ only by transpose of the measures, or by opposite signs, or the measures do not vanish in the absence of deformation. Such differences are not essential for the theory, although one should be aware of them. But we have also discovered a few strain measures which are incompatible with our Lagrangian stretch and wryness tensors. One should avoid such incompatible strain measures when analysing problems of physical importance using the micropolar continuum model.

2 Kinematics of the micropolar continuum

Let the body $\mathcal{B}$ consisting of material particles $X, Y, ...$ deform in the three-dimensional (3D) Euclidean physical space $\mathcal{E}$ whose translation vector space is E.

According to Cosserat and Cosserat (1909), Truesdell and Toupin (1960), Toupin (1964) and Eringen and Kafadar (1976), for example, each material particle of the micropolar continuum has six degrees of freedom of a rigid body.

In the reference (undeformed) placement $\kappa(\mathcal{B}) = B_\kappa \subset \mathcal{E}$ the material particle $X \in \mathcal{B}$ is given through its position vector $\mathbf{x} \in E$ relative to a point $o \in \mathcal{E}$ and by three orthonormal directors $\mathbf{h}_a \in E$, $a, b = 1, 2, 3$, fixing orientation of X in E (see Fig. 1). If $\mathbf{i}_a \in E$ are orthonormal base vectors of a common inertial frame $\{o, \mathbf{i}_a\}$ then $\mathbf{h}_a = \mathbf{H}\mathbf{i}_a$, where $\mathbf{H} = \mathbf{h}_a \otimes \mathbf{i}_a \in SO(3)$ (summed over the range of a) is the structure tensor of B_κ, the proper orthogonal one: $\mathbf{H}^{-1} = \mathbf{H}^T$, $\det \mathbf{H} = +1$. In the micropolar continuum the vectors $\mathbf{h}_a$ may also be viewed as the natural base vectors of the three-orthogonal system of arc-length coordinates s_a such that $\mathbf{h}_a = \partial \mathbf{x} / \partial s_a$.

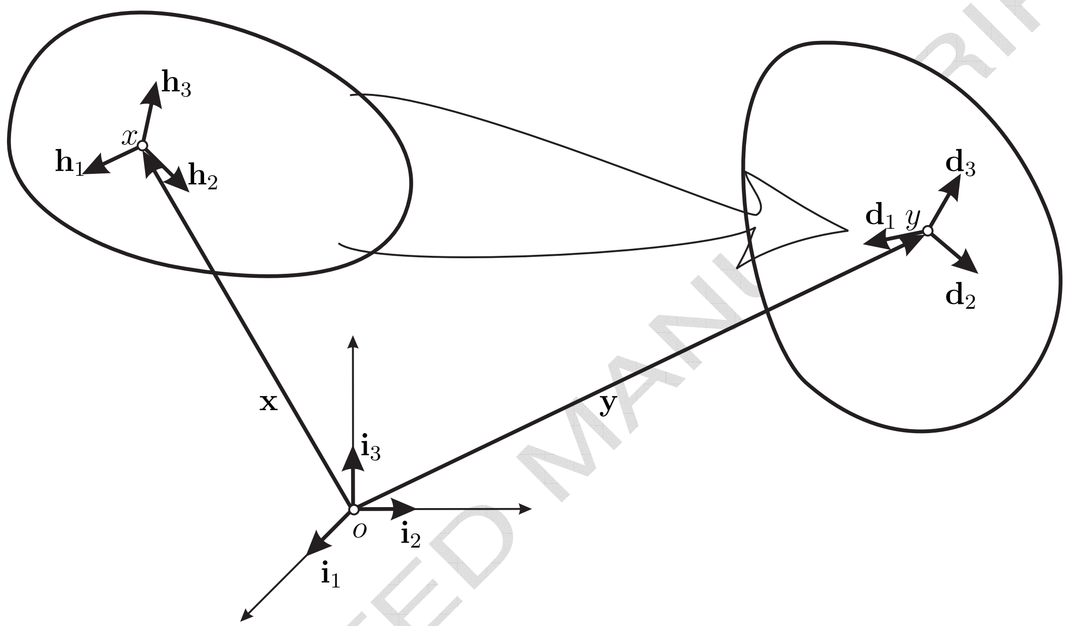

Figure 1. Micropolar body deformation.

In the actual (deformed) placement $\gamma(\mathcal{B}) = B_\gamma = \chi(B_\kappa) \subset \mathcal{E}$ the position of X becomes defined by the vector $\mathbf{y} \in E$, taken here for simplicity relative to the same point $o \in \mathcal{E}$, and by three orthonormal directors $\mathbf{d}_a \in E$, or by the structure tensor $\mathbf{D} = \mathbf{d}_a \otimes \mathbf{i}_a \in SO(3)$ of B_γ. As a result, the finite displacement of the micropolar continuum can be described by two following smooth mappings:

$$\mathbf{y} = \chi(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x}) \, , \quad \mathbf{d}_a = \mathbf{Q}(\mathbf{x}) \mathbf{h}_a \, , \tag{1}$$

where $\mathbf{u} \in E$ is the translation vector, and $\mathbf{Q} = \mathbf{D}\mathbf{H}^T = \mathbf{d}_a \otimes \mathbf{h}_a \in SO(3)$ is the proper orthogonal microrotation tensor: $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$. Two independent fields $\mathbf{u} = \mathbf{u}(\mathbf{x})$ and $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ describe translational and rotational degrees of freedom of the micropolar continuum, respectively.

The finite displacements (1) allow one to introduce two strain measures of the micropolar continuum which are different, in general, from only one strain tensor used in classical continuum mechanics as discussed, for example, by Truesdell and Toupin (1960), Truesdell and Noll (1965), or Wang and Truesdell (1973). In what follows we discuss three different ways of defining the two strain measures of the 3D micropolar continuum.

3 Strain measures by geometric approach

Within the geometric approach we define the strain measures by analysing difference of the fields describing position and orientation differentials of the material particles of the micropolar continuum in the 3D physical space.

Let C be a smooth curve in B_κ given by $x = x(s)$, where s is the arc length parameter. Then $\mathbf{x} = \mathbf{x}(s)$ and $\mathbf{H} = \mathbf{H}(s)$, and their differentials are

$$\begin{aligned} \mathrm{d}\mathbf{x} &= \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{x}\right)\mathrm{d}s = \mathbf{x}'\mathrm{d}s = (\mathrm{Grad}\,\mathbf{x})\mathrm{d}\mathbf{x}\,, \\ \mathrm{d}\mathbf{H} &= \left(\frac{\mathrm{d}}{\mathrm{d}s}\mathbf{H}\right)\mathrm{d}s = \mathbf{H}'\mathrm{d}s = (\mathrm{Grad}\,\mathbf{H})\mathrm{d}\mathbf{x}\,, \quad \mathrm{d}\mathbf{x} \in E\,, \\ \mathrm{Grad}\,\mathbf{x} &= \mathbf{I} \in E \otimes E\,, \quad \mathrm{Grad}\,\mathbf{H} \in SO(3) \otimes E\,, \end{aligned} \tag{2}$$

where $\mathbf{I}$ is the identity (metric) tensor of $E \otimes E$, and Grad is the gradient operator in B_κ.

In this paper, for the fixed origin $o \in \mathcal{E}$ the gradient of a vector field $\mathbf{v}(\mathbf{x}) \in E$ is the 2nd-order tensor field $\mathrm{Grad}\mathbf{v}(\mathbf{x}) \in E \otimes E$ and the gradient of the 2nd-order tensor field $\mathbf{A}(\mathbf{x}) \in E \otimes E$ is the 3rd-order tensor field $\mathrm{Grad}\mathbf{A}(\mathbf{x}) \in E \otimes E \otimes E$, both defined by the relations, see for example Ogden (1984),

$$\begin{aligned} [\mathrm{Grad}\mathbf{v}(\mathbf{x})]\,\mathbf{a} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{v}(\mathbf{x} + t\mathbf{a})|_{t=0}\,, \\ [\mathrm{Grad}\mathbf{A}(\mathbf{x})]\,\mathbf{a} &= \frac{\mathrm{d}}{\mathrm{d}t}\mathbf{A}(\mathbf{x} + t\mathbf{a})|_{t=0}\,, \quad \text{for any } t \in R\,,\ \mathbf{a} \in E\,. \end{aligned} \tag{3}$$

In components relative to $\mathbf{h}_a$ we have

$$\begin{aligned} &\mathbf{v} = v_a \mathbf{h}_a\,, \quad \mathbf{A} = A_{ab}\mathbf{h}_a \otimes \mathbf{h}_b\,, \\ &\mathrm{Grad}\,\mathbf{v} = \mathbf{v}_{,c} \otimes \mathbf{h}_c\,, \quad \mathrm{Grad}\,\mathbf{A} = \mathbf{A}_{,c} \otimes \mathbf{h}_c\,, \quad (.)_{,c} \equiv \partial(.)/\partial s_c\,. \end{aligned} \tag{4}$$

In particular, the gradients of products of the 2nd-order tensor $\mathbf{A}(\mathbf{x})$, $\mathbf{B}(\mathbf{x})$ and the vector $\mathbf{v}(\mathbf{x})$ fields in B_κ are given by

$$\mathrm{Grad}\,(\mathbf{A}\mathbf{v}) = (\mathbf{A}_{,c}\,\mathbf{v} + \mathbf{A}\mathbf{v}_{,c}\,) \otimes \mathbf{h}_c = \mathbf{v}\mathrm{Grad}\mathbf{A}^T + \mathbf{A}\mathrm{Grad}\mathbf{v}\,, \tag{5}$$

$$\begin{aligned}\mathrm{Grad}\,(\mathbf{AB}) &= (\mathbf{A},_c\mathbf{B}+\mathbf{AB},_c)\otimes\mathbf{h}_c = \left(\mathbf{B}^T\mathbf{A}^T,_c\right)^T\otimes\mathbf{h}_c + \mathbf{A}\,(\mathbf{B},_c\otimes\mathbf{h}_c)\\ &= \left(\mathbf{B}^T\mathrm{Grad}\,\mathbf{A}^T\right)^{\overset{1.2}{T}} + \mathbf{A}\mathrm{Grad}\,\mathbf{B}\,.\end{aligned} \tag{6}$$

Equivalent to (3) and (4) definitions of the gradient operator were used for example by Truesdell and Toupin (1960), Truesdell and Noll (1965), and Wang and Truesdell (1973).

However, using the operator $\nabla = \mathbf{h}_c\partial/\partial x_c$ alternative definitions of the gradient of $\mathbf{v}(\mathbf{x})$ and $\mathbf{A}(\mathbf{x})$ not equivalent to (3) leading to

$$\nabla\mathbf{v} = \mathbf{h}_c\otimes\mathbf{v},_c\ , \quad \nabla\mathbf{A} = \mathbf{h}_c\otimes\mathbf{A},_c\ , \tag{7}$$

were used, for example, in the books by Antman (2005), Lurie (1990, 2005), Naumenko and Altenbach (2007), and Zubov (1997). In this paper we shall not use these alternative definitions (7).

