On vectorially parameterized natural strain measures of the non-linear Cosserat continuum

W. Pietraszkiewicz [a,*], V. A. Eremeyev [b]

[a] *Institute of Fluid-Flow Machinery of the Polish Academy of Sciences, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland*

[b] *South Scientific Center of RASci & South Federal University, Milchakova str. 8a, 344090 Rostov on Don, Russia*

Abstract

The natural Lagrangian stretch and wryness tensors of the non-linear Cosserat continuum are expressed in terms of the general finite rotation vector. These expressions are then specialized for seven particular definitions of the rotation vectors known in the literature. It is expected that some of the vectorially parameterized strain measures derived here may be more convenient than others in specific applications.

Key words: Cosserat continuum, micropolar continuum, finite rotation vector, strain measures, wryness tensor

1 Introduction

In the recent paper by Pietraszkiewicz and Eremeyev (2009) we applied three different ways of defining the strain measures of the non-linear Cosserat continuum. We found in particular that the most natural definitions for the Lagrangian relative stretch $\mathbf{E}$ and wryness (or change of the microstructure curvature) $\boldsymbol{\Gamma}$ tensors are

$$\mathbf{E} = \mathbf{Q}^T(\mathbf{I} + \operatorname{Grad}\mathbf{u}) - \mathbf{I}, \quad \boldsymbol{\Gamma} = -\frac{1}{2}\boldsymbol{\epsilon} : \left(\mathbf{Q}^T \operatorname{Grad}\mathbf{Q}\right). \tag{1}$$

Here $\mathbf{u} \in E$ is the translation vector, $\mathbf{Q} \in SO(3)$ the proper orthogonal microrotation tensor, $\mathbf{I}$ the identity (metric) tensor in the undeformed configuration,

* Corresponding author. Tel: +48 58 6995263; fax: +48 58 3416144.
Email addresses: pietrasz@imp.gda.pl (W. Pietraszkiewicz), eremeyev.victor@gmail.com (V. A. Eremeyev).

$\boldsymbol{\epsilon} = -\mathbf{I} \times \mathbf{I}$ the 3rd-order skew Ricci tensor, $\times$ the vector product, and the double dot product : of two 3rd-order tensors $\mathbf{A}, \mathbf{B}$ represented in the orthonormal base $\mathbf{h}_a$, $a = 1, 2, 3$, is defined as $\mathbf{A} : \mathbf{B} = A_{amn} B_{mnb} \mathbf{h}_a \otimes \mathbf{h}_b$.

The orthonormal vectors $\mathbf{h}_a$ were interpreted by Pietraszkiewicz and Eremeyev (2009) as the natural base vectors of three-orthogonal system of curvilinear arc-length coordinates s_a such that $\mathbf{h}_a = \partial \mathbf{x} / \partial s_a \equiv \mathbf{x},_a$, where $\mathbf{x} \in E$ is the position vector of a material particle in the reference configuration of the body. Then gradients of the vector $\mathbf{v}(\mathbf{x}) \in E$ and 2nd-order tensor $\mathbf{T}(\mathbf{x}) \in E \otimes E$ fields were defined by $\operatorname{Grad} \mathbf{v} = \mathbf{v},_a \otimes \mathbf{h}_a$ and $\operatorname{Grad} \mathbf{T} = \mathbf{T},_a \otimes \mathbf{h}_a$, respectively.

While three components of $\mathbf{u}$ in (1) are all independent, nine components of $\mathbf{Q}$ in (1) are subjected to six constraints following from the orthogonality conditions $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$, so that only three rotational parameters of $\mathbf{Q}$ are independent.

In the literature, many techniques how to parameterize the rotation group $SO(3)$ were developed, see for example Rooney (1977), Guo (1981), Pietraszkiewicz and Badur (1983), Altman (1986), Atluri and Cazzani (1995), Borri et al. (2000), Geradin and Cardona (2001), and Chróścielewski et al. (2004). These parameterizations can roughly be classified as vectorial and non-vectorial ones. Various finite rotation vectors as well as the Cayley-Gibbs and exponential map parameters are examples of the vectorial parameterization, for they all have three independent scalar parameters as Cartesian components of a generalized vector in the 3D vector space E. The non-vectorial parameterizations are expressed either in terms of three scalar parameters that cannot be treated as vector components, such as Euler-type angles for example, or through more scalar parameters subjected to additional constraints, such as unit quaternions, Cayley-Klein parameters, or direction cosines. Each of these parameterizations has some advantages and drawbacks widely discussed in the literature.

The aim of this note is to express the strain measures (1) in terms of seven different vectorial parameters proposed in the literature. Each of these expressions may appear to be more convenient than others when solving specific problems of the non-linear Cosserat continuum.

2 The vectorial parameterization

The microrotation tensor $\mathbf{Q}$ represents the isometric and orientation-preserving transformation of the 3D vector space E into itself. By the Euler theorem such a transformation can be expressed in terms of the angle of rotation ϕ about the axis of rotation described by the eigenvector $\mathbf{e}$ corresponding to the real eigenvalue $+1$ of $\mathbf{Q}$ such that

$$\mathbf{Q}\mathbf{e} = +\mathbf{e}, \quad \cos\phi = \frac{1}{2}\left(\operatorname{tr}\mathbf{Q} - 1\right), \quad \sin\phi\,\mathbf{e} = \frac{1}{2}\operatorname{ax}\left(\mathbf{Q} - \mathbf{Q}^T\right), \tag{2}$$

where $\operatorname{tr}\mathbf{A}$ is the trace of the 2^{nd}-order tensor $\mathbf{A}$, and $\operatorname{ax}\mathbf{W}$ is the axial vector $\mathbf{w}$ of the skew 2^{nd}-order tensor $\mathbf{W}$ such that $\mathbf{W} = \mathbf{w} \times \mathbf{I} = \mathbf{I} \times \mathbf{w}$.

In terms of $\mathbf{e}$ and ϕ the microrotation tensor $\mathbf{Q}$ can be expressed by Gibbs (1901) formula, see for example Beatty (1977), Guo (1981), and Pietraszkiewicz and Badur (1983),

$$\mathbf{Q} = \cos\phi\,\mathbf{I} + (1 - \cos\phi)\,\mathbf{e} \otimes \mathbf{e} + \sin\phi\,\mathbf{e} \times \mathbf{I}. \tag{3}$$

In the vectorial parameterization of $\mathbf{Q}$ one introduces a scalar function $p(\phi)$ generating three components of the finite rotation vector $\mathbf{p}$ defined as, see for example Bauchau and Trainelli (2003),

$$\mathbf{p} = p(\phi)\,\mathbf{e}\,. \tag{4}$$

The generating function $p(\phi)$ in (4) has to be an odd function of ϕ with the limit behaviour $\lim\limits_{\phi\to 0}\frac{p(\phi)}{\phi} = \kappa$, where κ is a positive real normalization factor (usually 1 or $\frac{1}{2}$), and $p(0) = 0$. In terms of (4) the tensor $\mathbf{Q}$ and its transpose can be represented as

$$\begin{aligned} \mathbf{Q} &= \cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\,\mathbf{p}\otimes\mathbf{p} + \frac{\sin\phi}{p}\,\mathbf{p}\times\mathbf{I}\,, \\ \mathbf{Q}^T &= \cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\,\mathbf{p}\otimes\mathbf{p} - \frac{\sin\phi}{p}\,\mathbf{p}\times\mathbf{I}\,. \end{aligned} \tag{5}$$

The finite rotation vector (4) is the generalized vector. The composition of two successive rotations $\mathbf{Q}_3 = \mathbf{Q}_2\mathbf{Q}_1$, when expressed in terms of the corresponding vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ with angles of rotation ϕ_1, ϕ_2, ϕ_3, reads

$$\begin{aligned} \cos\frac{\phi_3}{2} &= \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1p_2}\,\mathbf{p}_1\cdot\mathbf{p}_2\,, \\ \frac{\sin\frac{\phi_3}{2}}{p_3}\,\mathbf{p}_3 &= \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1p_2}\left(\frac{p_2}{\cos\frac{\phi_2}{2}}\,\mathbf{p}_1 + \frac{p_1}{\cos\frac{\phi_1}{2}}\,\mathbf{p}_2 - \mathbf{p}_1\times\mathbf{p}_2\right)\,. \end{aligned} \tag{6}$$

The equation $(6)_1$ is used to compute ϕ_3, which also gives $\sin\frac{\phi_3}{2}$ and $p_3 = p(\phi_3)$. Then $(6)_2$ allows one to establish the vector $\mathbf{p}_3$.

Since $\mathbf{Q}^T\mathbf{Q}_{,a} = -(\mathbf{Q}^T\mathbf{Q}_{,a})^T$ is skew it can be expressed through the axial vector $\boldsymbol{\gamma}_a$,

$$\mathbf{Q}^T\mathbf{Q},_a = \boldsymbol{\gamma}_a \times \mathbf{I}, \quad \boldsymbol{\gamma}_a = -\frac{1}{2}\,\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathbf{Q},_a\right) = \phi,_a \mathbf{e} + \left[\sin\phi\,\mathbf{I} - (1-\cos\phi)\,\mathbf{e}\times\mathbf{I}\right]\mathbf{e},_a\,. \tag{7}$$

The vector $\boldsymbol{\gamma}_a$ describes the change of the reference microstructure curvature of the Cosserat continuum along the arc-length coordinate line s_a. It is analogous to the vector $\mathbf{k}_j$ of change of curvature of the curvilinear coordinate line θ^j in classical continuum mechanics defined as $\mathbf{R}^T\mathbf{R},_j = \mathbf{k}_j \times \mathbf{I}$ by Pietraszkiewicz and Badur (1983), where $\mathbf{R}$ was the rotation tensor following from the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$. But in the Cosserat continuum $\mathbf{Q}$ is the independent field not related to $\mathbf{u}$ and therefore $\mathbf{Q} \neq \mathbf{R}$, in general.

Differentiating the vector $\mathbf{p}$ in (4) along the coordinate line s_a we obtain the transformation relations

$$\phi,_a = \frac{1}{p'}\,p,_a\,, \quad \mathbf{e},_a = -\frac{1}{p^2}\,p,_a\,\mathbf{p} + \frac{1}{p}\,\mathbf{p},_a\,, \quad p' = \frac{dp}{d\phi}\,, \tag{8}$$

which introduced into (7) lead to

$$\boldsymbol{\gamma}_a = \frac{1}{p}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)p,_a\,\mathbf{p} + \frac{\sin\phi}{p}\,\mathbf{p},_a - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{p},_a\,. \tag{9}$$

Taking into account that $\mathbf{p}\cdot\mathbf{p},_a = pp,_a$, we have the identities

$$\mathbf{p} = \frac{1}{pp,_a}(\mathbf{p}\otimes\mathbf{p})\,\mathbf{p},_a\,, \quad \mathbf{p},_a = \mathbf{I}\mathbf{p},_a\,, \quad \mathbf{p}\times\mathbf{p},_a = (\mathbf{p}\times\mathbf{I})\,\mathbf{p},_a\,, \tag{10}$$

and the relation (9) can be given in the equivalent form

$$\boldsymbol{\gamma}_a = \mathbf{A}\mathbf{p},_a\,, \quad \mathbf{A} = \frac{\sin\phi}{p}\,\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p}\otimes\mathbf{p} - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{I}\,. \tag{11}$$

Substituting $(5)_2$ and (11) into (1), the natural Lagrangian stretch $\mathbf{E}$ and wryness $\boldsymbol{\Gamma}$ tensors can now be represented in terms of the finite rotation vector $\mathbf{p}$ by the general relations

$$\mathbf{E} = \left(\cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\,\mathbf{p}\otimes\mathbf{p} - \frac{\sin\phi}{p}\,\mathbf{p}\times\mathbf{I}\right)(\mathbf{I} + \mathrm{Grad}\,\mathbf{u}) - \mathbf{I}\,, \tag{12}$$

$$\boldsymbol{\Gamma} = \left[\frac{\sin\phi}{p}\,\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p}\otimes\mathbf{p} - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{I}\right]\mathrm{Grad}\,\mathbf{p}\,. \tag{13}$$