Since $\mathrm{d}(\mathbf{HH}^T) = \mathbf{0} = (\mathrm{d}\mathbf{H})\mathbf{H}^T + \mathbf{H}(\mathrm{d}\mathbf{H}^T)$, the tensor $(\mathrm{d}\mathbf{H})\mathbf{H}^T = -[(\mathrm{d}\mathbf{H})\mathbf{H}^T]^T$ is skew-symmetric and can be represented by its axial vector $\mathbf{b}$ depending linearly on $\mathrm{d}\mathbf{x}$, so that

$$\begin{gathered}(\mathrm{d}\mathbf{H})\mathbf{H}^T = \mathbf{b}\times\mathbf{I} = \mathbf{I}\times\mathbf{b}\,, \quad \mathbf{b} = \mathbf{B}\mathrm{d}\mathbf{x}\,,\\ \mathrm{d}\mathbf{h}_a = \mathbf{b}\times\mathbf{h}_a\,, \quad \mathbf{b} = \frac{1}{2}\mathbf{h}_a\times\mathrm{d}\mathbf{h}_a\,, \quad \mathbf{B} = \frac{1}{2}\mathbf{h}_a\times\mathrm{Grad}\,\mathbf{h}_a.\end{gathered} \tag{8}$$

Using (5) and the identity $\mathbf{v}\times\mathbf{A} = \boldsymbol{\epsilon} : (\mathbf{v}\otimes\mathbf{A})$ valid for any vector $\mathbf{v}$ and 2^{nd}-order tensor $\mathbf{A}$, for $\mathbf{B}$ in $(8)_2$ we obtain other representations

$$\mathbf{B} = \frac{1}{2}\mathbf{h}_a\times\left(\mathbf{h}_a\mathbf{H}\mathrm{Grad}\,\mathbf{H}^T\right) = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{H}\mathrm{Grad}\,\mathbf{H}^T\right), \tag{9}$$

where the 3^{rd}-order skew tensor $\boldsymbol{\epsilon} = -\mathbf{I}\times\mathbf{I}$, represented here in $\mathbf{h}_a$ base, is the Ricci tensor of the space $E\otimes E\otimes E$, and the double dot product : of two 3^{rd}-order tensors $\mathbf{A}$, $\mathbf{B}$ represented in the base $\mathbf{h}_a$ is defined as $\mathbf{A} : \mathbf{B} = A_{amn}B_{mnb}\mathbf{h}_a\otimes\mathbf{h}_b$. In (8) and (9), $\mathbf{B}\in E\otimes E$ is the microstructure curvature tensor in the undeformed (reference) placement of the micropolar continuum. Two tensors $\mathbf{I}$, $\mathbf{B}$ are the basic measures of local geometry of the reference placement B_κ.

In the actual (deformed) placement B_γ differentials of $\mathbf{y} = \mathbf{y}(s)$ and $\mathbf{D} = \mathbf{D}(s)$ along the corresponding material curve $D = \chi(C)$ are

$$\begin{aligned}&\mathrm{d}\mathbf{y} = \mathbf{y}'\mathrm{d}s = (\mathrm{grad}\,\mathbf{y})\mathrm{d}\mathbf{y} = (\mathrm{Grad}\,\mathbf{y})\mathrm{d}\mathbf{x} = \mathbf{F}\mathrm{d}\mathbf{x}\,,\\ &\mathrm{d}\mathbf{D} = \mathbf{D}'\mathrm{d}s = (\mathrm{grad}\,\mathbf{D})\mathrm{d}\mathbf{y} = (\mathrm{Grad}\,\mathbf{D})\mathrm{d}\mathbf{x}\,, \quad \mathrm{d}\mathbf{y}\in E\,,\\ &\mathrm{grad}\,\mathbf{y} = \mathbf{I}\in E\otimes E\,, \quad \mathrm{Grad}\,\mathbf{D}\in SO(3)\otimes E\,,\end{aligned} \tag{10}$$

where grad denotes the gradient operator in B_γ defined analogously to (3), and $\mathbf{F} = \mathrm{Grad}\,\mathbf{y}$ is the classical deformation gradient tensor. In the general curvilinear coordinates x^i of B_γ with the base vectors $\mathbf{g}_i = \partial\mathbf{y}/\partial x^i$, $i = 1, 2, 3$, gradient of the vector field $\mathbf{v}(\mathbf{y}) \in E$ takes the form $\mathrm{grad}\,\mathbf{v} = \mathbf{v}_{,i} \otimes \mathbf{g}^i$.

Again, the skew-symmetric tensor $(\mathrm{d}\mathbf{D})\mathbf{D}^T$ can be represented by its axial vector $\mathbf{c}$ depending linearly on $\mathrm{d}\mathbf{y}$, so that

$$
\begin{aligned}
(\mathrm{d}\mathbf{D})\mathbf{D}^T &= \mathbf{c} \times \mathbf{I} = \mathbf{I} \times \mathbf{c}\,, \quad \mathbf{c} = \mathbf{C}\mathrm{d}\mathbf{y}\,, \\
\mathrm{d}\mathbf{d}_a &= \mathbf{c} \times \mathbf{d}_a\,, \quad \mathbf{c} = \frac{1}{2}\mathbf{d}_a \times \mathrm{d}\mathbf{d}_a\,, \\
\mathbf{C} = \frac{1}{2}\mathbf{d}_a \times \mathrm{grad}\,\mathbf{d}_a &= \frac{1}{2}\mathbf{d}_a \times \left(\mathbf{d}_a\mathbf{D}\mathrm{grad}\,\mathbf{D}^T\right) = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{D}\mathrm{grad}\,\mathbf{D}^T\right),
\end{aligned} \tag{11}
$$

where $\mathbf{C} \in E \otimes E$ is the microstructure curvature tensor in the actual (deformed) placement of the micropolar continuum, and $\boldsymbol{\epsilon}$ is now represented in the $\mathbf{d}_a$ base. Two tensors $\mathbf{I}$, $\mathbf{C}$ are the basic measures of local geometry of the actual placement B_γ.

Since $\mathbf{Q}^T\mathbf{Q}_{,c} = -(\mathbf{Q}^T\mathbf{Q}_{,c})^T$ is skew it can be expressed through the axial vector $\boldsymbol{\gamma}_c$,

$$
\begin{aligned}
\mathbf{Q}^T\mathbf{Q}_{,c} &= \boldsymbol{\gamma}_c \times \mathbf{I} = \mathbf{I} \times \boldsymbol{\gamma}_c\,, \\
\boldsymbol{\gamma}_c &= -\frac{1}{2}\mathbf{h}_a \times \left(\mathbf{h}_a\mathbf{Q}^T\mathbf{Q}_{,c}\right) = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathbf{Q}_{,c}\right).
\end{aligned} \tag{12}
$$

This allows one to introduce the 2nd-order tensor

$$
\begin{aligned}
\boldsymbol{\Gamma} = \boldsymbol{\gamma}_c \otimes \mathbf{h}_c &= -\frac{1}{2}\mathbf{h}_a \times \left(\mathbf{h}_a\mathbf{Q}^T\mathrm{Grad}\,\mathbf{Q}\right) = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathrm{Grad}\,\mathbf{Q}\right), \\
\mathbf{Q}^T\mathrm{Grad}\mathbf{Q} &= \mathbf{I} \times \boldsymbol{\Gamma}.
\end{aligned} \tag{13}
$$

The tensor $\boldsymbol{\Gamma}$ characterizes uniquely the 3rd-order tensor $\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}$ skew with regard to first two tensor places. The tensor $\boldsymbol{\Gamma}$ is frequently called the wryness tensor in the literature, cf. Kafadar and Eringen (1971).

Using the chain rule $\mathrm{grad}\,\mathbf{d}_a = (\mathrm{Grad}\,\mathbf{d}_a)\,\mathbf{F}^{-1}$ with (1), (5) and (12), (13) the tensor $\mathbf{C}$ can now be represented by

$$
\begin{aligned}
\mathbf{C} &= \frac{1}{2}(\mathbf{Q}\mathbf{h}_a) \times [\mathrm{Grad}\,(\mathbf{Q}\mathbf{h}_a)]\,\mathbf{F}^{-1} \\
&= \frac{1}{2}\mathbf{Q}\left[\mathbf{h}_a \times \left(\mathbf{Q}^T\mathbf{Q}_{,c}\,\mathbf{h}_a \otimes \mathbf{h}_c\right)\right]\mathbf{F}^{-1} + \frac{1}{2}\mathbf{Q}\,(\mathbf{h}_a \times \mathrm{Grad}\mathbf{h}_a)\,\mathbf{F}^{-1} \\
&= \frac{1}{2}\mathbf{Q}\left\{\mathbf{h}_a \times \left[\mathbf{h}_a\left(\mathbf{Q}^T\mathbf{Q}_{,c}\right)^T \otimes \mathbf{h}_c\right]\right\}\mathbf{F}^{-1} + \mathbf{Q}\mathbf{B}\mathbf{F}^{-1} \\
&= -\frac{1}{2}\mathbf{Q}\left[\mathbf{h}_a \times \left(\mathbf{h}_a\mathbf{Q}^T\mathbf{Q}_{,c}\right) \otimes \mathbf{h}_c\right]\mathbf{F}^{-1} + \mathbf{Q}\mathbf{B}\mathbf{F}^{-1} \\
&= \mathbf{Q}\,(\boldsymbol{\Gamma} + \mathbf{B})\,\mathbf{F}^{-1}\,.
\end{aligned} \tag{14}
$$

The relative changes of lengths and orientations of the micropolar continuum during deformation are governed by differences of differentials (2) and (10) brought to the comparable orientation by the tensor $\mathbf{Q}$,

$$\begin{aligned}
&\mathrm{d}\mathbf{y} - \mathbf{Q}\mathrm{d}\mathbf{x} = \mathbf{X}\mathrm{d}\mathbf{x} = \mathbf{G}\mathrm{d}\mathbf{y}, \quad \mathbf{C}\mathrm{d}\mathbf{y} - \mathbf{Q}\mathbf{B}\mathrm{d}\mathbf{x} = \boldsymbol{\Phi}\mathrm{d}\mathbf{x} = \boldsymbol{\Delta}\mathrm{d}\mathbf{y}, \\
&\mathbf{X} = \mathbf{F} - \mathbf{Q}, \quad \mathbf{G} = \mathbf{I} - \mathbf{Q}\mathbf{F}^{-1} = \mathbf{X}\mathbf{F}^{-1}, \\
&\boldsymbol{\Phi} = \mathbf{C}\mathbf{F} - \mathbf{Q}\mathbf{B}, \quad \boldsymbol{\Delta} = \mathbf{C} - \mathbf{Q}\mathbf{B}\mathbf{F}^{-1} = \boldsymbol{\Phi}\mathbf{F}^{-1}.
\end{aligned} \tag{15}$$

Scalar products of each of $(15)_1$ by itself leads to the quadratic forms

$$\mathrm{d}\mathbf{x} \cdot \mathbf{X}^T\mathbf{X}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \cdot \mathbf{G}^T\mathbf{G}\mathrm{d}\mathbf{y}, \quad \mathrm{d}\mathbf{x} \cdot \boldsymbol{\Phi}^T\boldsymbol{\Phi}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \cdot \boldsymbol{\Delta}^T\boldsymbol{\Delta}\mathrm{d}\mathbf{y}. \tag{16}$$

However, the relative changes of lengths and orientations can also be calculated by the alternative back-rotated expressions

$$\begin{gathered}
\mathbf{Q}^T\mathrm{d}\mathbf{x} - \mathrm{d}\mathbf{x} = \mathbf{E}\mathrm{d}\mathbf{x} = \mathbf{Y}\mathrm{d}\mathbf{y}, \quad \mathbf{Q}^T\mathbf{C}\mathrm{d}\mathbf{y} - \mathbf{B}\mathrm{d}\mathbf{x} = \boldsymbol{\Gamma}\mathrm{d}\mathbf{x} = \boldsymbol{\Psi}\mathrm{d}\mathbf{y}, \\
\mathbf{E} = \mathbf{Q}^T\mathbf{F} - \mathbf{I} = \mathbf{Q}^T\mathbf{X}, \\
\mathbf{Y} = \mathbf{Q}^T - \mathbf{F}^{-1} = \mathbf{E}\mathbf{F}^{-1} = \mathbf{Q}^T\mathbf{G} = \mathbf{Q}^T\mathbf{X}\mathbf{F}^{-1}, \\
\boldsymbol{\Gamma} = \mathbf{Q}^T\mathbf{C}\mathbf{F} - \mathbf{B} = \mathbf{Q}^T\boldsymbol{\Phi}, \\
\boldsymbol{\Psi} = \mathbf{Q}^T\mathbf{C} - \mathbf{B}\mathbf{F}^{-1} = \boldsymbol{\Gamma}\mathbf{F}^{-1} = \mathbf{Q}^T\boldsymbol{\Delta} = \mathbf{Q}^T\boldsymbol{\Phi}\mathbf{F}^{-1}.
\end{gathered} \tag{17}$$