3 Particular finite rotation vectors

Among definitions of $\mathbf{p}$ used most often in the literature let us mention the finite rotation vectors defined as

$$\boldsymbol{\theta} = 2\tan\frac{\phi}{2}\,\mathbf{e}\,, \quad \boldsymbol{\phi} = \phi\,\mathbf{e}\,, \quad \boldsymbol{\varpi} = \sin\phi\,\mathbf{e}\,, \quad \boldsymbol{\rho} = \tan\frac{\phi}{2}\,\mathbf{e}\,, \tag{14}$$

where the generating functions are $\theta = 2\tan\frac{\phi}{2}$, ϕ, $\varpi = \sin\phi$, and$\rho = \tan\frac{\phi}{2}$, respectively. Within the non-linear Cosserat continuum the Cayley-Gibbs vector $\boldsymbol{\theta}$ was used for example by Shkutin (1980), Badur and Pietraszkiewicz (1986), Zubov (1997), and Nikitin and Zubov (1998), while the linear vector $\boldsymbol{\phi}$ (called also the exponential map) by Kafadar and Eringen (1971), Nistor (2002), and Ramezani and Naghdabadi (2007). The vector $\boldsymbol{\theta}$ was used in the non-linear theory of plates, see for example Hodges et al. (1993), and in the non-linear theory of composite beams by Hodges (2006), where the extensive review of the literature was given. In the non-linear theory of Cosserat-type shells and the classical continuum mechanics the vector $\boldsymbol{\varpi}$ was found to be convenient in papers by Pietraszkiewicz (1979), and Pietraszkiewicz and Badur (1983), while the Rodrigues rotation vector $\boldsymbol{\rho}$ was willingly used in analytical mechanics of rigid-body motion, see for example Pars (1965).

Less popular in the literature till now is the Euler-Rodrigues vector $\boldsymbol{\sigma}$, the Wiener-Milenkovic vector $\boldsymbol{\mu}$, and the Bauchau-Trainelli vector $\boldsymbol{\beta}$ defined by

$$\boldsymbol{\sigma} = 2\sin\frac{\phi}{2}\,\mathbf{e}\,, \quad \boldsymbol{\mu} = 4\tan\frac{\phi}{4}\,\mathbf{e}\,, \quad \boldsymbol{\beta} = 4\sin\frac{\phi}{4}\,\mathbf{e}\,, \tag{15}$$

whose generating functions are $\sigma = 2\sin\frac{\phi}{2}$, $\mu = 4\tan\frac{\phi}{4}$, and $\beta = 4\sin\frac{\phi}{4}$, respectively.

Introducing (14) and (15) into (12) and (13) and using appropriate trigonometric identities, after complex but elementary transformations we obtain the formulae for $\mathbf{E}$ and $\boldsymbol{\Gamma}$ expressed in terms of the corresponding finite rotation vectors. These formulae are given in Tables 1 and 2.

With all the vectorial parameterizations the singularities occur for some values of ϕ following from singularities of the generating functions $p(\phi)$, when $p \to \infty$, from singularities of the inverse relations $\mathbf{p} = \mathbf{p}(\mathbf{Q})$, as well as from singularities of $\mathbf{A}$ and $\mathbf{A}^{-1}$, see Bauchau and Trainelli (2003). Hence, we also indicate in Tables 1 and 2 the ranges of validity of ϕ for the analysis to be singular-free while using these strain measures in problems of the Cosserat continuum. When in applications there appear arbitrary values of the rotation angle ϕ, one needs at least five independent scalar parameters to parameterize the rotation group SO(3) in the globally one-to-one and singular-free manner, see for example Hopf (1940), Stuelpnagel (1964), and Perelyaev (2006). For the finite rotation vectors $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$, Bauchau and Trainelli (2003) described procedures how to handle arbitrary rotations by combining appropriate update and rescaling operations.

With the vectors $\boldsymbol{\theta}$, $\boldsymbol{\rho}$, $\boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}$, $\boldsymbol{\Gamma}$ in Tables 1 and 2 do not contain any trigonometric expressions of ϕ. This might suggest some convenience in further purely algebraic transformations. With the vectors $\boldsymbol{\phi}$, $\boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}$, $\boldsymbol{\Gamma}$ have broader range of singular-free behaviour. When $|\phi| < \pi$ the values of $\mu(\phi)$ and $\beta(\phi)$ are not much different from ϕ, that is $\mu(\phi) \approx \phi \approx \beta(\phi)$. In the limit the sin-type generating functions ϖ, σ, β converge to ϕ from below, while the tan-type ones θ, ρ, μ, from above.

Table 1
The natural Lagrangian stretch tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\mathbf{E}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\frac{\theta^2}{4}}\left[\left(1-\frac{\theta^2}{4}\right)\mathbf{I}+\frac{1}{2}\boldsymbol{\theta}\otimes\boldsymbol{\theta}-\boldsymbol{\theta}\times\mathbf{I}\right](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\phi} \equiv \phi\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\cos\phi\,\mathbf{I}+\frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\otimes\boldsymbol{\phi}-\frac{\sin\phi}{\phi}\boldsymbol{\phi}\times\mathbf{I}\right)(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\phi\,\mathbf{I}+\frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\otimes\boldsymbol{\varpi}-\boldsymbol{\varpi}\times\mathbf{I}\right)(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\rho^2}\left[(1-\rho^2)\mathbf{I}+2\boldsymbol{\rho}\otimes\boldsymbol{\rho}-2\boldsymbol{\rho}\times\mathbf{I}\right](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\left[\left(1-\frac{1}{2}\sigma^2\right)\mathbf{I}+\frac{1}{2}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma}-\cos\frac{\phi}{2}\boldsymbol{\sigma}\times\mathbf{I}\right](\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\frac{1}{\left(1+\frac{\mu^2}{16}\right)^2}\left\{\left[1-\frac{\mu^2}{16}\left(\frac{3}{8}-\frac{\mu^2}{16}\right)\right]\mathbf{I}+\frac{1}{2}\boldsymbol{\mu}\otimes\boldsymbol{\mu}-\left(1-\frac{\mu^2}{16}\right)\boldsymbol{\mu}\otimes\mathbf{I}\right\}(\mathbf{I}$ $+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left\{\left[1-\frac{\beta^2}{2}\left(1-\frac{\beta^2}{16}\right)\right]\mathbf{I}+\frac{1}{2}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\otimes\boldsymbol{\beta}\right.$ $\left.-\sqrt{1-\frac{\beta^2}{16}}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\times\mathbf{I}\right\}(\mathbf{I}+\mathrm{Grad}\,\mathbf{u})-\mathbf{I}$

When the values of $\mathbf{u}$ and ϕ as well as their spatial gradients are infinitesimal

$$\|\mathbf{u}\| \ll 1\,, \quad \|\mathrm{Grad}\,\mathbf{u}\| \ll 1\,, \quad |\phi| \ll 1\,, \quad \|\mathrm{Grad}\,\phi\| \ll 1,$$

we also have $\sin\phi \approx \phi$, $\cos\phi \approx 1$, and $p(\phi) \approx \kappa\phi$. Then from (3), (15) and (14) it follows that

$$\mathbf{p} \approx \kappa\boldsymbol{\vartheta}\,, \quad \mathbf{Q} \approx \mathbf{I}+\boldsymbol{\vartheta}\times\mathbf{I}\,,$$

where $\boldsymbol{\vartheta} = \phi\,\mathbf{e}$ is now the infinitesimal rotation vector. Then from (12) and (13) we

Table 2
The natural Lagrangian wryness tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\mathbf{\Gamma}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\dfrac{1}{1+\frac{\theta^2}{4}}\left(\mathbf{I} - \dfrac{1}{2}\boldsymbol{\theta}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\theta}$
$\boldsymbol{\phi} \equiv \phi\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\dfrac{\sin\phi}{\phi}\mathbf{I} + \dfrac{\phi - \sin\phi}{\phi^3}\boldsymbol{\phi}\otimes\boldsymbol{\phi} - \dfrac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\phi}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$\left[\mathbf{I} + \dfrac{1}{\varpi^2}\left(\dfrac{1}{\cos\phi} - 1\right)\boldsymbol{\varpi}\otimes\boldsymbol{\varpi} - \dfrac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\varpi}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\dfrac{2}{1+\rho^2}\left(\mathbf{I} - \boldsymbol{\rho}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\rho}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\dfrac{\phi}{2}\mathbf{I} - \dfrac{1}{4\cos\frac{\phi}{2}}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma} - \dfrac{1}{2}\boldsymbol{\sigma}\times\mathbf{I}\right)\mathrm{Grad}\,\boldsymbol{\sigma}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\dfrac{1}{\left(1+\frac{\mu^2}{16}\right)^2}\left[\left(1 - \dfrac{\mu^2}{16}\right)\mathbf{I} + \dfrac{1}{8}\boldsymbol{\mu}\otimes\boldsymbol{\mu} - \dfrac{1}{2}\boldsymbol{\mu}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\mu}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left[\sqrt{1-\dfrac{\beta^2}{16}}\left(1-\dfrac{\beta^2}{8}\right)\mathbf{I} + \dfrac{1-\left(1-\frac{\beta^2}{8}\right)\left(1-\frac{\beta^2}{16}\right)}{\beta^2\sqrt{1-\frac{\beta^2}{16}}}\boldsymbol{\beta}\otimes\boldsymbol{\beta} - \dfrac{1}{2}\left(1-\dfrac{\beta^2}{16}\right)\boldsymbol{\beta}\times\mathbf{I}\right]\mathrm{Grad}\,\boldsymbol{\beta}$

obtain

$$\mathbf{E} \approx \boldsymbol{\varepsilon} \equiv \mathrm{Grad}\,\mathbf{u} - \boldsymbol{\vartheta}\times\mathbf{I}\,, \quad \mathbf{\Gamma} \approx \boldsymbol{\gamma} \equiv \mathrm{Grad}\,\boldsymbol{\vartheta}\,. \tag{16}$$

The infinitesimal strain measures $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$ or their transpose were used in many papers and books in the field of linear Cosserat continuum. Let us mention here the books by Kröner (1968), Eringen (1999), Nowacki (1986), and Dyszlewicz (2004), where many references to other papers can be found.

4 Conclusions

Within the non-linear Cosserat continuum, introduction of the finite rotation vector gives the possibility to formulate the boundary-value problem in terms of displacement and finite rotation vectors as the primary unknown variables. In this note the natural Lagrangian stretch and wryness tensors derived by Pietraszkiewicz and Eremeyev (2009) have been expressed in terms of the general finite rotation vector. These expressions have then been specialized for seven different definitions of the rotation vectors known in the literature. Each of the particular forms of the strain measures has some advantages and drawbacks, and each of them may be more convenient than others in specific applications.

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* Highlighted Revision

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W. Pietraszkiewicz [a,*], V. A. Eremeyev [b]

[a] *Institute of Fluid-Flow Machinery of the Polish Academy of Sciences, ul. Gen. J. Fiszera 14, 80-952 Gdańsk, Poland*

[b] *South Scientific Center of RASci & South Federal University, Milchakova str. 8a, 344090 Rostov on Don, Russia*

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In the recent paper by Pietraszkiewicz and Eremeyev (2009) we applied three different ways of defining the strain measures of the non-linear Cosserat continuum. We found in particular that the most natural definitions for the Lagrangian relative stretch $\mathbf{E}$ and wryness (or change of the microstructure curvature) $\boldsymbol{\Gamma}$ tensors are

$$\mathbf{E} = \mathbf{Q}^T (\mathbf{I} + \operatorname{Grad} \mathbf{u}) - \mathbf{I}, \quad \boldsymbol{\Gamma} = -\frac{1}{2} \boldsymbol{\epsilon} : \left(\mathbf{Q}^T \operatorname{Grad} \mathbf{Q} \right). \tag{1}$$

Here $\mathbf{u} \in E$ is the translation vector, $\mathbf{Q} \in SO(3)$ the proper orthogonal micro-rotation tensor, $\mathbf{I}$ the identity (metric) tensor in the undeformed configuration,

* Corresponding author. Tel: +48 58 6995263; fax: +48 58 3416144.
Email addresses: pietrasz@imp.gda.pl (W. Pietraszkiewicz), eremeyev.victor@gmail.com (V. A. Eremeyev).