From (9), $(17)_5$ and the chain rule we obtain the following relations for $\boldsymbol{\Delta}$:

$$\boldsymbol{\Delta} = \mathbf{Q}\boldsymbol{\Gamma}\mathbf{F}^{-1} = -\frac{1}{2}\mathbf{d}_a \times (\mathbf{d}_a\mathbf{Q}^T \operatorname{grad}\mathbf{Q}) = -\frac{1}{2}\mathbf{Q}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{grad}\mathbf{Q}). \tag{18}$$

Scalar products of each of $(17)_1$ by itself give the alternative quadratic forms

$$\mathrm{d}\mathbf{x} \cdot \mathbf{E}^T\mathbf{E}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \cdot \mathbf{Y}^T\mathbf{Y}\mathrm{d}\mathbf{y}, \quad \mathrm{d}\mathbf{x} \cdot \boldsymbol{\Gamma}^T\boldsymbol{\Gamma}\mathrm{d}\mathbf{x} = \mathrm{d}\mathbf{y} \cdot \boldsymbol{\Psi}^T\boldsymbol{\Psi}\mathrm{d}\mathbf{y}. \tag{19}$$

From (16) and (19) it follows that each of the tensors $\mathbf{X}$, $\mathbf{E}$, or $\mathbf{G}$, $\mathbf{Y}$ and $\boldsymbol{\Phi}$, $\boldsymbol{\Gamma}$ or $\boldsymbol{\Delta}$, $\boldsymbol{\Psi}$ is the corresponding measure of deformation, stretch or orientation change of the non-linear micropolar continuum in the Lagrangian or Eulerian description, respectively.

The quadratic forms (16) and (19) do not change if $\mathbf{X}$, $\mathbf{E}$, $\boldsymbol{\Phi}$, $\boldsymbol{\Gamma}$ and their counterparts are replaced by $\mathbf{RX}$, $\mathbf{RE}$, $\mathbf{R}\boldsymbol{\Phi}$, $\mathbf{R}\boldsymbol{\Gamma}$, etc., respectively, where $\mathbf{R}$ is a proper orthogonal tensor. Hence, any so transformed tensor can also be regarded as the possible strain measure of the non-linear micropolar continuum. In particular when such a transformation with $\mathbf{R} = \mathbf{Q}^T$ is applied to the measures $\mathbf{X}$, $\mathbf{G}$, $\mathbf{\Phi}$, $\mathbf{\Delta}$ entering the quadratic form (16) the measures become $\mathbf{E}$, $\mathbf{Y}$, $\mathbf{\Gamma}$, $\mathbf{\Psi}$, i.e. those entering the quadratic form (19).

It follows from (15) and (17) that $\mathbf{X}$, $\mathbf{\Phi}$ (and $\mathbf{Y}$, $\mathbf{\Psi}$) are two-point tensors with the left leg associated with the deformed placement and the right leg with the undeformed one (and reverse for $\mathbf{Y}$, $\mathbf{\Psi}$). Such measures may also be called the deformation measures. The tensors $\mathbf{E}$, $\mathbf{\Gamma}$ are the relative Lagrangian strain measures, while the tensors $\mathbf{G}$, $\mathbf{\Delta}$ are the relative Eulerian strain measures.

Let us note some interesting features of the relative strain measures:

(1) All the measures are given in the common coordinate-free notation; their various component representations can easily be generated, if necessary.
(2) Definitions of the measures are valid for finite translations and rotations as well as for unrestricted stretches and changes of microstructure orientation of the micropolar body.
(3) The measures are expressed in terms of the rotation tensor $\mathbf{Q}$; for any specific parametrization of the rotation group $SO(3)$ by various finite rotation vectors, Euler angles, quaternions, etc. appropriate expressions for the measures can easily be found, if necessary.
(4) All the strain measures vanish in the rigid-body deformation $\mathbf{y} = \mathbf{O}\mathbf{x}+\mathbf{a}$, $\mathbf{D} = \mathbf{O}\mathbf{H}$ with a constant vector $\mathbf{a}$ and a constant proper orthogonal tensor $\mathbf{O}$ defined for the whole body.
(5) In the absence of deformation from the reference placement, that is when $\mathbf{F} = \mathbf{Q} = \mathbf{I}$, the relative strain measures identically vanish.
(6) The relative Lagrangian and Eulerian strain measures are not symmetric, in general: $\mathbf{E}^T \neq \mathbf{E}$, $\mathbf{\Gamma}^T \neq \mathbf{\Gamma}$, and $\mathbf{G}^T \neq \mathbf{G}$, $\mathbf{\Delta}^T \neq \mathbf{\Delta}$.

If the feature (5) is not required then instead of $\mathbf{E}$ and $\mathbf{\Gamma}$ we can use the following Lagrangian strain measures:

$$\mathbf{U} = \mathbf{Q}^T\mathbf{F} = \mathbf{E} + \mathbf{I}\,, \quad \mathbf{\Pi} = \mathbf{Q}^T\mathbf{C}\mathbf{F} = \mathbf{\Gamma} + \mathbf{B}\,,$$
$$\mathbf{\Pi} = \frac{1}{2}\mathbf{h}_a \times \left(\mathbf{h}_a\mathbf{H}\mathrm{Grad}\,\mathbf{H}^T - \mathbf{h}_a\mathbf{Q}^T\mathrm{Grad}\,\mathbf{Q}\right) = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{H}\mathrm{Grad}\,\mathbf{H}^T - \mathbf{Q}^T\mathrm{Grad}\,\mathbf{Q}\right). \tag{20}$$

While $\mathbf{U}$ in $(20)_1$ is still very simple, the formula $(20)_2$ for $\mathbf{\Pi}$ in terms of $\mathbf{H}$ and $\mathbf{Q}$ becomes quite complex, in general. This a why the wryness tensor $\mathbf{\Pi}$ was introduced only in one paper by Shkutin (1980) as $\mathbf{\Pi}^T$, see Chapter 6.

Applying the relative changes $(15)_1$ and $(17)_1$ our relative Lagrangian strain measures $\mathbf{E}$, $\mathbf{\Gamma}$ and their Eulerian counterparts $\mathbf{G}$, $\mathbf{\Delta}$ are defined uniquely. Hence, the measures $\mathbf{U}$, $\mathbf{\Pi}$ and their Eulerian counterparts (which are not discussed here) are defined uniquely as well. In our purely geometric approach there is no need for discussion whether these measures might be defined as transposed tensors or with opposite signs. The derivation process itself is concise, elegant and direct.

In most of the papers reviewed in Table 1 of Chapter 6 the strain measures were introduced into the non-linear micropolar continuum simply by definition, without detailed derivation of those measures. Some papers refer directly to the original book by Cosserat and Cosserat (1909), where the strain measures were derived in part by the geometric approach in an awkward notation through components in the common Cartesian frame. Nowadays such an approach is difficult to follow and fully understand. The results of Besdo (1974), where some elements of the geometric approach were used, seem to be incompatible with our strain measures (see discussion in Chapter 6). Most of the authors introducing the strain measures refer to Kafadar and Eringen (1971), who used the principle of material frame-indifference of the polar-elastic body to define the strain measures identified as $\mathbf{U}^T$ and $\mathbf{\Gamma}$ in our geometric approach. Unfortunately, their derivation process is not complete as well (see again discussion in Chapter 6). Although Merlini (1997) proposed two-point deformation measures, for the polar-elastic body he used the back-rotated strain measures coinciding with our $\mathbf{E}$ and $\mathbf{\Gamma}$. It seems that the derivation of the strain measures by geometric approach presented here is the most complete one in the literature.

4 Principle of virtual work and work-conjugate strain measures

Already Reissner (1973) noted that the internal structure of two local equilibrium equations of the micropolar elastic body requires specific two strain measures expressed through two independent translation and rotation vectors as the only field variables. This allowed him to define Cartesian components of such strain measures which may be identified as our stretch $\mathbf{U}^T$ and wryness $\mathbf{\Pi}^T$ tensors (see Chapter 6). In this Chapter we develop this idea in the general case of the non-linear micropolar continuum using the coordinate-free approach.

The local coordinate-free form of the equilibrium conditions (65) for the micropolar continuum is derived in the Appendix. Let us introduce in B_γ two arbitrary smooth vector fields $\mathbf{v}$, $\boldsymbol{\omega} \in E$. Then (65) generate the integral identity

$$\begin{aligned}\iiint\limits_{B_\kappa} \Big\{(\operatorname{Div}\mathbf{T} + \mathbf{f})\cdot\mathbf{v} + \Big[\operatorname{Div}\mathbf{M} - \operatorname{ax}\Big(\mathbf{F}\mathbf{T} - \mathbf{T}^T\mathbf{F}^T\Big) + \mathbf{m}\Big]\cdot\boldsymbol{\omega}\Big\}\, \mathrm{d}v \\ - \iint\limits_{\partial B_{\kappa f}} \{(\mathbf{n}\mathbf{T} - \mathbf{t}^*)\cdot\mathbf{v} + (\mathbf{n}\mathbf{M} - \mathbf{m}^*)\cdot\boldsymbol{\omega}\}\, \mathrm{d}a = 0\,. \end{aligned} \tag{21}$$

Let us apply the relation (61) to represent terms with divergence in (21),

$$\begin{aligned}(\mathrm{Div}\,\mathbf{T})\cdot\mathbf{v} &= \mathrm{Div}\,(\mathbf{T}\mathbf{v}) - \mathbf{T}^T : (\mathrm{Grad}\,\mathbf{v})\ ,\\ (\mathrm{Div}\,\mathbf{M})\cdot\boldsymbol{\omega} &= \mathrm{Div}\,(\mathbf{M}\boldsymbol{\omega}) - \mathbf{M}^T : (\mathrm{Grad}\,\boldsymbol{\omega})\ .\end{aligned} \tag{22}$$

The axial term in (21) can be transformed as follows:

$$\begin{aligned}&-\mathrm{ax}\left(\mathbf{FT} - \mathbf{T}^T\mathbf{F}^T\right)\cdot\boldsymbol{\omega} = [\boldsymbol{\epsilon} : (\mathbf{FT})]\cdot\boldsymbol{\omega} = -\boldsymbol{\omega}\cdot[(\mathbf{I}\times\mathbf{I}) : (\mathbf{FT})]\\ &= -\left(\boldsymbol{\omega}\times\mathbf{I}\right) : (\mathbf{FT}) = -\boldsymbol{\Omega} : (\mathbf{FT}) = +\mathbf{T}^T : (\boldsymbol{\Omega}\mathbf{F})\ ,\end{aligned} \tag{23}$$

where the skew tensor $\boldsymbol{\Omega} = \mathbf{I}\times\boldsymbol{\omega} = \boldsymbol{\omega}\times\mathbf{I}$, $\boldsymbol{\omega} = \mathrm{ax}(\boldsymbol{\Omega})$ has been introduced.