$\boldsymbol{\epsilon} = -\mathbf{I} \times \mathbf{I}$ the 3rd-order skew Ricci tensor, $\times$ the vector product, and the double dot product : of two 3rd-order tensors $\mathbf{A}, \mathbf{B}$ represented in the orthonormal base $\mathbf{h}_a$, $a = 1, 2, 3$, is defined as $\mathbf{A} : \mathbf{B} = A_{amn} B_{mnb} \mathbf{h}_a \otimes \mathbf{h}_b$.

The orthonormal vectors $\mathbf{h}_a$ were interpreted by Pietraszkiewicz and Eremeyev (2009) as the natural base vectors of three-orthogonal system of curvilinear arc-length coordinates s_a such that $\mathbf{h}_a = \partial\mathbf{x}/\partial s_a \equiv \mathbf{x},_a$, where $\mathbf{x} \in E$ is the position vector of a material particle in the reference configuration of the body. Then gradients of the vector $\mathbf{v}(\mathbf{x}) \in E$ and 2nd-order tensor $\mathbf{T}(\mathbf{x}) \in E \otimes E$ fields were defined by $\operatorname{Grad} \mathbf{v} = \mathbf{v},_a \otimes \mathbf{h}_a$ and $\operatorname{Grad} \mathbf{T} = \mathbf{T},_a \otimes \mathbf{h}_a$, respectively.

While three components of $\mathbf{u}$ in (1) are all independent, nine components of $\mathbf{Q}$ in (1) are subjected to six constraints following from the orthogonality conditions $\mathbf{Q}^{-1} = \mathbf{Q}^T$, $\det \mathbf{Q} = +1$, so that only three rotational parameters of $\mathbf{Q}$ are independent.

In the literature, many techniques how to parameterize the rotation group $SO(3)$ were developed, see for example Rooney (1977), Guo (1981), Pietraszkiewicz and Badur (1983), Altman (1986), Atluri and Cazzani (1995), Borri et al. (2000), Geradin and Cardona (2001), and Chróścielewski et al. (2004). These parameterizations can roughly be classified as vectorial and non-vectorial ones. Various finite rotation vectors as well as the Cayley-Gibbs and exponential map parameters are examples of the vectorial parameterization, for they all have three independent scalar parameters as Cartesian components of a generalized vector in the 3D vector space E. The non-vectorial parameterizations are expressed either in terms of three scalar parameters that cannot be treated as vector components, such as Euler-type angles for example, or through more scalar parameters subjected to additional constraints, such as unit quaternions, Cayley-Klein parameters, or direction cosines. Each of these parameterizations has some advantages and drawbacks widely discussed in the literature.

The aim of this note is to express the strain measures (1) in terms of seven different vectorial parameters proposed in the literature. Each of these expressions may appear to be more convenient than others when solving specific problems of the non-linear Cosserat continuum.

2 The vectorial parameterization

The microrotation tensor $\mathbf{Q}$ represents the isometric and orientation-preserving transformation of the 3D vector space E into itself. By the Euler theorem such a transformation can be expressed in terms of the angle of rotation ϕ about the axis of rotation described by the eigenvector $\mathbf{e}$ corresponding to the real eigenvalue +1 of $\mathbf{Q}$ such that

$$\mathbf{Q}\mathbf{e} = +\,\mathbf{e}, \quad \cos\phi = \frac{1}{2}\left(\operatorname{tr}\mathbf{Q} - 1\right), \quad \sin\phi\,\mathbf{e} = \frac{1}{2}\operatorname{ax}\left(\mathbf{Q} - \mathbf{Q}^T\right), \tag{2}$$

where $\operatorname{tr}\mathbf{A}$ is the trace of the 2^{nd}-order tensor $\mathbf{A}$, and $\operatorname{ax}\mathbf{W}$ is the axial vector $\mathbf{w}$ of the skew 2^{nd}-order tensor $\mathbf{W}$ such that $\mathbf{W} = \mathbf{w} \times \mathbf{I} = \mathbf{I} \times \mathbf{w}$.

In terms of $\mathbf{e}$ and ϕ the microrotation tensor $\mathbf{Q}$ can be expressed by Gibbs (1901) formula, see for example Beatty (1977), Guo (1981), and Pietraszkiewicz and Badur (1983),

$$\mathbf{Q} = \cos\phi\,\mathbf{I} + (1 - \cos\phi)\,\mathbf{e} \otimes \mathbf{e} + \sin\phi\,\mathbf{e} \times \mathbf{I}. \tag{3}$$

In the vectorial parameterization of $\mathbf{Q}$ one introduces a scalar function $p(\phi)$ generating three components of the finite rotation vector $\mathbf{p}$ defined as, see for example Bauchau and Trainelli (2003),

$$\mathbf{p} = p(\phi)\,\mathbf{e}\,. \tag{4}$$

The generating function $p(\phi)$ in (4) has to be an odd function of ϕ with the limit behaviour $\lim\limits_{\phi \to 0} \frac{p(\phi)}{\phi} = \kappa$, where κ is a positive real normalization factor (usually 1 or $\frac{1}{2}$), and $p(0) = 0$. In terms of (4) the tensor $\mathbf{Q}$ and its transpose can be represented as

$$\begin{aligned} \mathbf{Q} &= \cos\phi\,\mathbf{I} + \frac{1 - \cos\phi}{p^2}\,\mathbf{p} \otimes \mathbf{p} + \frac{\sin\phi}{p}\,\mathbf{p} \times \mathbf{I}\,, \\ \mathbf{Q}^T &= \cos\phi\,\mathbf{I} + \frac{1 - \cos\phi}{p^2}\,\mathbf{p} \otimes \mathbf{p} - \frac{\sin\phi}{p}\,\mathbf{p} \times \mathbf{I}\,. \end{aligned} \tag{5}$$

The finite rotation vector (4) is the generalized vector. The composition of two successive rotations $\mathbf{Q}_3 = \mathbf{Q}_2\mathbf{Q}_1$, when expressed in terms of the corresponding vectors $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{p}_3$ with angles of rotation ϕ_1, ϕ_2, ϕ_3, reads

$$\begin{aligned} \cos\frac{\phi_3}{2} &= \cos\frac{\phi_1}{2}\cos\frac{\phi_2}{2} - \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1 p_2}\,\mathbf{p}_1 \cdot \mathbf{p}_2\,, \\ \frac{\sin\frac{\phi_3}{2}}{p_3}\,\mathbf{p}_3 &= \frac{\sin\frac{\phi_1}{2}\sin\frac{\phi_2}{2}}{p_1 p_2}\left(\frac{p_2}{\cos\frac{\phi_2}{2}}\,\mathbf{p}_1 + \frac{p_1}{\cos\frac{\phi_1}{2}}\,\mathbf{p}_2 - \mathbf{p}_1 \times \mathbf{p}_2\right). \end{aligned} \tag{6}$$

The equation $(6)_1$ is used to compute ϕ_3, which also gives $\sin\frac{\phi_3}{2}$ and $p_3 = p(\phi_3)$. Then $(6)_2$ allows one to establish the vector $\mathbf{p}_3$.

Since $\mathbf{Q}^T\mathbf{Q}_{,a} = -(\mathbf{Q}^T\mathbf{Q}_{,a})^T$ is skew it can be expressed through the axial vector $\boldsymbol{\gamma}_a$,

$$\mathbf{Q}^T\mathbf{Q}_{,a} = \boldsymbol{\gamma}_a \times \mathbf{I}, \quad \boldsymbol{\gamma}_a = -\frac{1}{2}\,\boldsymbol{\epsilon} : \left(\mathbf{Q}^T\mathbf{Q}_{,a}\right) = \phi_{,a}\,\mathbf{e} + \Big[\sin\phi\,\mathbf{I} - (1-\cos\phi)\,\mathbf{e}\times\mathbf{I}\Big]\,\mathbf{e}_{,a}\,. \tag{7}$$

The vector $\boldsymbol{\gamma}_a$ describes the change of the reference microstructure curvature of the Cosserat continuum along the arc-length coordinate line s_a. It is analogous to the vector $\mathbf{k}_j$ of change of curvature of the curvilinear coordinate line θ^j in classical continuum mechanics defined as $\mathbf{R}^T\mathbf{R}_{,j} = \mathbf{k}_j \times \mathbf{I}$ by Pietraszkiewicz and Badur (1983), where $\mathbf{R}$ was the rotation tensor following from the polar decomposition $\mathbf{F} = \mathbf{RU} = \mathbf{VR}$. But in the Cosserat continuum $\mathbf{Q}$ is the independent field not related to $\mathbf{u}$ and therefore $\mathbf{Q} \neq \mathbf{R}$, in general.

Differentiating the vector $\mathbf{p}$ in (4) along the coordinate line s_a we obtain the transformation relations

$$\phi_{,a} = \frac{1}{p'}\,p_{,a}\,, \quad \mathbf{e}_{,a} = -\frac{1}{p^2}\,p_{,a}\,\mathbf{p} + \frac{1}{p}\,\mathbf{p}_{,a}\,, \quad p' = \frac{dp}{d\phi}\,, \tag{8}$$

which introduced into (7) lead to

$$\boldsymbol{\gamma}_a = \frac{1}{p}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)p_{,a}\,\mathbf{p} + \frac{\sin\phi}{p}\,\mathbf{p}_{,a} - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{p}_{,a}\,. \tag{9}$$

Taking into account that $\mathbf{p}\cdot\mathbf{p}_{,a} = pp_{,a}$, we have the identities

$$\mathbf{p} = \frac{1}{pp_{,a}}(\mathbf{p}\otimes\mathbf{p})\,\mathbf{p}_{,a}\,, \quad \mathbf{p}_{,a} = \mathbf{I}\mathbf{p}_{,a}\,, \quad \mathbf{p}\times\mathbf{p}_{,a} = (\mathbf{p}\times\mathbf{I})\,\mathbf{p}_{,a}\,, \tag{10}$$

and the relation (9) can be given in the equivalent form

$$\boldsymbol{\gamma}_a = \mathbf{A}\mathbf{p}_{,a}\,, \quad \mathbf{A} = \frac{\sin\phi}{p}\,\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p}\otimes\mathbf{p} - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{I}\,. \tag{11}$$

Substituting $(5)_2$ and (11) into (1), the natural Lagrangian stretch $\mathbf{E}$ and wryness $\boldsymbol{\Gamma}$ tensors can now be represented in terms of the finite rotation vector $\mathbf{p}$ by the general relations

$$\mathbf{E} = \left(\cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{p^2}\,\mathbf{p}\otimes\mathbf{p} - \frac{\sin\phi}{p}\,\mathbf{p}\times\mathbf{I}\right)(\mathbf{I} + \mathrm{Grad}\,\mathbf{u}) - \mathbf{I}\,, \tag{12}$$

$$\boldsymbol{\Gamma} = \left[\frac{\sin\phi}{p}\,\mathbf{I} + \frac{1}{p^2}\left(\frac{1}{p'} - \frac{\sin\phi}{p}\right)\mathbf{p}\otimes\mathbf{p} - \frac{1-\cos\phi}{p^2}\,\mathbf{p}\times\mathbf{I}\right]\mathrm{Grad}\,\mathbf{p}\,. \tag{13}$$