The second terms in (22) when used in (21) can be transformed by the divergence theorem

$$\iiint\limits_{B_\kappa} \mathrm{Div}\,(\mathbf{T}\mathbf{v})\,\mathrm{d}v = \iint\limits_{\partial B_\kappa} (\mathbf{nT})\cdot\mathbf{v}\,\mathrm{d}a\ ,\quad \iiint\limits_{B_\kappa} \mathrm{Div}\,(\mathbf{M}\boldsymbol{\omega})\,\mathrm{d}v = \iint\limits_{\partial B_\kappa} (\mathbf{nM})\cdot\boldsymbol{\omega}\,\mathrm{d}a\ . \tag{24}$$

When (22), (23) and (24) are introduced into (21) this identity becomes

$$\begin{aligned}&\iiint\limits_{B_\kappa}\left[\mathbf{T}^T : (\mathrm{Grad}\,\mathbf{v} - \boldsymbol{\Omega}\mathbf{F}) + \mathbf{M}^T : \mathrm{Grad}\,\boldsymbol{\omega}\right]\mathrm{d}v\\ &\quad= \iiint\limits_{B_\kappa}\left(\mathbf{f}\cdot\mathbf{v} + \mathbf{m}\cdot\boldsymbol{\omega}\right)\,\mathrm{d}v + \iint\limits_{\partial B_{\kappa f}}\left(\mathbf{t}^*\cdot\mathbf{v} + \mathbf{m}^*\cdot\boldsymbol{\omega}\right)\,\mathrm{d}a\\ &\qquad+ \iint\limits_{\partial B_{\kappa d}}\left[(\mathbf{nT})\cdot\mathbf{v} + (\mathbf{nM})\cdot\boldsymbol{\omega}\right]\,\mathrm{d}a\ .\end{aligned} \tag{25}$$

The vector field $\mathbf{v}$ may be interpreted, in particular, as the kinematically admissible virtual translation $\mathbf{v} \equiv \delta\mathbf{y}$ and the vector field $\boldsymbol{\omega}$ as the kinematically admissible virtual rotation $\boldsymbol{\omega} \equiv \mathrm{ax}(\delta\mathbf{Q}\mathbf{Q}^T)$ in B_γ, such that $\mathbf{v} = \boldsymbol{\omega} = \mathbf{0}$ on $\partial B_{\kappa d}$, where δ is the symbol of virtual change (variation). Then the last surface integral in (25) identically vanishes, two integrals in the second row of (25) describe the external virtual work, while the first volume integral in (25) describes the internal virtual work performed by the stress measures on the work-conjugate virtual strain measures. In this interpretation the formula (25) represents the principle of virtual work in the non-linear micropolar continuum.

But for such $\mathbf{v}$ and $\boldsymbol{\omega}$,

$$\begin{aligned}\delta\mathbf{F} = \delta\,(\mathrm{Grad}\,\mathbf{y}) = \mathrm{Grad}\ (\delta\mathbf{y}) = \mathrm{Grad}\,\mathbf{v}\ ,\\ (\delta\mathbf{Q})\mathbf{Q}^T = -\mathbf{Q}\left(\delta\mathbf{Q}^T\right) = \boldsymbol{\omega}\times\mathbf{I} = \boldsymbol{\Omega}\ ,\quad \delta\mathbf{Q}^T = -\mathbf{Q}^T\boldsymbol{\Omega}\ ,\end{aligned} \tag{26}$$

and from $(17)_2$ we obtain

$$\delta\mathbf{E} = \delta\mathbf{U} = (\delta\mathbf{Q}^T)\mathbf{F} + \mathbf{Q}^T\delta\mathbf{F} = -\mathbf{Q}^T\boldsymbol{\Omega}\mathbf{F} + \mathbf{Q}^T\text{Grad}\,\mathbf{v} = \mathbf{Q}^T\left(\text{Grad}\,\mathbf{v} - \boldsymbol{\Omega}\mathbf{F}\right). \tag{27}$$

Since $\mathbf{C} = \frac{1}{2}\mathbf{d}_a \times (\text{Grad}\,\mathbf{d}_a)\mathbf{F}^{-1}$, we can apply the relations $\delta\mathbf{d}_a = \boldsymbol{\omega} \times \mathbf{d}_a$ and $\delta\mathbf{F}^{-1} = -\mathbf{F}^{-1}(\delta\mathbf{F})\mathbf{F}^{-1}$ following from $\mathbf{d}_a = \mathbf{Q}\mathbf{h}_a$ and $\mathbf{F}^{-1}\mathbf{F} = \mathbf{0}$, respectively, and obtain

$$\delta\mathbf{C} = \frac{1}{2}(\boldsymbol{\omega} \times \mathbf{d}_a) \times (\text{Grad}\,\mathbf{d}_a)\mathbf{F}^{-1} + \frac{1}{2}\mathbf{d}_a \times \text{Grad}\,(\boldsymbol{\omega} \times \mathbf{d}_a)\mathbf{F}^{-1} - \mathbf{C}(\text{Grad}\,\mathbf{v})\mathbf{F}^{-1}. \tag{28}$$

The virtual change of $\boldsymbol{\Gamma}$ in $(17)_4$ together with (28) leads to

$$\begin{aligned} \delta\boldsymbol{\Gamma} = \delta\boldsymbol{\Pi} &= (\delta\mathbf{Q})^T\mathbf{C}\mathbf{F} + \mathbf{Q}^T(\delta\mathbf{C})\mathbf{F} + \mathbf{Q}^T\mathbf{C}(\delta\mathbf{F}) \\ &= -\mathbf{Q}^T(\boldsymbol{\omega} \times \mathbf{I})\mathbf{C}\mathbf{F} + \mathbf{Q}^T(\delta\mathbf{C})\mathbf{F} + \mathbf{Q}^T\mathbf{C}\,\text{Grad}\,\mathbf{v} \\ &= \frac{1}{2}\mathbf{Q}^T\left[-\boldsymbol{\omega} \times (\mathbf{d}_a \times \text{Grad}\,\mathbf{d}_a) + (\boldsymbol{\omega} \times \mathbf{d}_a) \times \text{Grad}\,\mathbf{d}_a \right. \\ &\qquad \left. + \mathbf{d}_a \times \text{Grad}\,(\boldsymbol{\omega} \times \mathbf{d}_a)\right]. \end{aligned} \tag{29}$$

But we have the identities

$$\begin{aligned} -\boldsymbol{\omega} \times (\mathbf{d}_a \times \text{Grad}\,\mathbf{d}_a) + (\boldsymbol{\omega} \times \mathbf{d}_a) \times \text{Grad}\,\mathbf{d}_a &= -\mathbf{d}_a \times (\boldsymbol{\omega} \times \text{Grad}\,\mathbf{d}_a), \\ \mathbf{d}_a \times \text{Grad}\,(\boldsymbol{\omega} \times \mathbf{d}_a) &= -\mathbf{d}_a \times (\mathbf{d}_a \times \text{Grad}\,\boldsymbol{\omega}) + \mathbf{d}_a \times (\boldsymbol{\omega} \times \text{Grad}\,\mathbf{d}_a), \\ -\mathbf{d}_a \times (\mathbf{d}_a \times \text{Grad}\,\boldsymbol{\omega}) = -\mathbf{d}_a(\mathbf{d}_a \cdot \text{Grad}\,\boldsymbol{\omega}) &+ (\mathbf{d}_a \cdot \mathbf{d}_a)\text{Grad}\,\boldsymbol{\omega} = 2\text{Grad}\,\boldsymbol{\omega}. \end{aligned} \tag{30}$$

Introducing (30) into $(29)_3$ we finally obtain

$$\delta\boldsymbol{\Gamma} = \delta\boldsymbol{\Pi} = \mathbf{Q}^T\text{Grad}\,\boldsymbol{\omega}. \tag{31}$$

It follows from (25) with (27) and (31) that the internal virtual work density under the first volume integral of $(25)_1$ can now be given by the expressions

$$\sigma = \mathbf{T}^T : (\mathbf{Q}\delta\mathbf{E}) + \mathbf{M}^T : (\mathbf{Q}\delta\boldsymbol{\Gamma}) = \mathbf{S} : \delta\mathbf{E} + \mathbf{K} : \delta\boldsymbol{\Gamma} = \mathbf{S} : \delta\mathbf{U} + \mathbf{K} : \delta\boldsymbol{\Pi}, \tag{32}$$

where

$$\mathbf{S} = \mathbf{Q}^T\mathbf{T}^T, \quad \mathbf{K} = \mathbf{Q}^T\mathbf{M}^T, \tag{33}$$

are the stress and couple-stress tensors whose natural components are referred entirely to the reference (undeformed) placement. The stress measures $\mathbf{S}$, $\mathbf{K}$ are work-conjugate to the respective relative Lagrangian strain measures $\mathbf{E}$, $\boldsymbol{\Gamma}$ and also to $\mathbf{U}$, $\boldsymbol{\Pi}$. These pairs of stress and strain measures are most convenient in the discussion of constitutive equations of the micropolar continuum.

The alternative way of introducing the strain measures presented in this Chapter confirms correctness of the Lagrangian strain measures defined in $(17)_{2,4}$ and $(20)_1$. Additionally, such an approach allows one to analyse other possible work-conjugate pairs of the stress and strain measures within the non-linear micropolar continuum. Some of such pairs have recently been discussed by Ramezani and Naghdabadi (2007).

5 Invariance of strain energy density of the polar-elastic body under superposed rigid-body deformations

In this Chapter we confine our attention to the simplest micropolar body – the polar-elastic body. In this case the constitutive relations are defined through the strain energy density W_κ per unit volume of the undeformed placement κ. At any point $x \in B_\kappa$, labelled by the undeformed position vector $\mathbf{x}$ and the microstructure curvature tensor $\mathbf{B}$, the density W_κ can be assumed to depend, in general, on the deformed position vector $\mathbf{y}$, the deformation gradient tensor $\mathbf{F}$, the microrotation tensor $\mathbf{Q}$, and its gradient $\mathrm{Grad}\,\mathbf{Q}$:

$$W_\kappa = W_\kappa(\mathbf{y}, \mathbf{F}, \ \mathbf{Q}, \ \mathrm{Grad}\,\mathbf{Q}; \ \mathbf{x}, \mathbf{B}) \ . \tag{34}$$

As any constitutive relation, the form of W_κ in (34) should satisfy the principle of material frame-indifference or the principle of objectivity formulated in the form suitable for classical continuum mechanics by Noll (1958), see Truesdell and Noll (1965). There has been an extensive discussion in the literature about the proper understanding of this principle, because its different formulations seem to reflect different physical contents. See for example recent papers by Murdoch (2003), Muschik and Restuccia (2002), Bertram and Svendsen (2001), Svendsen and Bertram (1999) and the book by Bertram (2005). In particular, Svendsen and Bertram (1999) found that the principle of material frame-indifference contains in fact three independent postulates: the principle of invariance under Euclidean transformations, the principle of invariance under superposed rigid-body motions, and the principle of form-invariance of the constitutive equations under change of observer. If any two of them are satisfied the third one becomes satisfied as well. Hence, from the material frame-indifference it follows, in particular, that W_κ should be invariant under superposed rigid-body deformations.