3 Particular finite rotation vectors

Among definitions of $\mathbf{p}$ used most often in the literature let us mention the finite rotation vectors defined as

$$\boldsymbol{\theta} = 2\tan\frac{\phi}{2}\,\mathbf{e}\,, \quad \boldsymbol{\phi} = \phi\,\mathbf{e}\,, \quad \boldsymbol{\varpi} = \sin\phi\,\mathbf{e}\,, \quad \boldsymbol{\rho} = \tan\frac{\phi}{2}\,\mathbf{e}\,, \tag{14}$$

where the generating functions are $\theta = 2\tan\frac{\phi}{2}$, ϕ, $\varpi = \sin\phi$, and $\rho = \tan\frac{\phi}{2}$, respectively. Within the non-linear Cosserat continuum the Cayley-Gibbs vector $\boldsymbol{\theta}$ was used for example by Shkutin (1980), Badur and Pietraszkiewicz (1986), Zubov (1997), and Nikitin and Zubov (1998), while the linear vector $\boldsymbol{\phi}$ (called also the exponential map) by Kafadar and Eringen (1971), Nistor (2002), and Ramezani and Naghdabadi (2007). The vector $\boldsymbol{\theta}$ was used in the non-linear theory of plates, see for example Hodges et al. (1993), and in the non-linear theory of composite beams by Hodges (2006), where the extensive review of the literature was given. In the non-linear theory of Cosserat-type shells and the classical continuum mechanics the vector $\boldsymbol{\varpi}$ was found to be convenient in papers by Pietraszkiewicz (1979), and Pietraszkiewicz and Badur (1983), while the Rodrigues rotation vector $\boldsymbol{\rho}$ was willingly used in analytical mechanics of rigid-body motion, see for example Pars (1965).

Less popular in the literature till now is the Euler-Rodrigues vector $\boldsymbol{\sigma}$, the Wiener-Milenkovic vector $\boldsymbol{\mu}$, and the Bauchau-Trainelli vector $\boldsymbol{\beta}$ defined by

$$\boldsymbol{\sigma} = 2\sin\frac{\phi}{2}\,\mathbf{e}\,, \quad \boldsymbol{\mu} = 4\tan\frac{\phi}{4}\,\mathbf{e}\,, \quad \boldsymbol{\beta} = 4\sin\frac{\phi}{4}\,\mathbf{e}\,, \tag{15}$$

whose generating functions are $\sigma = 2\sin\frac{\phi}{2}$, $\mu = 4\tan\frac{\phi}{4}$, and $\beta = 4\sin\frac{\phi}{4}$, respectively.

Introducing (14) and (15) into (12) and (13) and using appropriate trigonometric identities, after complex but elementary transformations we obtain the formulae for $\mathbf{E}$ and $\boldsymbol{\Gamma}$ expressed in terms of the corresponding finite rotation vectors. These formulae are given in Tables 1 and 2.

With all the vectorial parameterizations the singularities occur for some values of ϕ following from singularities of the generating functions $p(\phi)$, when $p \to \infty$, from singularities of the inverse relations $\mathbf{p} = \mathbf{p}(\mathbf{Q})$, as well as from singularities of $\mathbf{A}$ and $\mathbf{A}^{-1}$, see Bauchau and Trainelli (2003). Hence, we also indicate in Tables 1 and 2 the ranges of validity of ϕ for the analysis to be singular-free while using these strain measures in problems of the Cosserat continuum. When in applications there appear arbitrary values of the rotation angle ϕ, one needs at least five independent scalar parameters to parameterize the rotation group SO(3) in the globally one-to-one and singular-free manner, see for example Hopf (1940), Stuelpnagel (1964), and Perelyaev (2006). For the finite rotation vectors $\boldsymbol{\mu}$ and $\boldsymbol{\beta}$, Bauchau and Trainelli (2003) described procedures how to handle arbitrary rotations by combining appropriate update and rescaling operations.

With the vectors $\boldsymbol{\theta}$, $\boldsymbol{\rho}$, $\boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}$, $\boldsymbol{\Gamma}$ in Tables 1 and 2 do not contain any trigonometric expressions of ϕ. This might suggest some convenience in further purely algebraic transformations. With the vectors $\boldsymbol{\phi}$, $\boldsymbol{\mu}$, or $\boldsymbol{\beta}$ the formulae for $\mathbf{E}$, $\boldsymbol{\Gamma}$ have broader range of singular-free behaviour. When $|\phi| < \pi$ the values of $\mu(\phi)$ and $\beta(\phi)$ are not much different from ϕ, that is $\mu(\phi) \approx \phi \approx \beta(\phi)$. In the limit the sin-type generating functions ϖ, σ, β converge to ϕ from below, while the tan-type ones θ, ρ, μ, from above.

Table 1
The natural Lagrangian stretch tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\mathbf{E}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\frac{\theta^2}{4}}\left[\left(1-\frac{\theta^2}{4}\right)\mathbf{I} + \frac{1}{2}\boldsymbol{\theta}\otimes\boldsymbol{\theta} - \boldsymbol{\theta}\times\mathbf{I}\right](\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\phi} \equiv \phi\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\otimes\boldsymbol{\phi} - \frac{\sin\phi}{\phi}\boldsymbol{\phi}\times\mathbf{I}\right)(\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\phi\,\mathbf{I} + \frac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\otimes\boldsymbol{\varpi} - \boldsymbol{\varpi}\times\mathbf{I}\right)(\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\frac{1}{1+\rho^2}\left[(1-\rho^2)\mathbf{I} + 2\boldsymbol{\rho}\otimes\boldsymbol{\rho} - 2\boldsymbol{\rho}\times\mathbf{I}\right](\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\left[\left(1-\frac{1}{2}\sigma^2\right)\mathbf{I} + \frac{1}{2}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma} - \cos\frac{\phi}{2}\boldsymbol{\sigma}\times\mathbf{I}\right](\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\frac{1}{\left(1+\frac{\mu^2}{16}\right)^2}\left\{\left[1-\frac{\mu^2}{16}\left(\frac{3}{8}-\frac{\mu^2}{16}\right)\right]\mathbf{I} + \frac{1}{2}\boldsymbol{\mu}\otimes\boldsymbol{\mu} - \left(1-\frac{\mu^2}{16}\right)\boldsymbol{\mu}\otimes\mathbf{I}\right\}(\mathbf{I}$ $+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left\{\left[1-\frac{\beta^2}{2}\left(1-\frac{\beta^2}{16}\right)\right]\mathbf{I} + \frac{1}{2}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\otimes\boldsymbol{\beta}\right.$ $\left.-\sqrt{1-\frac{\beta^2}{16}}\left(1-\frac{\beta^2}{8}\right)\boldsymbol{\beta}\times\mathbf{I}\right\}(\mathbf{I}+\operatorname{Grad}\mathbf{u}) - \mathbf{I}$