In classical continuum mechanics two deformations $\chi(\mathbf{x})$ and $\chi^*(\mathbf{x})$ of the body differ by a rigid-body transformation if they are related as $\chi^*(\mathbf{x}) = \mathbf{O}\chi(\mathbf{x}) + \mathbf{a}$, where $\mathbf{a}$ is a constant vector and $\mathbf{O}$ a constant rotation tensor, both defined for the whole body. Corresponding deformation gradients are related as $\mathbf{F}^*(\mathbf{x}) = \mathbf{OF}(\mathbf{x})$. However, in micropolar continuum mechanics $\mathbf{Q}^*(\mathbf{x})$ cannot be found from the rigid-body transformation, because $\mathbf{Q}(\mathbf{x})$ is an independent field not expressible by $\chi(\mathbf{x})$. Therefore, after Kafadar and Eringen (1971) and Le and Stumpf (1998) we *assume* that under the rigid-body transformation the directors $\mathbf{d}_a$ are rotated as $\mathbf{y}$: $\mathbf{d}_a^*(\mathbf{x}) = \mathbf{O}\mathbf{d}_a(\mathbf{x})$, or $\mathbf{Q}^*(\mathbf{x}) = \mathbf{O}\mathbf{Q}(\mathbf{x})$. In other words, we *assume* that $\mathbf{d}_a$ are objective vectors. Applying (6) we also obtain that $\mathrm{Grad}\mathbf{Q}^*(\mathbf{x}) = \mathbf{O}\mathrm{Grad}\mathbf{Q}(\mathbf{x})$. Then the principle of invariance under the superposed rigid-body motion requires the values of W_κ to be the same for both deformations $\chi(\mathbf{x})$ and $\chi^*(\mathbf{x})$,

$$W_\kappa(\mathbf{y},\, \mathbf{F},\, \mathbf{Q},\, \mathrm{Grad}\,\mathbf{Q};\, \mathbf{x}, \mathbf{B}) = W_\kappa(\mathbf{O}\mathbf{y} + \mathbf{a}, \mathbf{O}\mathbf{F},\, \mathbf{O}\mathbf{Q},\, \mathbf{O}\mathrm{Grad}\,\mathbf{Q};\, \mathbf{x}, \mathbf{B})\,. \quad (35)$$

Since $\mathbf{a}$ and $\mathbf{O}$ are arbitrary, in order to assure invariance of W_κ in (35) the density should not depend on $\mathbf{y}$ and $\mathbf{Q}$. Then, if $\mathbf{O} \equiv \mathbf{Q}^T$, the function W_κ can be reduced to

$$W_\kappa = W_\kappa(\mathbf{Q}^T\mathbf{F},\, \mathbf{Q}^T\mathrm{Grad}\,\mathbf{Q};\, \mathbf{x}, \mathbf{B})\,, \quad (36)$$

which by $(13)_2$ and $(17)_2$ becomes equivalent to

$$W_\kappa = W_\kappa\,(\mathbf{E} + \mathbf{I},\, \mathbf{I} \times \mathbf{\Gamma};\, \mathbf{x}, \mathbf{B}) = \overline{W}_\kappa\,(\mathbf{E},\, \mathbf{\Gamma};\, \mathbf{x}, \mathbf{B})\;. \quad (37)$$

As a result of this discussion we again confirm that the relative Lagrangian strain measures $\mathbf{E}$, $\mathbf{\Gamma}$ (or the ones $\mathbf{U}$, $\mathbf{\Pi}$) are required to be the independent fields in the elastic strain energy density in order it to be invariant under superposed rigid-body deformations.

6 Discussion and comparative review of some other Lagrangian non-linear strain measures

The geometrical approach discussed in Chapter 3 generates many different strain measures related to each other by proper orthogonal transformations. Among these measures are the relative Lagrangian stretch $\mathbf{E}$ and wryness $\mathbf{\Gamma}$ tensor having several distinctive features. Additionally, the structure of equilibrium conditions discussed in Chapter 4 and invariance of the strain energy density of the polar-elastic body analysed in Chapter 5 both require the Lagrangian strain measures $\mathbf{E}$, $\mathbf{\Gamma}$ or $\mathbf{U}$, $\mathbf{\Pi}$. Taking together the results of the three ways of introducing the measures, the relative tensors $\mathbf{E}$ and $\mathbf{\Gamma}$ seem to be the most appropriate Lagrangian strain measures for the non-linear micropolar continuum. We shall call them the *natural* stretch and wryness tensor, respectively.

Let us review some definitions of the Lagrangian strain measures proposed in the representative literature on non-linear micropolar continuum and compare them with our natural measures $\mathbf{E}$, $\boldsymbol{\Gamma}$ or the measures $\mathbf{U}$, $\boldsymbol{\Pi}$.

The paper by Kafadar and Eringen (1971) is among the most referred to in the literature. The authors used two independent systems of curvilinear coordinates: X^K in B_κ with the reference base vectors $\mathbf{G}_K$, K=1,2,3, and x^k in B_γ with the spatial base vectors $\mathbf{g}_k$, $k = 1,2,3$. The deformation was described by three deformation functions $x^k = x^k(X^L)$ and the change of orientation by nine components $Q^k_{.K} = Q^k_{.K}(X^L)$ in the tensor basis $\mathbf{g}_k \otimes \mathbf{G}^K$ of the proper orthogonal tensor field $\mathbf{Q} = \mathbf{Q}(\mathbf{x})$ satisfying the constraints $\mathbf{Q}^T = \mathbf{Q}^{-1}$, $\det \mathbf{Q} = +1$. Requiring the strain energy density W_κ of the polar-elastic body to remain form-invariant under a rigid-body motion, three first-order PDE were derived, see their formula (25). These equations were regarded as the statement of objectivity for the polar elasticity. Then the authors stated without further details that ..."After lengthy manipulations it may be shown that the general solution of (25) is " ... $W_\kappa = W_\kappa(\mathfrak{E}_{KL}, \Gamma_{KL})$, where

$$\mathfrak{E}_{KL} = x^k{}_{,K}\, Q_{kL}\ , \quad \Gamma_{KL} = \frac{1}{2}\epsilon_{KMN} Q^{kM}_{\cdot\cdot;L} Q_k^{\cdot N}\ , \tag{38}$$

and ; means the covariant differentiation in the reference metric $G_{KL} = \mathbf{G}_K \cdot \mathbf{G}_L$. The same components of the strain measures (38) were used by Maugin (1974) and in Cartesian coordinates by Pabst (2005).

Identifying that $x^k{}_{,K}\, \mathbf{g}_k \otimes \mathbf{G}^K = \operatorname{Grad} \mathbf{y} = \mathbf{F}$ and $Q^k_{.K} \mathbf{g}_k \otimes \mathbf{G}^K = \mathbf{Q}$, the fields $\mathfrak{E}_{KL}$ are just components in the tensor basis $\mathbf{G}^K \otimes \mathbf{G}^L$ of the Lagrangian stretch tensor $\mathbf{F}^T\mathbf{Q}$, that is the tensor $\mathbf{U}^T$ given in $(20)_1$.

To identify Γ_{KL} in $(38)_2$ let us note that by extending the components into the coordinate-free form we can perform the following transformations:

$$\begin{aligned}
\boldsymbol{\Gamma} &= \frac{1}{2}\epsilon_{KMN} Q^{kM}_{\cdot\cdot;L} Q_k^{\cdot N} \mathbf{G}^K \otimes \mathbf{G}^L \\
&= -\frac{1}{2}\epsilon_{NMK} \mathbf{G}^K \left(Q_k^{\cdot N} Q^{kM}_{\cdot\cdot;L} \right) \otimes \mathbf{G}^L \\
&= -\frac{1}{2}\mathbf{G}_N \times \left(\delta^N_P Q_k^{\cdot P} \delta^k_j Q^{jM}_{\cdot\cdot;L} \right) \mathbf{G}_M \otimes \mathbf{G}^L \\
&= -\frac{1}{2}\mathbf{G}_N \times \left[\mathbf{G}^N \cdot \left(\mathbf{G}_P \otimes Q_k^{\cdot P} \mathbf{g}^k \right) \cdot \left(\mathbf{g}_j \otimes Q^{jM}_{\cdot\cdot;L} \mathbf{G}_M \otimes \mathbf{G}^L \right) \right] \\
&= -\frac{1}{2}\mathbf{G}_N \times \left(\mathbf{G}^N \mathbf{Q}^T \operatorname{Grad} \mathbf{Q} \right) \\
&= -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T \operatorname{Grad} \mathbf{Q} \right)\ .
\end{aligned} \tag{39}$$

In particular, we are always able to introduce in B_κ such a system of coordinates X^K in which the natural base vectors $\mathbf{G}_K$ would coincide locally with the reference orthonormal directors $\mathbf{h}_a$ of the orthogonal arc-length coordinates s_a. Then (39) becomes identical with the tensor $\boldsymbol{\Gamma}$ in (13). Therefore,

Γ_{KL} of Kafadar and Eringen (1971) are components of our $\mathbf{\Gamma}$ in the tensor basis $\mathbf{G}^K \otimes \mathbf{G}^L$ indeed.

Stojanović (1972) used three non-complanar and non-orthonormal directors $\mathbf{d}_{(\alpha)}$, $\alpha = 1, 2, 3$, rigidly rotated by the tensor $\mathbf{Q}$ from the fields $\mathbf{D}_{(\alpha)}$ in the reference placement B_κ. Introducing two independent curvilinear coordinate systems as in Kafadar and Eringen (1971) it was assumed that $\mathbf{D}_{(\alpha)}$ are parallel vectors satisfying $\mathbf{D}_{(\alpha)},_L = D^K_{(\alpha);L}\mathbf{G}_K = \mathbf{0}$. Thus the initial microstructure curvature tensor $\mathbf{B}$ was ignored by definition. The directors $\mathbf{d}_{(\alpha)}[\mathbf{y}(\mathbf{x})] = \mathbf{Q}\mathbf{D}_{(\alpha)}(\mathbf{x})$ together with the position vectors in the deformed placement $\mathbf{y}(\mathbf{x})$ were considered as the basic independent field variables. Requiring objectivity of the strain energy density $W_\kappa = W_\kappa\left(\mathbf{F},\, \mathbf{d}_{(\alpha)},\, \mathrm{Grad}\,\mathbf{d}_{(\alpha)};\, \mathbf{x}\right)$ of the polar-elastic material and its consistency with thermodynamics it was found (see his Eqn. (4.23)) that in quasi-static problems W_κ should be of the form $W_\kappa = W_\kappa\,(C_{KL},\, F_{KL};\, \mathbf{x})$, where

$$C_{KL} = C_{LK} = g_{mn}x^m_{.;K}x^n_{.;L}\,, \quad F_{KL} = g_{mn}x^m_{.;K}\Phi^n_{.L}\,,$$
$$\Phi^n_{.L} = \frac{1}{2}\epsilon^{nij}Q_{iN}Q^N_{.j;L}\,. \tag{40}$$

Here C_{KL} are components in $\mathbf{G}^K \otimes \mathbf{G}^L$ of the Green type symmetric strain tensor $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ used in the classical continuum mechanics, which in our case can also be interpreted through our stretch tensor $\mathbf{U}$ defined in $(20)_1$ as $\mathbf{C} = \mathbf{U}^T\mathbf{U}$.

The components $Q_{iN}Q^N_{.j;L}$ in (40) correspond to $\mathbf{Q}\mathrm{Grad}\,\mathbf{Q}^T$ and those $\Phi^n_{.L}$ to $\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}\mathrm{Grad}\mathbf{Q}^T\right)$ in the coordinate-free notation, so that F_{KL} are components in $\mathbf{G}^K \otimes \mathbf{G}^L$ of the tensor $\mathbf{F}^T\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}\mathrm{Grad}\mathbf{Q}^T\right)$. Let us perform the following transformations:

$$\begin{aligned}\mathbf{Q}\mathrm{Grad}\mathbf{Q}^T &= \mathbf{Q}\mathbf{Q}^T,_L \otimes \mathbf{G}^L = -\mathbf{Q},_L \mathbf{Q}^T \otimes \mathbf{G}^L \\ &= -\mathbf{Q}\left(\mathbf{Q}^T\mathbf{Q},_L\right)\mathbf{Q}^T \otimes \mathbf{G}^L = -\mathbf{Q}\left(\mathbf{I} \times \boldsymbol{\gamma}_L\right) \\ &= \mathbf{I} \times \left(-\mathbf{Q}\mathbf{\Gamma}\right)\mathbf{Q}^T \otimes \mathbf{G}^L\,,\end{aligned}$$

so that $\mathbf{Q}\mathbf{\Gamma} = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}\mathrm{Grad}\mathbf{Q}^T\right)$. Therefore, the bending measure of Stojanović (1972) coincides with our $\mathbf{F}^T\mathbf{Q}\mathbf{\Gamma}$.

Besdo (1974) used the curvilinear convected coordinates ξ^i, $i = 1, 2, 3$, and three base vectors: $\mathbf{g}_i$ in the actual (deformed) placement, $\tilde{\mathbf{g}}_i$ in the reference (undeformed) placement identified with the reference directors in B_κ, and $\hat{\mathbf{g}}_i$ identified with the directors in B_γ which are rotated from $\tilde{\mathbf{g}}_i$ by the finite rotation vector $\boldsymbol{\phi} = \phi\,\mathbf{e}$, where ϕ is the angle of rotation about the axis described by the unit vector $\mathbf{e}$. Then the mixed components of three strain measures of the micropolar continuum were defined as (see Besdo (1974), formulae (5.6) and (5.7))

$$\begin{aligned} \varepsilon^i_{.j} = \tilde{\mathbf{g}}^i \cdot \hat{\mathbf{g}}_j - \delta^i_j \,, \quad \gamma^i_{.j} = \delta^i_j - \hat{\mathbf{g}}^i \cdot \tilde{\mathbf{g}}_j \,, \\ \kappa^i_{.j} = \frac{1}{2}\epsilon^{ikm} \left(\mathbf{g}_{k,j} \cdot \mathbf{g}_m - \hat{\mathbf{g}}_{k,j} \cdot \hat{\mathbf{g}}_m \right) \,, \end{aligned} \tag{41}$$

where $(..)_{,j}$ is the partial derivative relative to ξ^j.

In the undeformed basis $\tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}^j$ the Lagrangian strain measures were defined by Besdo (1974), formulae (5.9), as

$$\tilde{\boldsymbol{\varepsilon}} = \varepsilon^i_{.j} \tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}^j = \hat{\mathbf{g}}_j \otimes \tilde{\mathbf{g}}^j - \mathbf{I} \,, \quad \tilde{\boldsymbol{\gamma}} = \gamma^i_{.j} \tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}^j \,, \quad \tilde{\boldsymbol{\kappa}} = \kappa^i_{.j} \tilde{\mathbf{g}}_i \otimes \tilde{\mathbf{g}}^j \,. \tag{42}$$

The stretch measure $\tilde{\boldsymbol{\varepsilon}}$ can alternatively be written as $\tilde{\boldsymbol{\varepsilon}} = \mathbf{Q} - \mathbf{I}$ which is not compatible with our $\mathbf{E}$ defined in $(17)_1$. The second Lagrangian stretch measure $\tilde{\boldsymbol{\gamma}}$ is not present at all in our approach. In the coordinate-free notation we have $\tilde{\boldsymbol{\gamma}} = \mathbf{I} - \mathbf{Q}^T = -\tilde{\boldsymbol{\varepsilon}}^T$, which means that $\tilde{\boldsymbol{\gamma}}$ is not an independent stretch measure indeed.

The wryness measure $\tilde{\boldsymbol{\kappa}}$ in (42) with $(41)_2$ can be written in the coordinate-free form in terms of our tensor fields as (we omit here those complex transformations)

$$\tilde{\boldsymbol{\kappa}} = \mathbf{F} \left[\frac{1}{2} \boldsymbol{\epsilon} : \left(\mathbf{F}^{-1} \mathrm{Grad} \mathbf{F} \right) + \mathbf{B} \right] - \mathbf{Q} \left(\boldsymbol{\Gamma} + \mathbf{B} \right) \,. \tag{43}$$

Since the first term in (43) contains the deformation gradient $\mathbf{F}$ it is difficult to establish the geometric meaning of $\tilde{\boldsymbol{\kappa}}$.

Shkutin (1980, 1988), whose results we translate into a more understandable notation of Pietraszkiewicz and Badur (1983), used convected curvilinear coordinates θ^i and three base vectors: the undeformed $\mathbf{g}_i$ associated with B_κ, the deformed $\bar{\mathbf{g}}_i$ associated with B_γ, and the rotated $\mathbf{d}_i$ obtained from $\mathbf{g}_i$ by the rotation performed with the finite rotation vector $\boldsymbol{\theta} = 2 \tan \phi/2 \, \mathbf{e}$. Shkutin (1980), by his formulae (1.4), (1.6) and (3.9), introduced two strain measures with components

$$\varepsilon_{ij} = \left(\bar{\mathbf{g}}_i - \mathbf{d}_i \right) \cdot \mathbf{d}_j \,, \quad l_{ij} = \frac{1}{2} \left(\mathbf{d}^k \times \mathbf{d}_{k,i} \right) \cdot \mathbf{d}_j \,. \tag{44}$$

We can extend the components ε_{ij} into the coordinate-free Lagrangian stretch tensor using $\mathbf{Q}$ instead of $\boldsymbol{\theta}$:

$$\begin{aligned} \boldsymbol{\varepsilon} &= \varepsilon_{ij} \mathbf{g}^i \otimes \mathbf{g}^j = \left[\left(\mathbf{F} \mathbf{g}_i - \mathbf{Q} \mathbf{g}_i \right) \cdot \left(\mathbf{Q} \mathbf{g}_j \right) \right] \mathbf{g}^i \otimes \mathbf{g}^j \\ &= \left[\mathbf{g}_i \left(\mathbf{F}^T \mathbf{Q} - \mathbf{I} \right) \mathbf{g}_j \right] \mathbf{g}^i \otimes \mathbf{g}^j = \mathbf{F}^T \mathbf{Q} - \mathbf{I} \,, \end{aligned}$$

and $\boldsymbol{\varepsilon}$ here coincides with our $\mathbf{E}^T$ in $(17)_1$.

Extending analogously the components l_{ij} into the coordinate-free form we obtain

$$\begin{aligned} \mathbf{l}^T = l_{ij}\mathbf{g}^j \otimes \mathbf{g}^i &= \frac{1}{2}\left\{\left[\left(\mathbf{Q}\mathbf{g}^k\right) \times (\mathbf{Q}\mathbf{g}_k)_{,i}\right] \cdot (\mathbf{Q}\mathbf{g}_j)\right\} \mathbf{g}^j \otimes \mathbf{g}^i \\ &= \frac{1}{2}\left\{\left[\mathbf{g}^k \times \left(\mathbf{Q}^T\mathbf{Q}_{,i}\,\mathbf{g}_k + \mathbf{g}_{k,i}\right)\right] \cdot \mathbf{g}_j\right\} \mathbf{g}^j \otimes \mathbf{g}^i \\ &= -\frac{1}{2}\mathbf{g}^k \times \left(\mathbf{g}_k\mathbf{Q}^T \mathrm{Grad}\mathbf{Q}\right) + \frac{1}{2}\mathbf{g}^k \times \mathrm{Grad}\mathbf{g}_k \\ &= \boldsymbol{\Gamma} + \mathbf{B} = \boldsymbol{\Pi}\,. \end{aligned} \tag{45}$$

Thus, l_{ij} are just components in $\mathbf{g}^i \otimes \mathbf{g}^j$ of our $\boldsymbol{\Pi}^T$ defined in $(20)_1$.

Badur and Pietraszkiewicz (1986), by their formulas (2.4), defined the strain measures by

$$\mathcal{U} = \mathcal{R}^T\mathbf{F}, \quad \mathcal{K} = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathcal{R}^T \mathrm{Grad}\mathcal{R}\right), \tag{46}$$

with $\mathcal{R}$ coinciding with our $\mathbf{Q}$. Hence, the stretch tensor $\mathcal{U}$ is identical with $\mathbf{U}$ in (20). The wryness tensor $\mathcal{K}$ coincides with $-\boldsymbol{\Gamma}$ defined in (13).

Reissner (1987) formulated the strain measures in the common Cartesian frame assuming that $\mathbf{h}_a \equiv \mathbf{i}_a$, $\mathbf{d}_a = \mathbf{Q}\mathbf{i}_a$, and using the convected initially Cartesian coordinate system x_a in which $\mathbf{i}_{a,c} = \mathbf{0}$ and the initial microstructure tensor $\mathbf{B} \equiv \mathbf{0}$. In our notation his definitions of Cartesian components of the strain measures are (see his Eqn. (4) and (9))

$$e_{ab} = \mathbf{y}_{,a} \cdot \mathbf{d}_b - \delta_{ab}\,, \quad k_{ab} = \frac{1}{2}\epsilon_{bmn}\mathbf{d}_{m,a} \cdot \mathbf{d}_n\,. \tag{47}$$

In the Cartesian tensor basis $\mathbf{i}_a \otimes \mathbf{i}_b$ the stretch tensor $(47)_1$ takes the coordinate-free form $\mathbf{e} = \mathbf{F}^T\mathbf{Q} - \mathbf{I}$ which can be identified with our $\mathbf{E}^T$ introduced in $(17)_1$. To identify the meaning of k_{ab} we perform the following transformations:

$$\begin{aligned} \mathbf{k}^T &= k_{ab}\mathbf{i}_b \otimes \mathbf{i}_a = \frac{1}{2}\mathbf{i}_b\epsilon_{bmn}\left[(\mathbf{Q}\mathbf{i}_n) \cdot (\mathbf{Q}\mathbf{i}_m)_{,a}\right] \otimes \mathbf{i}_a \\ &= \frac{1}{2}\mathbf{i}_b\epsilon_{bmn}\left(\mathbf{i}_n\mathbf{Q}^T\mathbf{Q}_{,a}\,\mathbf{i}_m\right) \otimes \mathbf{i}_a = -\frac{1}{2}\mathbf{i}_b\epsilon_{bnm}\left(Q_{pn}Q_{pm,a}\right) \otimes \mathbf{i}_a \\ &= -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T \mathrm{Grad}\mathbf{Q}\right)\,. \end{aligned} \tag{48}$$

Thus, the components k_{ab} of the Reissner wryness tensor can be identified with the Cartesian components of our wryness tensor $\boldsymbol{\Gamma}^T \equiv \boldsymbol{\Pi}^T$.

Zubov (1990) introduced the following Lagrangian strain measures:

$$\mathbf{U} = (\nabla\mathbf{y})\mathbf{Q}\,, \quad -\mathbf{L} \times \mathbf{I} = \left(\nabla\mathbf{Q}^T\right)\mathbf{Q}\,, \tag{49}$$

where the gradient operator was defined as in (7). Taking into account that $\nabla\mathbf{y} = (\mathrm{Grad}\mathbf{y})^T \equiv \mathbf{F}^T$, the stretch tensor $\mathbf{U}$ in $(49)_1$ is just our $\mathbf{U}^T$ in $(20)_1$.

To interpret the wryness tensor $\mathbf{L}$ in $(49)_2$ let us represent it in the undeformed base $\mathbf{h}_a$ leading to

$$-\mathbf{L}\times\mathbf{I}=\mathbf{h}_a\otimes\mathbf{Q}^T{}_{,a}\mathbf{Q}=-\mathbf{h}_a\otimes\mathbf{Q}^T\mathbf{Q}_{,a}=-\mathbf{h}_a\otimes\boldsymbol{\gamma}_a\times\mathbf{I}=-\boldsymbol{\Gamma}^T\times\mathbf{I}\,.$$

Thus, the Lagrangian bending measure $\mathbf{L}$ of Zubov (1990) is just $\boldsymbol{\Gamma}^T$ in our approach. The strain measures (49) were then used by Zubov and Eremeev (1996), Zubov (1997), and Yeremeyev and Zubov (1999).