When the values of $\mathbf{u}$ and ϕ as well as their spatial gradients are infinitesimal

$$\|\mathbf{u}\| \ll 1\,, \quad \|\operatorname{Grad}\mathbf{u}\| \ll 1\,, \quad |\phi| \ll 1\,, \quad \|\operatorname{Grad}\phi\| \ll 1,$$

we also have $\sin\phi \approx \phi$, $\cos\phi \approx 1$, and $p(\phi) \approx \kappa\phi$. Then from (3), (15) and (14) it follows that

$$\mathbf{p} \approx \kappa\boldsymbol{\vartheta}\,, \quad \mathbf{Q} \approx \mathbf{I} + \boldsymbol{\vartheta}\times\mathbf{I}\,,$$

where $\boldsymbol{\vartheta} = \phi\,\mathbf{e}$ is now the infinitesimal rotation vector. Then from (12) and (13) we

Table 2
The natural Lagrangian wryness tensor for different finite rotations vectors

$\mathbf{p}$	$\phi \in$	$\mathbf{\Gamma}$
$\boldsymbol{\theta} \equiv 2\tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\dfrac{1}{1+\frac{\theta^2}{4}}\left(\mathbf{I} - \dfrac{1}{2}\boldsymbol{\theta}\times\mathbf{I}\right)\operatorname{Grad}\boldsymbol{\theta}$
$\boldsymbol{\phi} \equiv \phi\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left(\dfrac{\sin\phi}{\phi}\mathbf{I} + \dfrac{\phi-\sin\phi}{\phi^3}\boldsymbol{\phi}\otimes\boldsymbol{\phi} - \dfrac{1-\cos\phi}{\phi^2}\boldsymbol{\phi}\times\mathbf{I}\right)\operatorname{Grad}\boldsymbol{\phi}$
$\boldsymbol{\varpi} \equiv \sin\phi\,\mathbf{e}$	$(-\pi, \pi)$	$\left[\mathbf{I} + \dfrac{1}{\varpi^2}\left(\dfrac{1}{\cos\phi} - 1\right)\boldsymbol{\varpi}\otimes\boldsymbol{\varpi} - \dfrac{1-\cos\phi}{\varpi^2}\boldsymbol{\varpi}\times\mathbf{I}\right]\operatorname{Grad}\boldsymbol{\varpi}$
$\boldsymbol{\rho} \equiv \tan\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\dfrac{2}{1+\rho^2}\left(\mathbf{I} - \boldsymbol{\rho}\times\mathbf{I}\right)\operatorname{Grad}\boldsymbol{\rho}$
$\boldsymbol{\sigma} \equiv 2\sin\frac{\phi}{2}\,\mathbf{e}$	$(-\pi, \pi)$	$\left(\cos\dfrac{\phi}{2}\mathbf{I} - \dfrac{1}{4\cos\frac{\phi}{2}}\boldsymbol{\sigma}\otimes\boldsymbol{\sigma} - \dfrac{1}{2}\boldsymbol{\sigma}\times\mathbf{I}\right)\operatorname{Grad}\boldsymbol{\sigma}$
$\boldsymbol{\mu} \equiv 4\tan\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\dfrac{1}{\left(1+\frac{\mu^2}{16}\right)^2}\left[\left(1 - \dfrac{\mu^2}{16}\right)\mathbf{I} + \dfrac{1}{8}\boldsymbol{\mu}\otimes\boldsymbol{\mu} - \dfrac{1}{2}\boldsymbol{\mu}\times\mathbf{I}\right]\operatorname{Grad}\boldsymbol{\mu}$
$\boldsymbol{\beta} \equiv 4\sin\frac{\phi}{4}\,\mathbf{e}$	$(-2\pi, 2\pi)$	$\left[\sqrt{1-\dfrac{\beta^2}{16}}\left(1-\dfrac{\beta^2}{8}\right)\mathbf{I} + \dfrac{1-\left(1-\frac{\beta^2}{8}\right)\left(1-\frac{\beta^2}{16}\right)}{\beta^2\sqrt{1-\frac{\beta^2}{16}}}\boldsymbol{\beta}\otimes\boldsymbol{\beta} - \dfrac{1}{2}\left(1-\dfrac{\beta^2}{16}\right)\boldsymbol{\beta}\times\mathbf{I}\right]\operatorname{Grad}\boldsymbol{\beta}$

obtain

$$\mathbf{E} \approx \boldsymbol{\varepsilon} \equiv \operatorname{Grad}\mathbf{u} - \boldsymbol{\vartheta}\times\mathbf{I}\,, \quad \mathbf{\Gamma} \approx \boldsymbol{\gamma} \equiv \operatorname{Grad}\boldsymbol{\vartheta}\,. \tag{16}$$

The infinitesimal strain measures $\boldsymbol{\varepsilon}$, $\boldsymbol{\gamma}$ or their transpose were used in many papers and books in the field of linear Cosserat continuum. Let us mention here the books by Kröner (1968), Eringen (1999), Nowacki (1986), and Dyszlewicz (2004), where many references to other papers can be found.

4 Conclusions

Within the non-linear Cosserat continuum, introduction of the finite rotation vector gives the possibility to formulate the boundary-value problem in terms of displacement and finite rotation vectors as the primary unknown variables. In this note the natural Lagrangian stretch and wryness tensors derived by Pietraszkiewicz and Eremeyev (2009) have been expressed in terms of the general finite rotation vector. These expressions have then been specialized for seven different definitions of the rotation vectors known in the literature. Each of the particular forms of the strain measures has some advantages and drawbacks, and each of them may be more convenient than others in specific applications.

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