To describe orientation of the material particles Dłużewski (1993) used three Euler angles ϕ^α, $\alpha=1,2,3$, treated as angular coordinates of the vector $\boldsymbol{\phi}=(\phi^\alpha)$ in the object orientation space R being the constant curvature space. Deformation of the polar continuum was described by two maps $\mathbf{y}=\mathbf{y}(\mathbf{x})$ and $\boldsymbol{\phi}=(\phi^\alpha)(\mathbf{x})$, and the strain measures were defined as

$$\mathfrak{C}=\mathbf{Q}^T\mathbf{F}\,,\quad \boldsymbol{\Gamma}=\mathbf{Q}^T\mathrm{Grad}\boldsymbol{\phi}\,. \tag{50}$$

The stretch tensor $\mathfrak{C}$ here coincides with our $\mathbf{U}$ in $(20)_1$. However, the wryness tensor $\boldsymbol{\Gamma}$ in $(50)_2$ is difficult to interpret in terms of our $\boldsymbol{\Gamma}$ in (13) or $(17)_2$ due to the use of the unconventional orientation space R by Dłużewski (1993).

Merlini (1997), formula (1), introduced the two-point deformation measures of the micropolar continuum, called the linear and angular strain, respectively, by

$$\boldsymbol{\chi}=\mathbf{F}-\mathbf{Q}\,,\quad \boldsymbol{\omega}=\mathbf{Q}\,\mathrm{ax}\left(\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)\,, \tag{51}$$

where the axial tensor $\mathbf{A}$ of $\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}$ was defined to satisfy $\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}=\mathbf{I}\times\mathbf{A}$. According to the relation (13), $\mathbf{A}$ here coincides with our $\boldsymbol{\Gamma}$ and we obtain

$$\boldsymbol{\omega}=-\mathbf{Q}\frac{1}{2}\boldsymbol{\epsilon}:\left(\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)\,. \tag{52}$$

Thus, the two-point tensors $\boldsymbol{\chi}$ and $\boldsymbol{\omega}$ here are just $\mathbf{QE}$ and $\mathbf{Q}\boldsymbol{\Gamma}$ in terms of our natural strain measures, respectively. But in the strain-energy density of polar-elastic body Merlini (1997) used the back-rotated strain measures $\boldsymbol{\varepsilon}=\mathbf{Q}^T\boldsymbol{\chi}$ and $\boldsymbol{\beta}=\mathbf{Q}^T\boldsymbol{\omega}$, called extension and distortion, which coincide with our strain measures $\mathbf{E}$ and $\boldsymbol{\Gamma}$, respectively.

Steinmann and Stein (1997) in their Section 3 introduced the non-symmetric strain measures of the non-linear micropolar continuum to be $\overline{\mathbf{U}}=\mathbf{Q}^T\mathbf{F}$ and $\overline{\mathbf{K}}=\mathrm{ax}\left(\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)$. The stretch tensor $\overline{\mathbf{U}}$ coincides with our tensor $\mathbf{U}$ defined in $(20)_1$. The axial tensor of the 3rd-order tensor $\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}$ was defined by Steinmann and Stein (1997) again as satisfying the relation $\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}=\mathbf{I}\times\overline{\mathbf{K}}$, and for the axial tensor they obtained $\overline{\mathbf{K}}=-\frac{1}{2}\mathbf{h}_a\times\left(\mathbf{h}_a\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)$ which coincides with our $\boldsymbol{\Gamma}$ defined in (13).

Nikitin and Zubov (1998) modified the strain measures (49) by defining them as follows:

$$\mathbf{U} = \mathbf{Q}^T\mathbf{F}\,, \quad \mathbf{Q}^T\mathbf{Q},_a \otimes \mathbf{h}_a = \mathbf{I}\times\mathbf{L}\,. \tag{53}$$

Now $\mathbf{U}$ from $(53)_1$ coincides with our $\mathbf{U}$ defined in $(20)_1$, while from (13) and (12) it follows that $\mathbf{L}$ in $(53)_2$ coincides with our $\mathbf{\Gamma}$. Nikitin and Zubov (1998) expressed $\mathbf{L}$ through the finite rotation vector $\boldsymbol{\theta} = 2\tan\phi/2\,\mathbf{e}$.

Grekova and Zhilin (2001) used the curvilinear convected coordinate system q^i, $i = 1, 2, 3$, with the base vectors $\mathbf{r}_i, \mathbf{r}^j$ in the reference placement B_κ. They introduced by definition the following Lagrangian strain measures:

$$\mathbf{A} = (\nabla\mathbf{y})\mathbf{Q}\,, \quad \mathbf{K} = \left(\mathbf{r}^j\otimes\boldsymbol{\phi}_i\right)\mathbf{Q}\,, \tag{54}$$

where

$$\boldsymbol{\phi}_i = \left(\mathbf{Q},_i\mathbf{Q}^T\right)\cdot\cdot\,\boldsymbol{\epsilon}/2 = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q},_i\mathbf{Q}^T\right) = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}\mathbf{Q},_i^T\right)$$

are the axial vectors of the skew tensors $\mathbf{Q},_i\mathbf{Q}^T$, that is $\mathbf{Q},_i\mathbf{Q}^T = \mathbf{I}\times\boldsymbol{\phi}_i$, and $\cdot\cdot$ means two subsequent contractions of the multiplied tensors.

The stretch tensor $\mathbf{A}$ in $(54)_1$ is just our $\mathbf{U}^T$ defined in $(20)_1$. To identify the meaning of $\mathbf{K}$ in $(54)_2$ let us remind that using (12) we obtain

$$\begin{aligned}\mathbf{I}\times\boldsymbol{\phi}_i &= \mathbf{Q}\left(\mathbf{Q}^T\mathbf{Q},_i\right)\mathbf{Q}^T = \mathbf{Q}\left(\mathbf{I}\times\boldsymbol{\gamma}_i\right)\mathbf{Q}^T = \mathbf{I}\times\mathbf{Q}\boldsymbol{\gamma}_i\,, \quad \boldsymbol{\phi}_i = \mathbf{Q}\boldsymbol{\gamma}_i\,,\\ \mathbf{K}^T &= \mathbf{Q}^T\left(\mathbf{Q}\boldsymbol{\gamma}_i\otimes\mathbf{r}^i\right) = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)\end{aligned}$$

Hence, $\mathbf{K}^T$ in (54) is equivalent to the wryness tensor $\mathbf{\Gamma}$ of Kafadar and Eringen (1971) and our (13). This definition of $\mathbf{K}$ was earlier introduced by Zhilin (1976) as the second deformation tensor of a directed surface.

Nistor (2002) used the initially Cartesian convected coordinates x_i, so that $\mathbf{h}_a \equiv \mathbf{i}_a$, and the components of the strain measures were defined in the common Cartesian frame as

$$c_{ij} = y_{k,i}Q_{kj}\,, \quad \gamma_{ij} = \frac{1}{2}\epsilon_{jmn}Q_{pn}Q_{pm,i}\,. \tag{55}$$

In the coordinate-free notation c_{ij} are the Cartesian components of the stretch tensor $\mathbf{c} = \mathbf{F}^T\mathbf{Q}$ which corresponds to our $\mathbf{U}^T$ in $(20)_1$. Performing transformations similar to (48) for the components γ_{ij} in $(55)_2$ we obtain

$$\boldsymbol{\gamma}^T = \gamma_{ij}\mathbf{i}_j\otimes\mathbf{i}_i = -\frac{1}{2}\mathbf{i}_j\epsilon_{jnm}Q_{pn}Q_{pm,i}\otimes\mathbf{i}_i = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathrm{Grad}\mathbf{Q}\right)\,.$$

Therefore, from (13) it follows that $\boldsymbol{\gamma}^T$ corresponds to our $\mathbf{\Gamma}\equiv\mathbf{\Pi}$, which also allows one to interpret γ_{ij} as the components k_{ab} defined in (47) by Reissner (1987).

Ramezani and Naghdabadi (2007) referring to Kafadar and Eringen (1971) introduced the coordinate-free form of two Lagrangian strain measures

$$\overline{\mathbf{U}} = \mathbf{F}^T\mathbf{Q}\,, \quad \mathbf{\Gamma} = \frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T \mathrm{Grad}\mathbf{Q}\right)\,. \tag{56}$$

The stretch tensor $\overline{\mathbf{U}}$ coincides with our $\mathbf{U}^T$, while the wryness tensor differs by sign from our $\mathbf{\Gamma}$ and the one of Kafadar and Eringen (1971).

From the review above summarised in Table 1 we can draw interesting conclusions. It is apparent that both strain measures introduced by Stojanović (1972) and Besdo (1974) are incompatible with our Lagrangian strain measures $\mathbf{E}, \mathbf{\Gamma}$ or $\mathbf{U}$, $\mathbf{\Pi}$. Also the wryness tensor defined by Dłużewski (1993) seems to differ from our tensor $\mathbf{\Gamma}$ in the way which is difficult to interpret. In all other papers summarised in Table 1 the strain measures are defined in the mixed way: the stretch tensor does not vanish in the reference placement while the wryness tensor does. The results by Shkutin (1980) are reversed: his stretch tensor is of the relative type while his wryness tensor does not vanish in the reference placement.

The stretch tensors proposed by Kafadar and Eringen (1971), Reissner (1987), Zubov (1990), Nistor (2002), and Ramezani and Naghdabadi (2007) are defined as transpose of our Lagrangian stretch tensor $\mathbf{U}$, while the stretch tensor of Shkutin (1980) coincides with transpose of our $\mathbf{E}$. Similarly, the wryness tensors by Reissner (1987) and Nistor (2002) coincide with transpose of our $\mathbf{\Gamma}$, the one by Shkutin (1980) is transpose of our $\mathbf{\Pi}$, while Badur and Pietraszkiewicz (1986), and Ramezani and Naghdabadi (2007) defined their wryness tensor with opposite sign to our $\mathbf{\Gamma}$. The wryness tensors defined by Kafadar and Eringen (1971), Steinmann and Stein (1997), and Nikitin and Zubov (1998) agree with our natural wryness tensor $\mathbf{\Gamma}$ defined in (13) and $(17)_4$. Only Merlini (1997) in the later part of his paper used the Lagrangian strain tensors coinciding with our natural strain measures $\mathbf{E}$, $\mathbf{\Gamma}$. Nobody as yet used both Lagrangian strain tensors coinciding with our strain measures $\mathbf{U}$, $\mathbf{\Pi}$.

7 Conclusions

We have discussed three different ways of defining the strain measures in the non-linear micropolar continuum.

The geometric approach has combined definitions of the relative changes of lengths and orientations of the body with appropriate quadratic forms in the Euclidean vector space. This has led to several two-point deformation measures as well as to the family of Lagrangian, global and relative strain measures

Table 1
Definitions of the stretch and wryness tensors

Paper	The stretch tensor	The wryness tensor
Kafadar and Eringen (1971)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Stojanović (1972)	$\mathbf{F}^T\mathbf{F}$	$\mathbf{F}^T\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}\operatorname{Grad}\mathbf{Q}^T)$
Besdo (1974)	$\mathbf{Q}-\mathbf{I}$	$\mathbf{F}\left[\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{F}^{-1}\operatorname{Grad}\mathbf{F}\right)+\mathbf{B}\right]$
		$-\mathbf{Q}\left(\boldsymbol{\Gamma}+\mathbf{B}\right)$
Shkutin (1980)	$\mathbf{F}^T\mathbf{Q}-\mathbf{I}$	$-\frac{1}{2}\left[\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})\right]^T$
		$+\mathbf{B}^T$
Badur and Pietraszkiewicz (1986)	$\mathbf{Q}^T\mathbf{F}$	$\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Reissner (1987)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}\left[\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})\right]^T$
Zubov (1990)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}\left[\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})\right]^T$
Dłużewski (1993)	$\mathbf{Q}^T\mathbf{F}$	$\mathbf{Q}^T \operatorname{Grad}\boldsymbol{\phi}$
Merlini (1997)	$\mathbf{F}-\mathbf{Q}$,	$-\mathbf{Q}\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$,
	$\mathbf{Q}^T\mathbf{F}-\mathbf{I}$	$-\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Steinmann and Stein (1997)	$\mathbf{Q}^T\mathbf{F}$	$-\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Nikitin and Zubov (1998)	$\mathbf{Q}^T\mathbf{F}$	$-\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Grekova and Zhilin (2001)	$\mathbf{F}^T\mathbf{Q}$	$\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
Nistor (2002)	$\mathbf{F}^T\mathbf{Q}$	$-\frac{1}{2}\left[\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})\right]^T$
Ramezani and Naghdabadi (2007)	$\mathbf{F}^T\mathbf{Q}$	$\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$
The present paper	$\mathbf{Q}^T\mathbf{F}-\mathbf{I}$	$-\frac{1}{2}\boldsymbol{\epsilon} : (\mathbf{Q}^T \operatorname{Grad}\mathbf{Q})$

and their Eulerian counterparts. All the measures are related to each other by orthogonal transformations. Due to several distinctive features of the relative Lagrangian and Eulerian strain measures combined with additional mechanical arguments presented in two other approaches, we have called such relative strain measures the natural ones.

In the alternative approach developed here global equilibrium conditions of forces and couples acting on an arbitrary part of the micropolar body have been regarded as primary relations. After formal transformations it has been proved that the back-rotated nominal stress and couple stress tensors are required to perform virtual work on corresponding variations of the Lagrangian strain measures derived by the geometric approach. Thus, we have independently confirmed that the structure of equilibrium conditions of the micropolar continuum requires the Lagrangian strain measures coinciding with the ones derived here.

Finally, we have confirmed once more that the invariance of the strain energy density of the polar-elastic body under superposed rigid-body deformations requires the density to be expressed through our Lagrangian strain measures as well.

Review of the representative literature in this field has shown that the Lagrangian strain measures were defined in some papers in the form incompatible with our Lagrangian strain measures. In most other papers the measures were defined either as transpose of our natural strain measures, or with opposite signs, or they did not vanish in the absence of deformation. One should be aware of those differences when analysing problems of physical importance using the micropolar continuum model.

We believe that in the present paper we have presented enough arguments to conclude that the relative stretch tensor $\mathbf{E}$ and the relative wryness tensor $\mathbf{\Gamma}$ introduced here by three different approaches are the most appropriate Lagrangian strain measures to be used in the non-linear micropolar continuum.

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Appendix. Local form of the equilibrium conditions

Let any part $\mathcal{P}$ of the micropolar body, identified with its sufficiently regular reference placement $P_\kappa = \kappa(\mathcal{P}) \subset B_\kappa$, be in an equilibrium state. Then in the referential description the global balances of forces and couples of P_κ take the form, see for example Eringen and Kafadar (1976), Eringen (1999), Lurie (2005),

$$\begin{aligned} &\iint\limits_{\partial P_\kappa} \mathbf{t}_{(n)}\,\mathrm{d}a + \iiint\limits_{P_\kappa} \mathbf{f}\,\mathrm{d}v = \mathbf{0}\,, \\ &\iint\limits_{\partial P_\kappa} \left(\mathbf{y}\times\mathbf{t}_{(n)} + \mathbf{m}_{(n)}\right)\mathrm{d}a + \iiint\limits_{P_\kappa} (\mathbf{y}\times\mathbf{f} + \mathbf{m})\,\mathrm{d}v = \mathbf{0}\,. \end{aligned} \tag{57}$$

Here $\mathbf{f}$ and $\mathbf{m}$ are the volume force and couple vectors applied at any point $y = \chi(x)$ of the deformed body, but measured per unit volume of P_κ, while $\mathbf{t}_{(n)}$ and $\mathbf{m}_{(n)}$ are the surface traction and couple vectors applied at any point of ∂P_γ, but measured per unit area of ∂P_κ, respectively.

If $\mathbf{n}$ is the unit vector externally normal to ∂P_κ, then using the Cauchy theorem the vectors $\mathbf{t}_{(n)}$ and $\mathbf{m}_{(n)}$ are expressible as linear functions of the respective stress $\mathbf{T}$ and couple-stress $\mathbf{M}$ tensors, called also the nominal type stress and couple-stress tensors in the literature, according to

$$\mathbf{t}_{(n)} = \mathbf{nT} \,, \quad \mathbf{m}_{(n)} = \mathbf{nM} \,. \tag{58}$$

This version of the Cauchy theorem follows a long tradition of defining the stress tensor in classical elasticity, see for example Love (1927), and Sneddon and Berry (1958). According to this tradition the first index of the stress tensor indicates direction of the normal to the cross section, on which acts the internal stress force vector, while the second index indicates direction of the component of the stress force.

The 2nd-order tensors $\mathbf{T}$ and $\mathbf{M}$ in (58) are mixed tensors whose left-hand sides are associated with the reference placement and right-hand sides with the deformed one. The transposed tensors $\mathbf{T}^T = \mathbf{T}_R$ and $\mathbf{M}^T = \mathbf{M}_R$ may be regarded as the 1st Piola-Kirchhoff type stress and couple-stress tensors, respectively. The form (58) of the Cauchy theorem was used, for example, by Eringen and Kafadar (1976), Atkin and Fox (1980), Billington (1986), Dai (2003), and Ramezani and Naghdabadi (2007).

The divergence of the 2nd-order tensor field $\mathbf{A}(\mathbf{x})$ on B_κ convenient to use with (58) is usually defined as the vector field $\text{Div}\mathbf{A}(\mathbf{x})$ satisfying

$$[\text{Div}\mathbf{A}(\mathbf{x})]\,\mathbf{a} = \text{Div}\,[\mathbf{A}(\mathbf{x})\mathbf{a}] \quad \forall \mathbf{a} \in E \,, \tag{59}$$

which in components relative to $\mathbf{h}_a$ takes the form

$$\text{Div}\mathbf{A} = \mathbf{h}_a \cdot \mathbf{A},_a = A_{ab},_a \mathbf{h}_b \,. \tag{60}$$

In particular, the divergence of product of the 2nd-order tensor $\mathbf{A}(\mathbf{x})$ and vector $\mathbf{v}(\mathbf{x})$ fields on B_κ is given by

$$\text{Div}\,(\mathbf{A}\mathbf{v}) = \mathbf{h}_a \cdot (\mathbf{A},_a \mathbf{v} + \mathbf{A}\mathbf{v},_a) = (\text{Div}\,\mathbf{A})\,\mathbf{v} + \mathbf{A}^T : (\text{Grad}\,\mathbf{v}) \,, \tag{61}$$

where the double dot product : of two 2nd-order tensors $\mathbf{A}, \mathbf{B}$ is defined by $\mathbf{A} : \mathbf{B} = \text{tr}\,(\mathbf{A}^T\mathbf{B}) = A_{ab}B_{ab}$.

According to Billington (1986), Section 1.10, the divergence theorems corresponding to the conventions (3), (58), (59) and (60) are

$$\begin{gathered}\iint\limits_{\partial P_\kappa} \mathbf{n}\mathrm{T}\,\mathrm{d}v = \iiint\limits_{P_\kappa} \text{Div}\mathbf{T}\,\mathrm{d}v \,, \quad \iint\limits_{\partial P_\kappa} \mathbf{nM}\,\mathrm{d}a = \iiint\limits_{P_\kappa} \text{Div}\mathbf{M}\,\mathrm{d}v \,, \\ \iint\limits_{\partial P_\kappa} (\mathbf{y} \times \mathbf{nT})\,\mathrm{d}a = \iiint\limits_{P_\kappa} \left[\mathbf{y} \times \text{Div}\mathbf{T} - \text{ax}\left(\mathbf{FT} - \mathbf{T}^T\mathbf{F}^T\right)\right]\mathrm{d}v \,,\end{gathered} \tag{62}$$

where $\text{ax}\,(\mathbf{A})$ is the axial vector of the skew 2nd-order tensor $\mathbf{A}$. In this paper we shall use the conventions (58)–(62) together with (3).

However, many authors used alternative forms of the Cauchy theorem $\mathbf{t}_{(n)} = \mathbf{T}_R\mathbf{n}\,,\ \mathbf{m}_{(n)} = \mathbf{M}_R\mathbf{n}$ and/or the alternative definition of divergence of the 2nd-order tensor field $\mathbf{A}(\mathbf{x})$ satisfying

$$\begin{gathered}[\mathrm{Div}\mathbf{A}(\mathbf{x})]\,\mathbf{a} = \mathrm{Div}\left[\mathbf{A}^T(\mathbf{x})\mathbf{a}\right] \qquad \forall \mathbf{a} \in E\,, \\ \mathrm{Div}\mathbf{A} = \mathbf{A},_b \cdot \mathbf{h}_b = A_{ab},_b\,\mathbf{h}_a\,,\end{gathered} \tag{63}$$

see for example Stojanović (1972), Wang and Truesdell (1973), Gurtin (1981), Marsden and Hughes (1983), Scarpetta (1989), or Dłużewski (1993). When these alternative conventions were applied, the corresponding divergence theorem would lead to, for example,

$$\iiint\limits_{P_\kappa} \mathrm{Div}\mathbf{T}_R\,\mathrm{d}v = \iint\limits_{\partial P_\kappa} \mathbf{T}_R\mathbf{n}\,\mathrm{d}a\,, \quad \iiint\limits_{P_\kappa} \mathrm{Div}\mathbf{M}_R\,\mathrm{d}v = \iint\limits_{\partial P_\kappa} \mathbf{M}_R\mathbf{n}\,\mathrm{d}a\,. \tag{64}$$

In this paper we shall not use these alternative conventions (63) and (64).

Let $\mathbf{t}^*(\mathbf{x})$ and $\mathbf{m}^*(\mathbf{x})$ be the external force and couple vector fields prescribed on the part $\partial B_{\gamma f}$, but measured per unit area of $\partial B_{\kappa f}$, respectively. Then using (58)–(62), from (57) after some transformations we obtain the local equilibrium equations and corresponding dynamic boundary conditions

$$\begin{aligned}&\mathrm{Div}\,\mathbf{T} + \mathbf{f} = \mathbf{0}, \quad \mathrm{Div}\,\mathbf{M} - \mathrm{ax}\left(\mathbf{FT} - \mathbf{T}^T\mathbf{F}^T\right) + \mathbf{m} = \mathbf{0} \quad \text{in } P_\kappa \subset B_\kappa\,, \\ &\mathbf{nT} - \mathbf{t}^* = \mathbf{0}\,, \quad \mathbf{nM} - \mathbf{m}^* = \mathbf{0} \quad \text{along } \partial P_{\kappa f} \subset \partial B_{\kappa f}\,,\end{aligned} \tag{65}$$

The corresponding kinematic boundary conditions are given by the relations

$$\mathbf{y} = \mathbf{y}^*\,, \quad \mathbf{Q} = \mathbf{Q}^* \quad \text{along } \partial P_{\kappa d} \subset \partial B_{\kappa d} = \partial B_\kappa \setminus \partial B_{\kappa f}\,, \tag{66}$$

where $\mathbf{y}^*$, $\mathbf{Q}^*$ are given functions of $\mathbf{x}$.

One can derive seven other formally different coordinate-free local forms of equilibrium conditions. Some of them following from other combinations of definitions of the gradient, divergence and/or Cauchy theorem are given by Maugin (1974), Scarpetta (1989), Lurie (1990), Zubov (1990, 1997), Steinmann and Stein (1997), Maugin (1998), Yeremeyev and Zubov (1999), and Dai (2003).